J. Math. Soc. Japan
Vol. 46, No. 1, 1994

Spectra and geodesic flows on nilmanifolds: Reductions of Hamiltonian systems and differential operators

By Ruishi KUWABARA

(Received Nov. 18, 1991)
(Revised Oct. 9, 1992)

0. Introduction.

Two compact Riemannian manifolds are said to be isospectral if their associated Laplace-Beltrami operators have the same spectra. In 1964 J. Milnor [**15**] first gave an example of a pair of isospectral 16-dimensional flat tori which are not isometric to each other. Later, other examples of pairs of isospectral but not isometric Riemannian manifolds were given by several authors. (See, e. g., the review paper [**2**].) In particular, T. Sunada [**17**] gave a general technique for constructing pairs of isospectral manifolds with a common finite Riemannian covering. In 1984 C. S. Gordon and E. Wilson [**9**] exhibited for the first time continuous one-parameter families of nonisometric, isospectral metrics (isospectral deformations of metrics). Their examples are constructed on solvmanifolds or nilmanifolds, i. e., manifolds whose universal Riemannian coverings are solvable or nilpotent Lie groups with left-invariant metrics. Their idea was further developed, involving Sunada's method, in the subsequent articles [**4**], [**5**].

In this paper we analyze the isospectral deformations on nilmanifolds constructed by Gordon, Wilson and DeTurck from the viewpoint of dynamical systems: classical Hamiltonian systems (geodesic flows) and quantized systems (Laplace-Beltrami operators). A key to our consideration is the reduction procedure of (classical) Hamiltonian systems formulated by Marsden and Weinstein [**14**] and its analog for differential operators (quantum systems). We decompose the system of geodesic flow and its quantization (the Laplace-Beltrami operator) into families of classical and quantum reduced systems, respectively. Another key point is the notion of (pseudo-)restricted-inner transformations of a Lie algebra, which is motivated by the notion of almost-inner derivations by Gordon and Wilson [**9**]. We make clear what occurs in the reduced systems under the deformations of Riemannian metrics generated by a (pseudo-)restricted-

This research was partially supported by Grant-in-Aid for Scientific Research (No. 03640163), Ministry of Education, Science and Culture.

inner transformation.

Section 1 is devoted to reviewing a formulation of dynamical systems on Lie groups with left-invariant metrics. In Section 2 we consider the Hamiltonian system (the system of geodesic flow) on the cotangent bundle over a nilmanifold. We obtain a family of reduced Hamiltonian systems of it by means of the momentum mapping associated to the symplectic action by the center of the nilpotent Lie group. The symplectic forms of the reduced phase spaces are represented in terms of the curvature form of a connection suitably defined on a principal torus bundle over the reduced configuration space. Corresponding to a subfamily of these reduced Hamiltonian systems, we introduce in Section 3 the Schrödinger operators (reduced quantum systems), which are defined by the connection on the principal torus bundle considered in § 2. It is found that the spectrum of the Laplace-Beltrami operator on the original nilmanifold consists of the union of the spectra of these Schrödinger operators on the reduced spaces. In Section 4 we introduce the notion of pseudo-restricted-inner transformations of a Lie algebra. This notion is, in a sense, a generalization of almost-inner derivations, which play a key role in Gordon-Wilson-DeTurck [4], [7], [9] to construct isospectral deformations of metrics. Next in Section 5 we show that each reduced system considered in §§ 2 and 3 is left invariant under the deformation of metric generated by a (pseudo-)restricted-inner transformation (Theorem 5.1 and Proposition 5.4). Finally, as an application of our discussions we give in § 6 an example of non-trivial isospectral deformation of connection on a line bundle over a Riemannian manifold.

1. Dynamical systems on Lie groups.

Let G be a Lie group endowed with a left invariant Riemannian metric. Consider the Hamiltonian dynamical system or the geodesic flow on the cotangent bundle T^*G with the Hamiltonian function defined by the Riemannian metric.

For an element g of G, let $L_g(R_g)$ denote the left (right) translation on G, and set $I_g = L_g \circ R_{g^{-1}}$ (the inner automorphism of G). As the differentials and their dual operators of these diffeomorphisms we define the following linear isomorphisms of the tangent and the cotangent spaces for each $h \in G$:

$$L_{g*}: T_hG \longrightarrow T_{gh}G\,, \qquad R_{g*}: T_hG \longrightarrow T_{hg}G\,,$$

$$L_g^*: T_{gh}^*G \longrightarrow T_h^*G\,, \qquad R_g^*: T_{hg}^*G \longrightarrow T_h^*G\,,$$

$$\mathrm{Ad}(g) = (I_{g*})_e : \mathfrak{g} \longrightarrow \mathfrak{g}\,, \qquad \mathrm{Ad}^*(g) = (I_g^*)_e : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*\,,$$

where $\mathfrak{g} = T_eG$ is the Lie algebra of G and $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$.

Now, we consider the cotangent bundle T^*G, and we have a bundle isomorphism $l: T^*G \to G\times\mathfrak{g}^*$ given by

$$T_h^*G \ni \xi \longmapsto (h,\ L_h^*\xi) \in G\times\mathfrak{g}^*.$$

Then,

$$l\circ L_g^*\circ l^{-1}(h,\ \mu) = (g^{-1}h,\ \mu), \qquad l\circ R_g^*\circ l^{-1}(h,\ \mu) = (hg^{-1},\ \mathrm{Ad}^*(g^{-1})\mu), \tag{1.1}$$

holds for $(h, \mu)\in G\times\mathfrak{g}^*$, and we denote these mappings on $G\times\mathfrak{g}^*$ by the same notations L_g^* and R_g^*. Let θ_0 be the canonical one form on the cotangent bundle T^*G, and set $\omega_0=-d\theta_0$. The two form ω_0 is the natural symplectic structure on T^*G. By virtue of the isomorphism l we define the forms $\theta=(l^{-1})^*\theta_0$ and $\omega=(l^{-1})^*\omega_0$ on $G\times\mathfrak{g}^*$. Thus we have the symplectic manifold $(G\times\mathfrak{g}^*, \omega)$. We call ω the *standard symplectic form* on $G\times\mathfrak{g}^*$.

PROPOSITION 1.1 (cf. [1, p. 315]). *Let* $(g, \mu)\in G\times\mathfrak{g}^*$, *and* $(v, \rho), (w, \sigma)\in T_{(g,\mu)}(G\times\mathfrak{g}^*)=T_gG\times\mathfrak{g}^*$. *Then,*

(1) $\theta(g, \mu)(v, \rho)=\mu(L_{g^{-1}*}v)$.

(2) $\omega(g, \mu)((v, \rho), (w, \sigma))=-\rho(L_{g^{-1}*}w)+\sigma(L_{g^{-1}*}v)+\mu([L_{g^{-1}*}v,\ L_{g^{-1}*}w])$.

Let $G\ni g\mapsto\langle\,,\rangle_g$ be a left-invariant Riemannian metric on G, which is uniquely defined by the inner product $\langle\,,\rangle=\langle\,,\rangle_e$ in $\mathfrak{g}$. Over each $g\in G$ we correspond $\xi\in T_g^*G$ to such $\xi^\#\in T_gG$ that $\xi(v)=\langle\xi^\#, v\rangle_g$ holds for every $v\in T_gG$. The function $H_0(\xi)=\langle\xi^\#, \xi^\#\rangle_g/2$ $(\xi\in T_g^*G)$ on (T^*G, ω_0) defines the Hamiltonian dynamical system whose flow is the geodesic flow. It is obvious that the function H_0 is invariant under the left translation L_g^* for every $g\in G$. Define the function $H=(l^{-1})^*H_0$ on $G\times\mathfrak{g}^*$. Then we have

$$H(g,\ \mu) = \frac{1}{2}\langle\mu,\ \mu\rangle^* := \frac{1}{2}\langle\mu^\#,\ \mu^\#\rangle \tag{1.2}$$

by means of (1.1). Let X_H is the Hamiltonian vector field on $(G\times\mathfrak{g}^*, \omega)$ defined by H, i.e. $\boldsymbol{i}_{X_H}\omega=dH$ ($\boldsymbol{i}_X$: the interior product with respect to X).

PROPOSITION 1.2. *Let* $(g, \mu)\in G\times\mathfrak{g}^*$. *Then,*

$$X_H(g,\ \mu) = (L_{g*}(\mu^\#),\ \mathrm{ad}^*(\mu^\#)\mu)\in T_gG\times\mathfrak{g}^*,$$

where $\mathrm{ad}^*(\mu^\#)$ *is the dual operator of* $\mathrm{ad}(\mu^\#): \mathfrak{g}\to\mathfrak{g}\,;\ w\mapsto[\mu^\#,\ w]$.

PROOF. Direct calculation using Proposition 1.1 and (1.2). ■

Note in the above proposition that the $\mathfrak{g}^*$-component of X_H is independent of $g\in G$.

