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Posinormal operators

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0. Introduction.

In this paper we study the properties of a large subclass of $\mathscr{B}(\mathscr{H})$, the set of all bounded linear operators $T: \mathscr{H} \rightarrow \mathscr{H}$ on a Hilbert space $\mathscr{H}$. As is customary, we refer to T^*T-TT^* as the *self-commutator of* T, denoted $[T^*, T]$. A self-adjoint operator P is *positive* if $\langle Pf, f\rangle \geqq 0$ for all $f \in \mathscr{H}$; the operator T is *normal* if $[T^*, T]=0$ and T is *hyponormal* if $[T^*, T]$ is positive. When T^* is hyponormal, we say T is *cohyponormal*; T is *seminormal* if T is hyponormal or cohyponormal. If T is the restriction of a normal operator to an invariant subspace, then T is *subnormal*.

If $A \in \mathscr{B}(\mathscr{H})$ is to belong to our subclass, then A must not be "too far" from normal; more precisely, there must exist an *interrupter* $S \in \mathscr{B}(\mathscr{H})$ such that $AA^*=A^*SA$, or equivalently, $[A^*, A]=A^*(I-S)A$. Two observations suggest the additional requirement that S be self-adjoint, even positive: (1) since AA^* is self-adjoint, each operator A in our subclass must satisfy $A^*S^*A=A^*SA$; (2) since $\langle SAf, Af\rangle=\langle A^*SAf, f\rangle=\|A^*f\|^2$ for all f, the interrupter S must be positive on Ran A (the range of A).

DEFINITION. If $A \in \mathscr{B}(\mathscr{H})$, then A is *posinormal* if there exists a positive operator $P \in \mathscr{B}(\mathscr{H})$ such that $AA^*=A^*PA$. $\mathscr{P}(\mathscr{H})$ will denote the set of all posinormal operators on $\mathscr{H}$. A is *coposinormal* if A^* is posinormal.

We note that if A is posinormal with interrupter P and V is an isometry (that is, $V^*V=I$), then, as one can easily check, VAV^* is posinormal with interrupter VPV^*. Consequently, posinormality is a unitary invariant (that is, if A is posinormal and T is unitarily equivalent to A, then T is also posinormal).

If the posinormal operator A is nonzero, the associated interrupter P must satisfy the condition $\|P\| \geqq 1$ since $\|A\|^2=\|AA^*\|=\|A^*PA\| \leqq \|A^*\|\|P\|\|A\|=\|P\|\|A\|^2$. We will make repeated use $\sqrt{P}$, whose existence is guaranteed by the functional calculus for (positive) self-adjoint operators. P need not be unique, as we will soon see; the following result gives a sufficient condition for the uniqueness of P.

THEOREM 0.1. *If A is posinormal with interrupter P and A has dense range, then P is unique.*

PROOF. Assume P_1 and P_2 both serve as interrupters for A. Then $A^*P_1A=AA^*=A^*P_2A$, so $A^*(P_1-P_2)A=0$. Since A has dense range, A^* is one to one and, consequently, $(P_1-P_2)A=0$. We again apply the fact that A has dense range to conclude that $P_1-P_2=0$.

COROLLARY 0.1. *If A has dense range and S serves as an interrupter for A, then A is posinormal and the interrupter S is positive and unique.*

A normal operator is (trivially) posinormal; we will soon see other examples—among them, posinormal operators that are not normal and, in some cases, not even hyponormal. For an example of an operator that is not posinormal, we consider U^*, the adjoint of the unilateral shift $U: l^2 \to l^2$; recall that U has matrix entries

$$u_{jk}=\begin{cases} 0 & \text{if } j\neq k+1 \\ 1 & \text{if } j=k+1\,. \end{cases}$$

Here $A=U^*$ cannot be posinormal since $AA^*=I$ while $A^*PA\neq I$ for all P (the trouble comes in the northwest corner: $0\neq 1$); a different proof will result from Corollary 2.3.

1. Examples.

The example which motivated this study is the Cesàro matrix

$$C=\begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix},$$

regarded as an operator on $\mathcal{H}=l^2$. The standard orthonormal basis for l^2 will be denoted by $\{e_n : n=0, 1, 2, \cdots\}$. If D is the diagonal operator with diagonal $\{(n+1)/(n+2) : n=0, 1, 2, \cdots\}$, then a routine computation verifies that

$$C^*DC = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} = CC^*.$$

So the Cesàro operator (on l^2) is posinormal with interrupter D. C is known to be hyponormal, even subnormal (see [8]). In [2], C is shown to be hyponormal by looking at determinants of finite sections of $[C^*, C]$. We include here a brief and different proof—one that takes advantage of the availability of D.

THEOREM 1.1. *C is hyponormal.*

PROOF. Since $I-D$ is a positive operator, we have $\langle [C^*, C]f, f\rangle = \langle (I-D)Cf, Cf\rangle \geqq 0$ for all f.

We have, in the Cesàro operator, an example of a nonnormal posinormal operator. The next proposition provides us with a large supply of additional examples, including the unilateral shift U; we will see more examples in sections 4 and 5.

PROPOSITION 1.1. *Every unilateral weighted shift with nonzero weights is posinormal.*

PROOF. In matrix form, the weighted shift $A=[a_{jk}]$ with nonzero weights w_k has entries

$$a_{jk} = \begin{cases} 0 & \text{if } j \neq k+1 \\ w_k & \text{if } j = k+1 . \end{cases}$$

Take P to be the diagonal matrix with diagonal entries $p_{00} \geqq 0$, $p_{11}=0$, and $p_{kk} = |w_{k-2}/w_{k-1}|^2$ for $k \geqq 2$. It is routine to verify that $AA^* = A^*PA$, as required. (Note: The freedom possible here for p_{00} illustrates the nonuniqueness of P when A does not have dense range.)

