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Borsuk-Ulam theorem and Stiefel manifolds

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

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Introduction.

There are several different, but equivalent versions of the classical Borsuk-Ulam theorem. One of them can be stated as follows:

THE CLASSICAL BORSUK-ULAM THEOREM. *Let S^n be the unit sphere in euclidean $(n+1)$-space $\boldsymbol{R}^{n+1}$. If $f: S^n \to \boldsymbol{R}^n$ is a $\boldsymbol{Z}_2$-map, i.e., satisfies $f(-x)=-f(x)$ for all $x \in S^n$, then $f^{-1}(0)$ is nonempty.*

Many authors have been contributing to generalizing and extending the Borsuk-Ulam theorem in various ways (see Steinlein [**10**]). Recently E. Fadell-S. Husseini and J. W. Jaworowski independently introduced an *ideal-valued cohomological index theory* and extended the theorem to maps of Stiefel manifolds, see [**2**], [**3**], [**4**] and [**5**].

Let $(\boldsymbol{R}^n)^k$ denote the cartesian product of k copies of $\boldsymbol{R}^n$. Any point of $(\boldsymbol{R}^n)^k$ is represented by a $(k \times n)$-matrix. Then the k-th orthogonal group $O(k)$ acts on $(\boldsymbol{R}^n)^k$ by matrix multiplication on the left. When $k \leqq n$, the Stiefel manifold $V_k(\boldsymbol{R}^n)$ of orthonormal k-frames in $\boldsymbol{R}^n$ can be considered a subspace of $(\boldsymbol{R}^n)^k$ on which $O(k)$ acts freely. In [**2**], [**3**], Fadell and Husseini considered $\boldsymbol{Z}_2^k$-maps $f: V_k(\boldsymbol{R}^n) \to (\boldsymbol{R}^{n-k})^k$ where $\boldsymbol{Z}_2^k = \boldsymbol{Z}_2 \times \cdots \times \boldsymbol{Z}_2$ (k times) is a subgroup of $O(k)$ which is diagonally imbedded, and they estimated the cohomological size of $f^{-1}(O)/\boldsymbol{Z}_2^k$ where O is the zero of $(\boldsymbol{R}^{n-k})^k$. In [**4**], [**5**], Jaworowski considered $O(2)$-maps $f: V_2(\boldsymbol{R}^n) \to (\boldsymbol{R}^l)^2$ and estimated the cohomological size of $f^{-1}(T)/O(2)$, where $T = \{A \in (\boldsymbol{R}^l)^2 \mid \operatorname{rank} A < 2\}$.

In the present paper we will consider more general class of maps of Stiefel manifolds and generalize their results. We will employ (mod 2) cup_1-*length*, denoted $\mathrm{cup}_1(X)$, as a measure of the cohomological size of a space X. $\mathrm{cup}_1(X)$ is defined to be the greatest number s such that there exist $x_1, \cdots, x_s \in H^1(X; \boldsymbol{Z}_2)$ with $x_1 \cup \cdots \cup x_s \neq 0$. The inequality $\mathrm{cup}_1(X) \geqq 0$ means X is at least nonempty. When $x_1, \cdots, x_s$ can be taken in any positive degrees, the usual *cup-length*, denoted $\mathrm{cup}(X)$, is defined. Then $\mathrm{cup}_1(X) \leqq \mathrm{cup}(X) < \mathrm{cat}(X)$, where cat$(X)$ denotes the Lusternik-Schnirelmann category of X. The inequality $\text{cup}_1(X)\geqq a\geqq 0$ implies $H^b(X;\boldsymbol{Z}_2)\neq 0$ for all b with $0\leqq b\leqq a$.

Given integers $k_1,\cdots,k_m>0$, we can diagonally imbed the product $O(k_1,\cdots,k_m)=O(k_1)\times\cdots\times O(k_m)$ into $O(k_1+\cdots+k_m)$. If $k_1+\cdots+k_m\leqq n$, $V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n)$ denotes the Stiefel manifold $V_{k_1+\cdots+k_m}(\boldsymbol{R}^n)$ with restricted $O(k_1,\cdots,k_m)$-action. $O(k_1,\cdots,k_m)$ acts also on a product space $(\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}$ as product action. Let $T_i=\{A\in(\boldsymbol{R}^{l_i})^{k_i}\mid \text{rank}\, A<k_i\}$. Then $T_1\times\cdots\times T_m$ is invariant under the action of $O(k_1+\cdots+k_m)$.

In sections 1-4 we will give some preliminaries on ideal-valued indices and calculate those of relevant spaces. We will show in section 5

THEOREM. *Let* $f:V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n)\to(\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}$ *be an* $O(k_1,\cdots,k_m)$-*map. Suppose*

$$l_i<n-\sum_{r=i+1}^{m}k_r$$

for all i *with* $1\leqq i\leqq m$. *Then*

$$\text{cup}_1(f^{-1}(T_1\times\cdots\times T_m)/O(k_1,\cdots,k_m))\geqq a\,,$$

where $a=mn-\sum_{i=2}^{m}(i-1)k_i-\sum_{i=1}^{m}\max\{k_i,l_i+1\}\geqq 0$. *In particular* $f^{-1}(T_1\times\cdots\times T_m)$ *is nonempty.*

If we take $m=1$, $k_1=1$ and $l_1=n-1$, then the theorem is just the classical Borsuk-Ulam theorem. If we take $k_1=\cdots=k_m=1$ and $l_1=\cdots=l_m=n-m$, then $T_1\times\cdots\times T_m$ consists only of zero and the theorem reduces to the case which Fadell and Husseini considered. If we take $m=1$ and $k_1=2$, then the theorem reduces to the case which Jaworowski considered. (But the estimation is weaker than Jaworowski's.)

Let $W_j=\{A\in(\boldsymbol{R}^l)^k\mid\text{rank}\,A\leqq j\}$ for any j. In section 6 we will discuss the cup_1-length of orbit spaces of $f^{-1}(W_j)$ for $O(k)$-maps $f:V_k(\boldsymbol{R}^n)\to(\boldsymbol{R}^l)^k$. In section 7 we will consider $O(k_1,\cdots,k_m)$-maps of products $V_{k_1}(\boldsymbol{R}^{n_1})\times\cdots\times V_{k_m}(\boldsymbol{R}^{n_m})$ of Stiefel manifolds. If we take $k_1=\cdots=k_m=1$, this reduces to the case of products of spheres which is considered in [2], [3]. In the last section 8 we will give some equivalent versions of the Borsuk-Ulam theorem for Stiefel manifolds which correspond to well-known equivalent versions of the classical Borsuk-Ulam theorem.

§1. Ideal-valued index.

In this section we will recall the definition and basic properties of ideal-valued index which was first introduced by Fadell and Husseini [2], [3] and independently by Jaworowski [4], [5].