DYNAMICAL SYSTEMS ON $\Gamma\backslash G$. Suppose G has a uniform discrete subgroup Γ. A left-invariant Riemannian metric on G induces a metric on the compact manifold $M=\Gamma\backslash G$. Corresponding to the isomorphism $T^*G\cong G\times\mathfrak{g}^*$, the isomorphism $T^*M\cong M\times\mathfrak{g}^*$ by left translations by G is induced, and the objects ω, H and X_H on $G\times\mathfrak{g}^*$ are identified with those on $M\times\mathfrak{g}^*$ because they are invariant under any left translation by $\gamma\in\Gamma$.

2. Reductions of the Hamiltonian systems on nilmanifolds.

Let $\mathfrak{g}$ be an n-dimensional nilpotent Lie algebra and $G=\exp\mathfrak{g}$ the corresponding Lie group, which is diffeomorphic to $\boldsymbol{R}^n$. Suppose G has a uniform discrete subgroup Γ, that is equivalent to the existence of a basis of $\mathfrak{g}$ relative to which the structure constants are rational (cf. [3]). We consider the Riemannian manifold $(M=\Gamma\backslash G, \langle\,,\rangle)$, where $\langle\,,\rangle$ is induced from a left invariant metric, also denoted by $\langle\,,\rangle$, on G, and the associated Hamiltonian system $\mathcal{H}=(T^*M\cong M\times\mathfrak{g}^*, \omega, H)$.

The group G has the non-empty center $Z=\exp\mathfrak{z}$, $\mathfrak{z}$ being the center of $\mathfrak{g}$. We put $r=\dim Z$. Let G_1 be the quotient group G/Z with the natural projection $\pi: G\to G_1=G/Z$. Then we have the following.

LEMMA 2.1. (1) *G_1 is a connected and simply connected nilpotent Lie group.*
(2) *$Z\cap\Gamma$ is a uniform discrete subgroup of Z, and $T=Z\cap\Gamma\backslash Z$ is isomorphic with the r-dimensional torus $S^1\times\cdots\times S^1=\{(e^{it_1},\cdots,e^{it_r}); 0\leqq t_j<2\pi\ (j=1,\cdots,r)\}$.*
(3) *$\Gamma_1=\pi(\Gamma)$ is a uniform discrete subgroup of G_1.*

PROOF. (1) is obvious. (2) We denote the inverse of exp by $\log: G\to\mathfrak{g}$. For $u, v\in\mathfrak{z}\cap\log\Gamma$, we have

$$u-v\in\mathfrak{z}\cap\log\Gamma$$

by the Campbell-Baker-Hausdorff formula. Hence, $\mathfrak{z}\cap\log\Gamma$ is a cocompact lattice in $\mathfrak{z}$, and we have the assertion. (3) This follows from the fact that $Z\cap\Gamma$ is a uniform discrete subgroup of Z (cf. [3, Lemma 5.1.4]). ■

By virtue of the above lemma we have a principal torus bundle

$$\hat{\pi}: M=\Gamma\backslash G\longrightarrow M_1=\Gamma_1\backslash G_1 \tag{2.1}$$

with the fiber T corresponding to $\pi: G\to G_1=G/Z$.

Now we consider the Hamiltonian system $\mathcal{H}$ and its reduction following Marsden-Weinstein [14]. Let $\Phi: Z\times(M\times\mathfrak{g}^*)\to M\times\mathfrak{g}^*$ be the action of Z on $M\times\mathfrak{g}^*$ defined by

$$\Phi_k([g],\mu)=L^*_{k^{-1}}([g],\mu)=([kg],\mu),$$

for $k\in Z$, $[g]\in M=\Gamma\backslash G$ with $g\in G$, and $\mu\in\mathfrak{g}^*$. Here we recall that $kg=gk$ for every $k\in Z$ and $g\in G$. It is easily shown that the action Φ is symplectic,

i. e., $\Phi_k^*\omega=\omega$ holds. Let $\mathfrak{z}^*$ be the dual space of the center $\mathfrak{z}$. Both $\mathfrak{z}$ and $\mathfrak{z}^*$ are isomorphic with $\boldsymbol{R}^r$.

LEMMA 2.2. *An* Ad*-*equivariant momentum mapping* $J: M\times\mathfrak{g}^*\to\mathfrak{z}^*$ *for the symplectic action* Φ *is given by*

$$J([g],\ \mu)(v)=\mu(\mathrm{Ad}(g^{-1})v)=\mu(v)$$

for $v\in\mathfrak{z}\subset\mathfrak{g}$.

PROOF. For each $v\in\mathfrak{z}$ let v^* denote the corresponding vector field on $M\times\mathfrak{g}^*$ relative to the action Φ. Let $\hat{J}$ is the dual map from $\mathfrak{z}$ to the space of smooth functions on $M\times\mathfrak{g}^*$. Then

$$d(\hat{J}(v))=\boldsymbol{i}_{v*}\omega \qquad (v\in\mathfrak{z}),$$

$$J\circ\Phi_k=\mathrm{Ad}^*(k^{-1})\circ J \qquad (k\in Z)$$

must hold. This is easily checked by straightforward calculations. ■

We construct the family of reduced phase spaces corresponding to the momentum mapping J. We notice that every $\kappa\in\mathfrak{z}^*$ is a regular value of J, and consider the submanifold

$$\begin{aligned}J^{-1}(\kappa)&=\{([g],\ \mu)\in M\times\mathfrak{g}^*;\ \mu(v)=\kappa(v)\quad\text{for all } v\in\mathfrak{z}\}\\&=\{([g],\ \mu_0+\mu_1)\in M\times\mathfrak{g}^*;\ \mu_1\in\mathfrak{z}^\perp\},\end{aligned}$$

where μ_0 is a fixed vector in $\mathfrak{g}^*$ such that $\mu_0(v)=\kappa(v)$ for all $v\in\mathfrak{z}$, and

$$\mathfrak{z}^\perp=\{\mu_1\in\mathfrak{g}^*;\ \mu_1(v)=0\quad\text{for all } v\in\mathfrak{z}\}.$$

The isotropy group $Z_\kappa=\{k\in Z;\ \mathrm{Ad}^*(k)\kappa=\kappa\}$ is nothing but Z, and the reduced phase space $P_\kappa=J^{-1}(\kappa)/Z_\kappa$ is given by

$$P_\kappa=\{([g_1],\ \mu_0+\mu_1);\ [g_1]\in M_1=\Gamma_1\backslash G_1,\ \mu_1\in\mathfrak{z}^\perp\}. \tag{2.2}$$

The symplectic form ω_κ on P_κ is defined as a unique two form which satisfies

$$p_\kappa^*\omega_\kappa=i_\kappa^*\omega\,,$$

where $p_\kappa: J^{-1}(\kappa)\to P_\kappa$ is the natural projection and $i_\kappa: J^{-1}(\kappa)\to M\times\mathfrak{g}^*$ is the inclusion map. Noticing that $T_{([g_1],\mu)}P_\kappa\cong T_{[g_1]}M_1\times\mathfrak{z}^\perp=T_{g_1}G_1\times\mathfrak{z}^\perp$ ($\mu=\mu_0+\mu_1$), we take its two elements $\rho=(L_{g_1*}v_1,\ \nu)$ and $\sigma=(L_{g_1*}w_1,\ \tau)$ with $v_1,\ w_1\in\mathfrak{g}_1$ (the Lie algebra of G_1). Then we obtain from Proposition 1.1

$$\omega_\kappa([g_1],\ \mu)(\rho,\ \sigma)=-\nu(w)+\tau(v)+\mu([v,\ w]), \tag{2.3}$$

where v and w are elements in $\mathfrak{g}$ satisfying $\pi_*(v)=v_1$ and $\pi_*(w)=w_1$, respectively. Recall that ω_κ given by (2.3) is well-defined without depending on the choice of v and w.

The Hamiltonian H, conserved under the action of Z, induces the Hamiltonian H_κ on P_κ as

$$H_\kappa([g_1], \mu) = \frac{1}{2}\langle\mu, \mu\rangle^* . \tag{2.4}$$

Next, let us take the direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{z} \oplus W \tag{2.5}$$

as vector spaces such that the subspaces $\mathfrak{z}$ and W are orthogonal to each other with respect to the inner product $\langle , \rangle$ in $\mathfrak{g}$. Each vector $v \in \mathfrak{g}$ is written as $v = v_\mathfrak{z} + v_W$ with $v_\mathfrak{z} \in \mathfrak{z}$, $v_W \in W$. Corresponding to this decomposition we have a decomposition of $\mathfrak{g}^*$:

$$\mathfrak{g}^* = \mathfrak{z}^\perp \oplus W^\perp .$$

Noticing that $\mathfrak{z}^* \cong \mathfrak{g}^*/\mathfrak{z}^\perp$, we identify $\mathfrak{z}^*$ with the subspace $W^\perp$ of $\mathfrak{g}^*$ and have

$$\mathfrak{g}^* = \mathfrak{z}^\perp \oplus \mathfrak{z}^* .$$

Put $\mu_0 = \kappa \in \mathfrak{z}^* (= W^\perp)$, and we have

$$P_\kappa = \{([g_1], \kappa + \mu_1) \in M_1 \times \mathfrak{g}^* ; \mu_1 \in \mathfrak{z}^\perp\} .$$

From the projection $\pi : G \to G_1 = G/Z$ we have isomorphisms of vector spaces, $\pi^* : \mathfrak{g}_1^* \to \mathfrak{z}^\perp \subset \mathfrak{g}^*$ and $\pi_*|_W : W \to \mathfrak{g}_1$. Thus we have a diffeomorphism

$$\Psi_\kappa : P_\kappa \longrightarrow T^*M_1 \cong M_1 \times \mathfrak{g}_1^* ; ([g_1], \kappa + \mu_1) \longmapsto ([g_1], \mu_1) . \tag{2.6}$$

The decomposition (2.5) defines a connection $\tilde{\nabla}$ on the principal torus bundle (2.1) by giving the distribution of horizontal spaces $\mathrm{Hor}([g]) = L_{g*}(W)$ $(g \in G)$. Then, we have the following (see, e.g. [**10**]).