It is easy to see that if A is the unilateral weighted shift with weights w_k, then $[A^*, A]$ is the diagonal matrix with diagonal entries $\{w_0^2, w_1^2 - w_0^2, w_2^2 - w_1^2, \cdots\}$. If $\{w_k\}$ is increasing, then A is hyponormal. The special case when $w_0=2$ and $w_k=1$ for all $k \geqq 1$ provides an example of a posinormal operator that is neither hyponormal nor cohyponormal.

2. Posinormality versus hyponormality.

We have just seen that posinormality does not imply hyponormality, but our experience with the Cesàro matrix and the unilateral shift suggests the plausibility of the reverse implication. The next result, from [4], will help settle the question (see Corollary 2.1).

THEOREM (Douglas). *For $A, B \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent*:

(1) $\operatorname{Ran} A \subseteq \operatorname{Ran} B$;

(2) $AA^* \leqq \lambda^2 BB^*$ *for some* $\lambda \geqq 0$; *and*

(3) *there exists a* $T \in \mathcal{B}(\mathcal{H})$ *such that* $A = BT$.

Moreover, if (1), (2), *and* (3) *hold, then there is a unique operator T such that*

(a) $\|T\|^2 = \inf\{\mu \mid AA^* \leqq \mu BB^*\}$;

(b) $\operatorname{Ker} A = \operatorname{Ker} T$; *and*

(c) $\operatorname{Ran} T \subseteq (\operatorname{Ran} B^*)^-$.

The next result is an indication of the somewhat limited extent to which posinormal operators display behavior associated with hyponormal operators. Recall that a hyponormal operator T must satisfy the inequality $\|T^*f\| \leqq \|Tf\|$ for all f. Statement (a) of the following proposition gives us an analogous result for posinormal operators; this result, together with the above theorem of Douglas, will lead to a characterization of posinormality (see Theorem 2.1).

PROPOSITION 2.1. *If A is posinormal with (positive) interrupter P, then the following statements hold*:

(a) $\|A^*f\| = \|\sqrt{P}Af\| \leqq \|\sqrt{P}\|\|Af\|$ *for every f in* $\mathcal{H}$.

(b) $\|\sqrt{P}A\| = \|A\|$.

PROOF. (a) Since A is posinormal and P is positive, $\|A^*f\|^2 = \langle AA^*f, f\rangle = \langle A^*PAf, f\rangle = \|\sqrt{P}Af\|^2 \leqq \|\sqrt{P}\|^2\|Af\|^2$ for all f in $\mathcal{H}$.

(b) From (a) we see that $\|A^*\| = \|\sqrt{P}A\|$, and $\|A\| = \|A^*\|$ is universal.

We note that if A is posinormal, then condition (2) in the theorem above is satisfied with $\lambda = \|\sqrt{P}\|$ and $B = A^*$. If condition (3) in the theorem holds, then there is an operator $T \in \mathcal{B}(\mathcal{H})$ such that $A = A^*T$, so $A^* = T^*A$; consequently, A is posinormal with interrupter TT^*. Thus Douglas' theorem has led almost immediately to the following result.

THEOREM 2.1. *For $A \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent*:

(1) *A is posinormal*;

(2) $\operatorname{Ran} A \subseteq \operatorname{Ran} A^*$;

(3) $AA^* \leqq \lambda^2 A^* A$ *for some* $\lambda \geqq 0$; *and*

(4) *there exists a* $T \in \mathcal{B}(\mathcal{H})$ *such that* $A = A^* T$.

Moreover, if (1), (2), (3), *and* (4) *hold, then there is a unique operator* T *such that*

(a) $\|T\|^2 = \inf\{\mu \mid AA^* \leqq \mu A^* A\}$;

(b) $\operatorname{Ker} A = \operatorname{Ker} T$; *and*

(c) $\operatorname{Ran} T \subseteqq (\operatorname{Ran} A)^-$.

COROLLARY 2.1. *Every hyponormal operator is posinormal.*

PROOF. If A is hyponormal, then condition (3) is satisfied with $\lambda = 1$.

Let $[A] = \{TA : T \in \mathcal{B}(\mathcal{H})\}$, the left ideal in $\mathcal{B}(\mathcal{H})$ generated by A. If A is posinormal, then, because of (4), we have $A^* = T^* A$ for some bounded operator T, so $A^* \in [A]$. Conversely, if $A^* \in [A]$, then $A^* = KA$ for some $K \in \mathcal{B}(\mathcal{H})$, so A is posinormal with interrupter $P = K^* K$. In summary, we have the following corollary.

COROLLARY 2.2. *A is posinormal if and only if $A^* \in [A]$.*

We note that if A is hyponormal, then for some contraction κ, $A^* = \kappa A$ (see [**3**], p. 3). A straightforward computation shows that in the case of the Cesàro operator the contraction $\kappa = \kappa(C)$ takes the form $\kappa(C) = [k_{mn}]$ where

$$k_{mn} = \begin{cases} \dfrac{1}{n+2} & \text{if } m \leqq n \\ -\dfrac{n+1}{n+2} & \text{if } m = n+1 \\ 0 & \text{if } m > n+1\,. \end{cases}$$

It is not hard to verify that $\kappa(C)^* \kappa(C) = D$.

COROLLARY 2.3. *If A is posinormal, then* $\operatorname{Ker} A \subseteqq \operatorname{Ker} A^*$; *in particular,* $\operatorname{Ker} A$ *is a reducing subspace for the posinormal operator* A.

If we did not already know that U^* fails to be posinormal, we would know now, for $\ker U^* \neq \{0\}$ while $\ker U = \{0\}$. In fact, the adjoint of any unilateral weighted shift with nonzero weights will fail to be posinormal.

COROLLARY 2.4. *In order for a cohyponormal operator A to be posinormal it is necessary that* $\operatorname{Ker} A = \operatorname{Ker} A^*$.