All spaces considered are paracompact and Hausdorff. Let G be a compact Lie group and $EG\to BG$ a universal principal G-bundle. The *G-index* of a G-space X, denoted $\mathrm{Ind}^G X$, is an ideal in $H^*(BG;\boldsymbol{K})$ where $H^*(\ ;\boldsymbol{K})$ is the Alexander-Spanier cohomology with coefficients in some field $\boldsymbol{K}$. In this paper we will take $\boldsymbol{Z}_2$ as $\boldsymbol{K}$, and it will be suppressed from the notation. $\mathrm{Ind}^G X$ is defined to be the kernel of the homomorphism $c_X^*: H^*(BG)\to H^*(EG\times_G X)$ induced from a map $c_X: EG\times_G X\to BG$ which classifies the free diagonal G-action on $EG\times X$. If X is a free G-space, then $\mathrm{Ind}^G X$ coincides with the kernel of the homomorphism $H^*(BG)\to H^*(X/G)$ induced from a classifying map $X/G\to BG$ for the free G-action on X.

PROPOSITION 1.1 ([2], [3], [4], [5]). *If $f: X\to Y$ is a G-map, then*

$$\mathrm{Ind}^G X \supset \mathrm{Ind}^G Y$$

in $H^*(BG)$.

The property of the G-index described in the following proposition is fundamental in this paper.

PROPOSITION 1.2 ([2], [3], [4], [5]). *Let X and Y be G-spaces, and W a G-invariant closed subspace of Y. If $f: X\to Y$ is a G-map, then*

$$\mathrm{Ind}^G f^{-1}(W)\cdot \mathrm{Ind}^G(Y-W) \subset \mathrm{Ind}^G X$$

in $H^*(BG)$, where $\cdot$ represents the product of ideals.

Denote by X_1*X_2 the join of a G_1-space X_1 and a G_2-space X_2, and represent points of X_1*X_2 by $[(t, x_1), (1-t, x_2)]$, $x_1\in X_1$, $x_2\in X_2$ and $0\leqq t\leqq 1$ with the usual identifications. Then X_1*X_2 becomes a $G_1\times G_2$-space via the action

$$(g_1, g_2)[(t, x_1), (1-t, x_2)] = [(t, g_1x_1), (1-t, g_2x_2)]$$

for $(g_1, g_2)\in G_1\times G_2$. We obtain

PROPOSITION 1.3 ([2]). *Let X_1 and X_2 be as above. Then*

$$\mathrm{Ind}^{G_1\times G_2}X_1*X_2 \supset \mathrm{Ind}^{G_1}X_1\otimes \mathrm{Ind}^{G_2}X_2$$

in $H^*(B(G_1\times G_2))=H^*(BG_1)\otimes H^*(BG_2)$.

PROPOSITION 1.4 ([2]). *If $G_1\times G_2$ acts on X_1 by $(g_1, g_2)x_1=g_1x_1$, then we obtain*

$$\mathrm{Ind}^{G_1\times G_2}X_1 = \mathrm{Ind}^{G_1}X_1\otimes H^*(BG_2)$$

in $H^*(BG_1)\otimes H^*(BG_2)$.

§ 2. Indices of Stiefel manifolds.

In this section we describe the $O(k)$-index of an $O(k)$-manifold $V_k(\boldsymbol{R}^n)$ along the line of Jaworowski [4], [5]. The orbit space $V_k(\boldsymbol{R}^n)/O(k)$ is a Grassmann manifold $G_k(\boldsymbol{R}^n)$. $BO(k)=G_k(\boldsymbol{R}^\infty)$ is a classifying space for free $O(k)$-actions, and has cohomology ring

$$H^*(BO(k)) = \boldsymbol{Z}_2[w_1, w_2, \cdots, w_k],$$

where each w_i is the i-th Stiefel-Whitney class of the universal k-plane bundle over $BO(k)$. Let $w=1+w_1+w_2+\cdots$ be the total Stiefel-Whitney class and $\bar{w}=1+\bar{w}_1+\bar{w}_2+\cdots$ be its dual class defined by the relation $w\bar{w}=1$ in $\boldsymbol{Z}_2[w_1, w_2, \cdots]$. Let $\tilde{J}(k, l)$ be the ideal in $\boldsymbol{Z}_2[w_1, w_2, \cdots]$ generated by $\bar{w}_{l+1}, \bar{w}_{l+2}, \cdots, \bar{w}_{l+k}$, and $J(k, l)$ be the image of $\tilde{J}(k, l)$ through the projection $\boldsymbol{Z}_2[w_1, w_2, \cdots]\to\boldsymbol{Z}_2[w_1, \cdots, w_k]$. Then we have

PROPOSITION 2.1 ([4], [5]).

$$\operatorname{Ind}^{O(k)}V_k(\boldsymbol{R}^n) = J(k, n-k).$$

§ 3. $O(k_1, \cdots, k_m)$-indices (1).

Let $0\leqq k\leqq l$ be integers. Let $T=\{A\in(\boldsymbol{R}^l)^k \mid \operatorname{rank} A<k\}$. Then $U_k(\boldsymbol{R}^l)=(\boldsymbol{R}^l)^k-T$ is the space of all (not necessarily orthonormal) k-frames in $\boldsymbol{R}^l$, and is invariant under the action of $O(k)$.

LEMMA 3.1. *$U_k(\boldsymbol{R}^l)$ is $O(k)$-equivariantly deformable to $V_k(\boldsymbol{R}^l)$.*

PROOF. There are well-known identifications:

$$\begin{aligned} G_k(\boldsymbol{R}^l) &= V_k(\boldsymbol{R}^l)/O(k) = O(l)/O(k)\times O(l-k) = U_k(\boldsymbol{R}^l)/GL(k;\boldsymbol{R}) \\ &= GL(l;\boldsymbol{R})/GL(k;\boldsymbol{R})_*\times GL(l-k;\boldsymbol{R}), \end{aligned}$$

and

$$U_k(\boldsymbol{R}^l)/O(k) = GL(l;\boldsymbol{R})/O(k)_*\times GL(l-k;\boldsymbol{R}),$$

where $GL(k;\boldsymbol{R})$ is the k-th general linear group over $\boldsymbol{R}$, and

$$H_*\times K = \left\{\begin{pmatrix} A & O \\ * & B\end{pmatrix} \middle| \begin{matrix} A\in H \\ B\in K\end{matrix}\right\}.$$