LEMMA 2.3. (1) *The connection form $\tilde{\theta}$ of $\tilde{\nabla}$ is a $\mathfrak{z}$-valued one form on M given by*

$$\tilde{\theta}([g])(X) = (L_{g^{-1}*}X)_\mathfrak{z} = \sum_{k=1}^{r} \langle L_{g^{-1}*}X, v_k \rangle v_k ,$$

where $X \in T_{[g]}M$ and $\{v_k ; 1 \leq k \leq r\}$ is an orthonormal basis of $\mathfrak{z}$.

(2) *The curvature form $\tilde{\Theta}$ of $\tilde{\nabla}$ is a $\mathfrak{z}$-valued two form on M given by*

$$\tilde{\Theta}([g])(L_{g*}v, L_{g*}w) = -[v_W, w_W]_\mathfrak{z} = -[v, w]_\mathfrak{z}$$

for $v, w \in \mathfrak{g}$.

REMARK. If $\mathfrak{g}$ is two-step nilpotent, i.e., $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$, then the curvature $\tilde{\Theta}$ is determined only by the Lie algebra structure not depending on the inner product of $\mathfrak{g}$.

Take a local section s of the bundle (2.1), and put $\Theta=s^*\tilde{\Theta}$, which is a $\mathfrak{z}$-valued two form on an open set of M_1. Then, it is easy to see that Θ is given independently on s by

$$\Theta([g_1])(L_{g_1*}v_1, L_{g_1*}w_1) = -[v, w]_{\mathfrak{z}}$$

for v_1, $w_1 \in \mathfrak{g}_1$, where v and w are vectors in $\mathfrak{g}$ such that $\pi_*(v)=v_1$ and $\pi_*(w)=w_1$, respectively. Thus Θ is a two form globally defined on M_1, and we call it also the curvature form of $\tilde{\nabla}$.

Let $\hat{\Theta}$ be the pull-back of Θ to $M_1\times\mathfrak{g}_1^*$ by the projection $M_1\times\mathfrak{g}_1^*\to M_1$. The formula (2.3) is rewritten as

$$\omega_\kappa([g_1], \mu)(\rho, \sigma) = -\nu(w)+\tau(v)+\kappa([v, w]_{\mathfrak{z}})+\mu_1([v, w]_W),$$

where $\mu=\kappa+\mu_1$. Hence, the symplectic form ω_κ on P_κ is transformed by (2.6) to the two form $\Omega_\kappa^{(1)}$ on $M_1\times\mathfrak{g}_1^*$ given by

$$\Omega_\kappa^{(1)} = \omega^{(1)}-\kappa\hat{\Theta}\,, \tag{2.7}$$

where $\omega^{(1)}$ is the standard symplectic form on $M_1\times\mathfrak{g}_1^*$. The term $\kappa\hat{\Theta}$ induces a "magnetic field" term in the equations of motion on M_1 (see, e. g. [**16**]).

Let us introduce the inner product $\langle\,,\rangle_1$ in $\mathfrak{g}_1$ such that $\pi_*|_W: W\to\mathfrak{g}_1$ is an isometry. Then the Hamiltonian H_κ on P_κ is transformed to $H_\kappa^{(1)}$ on $M_1\times\mathfrak{g}_1^*$ as

$$H_\kappa^{(1)}([g_1], \mu_1) = \frac{1}{2}\langle\mu_1, \mu_1\rangle_1^*+\frac{1}{2}\langle\kappa, \kappa\rangle^*\,, \tag{2.8}$$

where $\langle\,,\rangle_1^*$ is the inner product in $\mathfrak{g}_1^*$ naturally introduced from $\langle\,,\rangle_1$ in $\mathfrak{g}_1$. As a consequence we have the following.

PROPOSITION 2.4. *The reduced Hamiltonian system* $(P_\kappa, \omega_\kappa, H_\kappa)$ $(\kappa\in\mathfrak{z}^*)$ *is isomorphic with* $\mathcal{H}_\kappa^{(1)}=(M_1\times\mathfrak{g}_1^*, \Omega_\kappa^{(1)}, H_\kappa^{(1)})$.

REMARK. $\mathcal{H}_0^{(1)}$ is just the standard Hamiltonian system (the system of geodesic flow) over the Riemannian manifold $(M_1, \langle\,\rangle_1)$.

3. Reductions of the Laplacian on nilmanifolds.

The quantum object corresponding to the Hamiltonian system $\mathcal{H}=(M\times\mathfrak{g}^*, \omega, H)$ is the Laplace-Beltrami operator Δ on the Riemannian manifold $(M, \langle\,,\rangle)$. As a quantum version of the Marsden-Weinstein reduction of $\mathcal{H}$, we consider a "reduction" of the Laplacian on $(M, \langle\,,\rangle)$ to the Schrödinger operators which are associated with the reduced classical Hamiltonian systems.

We introduce the line bundles associated with the principal torus bundle (2.1). Let Λ^* be the lattice in $\mathfrak{z}^*$ dual to $\Lambda=\mathfrak{z}\cap\log\Gamma$, i. e.,

$$\Lambda^* = \{\lambda \in \mathfrak{z}^*;\ \lambda(v)/2\pi \in \boldsymbol{Z} \quad \text{for any } v \in \Lambda\}.$$

For each $\lambda \in \Lambda^*$ let $\rho_\lambda : T \to \boldsymbol{C}^* = \boldsymbol{C} \setminus \{0\}$ be the representation of $T = Z \cap \Gamma \backslash Z$ on $\boldsymbol{C}$ defined by

$$\rho_\lambda(t) = e^{i\lambda(v)},$$

where $t = [\exp v] \in T$ with $v \in \mathfrak{z}$. Let $\pi_\lambda : E_\lambda \to M_1$ be the complex line bundle associated with the principal torus bundle (2.1) by the representation ρ_λ, that is, the quotient manifold of $M \times \boldsymbol{C}$ with respect to the equivalence relation

$$([g], z) \overset{\lambda}{\sim} ([gk], \rho_\lambda(-[k])z) \qquad (k \in Z,\ z \in \boldsymbol{C}).$$

For each $[g] \in M$, define $\chi_{[g]} : \boldsymbol{C} \to \pi_\lambda^{-1}(\pi([g])) \subset E_\lambda$ by $z \to [([g], z)]_\lambda$, and $\chi_{[g]}$ is a $\boldsymbol{C}$-linear isomorphism. Each fibre $\pi_\lambda^{-1}([g_1])$ ($[g_1] \in M_1$) of E_λ is endowed with the metric such that $\chi_{[g]} : \boldsymbol{C} \to \pi_\lambda^{-1}([g_1])$ is an isometry for $[g] \in M$ with $\hat{\pi}([g]) = [g_1]$. Thus, E_λ is a Hermitian line bundle. Let $C_\lambda^\infty(M)$ denote the set consisting of every C^∞ function f on M such that

$$f([gk]) = \rho_\lambda([k]) f([g]) \tag{3.1}$$

for every $[k] \in T$, which is called an equivariant function with respect to ρ_λ. For $s \in C^\infty(E_\lambda)$ (the space of C^∞ sections of E_λ), define a C^∞ function $\tilde{s} = \chi_\lambda(s)$ on M by $\tilde{s}([g]) = \chi_{[g]}^{-1}(s(\pi([g])))$. Then $\tilde{s}$ belongs to $C_\lambda^\infty(M)$ and χ_λ gives a one-to-one correspondence between $C^\infty(E_\lambda)$ and $C_\lambda^\infty(M)$.

The connection $\tilde{\nabla}$ on the principal bundle (2.1) induces the linear connection $\tilde{\nabla}^{(\lambda)}$ on E_λ which is defined as the covariant derivative:

$$\tilde{\nabla}_X^{(\lambda)} s = \chi_\lambda^{-1}(\tilde{X}\tilde{s}),$$

$s \in C^\infty(E_\lambda)$, X being a vector field on M_1 and $\tilde{X}$ the horizontal lift of X to M. Let us take a local trivialization of the bundle (2.1): $\hat{\pi}^{-1}(U) \cong U \times T$, U being an open set of M_1. Let F_0 be the local section defined by

$$F_0(x_1) = (x_1, e) \qquad (x_1 \in U),$$

e being the identity of T, and let s_0 be the local section of E_λ defined by

$$s_0(x_1) = [(F_0(x_1), 1)]_\lambda.$$

Let $\tilde{\theta}$ and Θ be the connection and the curvature forms of $\tilde{\nabla}$, respectively (see § 2). Then the following is easy to check.