While the Cesàro matrix $C = M_1$ is hyponormal, the remaining p-Cesàro matrices

$$M_p=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \left(\frac{1}{2}\right)^p & \left(\frac{1}{2}\right)^p & 0 & 0 & \cdots \\ \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

where $p>1$ are not (see [**11**]). Here we will use Corollary 2.2 to show that all of these operators are, however, posinormal. Define $B_p=[b_{mn}]$ by

$$b_{mn}=\begin{cases} 1-\left(\dfrac{n+1}{n+2}\right)^p & \text{if } m\leqq n \\ -\left(\dfrac{n+1}{n+2}\right)^p & \text{if } m=n+1 \\ 0 & \text{if } m>n+1 \end{cases}$$

We observe that $B_1=\kappa(C)$. To see that B_p is bounded when $p>1$, we note that this matrix can be decomposed as $B_p=Y+Z$ where $Y=[y_{mn}]$ satisfies $y_{mn}=b_{mn}$ when $m=n+1$ and $y_{mn}=0$ otherwise (so Y is a weighted shift) and Z is the upper triangular matrix whose entries on and above the main diagonal agree with those from B_p and whose other entries are all zero. We note that the entries of Z are all nonnegative. Since $1-(n+1)^p/(n+2)^p<p/(n+2)$ for all $p>1$ (see [**7**, Theorem 42, 2.15.3, page 40]), Z is entrywise dominated by pC^*, an operator known to be bounded; Y is clearly a bounded operator, and consequently B_p is also bounded and $\|B_p\|\leqq\|Y\|+\|Z\|\leqq 1+2p$. A routine computation gives $M_p{}^*=B_pM_p$, and the following theorem has been proved.

THEOREM 2.2. *M_p is posinormal for all $p\geqq 1$.*

We have seen that C is posinormal, but what about C^*? Corollary 2.2 will help us here also, for it can be verified that $C=BC^*$ when $B=C-U^*$, so $C\in[C^*]$; it can also be easily checked that $\kappa(C)B=I=B\kappa(C)$. While B^*B is the interrupter for the posinormal operator C^*, the matrix product in the other order takes on a much simpler form: BB^* is the diagonal matrix with diagonal $\{2, 3/2, 4/3, 5/4, \cdots\}$. These observations justify the next theorem and its corollary.

THEOREM 2.3. *C^* is posinormal with interrupter $P=B^*B=(C^*-U)(C-U^*)$.*

COROLLARY 2.5. $\|C-U^*\|=\sqrt{2}$.

Theorem 2.1 assures us that a hyponormal operator must be posinormal and, consequently, possess an interrupter. The next result gives a necessary and sufficient condition (on the interrupter) and a sufficient condition (on the norm of the interrupter) for a posinormal operator to be hyponormal.

THEOREM 2.4. *Assume A is posinormal with interrupter P.* (a) *Then A is hyponormal if and only if the restriction of $I-P$ to* Ran A *is a positive operator.* (b) *If $\|P\|=1$, then A is hyponormal.*

PROOF. (a) The assertion follows immediately from the fact that $\langle [A^*, A]f, f\rangle = \langle (I-P)Af, Af\rangle$ for all f in $\mathcal{H}$. (b) If $\|P\|=1$, then $\|\sqrt{P}\|=1$ also, so $\|A^*f\| \leqq \|\sqrt{P}\|\|Af\| = \|Af\|$, one of the equivalent conditions for the hyponormality of A (again see [**3**], p. 3).

An alternate approach yields a different proof of (b): If A is not hyponormal, then $\langle (A^*A - A^*PA)f, f\rangle < 0$ for some f not in Ker A. Hence $\|Af\|^2 < \langle PAf, Af\rangle \leqq \|PAf\|\|Af\|$, so $\|PAf\| > \|Af\| > 0$ for some f and consequently $\|P\| > 1$.

COROLLARY 2.6. *If A is hyponormal and has dense range, then the unique interrupter P associated with A must satisfy $\|P\|=1$.*

PROOF. Since A is hyponormal and the range of A is dense, we conclude from Theorem 2.4 (a) that $I-P$ is a positive operator (P is unique by Theorem 0.1). It follows that $\|P\| \leqq \|I\| = 1$, and since $\|P\| \geqq 1$ is universal for nonzero A, the proof is complete.

The next result shows how a positive interrupter P for a hyponormal operator can be used to construct "new" hyponormal operators from an "old" one.

THEOREM 2.5. *Assume A is posinormal with interrupter P. If $I \geqq P$ (that is, $I-P$ is a positive operator) and Q is a positive operator satisfying $I \geqq Q \geqq P$, then* (a) *the operator $Z = \sqrt{Q}A\sqrt{Q}$ is hyponormal and* (b) *$A^*QA - AQA^*$ is a positive operator.*

PROOF. (a)

$$\begin{aligned}[Z^*, Z] &= \sqrt{Q}A^*QA\sqrt{Q} - \sqrt{Q}A^*PA\sqrt{Q} + \sqrt{Q}A^*PA\sqrt{Q} - \sqrt{Q}AQA^*\sqrt{Q} \\ &= \sqrt{Q}A^*(Q-P)A\sqrt{Q} + \sqrt{Q}A(I-Q)A^*\sqrt{Q}.\end{aligned}$$

Therefore $\langle [Z^*, Z]f, f\rangle = \langle (Q-P)A\sqrt{Q}f, A\sqrt{Q}f\rangle + \langle (I-Q)A^*\sqrt{Q}f, A^*\sqrt{Q}f\rangle \geqq 0$ for all f, as needed. The proof of (b) is similar.

For an application of this theorem, we consider the following example. Let Q denote the diagonal matrix with diagonal entries $\{(n+1)a_n : n=0, 1, 2, \cdots\}$

where $a_n=(n+3)/(n+2)^2$. Since $(n+1)/(n+2)\leqq(n+1)a_n\leqq 1$ for all n and $CC^*=C^*DC$, we see that $Z=\sqrt{Q}C\sqrt{Q}$ is hyponormal and $C^*QC-CQC^*$ is positive. The same conclusion holds for the cases when $a_n=\ln(1+1/(n+1))$, $a_n=\sin(1/(n+1))$, and $a_n=\operatorname{Arctan}(1/(n+1))$.

The following result is a corollary to each of Theorem 2.4 (a) and Theorem 2.5 (a) separately. We encountered a special case (Theorem 1.1) earlier.