The canonical projection

$$p: U_k(\boldsymbol{R}^l)/O(k) \longrightarrow U_k(\boldsymbol{R}^l)/GL(k;\boldsymbol{R}) = V_k(\boldsymbol{R}^l)/O(k)$$

is a fibre bundle with fibre $GL(k;\boldsymbol{R})/O(k)$ (see Steenrod [9; § 7]). From the arguments of linear algebra $GL(k;\boldsymbol{R})/O(k)$ is identified with the k-th positive definite symmetric matrices, which is homeomorphic to $\boldsymbol{R}^{k(k+1)/2}$. Thus $GL(k;\boldsymbol{R})/O(k)$ is contractible, and p is a homotopy equivalence. Let $q: V_k(\boldsymbol{R}^l)/O(k)\to U_k(\boldsymbol{R}^l)/O(k)$ be a homotopy inverse of p. Let $\tilde{j}: V_k(\boldsymbol{R}^l)/O(k)\to U_k(\boldsymbol{R}^l)/O(k)$ be the map induced from the inclusion $j: V_k(\boldsymbol{R}^l)\subset U_k(\boldsymbol{R}^l)$. We see $p\tilde{j}=\mathrm{id}$ and $\tilde{j}p\simeq qp\tilde{j}pqp\simeq\mathrm{id}$. By the covering homotopy theorem (Palais [**7**; 2.4.3], Bredon [**1**; II.7.3]) we obtain an $O(k)$-map $\varphi: U_k(\boldsymbol{R}^l)\to U_k(\boldsymbol{R}^l)$ such that $\varphi(U_k(\boldsymbol{R}^l))\subset V_k(\boldsymbol{R}^l)$ and φ is $O(k)$-equivariantly homotopic to the identity of $U_k(\boldsymbol{R}^l)$. This shows that $U_k(\boldsymbol{R}^l)$ is $O(k)$-equivariantly deformable to $V_k(\boldsymbol{R}^l)$. □

We obtain the following by Propositions 1.1, 2.1 and Lemma 3.1.

PROPOSITION 3.2.

$$\mathrm{Ind}^{O(k)}U_k(\boldsymbol{R}^l)=J(k,\, l-k)\,.$$

Let $d(A)$ denote the sum of squares of determinants of all k-th square submatrices of $A\in(\boldsymbol{R}^l)^k$, here A is considered a $(k\times l)$-matrix. Then we obtain

LEMMA 3.3. (1) *$d(A)$ is $O(k)$-invariant, i.e., $d(A)=d(gA)$ for all $g\in O(k)$.*
(2) *$d(A)\neq 0$ if and only if* $\mathrm{rank}\, A=k$, *i.e., $A\in U_k(\boldsymbol{R}^l)$.*

Let $k_1, \cdots, k_m$ be positive integers and $l_1, \cdots, l_m$ nonnegative integers. For any i with $1\leqq i\leqq m$, let

$$T_i=\{A\in(\boldsymbol{R}^{l_i})^{k_i}\,|\,\mathrm{rank}\, A<k_i\}\,.$$

Then $T_1\times\cdots\times T_m$ is $O(k_1,\cdots,k_m)$-invariant and closed subspace of $(\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}$. Suppose $k_i\leqq l_i$ for all i and define a map

$$\alpha:(\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}-T_1\times\cdots\times T_m\longrightarrow U_{k_1}(\boldsymbol{R}^{l_1})*\cdots*U_{k_m}(\boldsymbol{R}^{l_m})$$

as follows. For $(A_1,\cdots,A_m)\in(\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}-T_1\times\cdots\times T_m$,

$$\alpha(A_1,\cdots,A_m)=[(d_1,A_1),\cdots,(d_m,A_m)]$$

where $d_i=d_i(A_1,\cdots,A_m)=d(A_i)/(d(A_1)+\cdots+d(A_m))$. If $A_i\notin U_{k_i}(\boldsymbol{R}^{l_i})$ then $d_i=0$ by Lemma 3.3. This shows the above definition is well-defined. Moreover it may be shown that α is an $O(k_1,\cdots,k_m)$-equivariant homotopy equivalence. Thus we see that if $k_i\leqq l_i$ for all i then

$$\begin{aligned}&\mathrm{Ind}^{O(k_1,\cdots,k_m)}((\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}-T_1\times\cdots\times T_m)\\&\quad=\mathrm{Ind}^{O(k_1,\cdots,k_m)}U_{k_1}(\boldsymbol{R}^{l_1})*\cdots*U_{k_m}(\boldsymbol{R}^{l_m})\,.\end{aligned}$$

If for some i, say $i=1$, $l_1<k_1$, then

$$\begin{aligned}&(\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}-T_1\times\cdots\times T_m\\&\quad=(\boldsymbol{R}^{l_1})^{k_1}\times((\boldsymbol{R}^{l_2})^{k_2}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}-T_2\times\cdots\times T_m)\,,\end{aligned}$$

and this has the same equivariant homotopy type as $(\boldsymbol{R}^{l_2})^{k_2}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}-T_2\times\cdots\times T_m$.

From the above arguments and Propositions 1.3, 1.4, 3.2 we obtain

PROPOSITION 3.4. *Let $k_1, \cdots, k_m$ be positive integers and $l_1, \cdots, l_m$ non-negative integers. Then*

$$\mathrm{Ind}^{O(k_1,\cdots,k_m)}((\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}-T_1\times\cdots\times T_m)\supset\bigotimes_{i=1}^{m}J(k_i, l_i-k_i)$$

in $H^*(BO(k_1, \cdots, k_m))=\otimes_{i=1}^{m}H^*(BO(k_i))$. Here we make the convention that $J(k_i, l_i-k_i)=H^*(BO(k_i))$ if $l_i<k_i$.

§4. $O(k_1, \cdots, k_m)$-indices (2).

In this section we will discuss the $O(k_1, \cdots, k_m)$-indices of $O(k_1, \cdots, k_m)$-manifolds $V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n)$ and $V_{k_1}(\boldsymbol{R}^{n_1})\times\cdots\times V_{k_m}(\boldsymbol{R}^{n_m})$. We first obtain

PROPOSITION 4.1. *If $x_i\in H^*(BO(k_i))$ does not belong to $\mathrm{Ind}^{O(k_i)}V_{k_i}(\boldsymbol{R}^{n-k_{i+1}-\cdots-k_m})$ for all i with $1\leqq i\leqq m$, then $x_1\otimes\cdots\otimes x_m$ does not belong to $\mathrm{Ind}^{O(k_1,\cdots,k_m)}V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n)$.*

PROOF. We will prove the assertion:

$$x_i\otimes\cdots\otimes x_m\notin\mathrm{Ind}^{O(k_i,\cdots,k_m)}V_{(k_i,\cdots,k_m)}(\boldsymbol{R}^n)$$

for all i.

This will be shown by downward induction on i. When $i=m$, this assertion is true by the assumption of the proposition. Then we assume

$$x_{i+1}\otimes\cdots\otimes x_m\notin\mathrm{Ind}^{O(k_{i+1},\cdots,k_m)}V_{(k_{i+1},\cdots,k_m)}(\boldsymbol{R}^n)\,.$$