LEMMA 3.1. (1) *The connection form $\theta^{(\lambda)}$ of $\tilde{\nabla}^{(\lambda)}$ with respect to the section s_0, i.e., $\tilde{\nabla}_X^{(\lambda)} s_0 = \theta^{(\lambda)}(X) s_0$ holds, is given by*

$$\theta^{(\lambda)} = \rho_{\lambda *} F_0^* \tilde{\theta} = i\lambda F_0^* \tilde{\theta}.$$

(2) *The curvature form,* $\Theta^{(\lambda)}(X, Y)=[\tilde{\nabla}_X^{(\lambda)}, \tilde{\nabla}_Y^{(\lambda)}]-\tilde{\nabla}_{[X,Y]}^{(\lambda)}$, *of* $\tilde{\nabla}^{(\lambda)}$ *is given by*

$$\Theta^{(\lambda)}=i\lambda\Theta .$$

(3) *The connection* $\tilde{\nabla}^{(\lambda)}$ *is compatible with the Hermitian structure in* E_λ.

From the connection $\tilde{\nabla}^{(\lambda)}$ on E_λ and the Riemannian metric $m_1=\langle\,,\rangle_1$ on M_1 we can naturally define a differential operator $L^{(\lambda)}$ called the *Bochner-Laplacian,* which is a second order, non-negative, formally self-adjoint elliptic operator acting on $C^\infty(E_\lambda)$ (see, e.g., [**11**]). A local expression of $L^{(\lambda)}$ is given by

$$L^{(\lambda)}=-\sum_{j,k} m_1^{jk}(\nabla_j+ia_j^{(\lambda)})(\nabla_k+ia_k^{(\lambda)}),$$

where ∇ is the Levi-Civita connection defined by m_1, and $\theta^{(\lambda)}=i\sum a_j^{(\lambda)}dx^j$. As the quantum object corresponding to the reduced Hamiltonian system $\mathcal{H}_\lambda^{(1)}=(M_1\times\mathfrak{g}_1^*, \Omega_\lambda^{(1)}, H_\lambda^{(1)})$ we take the differential operator $D^{(\lambda)}$ on E_λ (the Schrödinger operator with a "magnetic" vector potential) given by

$$D^{(\lambda)}=L^{(\lambda)}+|\lambda|^2 .$$

Note that $D^{(0)}$ is just the Laplace-Beltrami operator on $(M_1, \langle\,,\rangle_1)$.

We pay attention to the spectrum of the Laplace-Beltrami operator Δ on $(M, \langle\,,\rangle)$ (denoted by $\mathrm{Spec}(\Delta)$) and that of $D^{(\lambda)}$ on E_λ (denoted by $\mathrm{Spec}(D^{(\lambda)})$). Take a set of C^∞ vector fields $\{X_1, \cdots, X_p\}$ ($p=\dim M_1$) defined on a neighborhood of $x_1\in M_1$ such that $\langle X_j, X_k\rangle_1(x_1)=\delta_{jk}$. Then,

$$(L^{(\lambda)}s)(x_1)=-\Big[\sum_{j=1}^p(\tilde{\nabla}_{X_j}^{(\lambda)})^2 s\Big](x_1)=-\Big[\sum_{j=1}^p\chi_\lambda^{-1}(\tilde{X}_j^2\tilde{s})\Big](x_1)$$

for $s\in C^\infty(E_\lambda)$ with $\chi_\lambda(s)=\tilde{s}\in C_\lambda^\infty(M)$. If we set $\tilde{L}^{(\lambda)}=\chi_\lambda\circ L^{(\lambda)}\circ\chi_\lambda^{-1}$, $\tilde{L}^{(\lambda)}$ is a differential operator acting on $C_\lambda^\infty(M)$ and

$$\tilde{L}^{(\lambda)}=-\sum_{j=1}^p\tilde{X}_j^2=\Delta+\sum_{k=1}^r Z_k^2 ,$$

where $\{Z_k\,;1\leqq k\leqq r\}$ is an orthonormal system of vector fields on M defined by $Z_k([g])=L_{g*}v_k$ ($g\in G$) with $\{v_k\,;1\leqq k\leqq r\}$ being an orthonormal basis of $\mathfrak{z}$. For $\tilde{s}\in C_\lambda^\infty(M)$ we have from (3.1)

$$Z_k\tilde{s}=i\lambda(v_k)\tilde{s} ,$$

hence,

$$\tilde{L}^{(\lambda)}\tilde{s}=\Delta\tilde{s}-\sum_{k=1}^r\{\lambda(v_k)\}^2\tilde{s}=\Delta\tilde{s}-|\lambda|^2\tilde{s} .$$

Noticing that $\chi_\lambda: C^\infty(E_\lambda)\to C_\lambda^\infty(M)$ is $\boldsymbol{C}$-linear isomorphism, we get the following.

LEMMA 3.2. *The equation* $D^{(\lambda)}s=cs$ ($c\in\boldsymbol{C}$) *holds for* $s\in C^\infty(E_\lambda)$ *if and only* *if* $\Delta\tilde{s}=c\tilde{s}$ *holds for* $\tilde{s}=\chi_\lambda(s)\in C^\infty_\lambda(M)$.

The spaces $C^\infty(M)$ and $C^\infty(E_\lambda)$ are naturally endowed with inner products, and we denote their completions by $L^2(M)$ and $L^2(E_\lambda)$, respectively. Then,

LEMMA 3.3. (1) *The map* $\chi_\lambda: C^\infty(E_\lambda)\to C^\infty_\lambda(M)$ *is bi-continuous.*
(2) *The set* $\bigoplus_{\lambda\in\Lambda^*}C^\infty_\lambda(M)$ *is dense in* $L^2(M)$.

PROOF. (1) is obvious. (2) follows directly from the theory of Fourier series. ■

By means of Lemmas 3.2 and 3.3 we have the following.

PROPOSITION 3.4. $\mathrm{Spec}(\Delta)=\bigcup_{\lambda\in\Lambda^*}\mathrm{Spec}(D^{(\lambda)})$.

4. Pseudo-restricted-inner transformations of Lie algebras.

Let G, Γ, Z, $\mathfrak{g}$, $\mathfrak{z}$, $\cdots$ etc. be same as in §§ 2 and 3. For each $\kappa\in\mathfrak{z}^*(\cong\mathfrak{g}^*/\mathfrak{z}^\perp)\subset\mathfrak{g}^*$, we set

$$\mathfrak{z}_\kappa^\perp=\begin{cases}\{\kappa+\nu\,;\,\nu\in\mathfrak{z}^\perp\} & (\kappa\neq 0)\\ \mathfrak{z}^\perp\setminus\{0\} & (\kappa=0)\end{cases}$$

which is a subset of $\mathfrak{g}^*$.

DEFINITIONS AND NOTATION. A linear transformation ϕ of $\mathfrak{g}$, i.e., $\phi\in\mathfrak{gl}(\mathfrak{g})$, is called a *pseudo-restricted-inner transformation of* $\mathfrak{g}$ *relative to* $\kappa\in\mathfrak{z}^*$ if there exists a smooth map $Y_\kappa:\mathfrak{z}_\kappa^\perp\to\mathfrak{g}$ which satisfies

$$\phi^*\mu=\mathrm{ad}^*(Y_\kappa(\mu))\mu \qquad (\mu\in\mathfrak{z}_\kappa^\perp) \tag{4.1}$$

for the dual operator $\phi^*:\mathfrak{g}^*\to\mathfrak{g}^*$ of ϕ, and

$$\nu\{\tau(Y_\kappa(\mu))\}-\tau\{\nu(Y_\kappa(\mu))\}=0 \tag{4.2}$$

for every constant vector field ν, $\tau:\mathfrak{z}_\kappa^\perp\to\mathfrak{z}^\perp$; $\nu(\mu)=\nu$, $\tau(\mu)=\tau$. If, in particular, the map Y_κ can be taken to be constant on $\mathfrak{z}_\kappa^\perp$, we call ϕ a *restricted-inner transformation of* $\mathfrak{g}$ *relative to* κ.

We denote by $\mathrm{PRIT}(\mathfrak{g}\,;\kappa)$ (resp. $\mathrm{RIT}(\mathfrak{g}\,;\kappa)$) the set of all pseudo-restricted-inner (resp. restricted-inner) transformations of $\mathfrak{g}$ relative to κ. Furthermore, for a subset S of $\mathfrak{z}^*$ we set

$$\mathrm{PRIT}(\mathfrak{g}\,;S)=\bigcap_{\kappa\in S}\mathrm{PRIT}(\mathfrak{g}\,;\kappa),\qquad \mathrm{RIT}(\mathfrak{g}\,;S)=\bigcap_{\kappa\in S}\mathrm{RIT}(\mathfrak{g}\,;\kappa),$$

each element of which we call a *pseudo-restricted-inner transformation* and a *restricted-inner transformation*, respectively, of $\mathfrak{g}$ *relative to* S. In particular, put $\mathrm{PRIT}(\mathfrak{g})=\mathrm{PRIT}(\mathfrak{g}\,;\mathfrak{z}^*)$ (resp. $\mathrm{RIT}(\mathfrak{g})=\mathrm{RIT}(\mathfrak{g}\,;\mathfrak{z}^*)$), which is called the set of *pseudo-restricted-inner* (resp. *restricted-inner*) *transformations of* $\mathfrak{g}$.