COROLLARY 2.7. *If A is posinormal with interrupter P and $I\geqq P$, then A is hyponormal.*

In this section we have seen several properties of hyponormal operators which are shared by posinormal operators. One important difference in behavior will emerge soon. Hyponormal operators behave well with respect to scalar multiplication and translation in the following sense: if $\lambda\in\boldsymbol{C}$ and T is hyponormal, then λT and $T+\lambda$ are both hyponormal also. In the next section we will see that a weaker result holds for posinormal operators.

3. Invertibility, translates, and posispectrum.

We start this section by looking at the relationship between invertibility and posinormality. A posinormal operator need not be invertible (example: the unilateral shift), but the following theorem tells us that an invertible operator must be posinormal.

THEOREM 3.1. *Every invertible operator is posinormal.*

PROOF. If A is invertible, then $A^*=A^*(A^{-1}A)=(A^*A^{-1})A$, so $A^*\in[A]$.

COROLLARY 3.1. *Every invertible operator is coposinormal.*

COROLLARY 3.2. *Assume $A\in\mathcal{B}(\mathcal{H})$ and $\lambda\notin\sigma(A)$, the spectrum of A. Then $A-\lambda$ is posinormal.*

In [1] A. Brown introduced a class of operators T satisfying the condition that T^*T commutes with T; these operators have since been referred to as *quasinormal*. Routine computations indicate that if T is quasinormal and $\lambda\neq 0$, then (1) λT is quasinormal but (2) the translate $T+\lambda$ can be quasinormal only if T is normal. As noted previously, the result is different for T hyponormal; in that case both λT and $T+\lambda$ are also hyponormal. The following theorem considers the same questions for posinormal operators.

THEOREM 3.2. *Assume A is posinormal with interrupter P and $\lambda\neq 0$.*

(a) *Then λA is posinormal (with interrupter P).*

(b) *The translate $A+\lambda$ need not be posinormal.*

PROOF. (a) $(\lambda A)(\lambda A)^* = |\lambda|^2 AA^* = |\lambda|^2 A^*PA = (\lambda A)^*P(\lambda A)$.

(b) Consider the case where $A = U^* - 2$ and $\lambda = 2$ (recall that U^* is the adjoint of the unilateral shift). Since 2 is not in $\sigma(U^*)$ (see [5], Problem 82), A is posinormal. But $A+2 = U^*$ is not posinormal.

DEFINITION. For $A \in \mathcal{B}(\mathcal{H})$ the *posispectrum* of A, denoted $p(A)$, is the set $\{\lambda : A-\lambda \text{ is not posinormal}\}$. Corollary 3.2 makes it clear that $p(A)$ is a subset of $\sigma(A)$.

To illustrate, we consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ acting on $\boldsymbol{R}^2$; a straightforward computation shows that, for $\lambda \neq 0$, $A+\lambda$ is posinormal with interrupter

$$P(\lambda) = \begin{bmatrix} 1-|\lambda|^{-2}+|\lambda|^{-4} & -\lambda|\lambda|^{-4} \\ -\bar{\lambda}|\lambda|^{-4} & 1+|\lambda|^{-2} \end{bmatrix}.$$

However, A is not posinormal, so $p(A) = \{0\} = \sigma(A)$. A quick check will verify that $p(A^*) = \{0\}$ also.

PROPOSITION 3.1. *If A is hyponormal, then $p(A) = \emptyset$.*

PROOF. Since translates of a hyponormal operator are hyponormal, $A-\lambda$ is hyponormal and hence posinormal for every λ.

PROPOSITION 3.2. *If U is the unilateral shift, then $p(U) = \emptyset$ and $p(U^*) = \sigma(U^*) = \{\lambda : |\lambda| \leqq 1\}$.*

PROOF. The assertion about $p(U)$ follows from Proposition 3.1. To determine $p(U^*)$ requires more work. For an operator B to satisfy $U-\bar{\lambda} = B(U^*-\lambda)$, it is necessary that $-\lambda b_{00} = -\bar{\lambda}$ and $b_{0k} - \lambda b_{0,k+1} = 0$ for all $k \geqq 0$; consequently, for $\lambda \neq 0$, $b_{0k} = \bar{\lambda}(\lambda^{-k-1})$ for all $k \geqq 0$. This mean that $\|B^*e_0\|^2 = \sum_{k=0}^{\infty} |b_{0k}|^2 = \sum_{k=0}^{\infty} |\lambda|^{-2k} = +\infty$ when $0 < |\lambda| \leqq 1$, which cannot hold for any bounded operator B. We have already seen that U^* is not posinormal, so $\{\lambda : |\lambda| \leqq 1\} = \sigma(U^*) \supseteqq p(U^*) \supseteqq \{\lambda : |\lambda| \leqq 1\}$ and hence $p(U^*) = \{\lambda : |\lambda| \leqq 1\}$.

It can be verified that if $S = S(\lambda)$ is the diagonal matrix with diagonal entries $\{1-|\lambda|^{-2}, 1, 1, 1, \cdots\}$, then S serves as an interrupter for $U+\lambda$ when $\lambda \neq 0$. When $|\lambda| \geqq 1$, S is a positive interrupter; but when $|\lambda| < 1$, $\langle Se_0, e_0 \rangle = 1-|\lambda|^{-2} < 0$ and hence S is not positive. We know by Proposition 3.1 that $U+\lambda$ has a positive interrupter even when $|\lambda| < 1$; since $-\bar{\lambda} \in \pi_0(U^*)$, the point spectrum of U^*, for $|\lambda| < 1$, $\operatorname{Ran}(U+\lambda)$ is not dense, so the nonuniqueness of the interrupter in this case does not surprise us.

COROLLARY 3.3. *The set $\mathcal{P}(\mathcal{H})$ of posinormal operators is not closed in the operator norm topology on $\mathcal{B}(\mathcal{H})$.*

PROOF. Suppose λ_n is a decreasing sequence converging to 1. Then $U^*-\lambda_n$ converges in the operator norm to U^*-1, but U^*-1 is not posinormal while each $U^*-\lambda_n$ is posinormal.