There is a fibre bundle

$$V_{k_i}(\boldsymbol{R}^{n-k_{i+1}-\cdots-k_m})\xrightarrow{j_i}V_{(k_i,\cdots,k_m)}(\boldsymbol{R}^n)\xrightarrow{p_i}V_{(k_{i+1},\cdots,k_m)}(\boldsymbol{R}^n)\,,$$

where p_i is the projection to the last $k_{i+1}+\cdots+k_m$ vectors of $(k_i+\cdots+k_m)$-frames and j_i is the inclusion to the canonical fibre. There is a homotopy commutative diagram

$$\begin{array}{ccc}
V_{k_i}(\boldsymbol{R}^{n-k_{i+1}-\cdots-k_m})/O(k_i) & \xrightarrow{\alpha_1} & BO(k_i)\\
\tilde{j}_i\big\downarrow & & \big\downarrow j\\
V_{(k_i,\cdots,k_m)}(\boldsymbol{R}^n)/O(k_i, \cdots, k_m) & \xrightarrow{\alpha_2} & BO(k_i)\times\cdots\times BO(k_m)\\
\tilde{p}_i\big\downarrow & & \big\downarrow p\\
V_{(k_{i+1},\cdots,k_m)}(\boldsymbol{R}^n)/O(k_{i+1}, \cdots, k_m) & \xrightarrow{\alpha_3} & BO(k_{i+1})\times\cdots\times BO(k_m)\,,
\end{array}$$

where the vertical sequence on the left-hand side is the fibre bundle induced from the bundle above, α_1, α_2, α_3 are classifying maps for corresponding free actions, j is the inclusion to $BO(k_i)\times\{\mathrm{pt}\}$, and p is the projection. Since j^* and α_1^* are surjective on cohomology, $\tilde{j}_i^*$ is also surjective. Thus there exists a right inverse of $\tilde{j}_i^*$ as $\boldsymbol{Z}_2$-module homomorphism,

$$\theta : H^*(V_{k_i}(\boldsymbol{R}^{n-k_{i+1}-\cdots-k_m})/O(k_i)) \longrightarrow H^*(V_{(k_i,\cdots,k_m)}(\boldsymbol{R}^n)/O(k_i, \cdots, k_m)).$$

Moreover θ can be chosen so as to satisfy

$$\theta\alpha_1^*(x_i) = \alpha_2^*(x_i\otimes 1\otimes\cdots\otimes 1),$$

since $\tilde{j}_i^*\alpha_2^*(x_i\otimes 1\otimes\cdots\otimes 1)=\alpha_1^*(x_i)$. Applying the Leray-Hirsch theorem [8], we obtain an isomorphism

$$\begin{aligned}\Psi : H^*(V_{k_i}(\boldsymbol{R}^{n-k_{i+1}-\cdots-k_m})/O(k_i))\otimes H^*(V_{(k_{i+1},\cdots,k_m)}(\boldsymbol{R}^n)/O(k_{i+1}, \cdots, k_m))\\ \cong H^*(V_{(k_i,\cdots,k_m)}(\boldsymbol{R}^n)/O(k_i, \cdots, k_m))\end{aligned}$$

given by $\Psi(a\otimes b)=\theta(a)\cdot\tilde{p}_i^*(b)$. From the assumptions of the proposition and the induction we have

$$\alpha_1^*(x_i)\otimes\alpha_3^*(x_{i+1}\otimes\cdots\otimes x_m) \neq 0 .$$

Then we see

$$\begin{aligned}0 &\neq \Psi(\alpha_1^*(x_i)\otimes\alpha_3^*(x_{i+1}\otimes\cdots\otimes x_m))\\ &= \theta\alpha_1^*(x_i)\cdot\tilde{p}_i^*\alpha_3^*(x_{i+1}\otimes\cdots\otimes x_m)\\ &= \alpha_2^*(x_i\otimes 1\otimes\cdots\otimes 1)\cdot\alpha_2^*(1\otimes x_{i+1}\otimes\cdots\otimes x_m)\\ &= \alpha_2^*(x_i\otimes\cdots\otimes x_m) .\end{aligned}$$

This implies $x_i\otimes\cdots\otimes x_m\notin \mathrm{Ind}^{O(k_i,\cdots,k_m)}V_{(k_i,\cdots,k_m)}(\boldsymbol{R}^n)$. □

$V_{k_1}(\boldsymbol{R}^{n_1})\times\cdots\times V_{k_m}(\boldsymbol{R}^{n_m})$ is an $O(k_1, \cdots, k_m)$-manifold by product action. We obtain the following proposition by a similar way to the proof of Proposition 4.1.

PROPOSITION 4.2. *If $x_i\in H^*(BO(k_i))$ does not belong to $\mathrm{Ind}^{O(k_i)}V_{k_i}(\boldsymbol{R}^{n_i})$ for all i with $1\leqq i\leqq m$, then $x_1\otimes\cdots\otimes x_m$ does not belong to $\mathrm{Ind}^{O(k_1,\cdots,k_m)}V_{k_1}(\boldsymbol{R}^{n_1})\times\cdots\times V_{k_m}(\boldsymbol{R}^{n_m})$.*

§ 5. Maps of Stiefel manifolds.

THEOREM 5.1. *Let $k_1, \cdots, k_m$ be positive integers with $k_1+\cdots+k_m\leqq n$, and $l_1, \cdots, l_m$ nonnegative integers. Let*

$$f : V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n) \longrightarrow (\boldsymbol{R}^{l_1})^{k_1}\times\cdots\times(\boldsymbol{R}^{l_m})^{k_m}$$

be an $O(k_1, \cdots, k_m)$*-map. If*

$$l_i < n - \sum_{r=i+1}^{m} k_r \qquad \text{for all } i \text{ with } 1 \leqq i \leqq m, \tag{5.2}$$

then it follows

$$\mathrm{cup}_1(f^{-1}(T_1 \times \cdots \times T_m)/O(k_1, \cdots, k_m)) \geqq a,$$

where

$$T_i = \{A \in (\boldsymbol{R}^{l_i})^{k_i} \mid \mathrm{rank}\, A < k_i\},$$

and

$$a = mn - \sum_{i=2}^{m} (i-1)k_i - \sum_{i=1}^{m} \max\{k_i, l_i+1\} \geqq 0.$$

In particular $f^{-1}(T_1 \times \cdots \times T_m)$ *is nonempty.*

NOTE 5.3. (1) When $i=m$ in (5.2), $\sum_{r=i+1}^{m} k_r$ is understood to be zero.

(2) For i with $l_i < k_i$, (5.2) automatically follows from the assumption $k_1 + \cdots + k_m \leqq n$.

PROOF OF THEOREM 5.1. We see from Propositions 1.2 and 3.4

$$\mathrm{Ind}^{O(k_1, \cdots, k_m)} f^{-1}(T_1 \times \cdots \times T_m) \cdot \bigotimes_{i=1}^{m} J(k_i, l_i - k_i) \subset \mathrm{Ind}^{O(k_1, \cdots, k_m)} V_{(k_1, \cdots, k_m)}(\boldsymbol{R}^n) \tag{5.4}$$

in $\otimes_{i=1}^{m} H^*(BO(k_i))$. Let $w_j(i)$ and $\bar{w}_j(i)$ denote the j-th Stiefel-Whitney class and the j-th dual class in $H^*(BO(k_i))$, respectively. In what follows j may be negative in the notation $\bar{w}_j(i)$. In this case we make the convention $\bar{w}_j(i)=1$. Let

$$a_i = n - \sum_{r=i+1}^{m} k_r - \max\{k_i, l_i+1\}.$$

Note that a_i is nonnegative. We see

$$w_1(i)^{a_i} \bar{w}_{l_i - k_i + 1}(i) \notin \mathrm{Ind}^{O(k_i)} V_{k_i}(\boldsymbol{R}^{n - k_{i+1} - \cdots - k_m}),$$

since $\mathrm{Ind}^{O(k_i)} V_{k_i}(\boldsymbol{R}^{n-k_{i+1}-\cdots-k_m}) = J(k_i, n-k_i-\cdots-k_m)$ is generated by elements of degrees greater than $n-k_i-\cdots-k_m$. Thus it follows from Proposition 4.1

$$\bigotimes_{i=1}^{m} w_1(i)^{a_i} \bar{w}_{l_i-k_i+1}(i) \notin \mathrm{Ind}^{O(k_1, \cdots, k_m)} V_{(k_1, \cdots, k_m)}(\boldsymbol{R}^n).$$