Gordon-Wilson [9] and Gordon [7] introduced the notion of almost-inner derivations: An derivation ϕ of $\mathfrak{g}$ is called an *almost-inner derivation relative to* Γ if $\phi(X)\in[\mathfrak{g}, X]$ for any $X\in\log\Gamma$. We denote by $\mathrm{AID}(\mathfrak{g}\,;\Gamma)$ the set of all almost-inner derivations of $\mathfrak{g}$ relative to Γ, which is a Lie subalgebra of the Lie algebra consisting of all derivations of $\mathfrak{g}$. In particular, put $\mathrm{AID}(\mathfrak{g})=\mathrm{AID}(\mathfrak{g}\,;G)$, each element of which we call an *almost-inner derivation of* $\mathfrak{g}$.

Concerning the notions above, we have the following. The assertion (1) was pointed out by C. Gordon.

LEMMA 4.1. (1) $\mathrm{PRIT}(\mathfrak{g}\,;0)=\mathrm{RIT}(\mathfrak{g}\,;0)$.

(2) *If* ϕ *belongs to* $\mathrm{PRIT}(\mathfrak{g}\,;\Lambda^*)$, *then* $\phi(\mathfrak{z})=\{0\}$ *holds.*

(3) *If* $\mathfrak{g}$ *is a two-step nilpotent Lie algebra, then we have*

$$\mathrm{RIT}(\mathfrak{g})=\mathrm{PRIT}(\mathfrak{g})=\mathrm{AID}(\mathfrak{g}),$$

and

$$\mathrm{RIT}(\mathfrak{g}\,;\Lambda^*)=\mathrm{PRIT}(\mathfrak{g}\,;\Lambda^*)=\mathrm{AID}(\mathfrak{g}\,;\Gamma)\,.$$

PROOF. (1) Suppose ϕ belongs to $\mathrm{PRIT}(\mathfrak{g}\,;0)$, and $\phi^*\mu=\mathrm{ad}^*(Y(\mu))\mu$ holds. Since $\mathfrak{g}^{(1)}:=[\mathfrak{g},\mathfrak{g}]\neq\mathfrak{g}$, we can take $\mu_0\in\mathfrak{z}^{\perp}\setminus\{0\}$ such that $\mu_0(\mathfrak{g}^{(1)})=0$. Then, $\mathrm{ad}^*(Y)\mu_0=0$ for $\forall Y\in\mathfrak{g}$ and $\phi^*\mu_0=0$ hold. Put $Y_0=Y(\mu_0)$. We show $\phi^*\mu=\mathrm{ad}^*(Y_0)\mu$ for any $\mu\in\mathfrak{z}^{\perp}$. It is obvious for $\mu=c\mu_0$ ($c\in\boldsymbol{R}$). Choose a vector space complement V of $\boldsymbol{R}\mu_0$ in $\mathfrak{z}^{\perp}$. Define

$$\bar{Y}:V\longrightarrow\mathfrak{g}\,;\bar{Y}(\nu)=Y(\mu_0+\nu)\,.$$

Note that $\bar{Y}$ is continuous on V and $\bar{Y}(0)=Y_0$. We have for every $\nu\in V$

$$\mathrm{ad}^*(\bar{Y}(\nu))\nu=\mathrm{ad}^*(\bar{Y}(\nu))(\mu_0+\nu)=\phi^*(\mu_0+\nu)=\phi^*\nu\,.$$

Replacing ν with $c\nu$ in the formula above, we have $\phi^*(c\nu)=\mathrm{ad}^*(\bar{Y}(c\nu))(c\nu)$. Hence, we get

$$\phi^*\nu=\mathrm{ad}^*(\bar{Y}(c\nu))\nu\,,\qquad\text{for }\forall c\neq 0\,.$$

Let $c\to 0$ in this, and we have $\phi^*\nu=\mathrm{ad}^*(Y_0)\nu$ for any $\nu\in V$. Therefore, noticing the linearity, we have $\phi^*\mu=\mathrm{ad}^*(Y_0)\mu$ for any $\mu\in\mathfrak{z}^{\perp}$.

(2) For any $w\in\mathfrak{z}$ and any $\mu\in\bigcup_{\lambda\in\Lambda^*}\mathfrak{z}_\lambda^{\perp}$, we have

$$\mu(\phi(w))=(\phi^*\mu)(w)=\mu([Y(\mu),w])=0\,.$$

This leads $\phi(w)=0$ because we can choose a basis of $\mathfrak{g}^*$ consisting of vectors in $\bigcup_{\lambda\in\Lambda^*}\mathfrak{z}_\lambda^{\perp}$.

(3) First we show that if g is two-step nilpotent, each ϕ in $\mathrm{PRIT}(\mathfrak{g}\,;\Lambda^*)$ is a derivation. Note that

$$\phi^*\mu_1 = 0 \qquad \text{for } \forall \mu_1 \in \mathfrak{z}^\perp . \tag{4.3}$$

In fact, we have $(\phi^*\mu_1)(v)=\mu_1([Y(\mu_1), v])=0$ for any $v\in\mathfrak{g}$, because $[Y(\mu_1), v]$ is contained in $\mathfrak{z}$. It follows from (4.3) that $\phi(\mathfrak{g})\subset\mathfrak{z}$. Noticing this fact and (2) of this Lemma, we easily see that ϕ is a derivation. Next, suppose $\phi\in \mathrm{AID}(\mathfrak{g}\,;\Gamma)$ satisfies $\phi^*\mu_0=\mathrm{ad}^*(Y_0)\mu_0$ $(Y_0\in\mathfrak{g})$ for $\mu_0\in\mathfrak{z}_\kappa^\perp$. Then, we show that $\phi^*\mu=\mathrm{ad}^*(Y_0)\mu$ holds for any $\mu\in\mathfrak{z}_\kappa^\perp$. In fact, for $\mu=\mu_0+\mu_1$ $(\mu_1\in\mathfrak{z}^\perp)$ we have

$$\phi^*\mu = \phi^*\mu_0 = \mathrm{ad}^*(Y_0)\mu_0 = \mathrm{ad}^*(Y_0)(\mu_0+\mu_1) = \mathrm{ad}^*(Y_0)\mu .$$

Thus the first assertion is proved. Finally, let $\mathcal{L}$ be the lattice in $\mathfrak{g}$ generated by $\log\Gamma$ and set

$$\mathcal{L}^* = \{\mu\in\mathfrak{g}^*\,;\ \mu(v)/2\pi\in\boldsymbol{Z} \quad \text{for any } v\in\mathcal{L}\}.$$

Then, ϕ belongs to $\mathrm{AID}(\mathfrak{g}, \Gamma)$ if and only if for any $\mu\in\mathcal{L}^*$ there exists $Y=Y(\mu)$ $\in\mathfrak{g}$ such that $\phi^*\mu=\mathrm{ad}^*(Y)\mu$ ([7]). Obviously the set $\Lambda^*(\subset\mathfrak{z}^*)$ is equal to $\mathcal{L}^*$ $(\subset\mathfrak{g}^*)$ with the operation to be restricted to $\mathfrak{z}$. This shows the second assertion. ∎

EXAMPLE 4.2 (see [4], for details). Let $\mathfrak{g}$ be the six-dimensional Lie algebra with basis $\mathcal{B}=\{u_1, u_2, v_1, v_2, w_1, w_2\}$ and

$$[u_1, v_1] = [u_2, v_2] = w_1 , \qquad [u_1, v_2] = w_2 ,$$

all other brackets being zero. This is a two-step nilpotent Lie algebra with the center $\mathfrak{z}$ being generated by $\{w_1, w_2\}$. One way to realize it as a matrix algebra is to let $\sum_{i=1}^2 (x_iu_i+y_iv_i+z_iw_i)$ correspond to the 7×7 matrix

$$\left(\begin{array}{cccc|ccc} 0 & x_1 & x_2 & z_1 & & & \\ 0 & 0 & 0 & y_1 & & 0 & \\ 0 & 0 & 0 & y_2 & & & \\ 0 & 0 & 0 & 0 & & & \\ \hline & & & & 0 & x_1 & z_2 \\ & 0 & & & 0 & 0 & y_2 \\ & & & & 0 & 0 & 0 \end{array}\right)$$