Later in this section we will see that $\mathcal{P}(\mathcal{H})$ is also not an open set in the operator norm topology on $\mathcal{B}(\mathcal{H})$. Meanwhile, we continue computing posispectra.

PROPOSITION 3.3. *If A is a unilateral weighted shift with positive weights w_n such that $w_n \to 0$, then $p(A)=\emptyset$ and $p(A^*)=\{0\}$.*

PROOF. We already know from Proposition 1.1 that A is posinormal. From [**5**, Solution 96] we know that $\sigma(A)=\{0\}$ and, consequently, $A-\lambda$ is posinormal for every $\lambda \neq 0$. Therefore $p(A)=\emptyset$. We now show that $p(A^*)=\{0\}$: Since $\sigma(A^*)=\sigma(A)=\{0\}$, $A^*-\lambda$ is posinormal for every $\lambda \neq 0$; but A^* fails to be posinormal for the same reason U^* does.

For specificity, consider the case when $w_n=1/(n+1)$. Since $\{w_n\}$ is decreasing, A cannot be hyponormal. This example illustrates, since the spectral radius $r(A)=0$ while $\|A\|>0$, that a posinormal operator need not satisfy $\|A\|=r(A)$. This example also demonstrates that a compact posinormal operator need not be normal (or even hyponormal), need not have a compact interrupter (or multiplier), and need not have a posinormal adjoint.

Our two most recent propositions involve the computation of $p(A)$ in cases where A is particularly tame. The next proposition will sometimes aid in computing $p(A)$ in other cases.

PROPOSITION 3.4. (a) *If $\lambda \in \pi_0(A)$ but $\bar{\lambda} \notin \pi_0(A^*)$, then $\lambda \in p(A)$.*
(b) *If $\lambda \in \pi(A)$ but $\bar{\lambda} \notin \pi(A^*)$, then $\lambda \in p(A)$.*

PROOF. (a) Assume $\lambda \in \pi_0(A)$. Then $(A-\lambda)f=0$ for some nonzero $f \in \mathcal{H}$. If $A-\lambda$ were posinormal then we would have $(A-\lambda)^*f=0$ by Corollary 2.3, so $\bar{\lambda} \in \pi_0(A^*)$. (b) If $\lambda \in \pi(A)$, the approximate point spectrum, then there exists a sequence of unit vectors $f_n \in \mathcal{H}$ such that $\|(A-\lambda)f_n\| \to 0$. If $A-\lambda$ were posinormal, then we would have $(A-\lambda)^*=B(A-\lambda)$ for some $B \in \mathcal{B}(\mathcal{H})$; consequently, $\|(A-\lambda)^*f_n\| \leqq \|B\| \|(A-\lambda)f_n\| \to 0$, so $\bar{\lambda} \in \pi(A^*)$.

This result allows us to determine $p(A^*)$ if A is the unilateral weighted shift with weights $w_n=(1+1/(n+1))^2$. From [**5**, Solution 93] we know that $\pi_0(A)=\emptyset$ and $\pi_0(A^*)=\{\lambda : |\lambda| \leqq 1\}=\sigma(A^*)=\sigma(A)$. We apply Proposition 3.4 (a) to conclude that $p(A^*)=\{\lambda : |\lambda| \leqq 1\}$.

Since the posispectrum has turned out to be a closed set in each example presented so far, we might reasonably wonder if $p(A)$ is always closed. Proposition 3.4 (a) will help us, through partial determination of $p(C^*)$ (where C is the Cesàro operator), settle that question negatively. Since $\pi_0(C)=\emptyset$ while $\pi_0(C^*)=\{\lambda: |\lambda-1|<1\}$ (see [2]), we see that $\{\lambda: |\lambda-1|<1\}\subseteq p(C^*)\subseteq \sigma(C^*)=\{\lambda: |\lambda-1|\leqq 1\}$. Since we know that C^* is posinormal, we can improve our claim slightly: $\{\lambda: |\lambda-1|<1\}\subseteq p(C^*)\subseteq\{\lambda: |\lambda-1|\leqq 1\}\setminus\{0\}$; therefore $p(C^*)$ cannot be a closed set. This same example also provides us an opportunity to show that $\mathcal{P}(\mathcal{H})$ is not an open set in the operator norm topology on $\mathcal{B}(\mathcal{H})$; for any ε-ball $B_\varepsilon(C^*)=\{T\in\mathcal{B}(\mathcal{H}): \|C^*-T\|<\varepsilon\}$ contains the nonposinormal operators $C^*-\lambda$ for all real λ in $(0, \min\{\varepsilon, 2\})$.

We say that an operator A is *totally posinormal* if the translates $A+\lambda$ are posinormal for all λ. Proposition 3.1 tells us that any hyponormal operator will be totally posinormal, and Proposition 3.3 gives an example of a nonhyponormal totally posinormal operator. We observe that, as an immediate consequence of Proposition 3.4 (a), any totally posinormal operator A must satisfy $\pi_0(A)^*\subseteq \pi_0(A^*)$.

In [14] Stampfli and Wadhwa studied dominant operators; an operator $A\in \mathcal{B}(\mathcal{H})$ is *dominant* if $\operatorname{Ran}(A-\lambda)\subseteq\operatorname{Ran}(A-\lambda)^*$ for all $\lambda\in\sigma(A)$. As a consequence of Theorem 2.1, we have the following result.

PROPOSITION 3.5. *A is totally posinormal if and only if A is dominant.*

In an earlier paper [15] Wadhwa studied M-hyponormal operators; an operator $A\in\mathcal{B}(\mathcal{H})$ is *M-hyponormal* if there exists a real number M such that $\|(A-\lambda)^*f\|\leqq M\|(A-\lambda)f\|$ for all $f\in\mathcal{H}$ and all complex numbers λ. By [14] and [15], it is known that

$$\begin{aligned}\{\text{subnormal operators}\} &\subseteq \{\text{hyponormal operators}\}\\ &\subseteq \{M\text{-hyponormal operators}\}\\ &\subseteq \{\text{dominant operators}\}.\end{aligned}$$

As a consequence of Theorem 3.1 and Proposition 3.5, we have the following: $\{\text{dominant operators}\}\cup\{\text{invertible operators}\}\subseteq\{\text{posinormal operators}\}$.