Since

$$\bigotimes_{i=1}^{m} \bar{w}_{l_i-k_i+1}(i) \in \bigotimes_{i=1}^{m} J(k_i, l_i-k_i),$$

(5.4) implies

$$\bigotimes_{i=1}^{m} w_1(i)^{a_i} \notin \mathrm{Ind}^{O(k_1,\cdots,k_m)} f^{-1}(T_1 \times \cdots \times T_m)\,.$$

This shows

$$\mathrm{cup}_1(f^{-1}(T_1 \times \cdots \times T_m)/O(k_1, \cdots, k_m)) \geqq \sum_{i=1}^{m} a_i = a \geqq 0\,. \qquad \square$$

REMARK 5.5. The case of $m=1$ and $k_1=2$ in Theorem 5.1 is discussed in Jaworowski [4], [5].

REMARK 5.6. Let Ω_m denote the set of all permutations of $\{1, 2, \cdots, m\}$. An $O(k_1, \cdots, k_m)$-map

$$f: V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n) \longrightarrow (\boldsymbol{R}^{l_1})^{k_1} \times \cdots \times (\boldsymbol{R}^{l_m})^{k_m}$$

gives an $O(k_{\sigma(1)}, \cdots, k_{\sigma(m)})$-map

$$f_\sigma: V_{(k_{\sigma(1)},\cdots,k_{\sigma(m)})}(\boldsymbol{R}^n) \longrightarrow (\boldsymbol{R}^{l_{\sigma(1)}})^{k_{\sigma(1)}} \times \cdots \times (\boldsymbol{R}^{l_{\sigma(m)}})^{k_{\sigma(m)}}$$

for any $\sigma \in \Omega_m$. Since $f^{-1}(T_1 \times \cdots \times T_m/O(k_1, \cdots, k_m))$ and $f_\sigma^{-1}(T_{\sigma(1)} \times \cdots \times T_{\sigma(m)}/O(k_{\sigma(1)}, \cdots, k_{\sigma(m)}))$ are homeomorphic to each other, we obtain the following from Theorem 5.1:

If there exists $\sigma \in \Omega_m$ *such that*

$$l_{\sigma(i)} < n - \sum_{r=i+1}^{m} k_{\sigma(r)}$$

for all i *with* $1 \leqq i \leqq m$, *then*

$$\mathrm{cup}_1(f^{-1}(T_1 \times \cdots \times T_m)/O(k_i, \cdots, k_m)) \geqq a_\sigma\,,$$

where $a_\sigma = mn - \sum_{i=2}^{m}(i-1)k_{\sigma(i)} - \sum_{i=1}^{m} \max\{k_i, l_i+1\} \geqq 0$.

If we take $k_1 = \cdots = k_m = 1$, Remark 5.6 implies

COROLLARY 5.7. *Let* $l_1, \cdots, l_m$ *be nonnegative integers and suppose* $m \leqq n$. *Let* $f: V_{(1,\cdots,1)}(\boldsymbol{R}^n) \to \boldsymbol{R}^{l_1} \times \cdots \times \boldsymbol{R}^{l_m}$ *be a* $\boldsymbol{Z}_2^m$*-map. If there exists* $\sigma \in \Omega_m$ *such that* $l_{\sigma(i)} \leqq n-i$ *for all* i *with* $1 \leqq i \leqq m$, *then*

$$\mathrm{cup}_1(f^{-1}(O)/\boldsymbol{Z}_2^m) \geqq \frac{1}{2} m(2n-m-1) - \sum_{i=1}^{m} l_i \geqq 0\,,$$

where O *is the zero of* $\boldsymbol{R}^{l_1} \times \cdots \times \boldsymbol{R}^{l_m}$. *In particular* $f^{-1}(O)$ *is nonempty.*

NOTE 5.8. In case of $l_1 = \cdots = l_m = n-m$ the above corollary is just Fadell-Husseini [**2**; Theorem 5.5] and Fadell [**3**; Corollary 6.7].

REMARK 5.9. In connection with the remark given at the bottom of page

83 of Fadell-Husseini [2], we should note the following.

Suppose $l_i \leqq n$ and let $p_i : \boldsymbol{R}^n \to \boldsymbol{R}^{l_i}$ be the projection to the first l_i coordinates. Let $f : V_{(1,\dots,1)}(\boldsymbol{R}^n) \to \boldsymbol{R}^{l_1} \times \cdots \times \boldsymbol{R}^{l_m}$ be the restriction of $p_1 \times \cdots \times p_m$: $\boldsymbol{R}^n \times \cdots \times \boldsymbol{R}^n \to \boldsymbol{R}^{l_1} \times \cdots \times \boldsymbol{R}^{l_m}$. Then f is $\boldsymbol{Z}_2^m$-equivariant. Let $q_i : \boldsymbol{R}^n \to \{0\} \times \boldsymbol{R}^{n-l_i} \subset \boldsymbol{R}^n$ be the projection to the last $n-l_i$ coordinates. If there exists an m-frame $(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m) \in V_{(1,\dots,1)}(\boldsymbol{R}^n)$ such that $f(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m) = O$, then $\boldsymbol{v}_i = q_i(\boldsymbol{v}_i)$ for all i. We can choose $\sigma \in \Omega_m$ such that

$$n - l_{\sigma(1)} \leqq n - l_{\sigma(2)} \leqq \cdots \leqq n - l_{\sigma(m)} .$$

Then $\boldsymbol{v}_{\sigma(1)}, \cdots, \boldsymbol{v}_{\sigma(i)} \in \boldsymbol{R}^{n-l_{\sigma(i)}}$ and these vectors are linearly independent. This implies $i \leqq n - l_{\sigma(i)}$, or $l_{\sigma(i)} \leqq n-i$ for all i.

The contraposition of the above arguments shows that the condition $l_{\sigma(i)} \leqq n-i$ in Corollary 5.7 is the best possible for $\boldsymbol{Z}_2^m$-map $f : V_{(1,\dots,1)}(\boldsymbol{R}^n) \to \boldsymbol{R}^{l_1} \times \cdots \times \boldsymbol{R}^{l_m}$ to have zeros. This also means that we have a partial converse of Corollary 5.7.