Let $\phi:\mathfrak{g}\to\mathfrak{g}$ be the derivation defined by

$$\phi(v_2) = w_2 ,$$

with zero on the remaining elements of $\mathcal{B}$. Then, ϕ belongs to $\mathrm{RIT}(\mathfrak{g})$ and $\mathrm{AID}(\mathfrak{g})$. In fact, using the basis $\mathcal{B}^*=\{u_1^*, u_2^*, v_1^*, v_2^*, w_1^*, w_2^*\}$ of $\mathfrak{g}^*$ dual to $\mathcal{B}$, we have, for $\mu=\kappa_1w_1^*+\kappa_2w_2^*+\mu_1$ $(\mu_1\in\mathfrak{z}^\perp)$,

$$\phi^*\mu = \begin{cases} \mathrm{ad}^*(u_1)\mu & (\kappa_1=0) \\ \mathrm{ad}^*\left(\dfrac{\kappa_2}{\kappa_1}u_2\right)\mu & (\kappa_1\neq 0)\,. \end{cases}$$

EXAMPLE 4.3. Let $\mathfrak{g}$ be the $(n+3)$-dimensional $(n\geqq 2)$ Lie algebra with the basis $\mathcal{B}=\{u_1, u_2, v_j, w\,;\, 1\leqq j\leqq n\}$ satisfying

$$[u_1, v_j] = v_{j+1}\ (1\leqq j\leqq n-1),\quad [u_1, v_n] = w,\quad [u_1, w] = [u_1, u_2] = 0,$$

$$[u_2, v_j] = [u_1, [u_1, v_j]] = v_{j+2}\quad (1\leqq j\leqq n-2),$$

$$[u_2, v_{n-1}] = w,\qquad [u_2, v_n] = [u_2, w] = 0,$$

$$[v_j, v_k] = 0\qquad (1\leqq j,\ k\leqq n)\,.$$

This is an $(n+1)$-step nilpotent Lie algebra with the one dimensional center generated by w. One realization as a matrix algebra is obtained by letting $\sum_{i=1}^{2} x_iu_i+\sum_{j=1}^{n} y_jv_j+zw$ correspond to the $(n+2)\times(n+2)$ matrix

$$\begin{pmatrix} 0 & x_1 & x_2 & 0 & \cdots & 0 & z \\ & 0 & x_1 & x_2 & \cdots & 0 & y_n \\ & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & x_2 & y_3 \\ & & & & \ddots & x_1 & y_2 \\ & 0 & & & & 0 & y_1 \\ & & & & & & 0 \end{pmatrix}.$$

Let ϕ be the derivation of $\mathfrak{g}$ defined by

$$\phi(u_2) = w$$

with zero on the remaining elements of $\mathcal{B}$. Let $\mathcal{B}^*=\{u_1^*, u_2^*, v_j^*, w^*\}$ be the basis of $\mathfrak{g}^*$ dual to $\mathcal{B}$. For $\mu=\kappa w^*+\mu_1u_1^*+\mu_2u_2^*+\sum_{j=1}^{n}\nu_jv_j^*$, we have

$$\phi^*\mu = \begin{cases} \mathrm{ad}^*(cv_n)\mu & (\kappa=0) \\ \mathrm{ad}^*\left(-v_{n-1}+\dfrac{\nu_n}{\kappa}v_n\right)\mu & (\kappa\neq 0) \end{cases}$$

c being constant. Thus, ϕ belongs to $\mathrm{AID}(\mathfrak{g})$, $\mathrm{PRIT}(\mathfrak{g})$ and $\mathrm{RIT}(\mathfrak{g}\,;0)$, but does not belong to $\mathrm{RIT}(\mathfrak{g}\,;\kappa)$ $(\kappa\neq 0)$.

EXAMPLE 4.4 (cf. [6]). Let $\mathfrak{g}$ be the six-dimensional Lie algebra with the basis $\mathcal{B}=\{u_1, \cdots, u_5, w\}$ which satisfies that

$$[u_1, u_2] = u_3\,,\qquad [u_1, u_3] = u_4\,,$$

$$[u_1, u_4] = u_5\,,\qquad [u_2, u_3] = u_5\,,$$

$$[u_5, u_2] = w, \qquad [u_3, u_4] = w,$$

and all other brackets are zero. This is a five-step nilpotent Lie algebra with the one dimensional center generated by w. Let ϕ be the derivation of $\mathfrak{g}$ defined by

$$\phi(u_2) = u_5$$

with zero on the remaining elements of $\mathscr{B}$. Let $Y(\mu)$ be a vector in $\mathfrak{g}$ satisfying $\phi^*\mu = \mathrm{ad}^*(Y(\mu))\mu$. We put $\mu = \kappa w^* + \sum_{i=1}^{5} \mu_i u_i^*$ using the basis $\mathscr{B}^* = \{u_1^*, \cdots, u_5^*, w^*\}$ of $\mathfrak{g}^*$ dual to $\mathscr{B}$. If $\kappa \neq 0$, then we can take

$$Y(\mu) = \left(\frac{\mu_5}{\kappa}\right) u_5 .$$

If $\kappa = 0$, then we can take

$$Y(\mu) = \begin{cases} -u_3 + (\mu_4/\mu_5) u_4 & (\mu_5 \neq 0) \\ c_4 u_4 + c_5 u_5 & (\mu_5 = 0) \end{cases}$$

where c_4 and c_5 are constants. Thus, ϕ belongs to $\mathrm{AID}(\mathfrak{g})$ and $\mathrm{PRIT}(\mathfrak{g}; \kappa)$ for every $\kappa \neq 0$, but does not belong to $\mathrm{PRIT}(\mathfrak{g}; 0)$.

We conclude this section with the following question, which is interesting in connection with Theorem 5.1 below and Remark (2) following after it.

QUESTION. *Does every almost-inner derivation of a nilpotent Lie algebra* $\mathfrak{g}$ *belong to* $\mathrm{PRIT}(\mathfrak{g}; \kappa)$ *for* $\forall \kappa \in \mathfrak{z}^* \setminus \{0\}$*?*

5. Deformations of classical and quantum systems.

Let $\Phi_t = \exp(t\phi)$ $(t \in \boldsymbol{R})$ be a one parameter subgroup of linear isomorphisms of $\mathfrak{g}$. Then, we have the one parameter family

$$\langle v, w \rangle_t = \langle \Phi_t v, \Phi_t w \rangle \quad (v, w \in \mathfrak{g})$$

of inner products of $\mathfrak{g}$. Let us consider a one parameter family of Riemannian metrics on $M = \Gamma \backslash G$ induced from the left-invariant metrics on G defined by $\langle , \rangle_t$. Corresponding to the metrics $\langle , \rangle_t$ we have one parameter families of classical Hamiltonian systems, $\mathscr{H}_t = (M \times \mathfrak{g}^*, \omega, H_t)$ and $\mathscr{H}_{\kappa, t} = (P_\kappa, \omega_\kappa, H_{\kappa, t})$ $(\kappa \in \mathfrak{z}^*)$, and one parameter families of quantum systems, Δ_t and $D_t^{(\lambda)}$ on E_λ.

Concerning the classical systems $\mathscr{H}_{\kappa, t}$ we have the following.

THEOREM 5.1. *Suppose* $\Phi_t = \exp(t\phi)$ *with* ϕ *belonging to* $\mathrm{PRIT}(\mathfrak{g}; \kappa)$ $(\kappa \in \mathfrak{z}^*)$. *Then* $\mathscr{H}_{\kappa, t} \cong \mathscr{H}_{\kappa, 0}$ *as Hamiltonian systems for every t, that is, there exists a one parameter family* $\psi_t : P_\kappa \to P_\kappa$ *of diffeomorphisms such that*

$$\psi_t^* \omega_\kappa = \omega_\kappa , \tag{5.1}$$

and

$$\phi_t^* H_{\kappa,0} = H_{\kappa,t}. \tag{5.2}$$

PROOF. For the transformation ϕ belonging to PRIT$(\mathfrak{g}, \kappa)$, there exists $Y: \mathfrak{z}_\kappa^\perp \to \mathfrak{g}$ satisfying $-\phi^*\mu = \mathrm{ad}^*(Y(\mu))\mu$ for $\mu \in \mathfrak{z}_\kappa^\perp$. Put $Y^{(1)} = \pi_* \circ Y: \mathfrak{z}_\kappa^\perp \to \mathfrak{g}_1 = \mathfrak{g}/\mathfrak{z}$, and let V be a smooth vector field on P_κ defined by

$$V([g_1], \mu) = (L_{g_1*}(Y^{(1)}(\mu)), -\phi^*\mu).$$

Here we note that $-\phi^*\mu$ belongs to $\mathfrak{z}^\perp$. For the vector field V we show that

$$\boldsymbol{L}_V \omega_\kappa = 0, \tag{5.3}$$

$\boldsymbol{L}$ being the Lie derivative, and

$$V H_{\kappa,t} = H'_{\kappa,t}\left(= \frac{d}{dt} H_{\kappa,t}\right). \tag{5.4}$$

