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On the distribution of primes in short intervals

By Jerzy KACZOROWSKI and Alberto PERELLI

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1. Introduction.

In a recent paper [**P**], the second named author showed that an estimate of the form

$$J_1(N, H) = o(NH^2) \qquad \text{for} \quad H \geqq N^\theta, \tag{1}$$

where $0<\theta<1$ and

$$J_1(N, H) = \int_N^{2N} |\psi(x+H)-\psi(x)-H|^2 dx,$$

follows from an estimate of the form

$$\int_N^{2N} |E(x, T)|^2 dx = o\left(\frac{N^3}{T^2 L}\right) \qquad \text{for} \quad T \leqq N^{1-\theta} L. \tag{2}$$

Here $L=\log N$ and $E(x, T)$ denotes the remainder term in the classical explicit formula

$$\psi(x) = x - \sum_{|\gamma| \leqq T} \frac{x^\rho}{\rho} + E(x, T)$$

where $\rho=\beta+i\gamma$ runs over the non-trivial zeros of the Riemann zeta function. It is well known, see e.g. ch. 17 of Davenport [**D**], that

$$E(x, T) \ll \frac{x \log^2 x}{T}. \tag{3}$$

Since (2) is only a power of L stronger than the estimate which follows from (3), it may appear somewhat surprising that a bound of the form (2) implies a bound of the form (1), for every $0<\theta<1$.

In this paper we give a partial explanation of the above implication. Indeed, in [**K-P**] we obtain the following new form of the explicit formula. Let $0<\varepsilon<1/4$,

$$w(u) = \begin{cases} 1 & 0 \leqq u \leqq \frac{1}{2} \\ 2(1-u) & \frac{1}{2} \leqq u \leqq 1, \end{cases} \qquad \operatorname{sgn}(u) = \begin{cases} 1 & u>0 \\ 0 & u=0 \\ -1 & u<0 \end{cases}$$

$$G(x, T, n) = \frac{2}{T}\int_{T/2}^{T}\int_{\tau|\log(x/n)|}^{\infty} \frac{\sin u}{u} du\, d\tau,$$

$N^{\varepsilon} \leqq T \leqq N^{1-\varepsilon}$, $N/2 \leqq x \leqq 4N$ and $1 \leqq M \leqq L^{-9}\min(N^{1/16}, T^{1/5})$. Then

$$\psi(x) = x - \sum_{|\gamma| \leqq T} w\left(\frac{|\gamma|}{T}\right)\frac{x^{\rho}}{\rho} + R(x, T) \tag{4}$$

where

$$R(x, T) = \frac{1}{\pi}\sum_{x-NM/T<n\leqq x+NM/T} \Lambda(n)\,\mathrm{sgn}(x-n)G(x, T, n) + O_{\varepsilon}\left(\frac{N}{TM}\right), \tag{5}$$

see the Corollary in [**K-P**]. The main feature of the above explicit formula is that the error term consists of a "local problem" involving prime numbers plus a good error, thus providing a link between the quantities in (1) and (2). Using (4) and (5) we will show that estimates of the type (1) and (2) are in fact equivalent.

Let

$$\Delta_1(x, H) = \psi(x+H) - \psi(x) - H$$
$$\Delta_2(x, H) = \psi(x+H) - 2\psi(x) + \psi(x-H)$$

denote the "first" and "second" difference of primes in short intervals, and let

$$J_2(N, H) = \int_N^{2N} |\Delta_2(x, H)|^2 dx$$
$$I(N, T) = \int_N^{2N} |R(x, T)|^2 dx\,.$$

Since the function $G(x, T, n)$ behaves like $\min(1, (N/(T|n-x|))^2)$, see Lemma 1 of [**K-P**], it is clear from (5) that $R(x, T)$ is closely related to $-\Delta_2(x, H)$ with H around N/T. Therefore, since

$$\Delta_2(x, H) = \Delta_1(x, H) - \Delta_1(x-H, H),$$

one expects that a non-trivial bound for $J_1(N, H)$ implies a non-trivial bound for $I(N, T)$ with T around N/H. Indeed we have

THEOREM 1. *Let* $0<\varepsilon<1/4$, $N^{\varepsilon} \leqq T \leqq N^{1-\varepsilon}$ *and* $1 \leqq M \leqq N^{\varepsilon/6}$. *Then*

$$I(N, T) \ll_{\varepsilon} M^2 J_1(N, H) + \frac{N^3}{T^2}\left(\frac{\log(M+1)}{M}\right)^2,$$

where $H = N/TM$.