It has been noted that any hyponormal operator must be totally posinormal. Among the hyponormal operators, many are noncompact and possess a spectrum with positive planar measure; the Cesàro operator on l^2, for example, has spectrum $\{\lambda: |\lambda-1|\leqq 1\}$. On the other hand, Proposition 3.3 supplies an example of a compact nonhyponormal (in fact, nonseminormal) totally posinormal operator. A natural question at this point would be—Does there exist a nonseminormal, totally posinormal operator whose spectrum has positive planar measure? For the answer, consider the unilateral weighted shift with weights $w_0=1$, $w_1=2$, $w_k=1$ for all $k\geqq 2$ (the notation is that of Proposition 1.1); this example (see [15] for details) settles our question affirmatively because it is M-hyponormal, not seminormal, and similar to the unilateral shift.

Another natural question arises: Does there exist a nonnormal, totally posinormal operator with totally posinormal adjoint? The following example shows the answer is yes. Let $\{f_n\}_{-\infty}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$; define $Tf_n=2^{-|n|}f_{n+1}$ for $-\infty<n<\infty$. T is dominant, codominant, and nonnormal; for details, see [14].

We have seen that posinormality is not preserved under the taking of adjoints. The next theorem is a modest result in the same direction; its proof consists of a straightforward computation and will be omitted.

THEOREM 3.3. *If A is posinormal with an invertible interrupter P, then $B=\sqrt{P}A^*\sqrt{P}$ is posinormal with interrupter P^{-1}.*

We illustrate this theorem with an example. Since the Cesàro operator C on $\mathcal{H}=l^2$ is posinormal with invertible interrupter D, it follows from Theorem 3.3 that $B=\sqrt{D}C^*\sqrt{D}$ is posinormal with interrupter D^{-1}. We note that while C is hyponormal, B cannot be hyponormal since B has dense range (a consequence of the fact that C^* does) and $\|D^{-1}\|=2$ (see Corollary 2.6); in fact, B is cohyponormal (see Theorem 2.5), and hence B^* is posinormal. We also note that, since $\sqrt{D}B^*(\sqrt{D}^{-1})=DC$, the operator B^* is similar to $U^*CU=DC$; results from [12] applied to the terraced matrix $C_2\equiv DC$ allow us to conclude that $\pi_0(B^*)=\emptyset$, $\pi_0(B)=\{\lambda: |\lambda-1|<1\}$, $\sigma(B)=\{\lambda: |\lambda-1|\leqq 1\}$ and $\|B\|=2$ (see section 5 for more information on this and related matrices). By Proposition 3.4 (a), we have $\{\lambda: |\lambda-1|<1\}\subseteqq p(B)\subseteqq\{\lambda: |\lambda-1|\leqq 1\}$. We conclude our discussion of this example with the remark that B^* and DC, although similar, are not unitarily equivalent; justification of that negative claim is left to the reader.

We know that if A is invertible, then both A and A^* will be posinormal; furthermore, A^{-1} and $(A^{-1})^*$ will also be posinormal. The following theorem formalizes these relationships in terms of interrupters; its proof is straightforward and will be omitted.

THEOREM 3.4. *Assume A is invertible. If P serves as the interrupter for the posinormal operator A^*, then* (1) *P is invertible and* (2) *P^{-1} serves as the interrupter for the posinormal operator A^{-1}.*

The remaining results of this section address the following question: since $A-\lambda$ is posinormal for "many" values of λ, to what extent does the interrupter depend on λ? As we see here, if A is to be nonnormal, the dependence of P on λ is rather severe.

THEOREM 3.5. *Assume $A-\lambda$ is posinormal for four distinct complex values*

$\lambda=0$, λ_1, λ_2, and λ_3 where $\lambda_3=\lambda_1\lambda_2/(\lambda_1-\lambda_2)$, and assume that the same positive operator P functions as an interrupter for $A-\lambda$ in each of those four cases. Then A is normal.

PROOF. Since $(A-\lambda)(A-\lambda)^*=(A-\lambda)^*P(A-\lambda)$ for $\lambda=0$, λ_1, λ_2, and λ_3, we find that, for $k=1, 2$, and 3, $(A-\lambda_k)(A-\lambda_k)^*=(A-\lambda_k)^*P(A-\lambda_k)$ reduces to the equation

$$\frac{1}{2}(I-P)=\mathrm{Re}\left[\frac{1}{\lambda_k}(I-P)A\right].$$

Therefore $\mathrm{Re}[((\lambda_1-\lambda_2)/\lambda_1\lambda_2)(I-P)A]=\mathrm{Re}[(1/\lambda_2)(I-P)A]-\mathrm{Re}[(1/\lambda_1)(I-P)A]=0$, from which it follows that $(1/2)(I-P)=\mathrm{Re}[(1/\lambda_3)(I-P)A]=0$, and hence A is normal.

A slight modification of the above proof leads to the following result.

COROLLARY 3.4. *Assume $A-\lambda$ is posinormal for three distinct real values of λ and that the same positive operator P functions as an interrupter for $A-\lambda$ for each of those three values. Then A is normal.*

An even tighter result is obtained when the question is recast in terms of the multiplier B.

THEOREM 3.6. *Assume $A-\lambda$ is posinormal for two distinct values of λ, and assume that the same operator B functions as a multiplier for $A-\lambda$ for both of those values. Then A is normal.*

PROOF. Assume $(A-\lambda_1)^*=B(A-\lambda_1)$ and $(A-\lambda_2)^*=B(A-\lambda_2)$ where $\lambda_1\neq\lambda_2$. Then $(\bar{\lambda}_1-\bar{\lambda}_2)I=(\lambda_1-\lambda_2)B$, so $B=((\bar{\lambda}_1-\bar{\lambda}_2)/(\lambda_1-\lambda_2))I$. Therefore $P=B^*B=I$ serves as an interrupter for $A-\lambda$ when $\lambda=\lambda_1, \lambda_2$; it follows that A is normal.