In fact, for $\rho = (L_{g_1*}v_1, \nu)$, $\sigma = (L_{g_1*}w_1, \tau) \in T_{[g_1]}M_1 \times \mathfrak{z}^\perp$ $(v_1, w_1 \in \mathfrak{g}_1)$ we have

$$\begin{aligned}
&(\boldsymbol{L}_V \omega_\kappa)([g_1], \mu)(\rho, \sigma) \\
&= d(\boldsymbol{i}_V \omega_\kappa)([g_1], \mu)(\rho, \sigma) \\
&= \rho\{\omega_\kappa([g_1], \mu)(V, \sigma)\} - \sigma\{\omega_\kappa([g_1], \mu)(V, \rho)\} - \omega_\kappa([g_1], \mu)(V, [\rho, \sigma]),
\end{aligned}$$

where we regard in the last line ρ and σ as the vector fields $\rho([h_1], \zeta) = (L_{h_1*}v_1, \nu)$ and $\sigma([h_1], \zeta) = (L_{h_1*}w_1, \tau)$ respectively, on $P_\kappa \cong M_1 \times \mathfrak{z}^\perp$. Note that $[\rho, \sigma]_{([g_1], \mu)} = (L_{g_1*}[v_1, w_1], 0)$. By means of the formula (2.3) and the conditions for PRIT the above turns out to be

$$\begin{aligned}
\rho\{\phi^*\mu(w) + \tau(Y(\mu)) + \mu([Y(\mu), w])\} - \sigma\{\phi^*\mu(v) + \nu(Y(\mu)) + \mu([Y(\mu), v])\} \\
- \phi^*\mu([v, w]) - \mu([Y(\mu), [v, w]]) = 0.
\end{aligned}$$

Thus (5.3) is shown. As to (5.4) we have

$$H'_{\kappa,t}([g_1], \mu) = \frac{1}{2}\langle \mu^\#, \mu^\# \rangle'_t + \langle (\mu^\#)', \mu^\# \rangle_t.$$

Differentiate the equation $\langle \mu^\#, v \rangle_t = \mu(v)$ $(v \in \mathfrak{g})$ with respect to t, and we get $\langle (\mu^\#)', v \rangle_t = -\langle \mu^\#, v \rangle'_t$. On the other hand,

$$\langle v, v \rangle'_t = \frac{d}{dt}\langle \Phi_{t*}v, \Phi_{t*}v \rangle = 2\langle \phi(v), v \rangle_t$$

holds. Therefore we get

$$H'_{\kappa,t}([g_1], \mu) = -\frac{1}{2}\langle \mu^\#, \mu^\# \rangle'_t = -\langle \phi(\mu^\#), \mu^\# \rangle_t = -\mu(\phi(\mu^\#))$$
$$= -\phi^*\mu(\mu^\#) = -\langle \phi^*\mu, \mu \rangle_t^*$$
$$= (VH_{\kappa,t})([g_1], \mu).$$

Finally, let ψ_t be the diffeomorphisms satisfying the equation

$$\frac{d}{dt}\psi_t = V \circ \psi_t, \qquad \psi_0 = \text{identity}.$$

Then ψ_t satisfies the conditions (5.1) and (5.2). ∎

COROLLARY 5.2. *If ϕ belongs to* PRIT$(\mathfrak{g})$, *then $\mathcal{H}_{\kappa,t} \cong \mathcal{H}_{\kappa,0}$ holds for every $\kappa \in \mathfrak{z}^*$.*

REMARKS. (1) If ϕ is an inner derivation, then $\mathcal{H}_t \cong \mathcal{H}_0$ holds. In fact, if $\phi = \mathrm{ad}(Y)$ $(Y \in \mathfrak{g})$, then the right translation $R_{\exp(tY)}$ on G induces an isomorphism between $\mathcal{H}_t$ and $\mathcal{H}_0$. If $\phi \in \mathrm{PRIT}(\mathfrak{g})$ is not inner, $\mathcal{H}_t \cong \mathcal{H}_0$ does not hold, in general. C. Gordon [8] pointed out for ϕ in Example 4.2 that the geodesic flows of $\mathcal{H}_t$ $(t \in \boldsymbol{R})$ are not conjugate under any continuous family of homeomorphisms of $M \times (\mathfrak{g}^* \setminus \{0\})$.

(2) In the case of the derivation ϕ in Example 4.4, we have not $\mathcal{H}_{0,t} \cong \mathcal{H}_{0,0}$. The system $\mathcal{H}_0$ isomorphic with $\mathcal{H}_0^{(1)} = (M_1 \times \mathfrak{g}_1^*, \omega^{(1)}, H^{(1)})$, where the quotient Lie algebra $\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{z}$ is spanned by $\mathcal{B}_1 = \{\bar{u}_1, \cdots, \bar{u}_5\}$ with

$$[\bar{u}_1, \bar{u}_2] = \bar{u}_3, \qquad [\bar{u}_1, \bar{u}_3] = \bar{u}_4,$$
$$[\bar{u}_1, \bar{u}_4] = \bar{u}_5, \qquad [\bar{u}_2, \bar{u}_3] = \bar{u}_5,$$

and all other brackets being zero. Note that $\mathfrak{g}_1$ is a nilpotent Lie algebra with the center $\mathfrak{z}_1 = \mathrm{span}\{\bar{u}_5\}$. We can apply the procedure of reduction to the system $\mathcal{H}_0^{(1)}$, and obtain the family, $\mathcal{H}_{0,\kappa_1}^{(1)}$ $(\kappa_1 \in \mathfrak{z}_1^*)$, of reduced systems of $\mathcal{H}_0^{(1)}$. The derivation ϕ induces the derivation $\bar{\phi}$ of $\mathfrak{g}_1$ given by

$$\bar{\phi}(\bar{u}_2) = \bar{u}_5,$$

with zero on the remaining elements of $\mathcal{B}_1$. Then we can see that $\bar{\phi}$ belongs to PRIT$(\mathfrak{g}_1)$. Hence, we have $\mathcal{H}_{0,\kappa_1,t}^{(1)} \cong \mathcal{H}_{0,\kappa_1,0}^{(1)}$ for every $\kappa_1 \in \mathfrak{z}_1^*$.

Next, we consider the differential operators $D_t^{(\lambda)}$ on E_λ. The operator $D_t^{(\lambda)}$ corresponds to the geometric object $Q_{\lambda,t} = (M_1, \langle\, , \rangle_1; E_\lambda, \tilde{\nabla}_t^{(\lambda)})$. It is natural for us to say that $Q = (M_1, \langle\, , \rangle_1; E, \tilde{\nabla})$ and $Q' = (M_1, \langle\, , \rangle'_1; E, \tilde{\nabla}')$ on a Hermitian line bundle $E \to M_1$ are isomorphic to each other (denoted by $Q \cong Q'$) if there exists an automorphism of vector bundle, $\varphi: E \to E$, such that (i) φ preserves the Hermitian structure, (ii) the associated diffeomorphism $\bar{\varphi}$ of M_1 satisfies

$\langle\,,\rangle_1'=\bar{\varphi}^*\langle\,,\rangle_1$, and (iii) $\tilde{\nabla}'=\varphi^*\tilde{\nabla}$, i. e., $\tilde{\nabla}'_{X'}s'=\varphi^*(\tilde{\nabla}_X s)$ holds for s, $s'\in C^\infty(E)$ and the vector fields X, X' on M_1 such that $s'=\varphi^*s$ and $X=\bar{\varphi}_*(X')$, where

$$(\varphi^*s)(x_1):=\varphi^{-1}(s(\bar{\varphi}(x_1))) \quad (x_1\in M_1).$$

The following is immediately obtained.

LEMMA 5.3. *Let L and L' be the Bochner-Laplacians associated with Q and Q', respectively. If $Q\cong Q'$, then* $\mathrm{Spec}(L)=\mathrm{Spec}(L')$ *holds.*

For $\lambda\in\Lambda^*$, let $\mathfrak{z}_\lambda=\ker(\lambda)$, which is an ideal of $\mathfrak{z}$, and let Z_λ be the corresponding connected subgroup of Z. Set $G_\lambda=G/Z_\lambda$. Since λ belongs to Λ^*, $Z_\lambda\cap\Gamma\backslash Z_\lambda$ is compact and the projection of G onto G_λ carries Γ to a uniform discrete subgroup Γ_λ of G_λ. Then, $M_\lambda:=\Gamma_\lambda\backslash G_\lambda$ is a principal circle bundle over M_1. The functional λ, viewed on $\mathfrak{z}/\mathfrak{z}_\lambda$ defines a faithful unitary representation of the circle, and the associated line bundle is equivalent to E_λ. The inner product on $\mathfrak{g}$ induces an inner product on $\mathfrak{g}_\lambda:=\mathfrak{g}/\mathfrak{z}_\lambda$ like as that on $\mathfrak{g}_1$ considered in § 2. Moreover, we have the corresponding connection on the principal circle bundle $M_\lambda\to M_1$. It is not hard to see that the associated connection and the Bochner-Laplacian on the associated line bundle E_λ are equivalent to those we introduced formerly in § 3.

On the basis of the discussions above, we have the following.