If $l_i < k_i$ for all i, then $T_1 \times \cdots \times T_m = (\boldsymbol{R}^{l_1})^{k_1} \times \cdots \times (\boldsymbol{R}^{l_m})^{k_m}$ in Theorem 5.1 and Remark 5.6, and thus we have for all $\sigma \in \Omega_m$,

$$\begin{aligned} mn - \sum_{i=1}^{m} i k_{\sigma(i)} &\leqq \operatorname{cup}_1(V_{(k_1,\dots,k_m)}(\boldsymbol{R}^n)/O(k_1, \cdots, k_m)) \\ &\leqq \dim V_{(k_1,\dots,k_m)}(\boldsymbol{R}^n)/O(k_1, \cdots, k_m) . \end{aligned}$$

We see

$$\begin{aligned} &\dim V_{(k_1,\dots,k_m)}(\boldsymbol{R}^n)/O(k_1, \cdots, k_m) - (mn - \sum_{i=1}^{m} i k_{\sigma(i)}) \\ &= \sum_{i=1}^{m} (n - \sum_{r=1}^{m} k_{\sigma(r)})(k_{\sigma(i)} - 1) , \end{aligned}$$

and this equals zero if

$$(k_{\sigma(1)}, \cdots, k_{\sigma(m)}) = (1, \cdots, 1) \quad \text{or} \quad (n-m+1, 1, \cdots, 1) .$$

This implies

REMARK 5.10. (1)

$$\begin{aligned} \operatorname{cup}_1(V_{(1,\dots,1)}(\boldsymbol{R}^n)/\boldsymbol{Z}_2^m) &= \dim V_{(1,\dots,1)}(\boldsymbol{R}^n)/\boldsymbol{Z}_2^m \\ &= \frac{1}{2} m(2n-m-1) , \end{aligned}$$

where 1 is repeated m times in the notation $V_{(1,\dots,1)}(\boldsymbol{R}^n)$ above.

(2) If $(k_{\sigma(1)},\dots, k_{\sigma(m)}) = (n-m+1, 1, \cdots, 1)$ for some $\sigma \in \Omega_m$, then

$$\mathrm{cup}_1(V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n)/O(k_1,\ \cdots,\ k_m)) = \dim V_{(k_1,\cdots,k_m)}(\boldsymbol{R}^n)/O(k_1,\ \cdots,\ k_m)$$
$$= \frac{1}{2}(m-1)(2n-m).$$

§ 6. Inverse images of matrices with rank $\leqq j$.

Considering an $O(k)$-space $(\boldsymbol{R}^l)^k$, let $W_j=\{A\in(\boldsymbol{R}^l)^k \mid \mathrm{rank}\, A<j\}$. Then W_j is $O(k)$-invariant.

THEOREM 6.1. *Let* $f: V_k(\boldsymbol{R}^n)\to(\boldsymbol{R}^l)^k$ *be an* $O(k)$*-map.*

(1) *If* $0\leqq i<k\leqq n$ *and* $i\leqq l\leqq n-k+i$, *then*

$$\mathrm{cup}_1(f^{-1}(W_j)/H_{i+1}) \geqq \frac{1}{2}(k-i)(2n-2l-k+i-1) \geqq 0$$

for all $j\geqq i$, *where* H_{i+1} *is any subgroup of* $O(k)$ *conjugate to* $O(i+1, 1, \cdots, 1)$, 1 *repeated* $k-i-1$ *times. In particular* $f^{-1}(W_j)$ *is nonempty.*

(2) *If* $0<k\leqq n$ *and* $0\leqq l\leqq n-k$, *then*

$$\mathrm{cup}_k(f^{-1}(W_j)/O(k)) \geqq n-k-l \geqq 0$$

for all $j\geqq 0$.

Here $\mathrm{cup}_k(X)$ denote the longest length of nonzero monomial $x_1\cup\cdots\cup x_s$ in $H^*(X)$ with degree $x_i=k$ for all x_i.

PROOF OF THEOREM 6.1. (1) If $H_{i+1}=gO(i+1, 1, \cdots, 1)g^{-1}$ for $g\in O(k)$, the map $f^{-1}(W_j)/H_{i+1}\to f^{-1}(W_j)/O(i+1, 1, \cdots, 1)$ induced by the action of g is a homeomorphism. Thus it suffices to prove the case of $H_{i+1}=O(i+1, 1, \cdots, 1)$. Restricting $O(k)$-actions to $O(i+1, 1, \cdots, 1)$-actions and then considering f to be an $O(i+1, 1, \cdots, 1)$-map $V_{(i+1,1,\cdots,1)}(\boldsymbol{R}^n)\to(\boldsymbol{R}^l)^{i+1}\times\boldsymbol{R}^l\times\cdots\times\boldsymbol{R}^l$, we obtain from Theorem 5.1

$$\mathrm{cup}_1(f^{-1}(T_1\times\cdots\times T_{k-i})/O(i+1, 1, \cdots, 1)) \geqq \frac{1}{2}(k-i)(2n-2l-k+i-1),$$

where

$$T_1 = \{A\in(\boldsymbol{R}^l)^{i+1} \mid \mathrm{rank}\, A\leqq i\},$$
$$T_2 = \cdots = T_{k-i} = \{0\in\boldsymbol{R}^l\}.$$

Recalling the proof of Theorem 5.1, we see this estimation of cup_1 from the following fact:

$$\bigotimes_{r=1}^{k-i} w_1(r)^{a_r} \notin \mathrm{Ind}^{O(i+1,1,\cdots,1)} f^{-1}(T_1\times\cdots\times T_{k-i})$$

in $H^*(BO(i+1, 1, \cdots, 1)) = H^*(BO(i+1))\otimes H^*(BO(1))\otimes\cdots\otimes H^*(BO(1))$, where $a_r=n-k-l+i+r-1$. This implies

$$\bigotimes_{r=1}^{k-i} w_1(r)^{a_r} \notin \mathrm{Ind}^{O(i+1,1,\cdots,1)} f^{-1}(W_j)$$

since $f^{-1}(T_1 \times \cdots \times T_{k-i}) \subset f^{-1}(W_j)$, and hence

$$\mathrm{cup}_1(f^{-1}(W_j)/O(i+1, 1, \cdots, 1)) \geqq \sum_{r=1}^{k-i} a_r = \frac{1}{2}(k-i)(2n-2l-k+i-1).$$