4. Discrete generalized Cesàro operators.

In this brief section we consider the lower triangular matrices

$$A_\alpha=\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{\alpha}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{\alpha^2}{3} & \frac{\alpha}{3} & \frac{1}{3} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}, \qquad 0\leqq\alpha\leqq 1,$$

regarded as operators on l^2. These operators have been studied in [9, 10]. I-

[9] it was shown that A_α is not hyponormal when $0<\alpha<1$; this is in stark contrast to the fact that both $A_1=C$ and the diagonal operator A_0 are hyponormal (better yet: subnormal and self-adjoint, respectively). Here we will show that these operators all belong together, however, when classified in terms of posinormality. The case $\alpha=0$ is trivial, so we restrict our attention to $0<\alpha\leqq 1$. Define $B_\alpha=[b_{mn}]$ by

$$b_{mn}=\begin{cases}\alpha^{n-m}\left(1-\dfrac{n+1}{n+2}\alpha^2\right) & \text{if } m\leqq n\\ -\dfrac{n+1}{n+2}\alpha & \text{if } m=n+1\\ 0 & \text{if } m>n+1\,.\end{cases}$$

Note that when $\alpha=1$, B_α is the contraction (hence bounded) operator $\kappa(C)$ from section 2. We need to know that B_α is also bounded when $0<\alpha<1$. Once that is determined, a routine computation gives $A_\alpha{}^*=B_\alpha A_\alpha$, settling the question of posinormality for A_α.

To see that B_α is bounded, we consider the decomposition $B_\alpha=Y+Z$ where $Y=[y_{mn}]$ is a weighted shift satisfying $y_{mn}=b_{mn}$ when $m=n+1$ and $y_{mn}=0$ otherwise and Z is the upper triangular matrix whose entries on and above the main diagonal agree with those from B_α and whose other entries are all zero; then $\|Y\|\leqq 1$ and $\|Z\|\leqq\|T_\alpha\|$ where

$$T_\alpha=\begin{bmatrix}1 & \alpha & \alpha^2 & \alpha^3 & \cdots\\ 0 & 1 & \alpha & \alpha^2 & \cdots\\ 0 & 0 & 1 & \alpha & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdots\end{bmatrix}.$$

It remains to show that the Toeplitz matrix T_α or its adjoint is bounded. If $t_n=\alpha^n$ for $n=0, 1, 2, \cdots$, and $t_n=0$ for $n=-1, -2, -3, \cdots$, then $\sum_n |t_n|^2=(1-\alpha^2)^{-1}<\infty$, so the t_n's are Fourier coefficients of a function ϕ in $L^2(0, 1)$; the function ϕ is given by

$$\phi(x)=\sum_{n=0}^{\infty}\alpha^n e^{2\pi i n x}=(1-\alpha e^{2\pi i x})^{-1}\,.$$

Since ϕ is bounded (with $|\phi(x)|\leqq(1-\alpha)^{-1}$ for all x), the matrix $T_\alpha{}^*$ is bounded [see **6**, pp. 24-25].

THEOREM 4.1. *A_α is posinormal for all $\alpha\in[0, 1]$.*

5. Shift-conjugated Cesàro matrices.

In this section we consider the terraced matrix $C_{k+1}=(U^k)^*C(U^k)$ for positive integers k:

$$C_k=\begin{bmatrix} \frac{1}{k} & 0 & 0 & \cdots \\ \frac{1}{k+1} & \frac{1}{k+1} & 0 & \cdots \\ \frac{1}{k+2} & \frac{1}{k+2} & \frac{1}{k+2} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

Visually, C_{k+1} can be obtained from the Cesàro matrix $C=C_1$ by deleting the first k rows and columns from C. We note that in fact for all $k>0$ (and not just the positive integers) the matrix C_k gives a bounded operator on l^2: C_k can be expressed as D_kC where D_k is the diagonal matrix with diagonal $\{(1+n)/(k+n): n=0, 1, 2, \cdots\}$; it is clear by inspection that $\|C_k\|\leqq\|C\|=2$ for $k\geqq 1$ (the proof that $\|C\|=2$ appears in [**2**]); and for $0<k<1$, we have $\|C_k\|=\|D_kC\|\leqq\|D_k\|\|C\|=2/k$. Results from [**12**] and [**13**] justify the remaining assertions of the next theorem.

THEOREM 5.1. *For each $k>0$, C_k is a bounded operator on l^2; $\|C_k\|=2$ when $k\geqq 1$ and $\|C_k\|\leqq 2/k$ when $0<k<1$. Moreover, $\pi_0(C_k)=\emptyset$ unless $k<1$, in which case $\pi_0(C_k)=\{1/k\}$; $\pi_0(C_k^*)=\{\lambda: |\lambda-1|<1\}\cup\{1/k\}$; and $\sigma(C_k)=\{\lambda: |\lambda-1|\leqq 1\}\cup\{1/k\}$.*

We show that, for all $k>0$, C_k is posinormal with interrupter $P=[p_{mn}]$ whose entries are given by

$$p_{mn}=\begin{cases} \dfrac{n^2+(2k+1)n+k^2+1}{(n+k+1)^2} & \text{if } m=n \\ \dfrac{1-k}{(m+k+1)(n+k+1)} & \text{if } m\neq n\,. \end{cases}$$

We comment that working here with interrupters rather than multipliers puts us in better position to investigate hyponormality once posinormality is settled. Note that when $k=1$, P reduces to the diagonal operator D. To see that P is bounded, we observe that P can be decomposed as $P=L+R+L^*$ where R is the diagonal matrix with diagonal from P and L is the lower triangular matrix whose entries below the main diagonal agree with those from P and whose other entries are all zero; then $\|R\|\leqq 1$ and $\|L\|\leqq|k-1|\|C\|=2|k-1|$, so $\|P\|\leqq 1+4|k-1|$.