PROPOSITION 5.4. *Suppose $\Phi_t=\exp(t\phi)$ with ϕ belonging to* $\mathrm{RIT}(\mathfrak{g}\,;\lambda)$ $(\lambda\in\Lambda^*)$ *and $\phi(\mathfrak{z})=\{0\}$ holds. Then $Q_{\lambda,t}\cong Q_{\lambda,0}$ holds for every t.*

PROOF. Since $\phi(\mathfrak{z})=\{0\}$, ϕ induces the linear transformation ϕ_λ of $\mathfrak{g}_\lambda$. If we can show that ϕ_λ is an inner derivation of $\mathfrak{g}_\lambda$, the proof is completed. In fact, for $\phi_\lambda=\mathrm{ad}(Y)$ $(Y\in\mathfrak{g}_\lambda)$, we have the right translation $R_{\exp(tY)}$ on G_λ. It induces an isometry between the metrics of M_λ for $t=t$ and $t=0$, which, moreover, is commutative with the translations by Z/Z_λ. Thus we have an isomorphism between $Q_{\lambda,t}$ and $Q_{\lambda,0}$.

Note that $(\mathfrak{g}_\lambda)^*\cong(\mathfrak{z}_\lambda)^\perp\subset\mathfrak{g}^*$, and we identify the dual of ϕ_λ with the operator ϕ^* restricted on $(\mathfrak{z}_\lambda)^\perp$. Obviously we have $(\mathfrak{z}_\lambda)^\perp=\mathfrak{z}^\perp\oplus\boldsymbol{R}\lambda$. It is easy to see that if ϕ belongs to $\mathrm{RIT}(\mathfrak{g}\,;\lambda)$, then there exists $Y\in\mathfrak{g}$ such that $\phi^*\mu=\mathrm{ad}(Y)\mu$ for any $\mu\in(\mathfrak{z}_\lambda)^\perp$. Thus ϕ_λ is an inner derivation. ■

Recalling Lemma 4.1 (2), we have the following.

COROLLARY 5.5. *If ϕ belongs to* $\mathrm{RIT}(\mathfrak{g};\Lambda^*)$, *then $Q_{\lambda,t}\cong Q_{\lambda,0}$ for every $\lambda\in\Lambda^*$, and accordingly,* $\mathrm{Spec}(\Delta_t)=\mathrm{Spec}(\Delta_0)$ *holds.*

REMARK. It was shown that a derivation in $\mathrm{AID}(\mathfrak{g}\,;\Gamma)$ induces a (nontrivial) isospectral deformation of metric on $M=\Gamma\backslash G$ by Gordon-Wilson [**9**] on the basis of the Kirillov theory and by DeTurck-Gordon [**5**] on the basis of the trace formula. Our arguments for isospectrality in this section correspond to their research for the two-step nilmanifolds developed in [4], where the isospectrality is shown by constructing explicitly the unitary operators associated with the geometric transformations. Recently, Marhuenda [13] found that these intertwining operators belong to the space of Fourier integral operators associated with various pairwise intersecting Lagrangians (see also [18]).

6. Isospectral deformations of connections on line bundles.

In [12] we constructed on some fixed line bundle a pair of distinct connections with the same spectrum of the associated Bochner-Laplacians. The discussions in the preceding sections permit us to construct a non-trivial deformation of connection on a line bundle over a fixed Riemannian manifold. We recall that a deformation $\tilde{\nabla}_t$ of connection on the line bundle $E \to M_1$ said to be *trivial* if there is a one parameter family φ_t of automorphisms of E such that the associated diffeomorphisms $\bar{\varphi}_t : M_1 \to M_1$ are isometries, and $\tilde{\nabla}_t = \varphi_t^* \tilde{\nabla}_0$ holds.

Given a one parameter family $\Phi_t = \exp(t\phi)$ of linear isomorphisms of $\mathfrak{g}$ with ϕ belonging to $\mathrm{AID}(\mathfrak{g}; \Gamma)$. Then, Φ_t induces a (non-trivial) isospectral deformation of the Laplacian on $M = \Gamma \backslash G$, and moreover $\mathrm{Spec}(L_t^{(\lambda)}) = \mathrm{Spec}(L_0^{(\lambda)})$ for the Bochner-Laplacian $L_t^{(\lambda)}$ on the line bundle $(E_\lambda, \tilde{\nabla}_t^{(\lambda)})$ $(\lambda \in \Lambda^*)$ over $M_1 = \Gamma_1 \backslash G_1$. This follows from Proposition 3.4 and the continuous dependence of the eigenvalues under the deformation of the operators. If ϕ does not belong to $\mathrm{RIT}(\mathfrak{g}; \lambda)$ for some $\lambda \neq 0$ in Λ^*, then $(M_1, \langle\,,\rangle_{1,t}; E_\lambda, \tilde{\nabla}_t^{(\lambda)}) \cong (M_1, \langle\,,\rangle_{1,0}; E_\lambda, \tilde{\nabla}_0^{(\lambda)})$ does not necessarily hold. On the other hand, if ϕ belongs to $\mathrm{RIT}(\mathfrak{g}; 0)$, then we have $(M_1, \langle\,,\rangle_{1,t}) \cong (M_1, \langle\,,\rangle_{1,0})$ as Riemannian manifolds. Thus, we can expect to construct a non-trivial deformation of connection on the line bundle E_λ over $M_1 = \Gamma_1 \backslash G_1$ by finding a nilpotent Lie algebra $\mathfrak{g}$ and a linear transformation ϕ of $\mathfrak{g}$ which satisfy the following properties:

(i) ϕ belongs to $\mathrm{AID}(\mathfrak{g}; \Gamma)$,
(ii) ϕ belongs to $\mathrm{RIT}(\mathfrak{g}; 0)$, and
(iii) ϕ does not belong to $\mathrm{RIT}(\mathfrak{g}; \lambda)$ for some $\lambda \neq 0$ in Λ^*.

EXAMPLE 6.1. The Lie algebra $\mathfrak{g}$ and the derivation ϕ of Example 4.3 satisfy the above condition (i), (ii) and (iii) for every $\lambda \neq 0$ in Λ^*. Thus we have an isospectral deformation of connection on the line bundle E_λ over the Riemannian manifold $(M_1, \langle\,,\rangle_1)$. We need to check it to be non-trivial. We recall that the connection on the principal bundle M is uniquely determined by the Riemannian metric on M. It follows from Lemma 3.1 that there is a one-to-one correspondence between the sets of connections on M and those on E_λ $(\lambda \neq 0)$ because the fibre of M is S^1. Therefore, if there is a one parameter family φ_t of automorphisms of E_λ which are isomorphisms between $Q_{\lambda,0}$ and

$Q_{\lambda,t}$, then it induces a one parameter family $\tilde{\varphi}_t$ of diffeomorphisms of M such that $\tilde{\varphi}_t^*\langle\,,\rangle_0=\langle\,,\rangle_t$, that is, $\langle\,,\rangle_t$ is a trivial deformation of metric on M. However, this is impossible by virtue of Proposition 5.2 of [9], which asserts that any trivial deformation is induced from an inner derivation.

References

[1] R. Abraham and J. E. Marsden, Foundations of Mechanics, 2nd ed., Benjamin/Cummings, Reading MA, 1978.

[2] P. Bérard, Variétés riemanniennes isospectrales non isométriques, Séminaire BOURBAKI, 1988-89, **705**, 1989.

[3] L. Corwin and F. P. Greenleaf, Representations of nilpotent Lie groups and their applications. Part I: Basic theory and examples, Cambridge Univ. Press, 1989.

[4] D. M. DeTurck and C. S. Gordon, Isospectral deformation I: Riemannian structures on two-step nilspaces, Comm. Pure Appl. Math., **40** (1987), 367-387.

[5] D. M. DeTurck and C. S. Gordon, Isospectral deformation II: trace formulas, metrics, and potentials, Comm. Pure Appl. Math., **42** (1989), 1067-1095.

[6] D. DeTurck, H. Gluck, C. Gordon and D. Webb, How can a drum change shape, while sounding the same? Part II, to appear in Mechanics, Analysis and Geometry 200 years after Lagrange.

[7] C. S. Gordon, The Laplace spectra versus the length spectra of Riemannian manifolds, Contemp. Math., **51** (1986), 63-80.

[8] C. S. Gordon, You can't hear the geodesic flow, preprint.

[9] C. S. Gordon and E. N. Wilson, Isospectral deformations of compact solvmanifolds, J. Differential Geom., **19** (1984), 241-256.

[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I, John Wiley, New York, 1963.

[11] R. Kuwabara, On spectra of the Laplacian on vector bundles, J. Math. Tokushima Univ., **16** (1982), 1-23.

[12] R. Kuwabara, Isospectral connections on line bundles, Math. Z., **204** (1990), 465-473.

[13] F. Marhuenda, Microlocal analysis of some isospectral deformations, 1991, preprint.

[14] J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., **5** (1974), 121-130.

[15] J. Milnor, Eigenvalues of the Laplace operator on certain manifolds, Proc. Nat. Acad. Sci. U. S. A., **51** (1964), 542.

[16] R. Montgomery, Canonical formulations of a classical particle in a Yang-Mills field and Wong's equations, Lett. Math. Phys., **8** (1984), 59-67.

[17] T. Sunada, Riemannian coverings and isospectral manifolds, Ann. of Math., **121** (1985), 169-186.

[18] S. Zelditch, Isospectrality in the FIO category, J. Differential Geom., **35** (1992), 689-710.

Ruishi KUWABARA
Department of Mathematics
University of Tokushima
Minami-Josanjima, Tokushima 770
Japan