(2) Considering the case of $i=0$ in (1) above, we have

$$\bigotimes_{r=1}^{k} w_1(r)^{a_r} \notin \mathrm{Ind}^{O(1,\cdots,1)} f^{-1}(O)$$

in $H^*(BO(1, \cdots, 1)) = H^*(BO(1)) \otimes \cdots \otimes H^*(BO(1))$, k times, where $a_r = n-k-l+r-1$, and $O \in (\boldsymbol{R}^l)^k$ is the zero. Letting $a=n-k-l$ and $w=\otimes_{r=1}^k w_1(r)$, we see $w^a \notin \mathrm{Ind}^{O(1,\cdots,1)} f^{-1}(O)$. There is a homotopy commutative diagram

$$\begin{array}{ccc}
f^{-1}(O)/O(1, \cdots, 1) & \xrightarrow{\ \alpha_1\ } & BO(1) \times \cdots \times BO(1) \\
\downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \varepsilon} \\
f^{-1}(O)/O(k) & \xrightarrow{\ \alpha_2\ } & BO(k) \\
\downarrow{\scriptstyle \beta_2} & \nearrow{\scriptstyle \alpha_3} & \\
f^{-1}(W_j)/O(k) & &
\end{array}$$

where α_1, α_2, and α_3 are classifying maps for corresponding free actions, β_1 and ε are induced from the inclusion $O(1, \cdots, 1) \subset O(k)$, and β_2 is induced from the inclusion $f^{-1}(O) \subset f^{-1}(W_j)$. From Milnor-Stasheff [**6**; § 7] we see $\varepsilon^*(w_k)=w$, where $w_k \in H^*(BO(k))$ is the k-th Stiefel-Whitney class. Since $\alpha_1^* \varepsilon^*(w_k^a) = \alpha_1^*(w^a) \neq 0$, $\alpha_3^*(w_k^a) \neq 0$ in $H^*(f^{-1}(W_j)/O(k))$. Hence

$$\mathrm{cup}_k(f^{-1}(W_j)/O(k)) \geqq a = n-k-l. \qquad \square$$

REMARK 6.2. Given a permutation $\sigma \in \Omega_k$, then we have an isomorphism $\varphi_\sigma \colon O(k) \to O(k)$ defined by

$$(a_{ij}) \longmapsto (a_{\sigma(i)\sigma(j)}).$$

There exists $g_\sigma \in O(k)$ such that $\varphi_\sigma(g) = g_\sigma g g_\sigma^{-1}$ for all $g \in O(k)$. Thus in Theorem 6.1, H_{i+1} can be taken to be $\varphi_\sigma(O(i+1, 1, \cdots, 1))$.

The following proposition will give a relation between $\mathrm{cup}_1(f^{-1}(W_j)/H_i)$ $(1 \leqq i \leqq k)$.

PROPOSITION 6.3. *Let X be a free $O(k)$-space. If $1 \leqq i_1 \leqq i_2 \leqq k$, then we obtain*

$$\mathrm{cup}_1(X/H_{i_1}) = \mathrm{cup}_1(X/H_{i_2}) + \frac{1}{2}(i_2 - i_1)(i_1 + i_2 - 1),$$

where H_i is a subgroup of $O(k)$ conjugate to $O(i, 1, \cdots, 1)$, 1 repeated $k-i$ times.

PROOF. It suffices to prove

$$\mathrm{cup}_1(X/O(i, 1, \cdots, 1)) = \mathrm{cup}_1(X/O(k)) + \frac{1}{2}(k-i)(k+i-1)$$

(cf. the top part of the proof of Theorem 6.1). The space of right cosets, $O(i) \times I_{k-i} \backslash O(k)$, is identified with $V_{k-i}(\boldsymbol{R}^k)$, where I_{k-i} is the $(k-i)$-th unit matrix. Then we obtain a fibre bundle

$$V_{k-i}(\boldsymbol{R}^k) \xrightarrow{\iota} X/O(i) \times I_{k-i} \xrightarrow{\beta} X/O(k) \tag{6.4}$$

(see Bredon [1; p. 113]). We give an action of $O(1, \cdots, 1) = \boldsymbol{Z}_2^{k-i}$ to $V_{k-i}(\boldsymbol{R}^k)$ in such a way $V_{k-i}(\boldsymbol{R}^k) = V_{(1,\dots,1)}(\boldsymbol{R}^k)$. $X/O(i) \times I_{k-i}$ has the free $\boldsymbol{Z}_2^{k-i}$-action such that its orbit space is $X/O(i, 1, \cdots, 1)$, and then ι is $\boldsymbol{Z}_2^{k-i}$-equivariant. There is a diagram

$$\begin{array}{ccc}
V_{(1,\dots,1)}(\boldsymbol{R}^k)/O(1, \cdots, 1) & \xrightarrow{\alpha_1} & BO(1) \times \cdots \times BO(1) \\
\tilde{\iota} \downarrow & & \downarrow \iota' \\
X/O(i, 1, \cdots, 1) & \xrightarrow{\alpha_2} & BO(i) \times BO(1) \times \cdots \times BO(1) \\
\tilde{\beta} \downarrow & & \\
X/O(k) & &
\end{array}$$

where the vertical sequence on the left-hand side is a fibre bundle given by passing (6.4) to orbit spaces, α_1 and α_2 are classifying maps for free actions, and ι' is the inclusion to $\{\mathrm{pt}\} \times BO(1) \times \cdots \times BO(1)$. Then the square is homotopy commutative. $\tilde{\iota}^*$ is surjective, since α_1^* is surjective as shown in Fadell-Husseini [2; p. 78] and ι'^* is also surjective. Let

$$\theta : H^*(V_{(1,\dots,1)}(\boldsymbol{R}^k)/O(1, \cdots, 1)) \longrightarrow H^*(X/O(i, 1, \cdots, 1))$$

be a right inverse of $\tilde{\iota}^*$ as module homomorphism. Leray-Hirsch theorem gives an isomorphism of modules,

$$H^*(V_{(1,\dots,1)}(\boldsymbol{R}^k)/O(1, \cdots, 1)) \otimes H^*(X/O(k)) \cong H^*(X/O(i, 1, \cdots, 1))$$

given by $x \otimes y \mapsto \theta(x) \cdot \tilde{\beta}^*(y)$. Since

$$\mathrm{cup}_1(V_{(1,\dots,1)}(\boldsymbol{R}^k)/O(1, \cdots, 1)) = \frac{1}{2}(k-i)(k+i-1)$$

by Remark 5.10 (1), there exist $x_1, \cdots, x_a \in H^1(V_{(1,\cdots,1)}(\boldsymbol{R}^k)/O(1, \cdots, 1))$ such that $x_1 \cdots x_a \neq 0$, where $a=(k-i)(k+i-1)/2$. We may assume $\theta(x_1 \cdots x_a)=\theta(x_1) \cdots \theta(x_a)$. This implies

$$\mathrm{cup}_1(X/O(i, 1, \cdots, 1)) = \mathrm{cup}_1(X/O(k))+\frac{1}{2}(k-i)(k+i-1) \qquad \square$$

REMARK 6.5. We have a partial converse of Theorem 6.1:
Let $0 \leqq i < k \leqq n$ and suppose $f^{-1}(W_i) \neq \emptyset$ for all $O(k)$-map $f: V_k(\boldsymbol{R}^n) \to (\boldsymbol{R}^l)^k$. Then $l \leqq n-k+i$.