One can check that $PC_k \equiv [\alpha_{mn}]$ has matrix entries satisfying

$$\alpha_{mn} = \begin{cases} \dfrac{n+1}{(m+k+1)(n+k)} & \text{if } m \geqq n \\ \dfrac{1-k}{(m+k+1)(n+k)} & \text{if } m < n \end{cases};$$

using these entries, it is not hard to verify that $C_k C_k^* = C_k^* P C_k$. In order to see that C_k is posinormal, it remains to show that P is positive; it suffices to show that P_N, the N^{th} finite section of P (involving rows $m=0, 1, \cdots, N$ and columns $n=0, 1, \cdots, N$), has positive determinant for each positive integer N. For columns $n=1, 2, \cdots, N$, we multiply the n^{th} column from P_N by $(k+n+1)/\cdot(k+n)$ and then subtract from the $(n-1)^{st}$ column. Call the new matrix P_N' and note that $\det P_N' = \det P_N$. We now work with the rows of P_N': For $m=1, 2, \cdots, N$, we multiply the m^{th} row from P_N' by $(k+m+1)/(k+m)$ and then subtract from the $(m-1)^{st}$ row. The resulting matrix is tridiagonal and also has the same determinant at P_N; that new matrix is constantly -1 on the two off-diagonals and is almost constantly 2 on the main diagonal — the only exception is the last entry: $(k^2+2Nk+N^2+N+1)/(k+N+1)^2$. To finish our computation, we work this tridiagonal matrix into triangular form: Multiply each row $m=0, 1, \cdots, N-1$ by $(m+1)/(m+2)$ and add to the $(m+1)^{st}$ row. The new matrix is triangular and has diagonal $\{2, 3/2, 4/3, \cdots, (N+1)/N, (N+k^2+1)/\cdot(N+1)(N+k+1)^2\}$; from this we conclude that $\det P_N = (N+k^2+1)/(N+k+1)^2$.

We note that the positivity (and uniqueness) of P could have been demonstrated more briefly using the fact that C_k has dense range; however, our computational procedure provides a springboard for investigating the positivity of $I-P$. To see when $I-P$ is positive, we compute $\det(I-P)_N$ where $(I-P)_N$ is the N^{th} finite section of $I-P$. Following exactly the same sequence of column and row operations we used for P_N, we arrive at a tridiagonal matrix of the following form:

$$Y_N = \begin{bmatrix} d_0 & a_0 & 0 & \cdots & 0 \\ a_0 & d_1 & a_1 & \cdots & 0 \\ 0 & a_1 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & d_{N-1} & a_{N-1} \\ 0 & 0 & \cdots & a_{N-1} & d_N \end{bmatrix}$$

where $a_n = -1/(k+n+1)$, $d_n = (2k+2n+3)/(k+n+1)^2$ $(0 \leqq n \leqq N-1)$, and $d_N = (2k+N)/(N+k+1)^2$. In transforming Y_N into a triangular matrix with the same determinant, we find that the new matrix has diagonal entries δ_n which are given by a recursion formula: $\delta_0=d_0$, $\delta_n=d_n-a_{n-1}{}^2/\delta_{n-1}$ $(1\leqq n\leqq N)$. An induction argument shows that $\delta_n\geqq(n+k+2)/(n+k+1)^2$ for $0\leqq n\leqq N-1$; since d_N departs from the pattern set by the earlier d_n's, δ_N must be handled separately: $\delta_N=d_N-a_{N-1}{}^2/\delta_{N-1}\geqq(k-1)/(N+k+1)^2$. So $\det(I-P)_N=\prod_{j=0}^N\delta_j>0$ for $k>1$.

The computation just completed tells us that C_k is hyponormal when $k>1$. Further calculations (we omit the details) reveal an exact value for the determinant:

$$\det(I-P)_N=\left[\prod_{j=0}^{N}\frac{1}{j+k+1}\right]\left[(k-1)\sum_{j=0}^{N-1}\frac{1}{j+k+1}+\frac{2k+N}{N+k+1}\right].$$

For $k<1$, $\det(I-P)_N$ is eventually negative, so C_k is not hyponormal in this case. We summarize the main results of this section in the following theorem.

THEOREM 5.2. *C_k is posinormal for all $k>0$; C_k is hyponormal if and only if $k\geqq1$.*

The fact that C_k is not hyponormal when $k=1/2$ has a curious consequence, especially in light of our knowledge of the Cesàro matrix.

COROLLARY 5.1. *The matrix*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix},$$

regarded as an operator on $\mathcal{H}=l^2$, is not hyponormal.

Before leaving Theorem 5.2, we note that [**13**, Lemma 4] shows, using an entirely different approach from that presented here, that C_k is hyponormal for $k>1$.

Earlier we saw that the Cesàro operator on l^2 is coposinormal (see Theorem 2.3). The next theorem settles the remaining cases.

THEOREM 5.3. *C_k is coposinormal for all $k>0$.*

PROOF. Define $B=[b_{mn}]$ by

$$b_{mn} = \begin{cases} \dfrac{k}{k+m} & \text{if } n=0 \\ \dfrac{1}{k+m} & \text{if } 0<n\leqq m \\ -1 & \text{if } n=m+1 \\ 0 & \text{if } n>m+1\,. \end{cases}$$

Then $B+U^*$ is a lower triangular matrix with nonnegative entries, all dominated by the corresponding entries from $(\max\{1, k\})C_k$; it follows that $B\in \mathcal{B}(l^2)$. A routine computation verifies that $C_k=BC_k{}^*$, and the proof is complete.

We note that when $0<k<1$, the operator C_k illustrates that it is possible for an operator having spectrum with positive planar measure to be both posinormal and coposinormal without being seminormal.

6. Questions and comments.

In closing, we cite some questions for possible further study.

(1) The development of analytic models for hyponormal and subnormal operators has resulted in some interesting connections with function theory. Is a similar development possible for posinormal operators?

(2) How much more can be said about the posispectrum, and what can be said about operators satisfying $p(A)=p(A^*)$?

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