For the proof it suffices to show the existence of an $O(k)$-map $f: V_k(\boldsymbol{R}^n) \to (\boldsymbol{R}^l)^k$ such that $f^{-1}(W_i)=\emptyset$ if $n-k+i<l$. Considering $A \in V_k(\boldsymbol{R}^n)$ to be a $(k \times n)$-matrix, let $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ be the column vectors of A, i.e., $A=(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n)$. If $n \leqq l$, we define $f(A)=(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n, \boldsymbol{0}, \cdots, \boldsymbol{0})$, $\boldsymbol{0}$ repeated $l-n$ times. Then f is $O(k)$-equivariant and rank $f(A)=k$. Thus $f^{-1}(W_i)=\emptyset$. If $l<n$, we define $f(A)=(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_l)$. Then f is also $O(k)$-equivariant, rank $f(A) \geqq k-(n-l)>i$, and hence $f^{-1}(W_i)=\emptyset$.

§7. Maps of products spaces.

THEOREM 7.1. *Let k_i, n_i, l_i $(1 \leqq i \leqq m)$ be integers with $0<k_i \leqq n_i$, and*

$$f: V_{k_1}(\boldsymbol{R}^{n_1}) \times \cdots \times V_{k_m}(\boldsymbol{R}^{n_m}) \longrightarrow (\boldsymbol{R}^{l_1})^{k_1} \times \cdots \times (\boldsymbol{R}^{l_m})^{k_m}$$

an $O(k_1, \cdots, k_m)$-map. If $0 \leqq l_i < n_i$ for all i, then we obtain

$$\mathrm{cup}_1(f^{-1}(T_1 \times \cdots \times T_m)/O(k_1, \cdots, k_m)) \geqq \sum_{i=1}^m n_i - \sum_{i=1}^m \max\{k_i, l_i+1\} \geqq 0,$$

where $T_i=\{A \in (\boldsymbol{R}^{l_i})^{k_i} \mid \mathrm{rank}\, A<k_i\}$.

PROOF. From Propositions 1.2 and 3.4 we have

$$\mathrm{Ind}^{O(k_1,\cdots,k_m)} f^{-1}(T_1 \times \cdots \times T_m) \cdot \bigotimes_{i=1}^m J(k_i, l_i-k_i)$$
$$\subset \mathrm{Ind}^{O(k_1,\cdots,k_m)} V_{k_1}(\boldsymbol{R}^{n_1}) \times \cdots \times V_{k_m}(\boldsymbol{R}^{n_m})$$

in $\otimes_{i=1}^m H^*(BO(k_i))$. Let $w_j(i)$ and $\bar{w}_j(i)$ be as in the proof of Theorem 5.1. Letting $a_i=n_i-\max\{k_i, l_i+1\}$, from Proposition 4.2 we see

$$\bigotimes_{i=1}^m w_1(i)^{a_i} \bar{w}_{l_i-k_i+1}(i) \notin \mathrm{Ind}^{O(k_1,\cdots,k_m)} V_{k_1}(\boldsymbol{R}^{n_1}) \times \cdots \times V_{k_m}(\boldsymbol{R}^{n_m}).$$

Since

$$\bigotimes_{i=1}^m \bar{w}_{l_i-k_i+1}(i) \in \bigotimes_{i=1}^m J(k_i, l_i-k_i),$$

we see

$$\bigotimes_{i=1}^{m} w_1(i)^{a_i} \notin \mathrm{Ind}^{O(k_1,\cdots,k_m)} f^{-1}(T_1\times\cdots\times T_m).$$

This implies

$$\mathrm{cup}_1(f^{-1}(T_1\times\cdots\times T_m)/O(k_1,\cdots,k_m)) \geqq \sum_{i=1}^{m} a_i \geqq 0.$$

This proves the theorem. □

If we take $k_1=\cdots=k_m=1$ in Theorem 7.1, we obtain

COROLLARY 7.2. *Let* $f: S^{n_1-1}\times\cdots\times S^{n_m-1}\to \boldsymbol{R}^{l_1}\times\cdots\times \boldsymbol{R}^{l_m}$ *be a* $\boldsymbol{Z}_2^m$*-map. If* $l_i<n_i$ *for all* i*, then we obtain*

$$\mathrm{cup}_1(f^{-1}(O)/\boldsymbol{Z}_2^m) \geqq \sum_{i=1}^{m}(n_i-l_i)-m,$$

where $O\in \boldsymbol{R}^{l_1}\times\cdots\times \boldsymbol{R}^{l_m}$ *is the zero.*

If $(n_1, n_2, \cdots, n_m)=(n, n-1, \cdots, n-m+1)$ and $l_1=\cdots=l_m=n-m$ in the above corollary, then the corollary is Fadell-Husseini [**2**; Theorems 5.1, 5.2] and Fadell [**3**; Corollary 6.2].

§ 8. Several equivalent versions.

We conclude this paper by giving several equivalent versions of Borsuk-Ulam theorem for Stiefel manifolds. Theorem 5.1 (or Theorems 6.1, 7.1) gives the following as a special case:

THEOREM 8.1. *If* $f: V_k(\boldsymbol{R}^{n+1})\to(\boldsymbol{R}^n)^k$ *is an* $O(k)$*-map, then* $f^{-1}(T)$ *is non-empty, where* $T=\{A\in(\boldsymbol{R}^n)^k \mid \mathrm{rank}\, A<k\}$.

We see from Lemma 3.1 that Theorem 8.1 is equivalent to

THEOREM 8.2. *There does not exist an* $O(k)$*-map* $V_k(\boldsymbol{R}^{n+1})\to V_k(\boldsymbol{R}^n)$.

Let $f: V_k(\boldsymbol{R}^{n+1})\to(\boldsymbol{R}^n)^k$ be an arbitrary map, and define its *average* with respect to a Haar measure in $O(k)$, $\mathrm{Av}\, f: V_k(\boldsymbol{R}^{n+1})\to(\boldsymbol{R}^n)^k$, by

$$\mathrm{Av}\, f(x)=\int_{g\in O(k)} g^{-1}f(gx)dg$$

for $x\in V_k(\boldsymbol{R}^{n+1})$. Then $\mathrm{Av}\, f$ is $O(k)$-equivariant, and $\mathrm{Av}\, f=f$ if f is already $O(k)$-equivariant.

We have one more equivalent version:

THEOREM 8.3. *If* $f: V_k(\boldsymbol{R}^{n+1})\to(\boldsymbol{R}^n)^k$ *is an arbitrary map, then there exists* $x\in V_k(\boldsymbol{R}^{n+1})$ *with* $\mathrm{rank\,Av}\, f(x)<k$.

If we take $k=1$ in Theorems 8.1, 8.2, 8.3, then the theorems reduce to the well-known versions of the classical Borsuk-Ulam theorem:

(1) *If* $f: S^n \to \boldsymbol{R}^n$ *is a* $\boldsymbol{Z}_2$*-map, then* $f^{-1}(0)$ *is nonempty.*

(2) *There does not exist a* $\boldsymbol{Z}_2$*-map* $S^n \to S^{n-1}$.

(3) *If* $f: S^n \to \boldsymbol{R}^n$ *is an arbitrary map, then there exists* $x \in S^n$ *with* $f(x)=f(-x)$, *i.e.,* $(f(x)-f(-x))/2=0$, *which is the average on* $\boldsymbol{Z}_2$.

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