In the opposite direction, one cannot in general obtain non-trivial bounds for $\Delta_1(x, H)$ from non-trivial bounds for $\Delta_2(x, H)$. For example, Maier's [**Ma**] construction of short intervals containing more (resp. less) primes than expected apparently does not affect the behaviour of $\Delta_2(x, H)$. Moreover, it is not difficult to construct a sequence of integers, whose global behaviour reflects the behaviour of primes, for which $J_2(N, H)=o(NH^2)$ but $J_1(N, H)=\Omega(NH^2)$ for suitable values of H. Such an example will be given in section 4. However, the following result provides an implication from $I(N, T)$ to $J_1(N, H)$. Given $0<\varepsilon<1/4$ and $H, K\geqq 1$ let

$$J=\left[\frac{\log(L^{1/2}KN^{1-\varepsilon/2}/H)}{\log 2}\right]$$

and, for $j=1, \cdots, J$, let

$$H_j=\frac{2^j H}{4L^{1/2}K}, \qquad T_j=\frac{N}{100H_j}.$$

THEOREM 2. *Let* $0<\varepsilon<1/4$, $N^{\varepsilon}\leqq H\leqq N^{1-\varepsilon}$ *and* $1\leqq K\leqq \exp(cL^{1/4})$, $c>0$ *an absolute constant. Then*

$$J_1(N, H) \ll_{\varepsilon} H^2 \sum_{j=1}^{J} H_j^{-2} I(N, T_j)+\frac{NH^2}{K^2}.$$

In view of the example in sect. 4 and the above mentioned relation between $I(N, T)$ and $J_2(N, H)$, we see that the result in Theorem 2 depends on the particular structure of the primes.

Define

$$\theta_1=\inf\{\theta\in(0, 1): J_1(N, H)\ll_A NH^2L^{-A} \quad \text{for } H\geqq N^{\theta} \text{ and } A>0\}$$

$$\theta_2=\sup\left\{\theta\in(0, 1): I(N, T)\ll_A \frac{N^3}{T^2}L^{-A} \quad \text{for } T\leqq N^{\theta} \text{ and } A>0\right\}.$$

From Theorems 1 and 2 we easily obtain

COROLLARY. $\theta_1+\theta_2=1$.

One expects that $\theta_1=0$ and $\theta_2=1$, which would follow from the Density Hypothesis, see e.g. [**P**]. From an unconditional viewpoint, we only know that $\theta_1\leqq 1/6$, see e.g. [**H-B**], and hence $\theta_2\geqq 5/6$.

We finally remark that the non-trivial bound $J_1(N, H)=o(NH^2)$ implies, via Theorem 1, the non-trivial bound $I(N, T)=o(N^3/T^2)$, where H and T are suitably related. In the opposite direction, from Theorem 2 we see that in order to obtain the non-trivial bound $J_1(N, H)=o(NH^2)$ we need to assume the estimate $I(N, T)=o(N^3/T^2L)$, where H and T are again suitably related. This appears to be a defect of our method, due to the dissection argument at the beginning of the proof of Theorem 2. It would be desirable to obtain $J_1(N, H)=o(NH^2)$ from $I(N, T)=o(N^3/T^2)$.

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2. Proof of Theorem 1.

In the sequel c will denote a positive absolute constant, whose value will not necessarily be the same at each occurrence.

LEMMA 1. *Let* $1\leqq M^4\leqq T\leqq NM^{-2}$, $N/2\leqq x\leqq 4N$ *and*

$$\Sigma=\sum_{x-NM/T<n\leqq x+NM/T}\operatorname{sgn}(x-n)G(x,T,n).$$

Then

$$\Sigma\ll\frac{N}{TM}.$$

PROOF. Clearly

$$\int_{M^2}^{\infty}\frac{\sin u}{u}du\ll\frac{1}{M^2}.$$

Hence, since $M^2>\tau|\log(x/n)|$ for $T/2\leqq\tau\leqq T$ and $|n-x|\leqq NM/T$, we have that

$$\begin{aligned}\Sigma&=\frac{2}{T}\int_{T/2}^{T}\Big(\sum_{x-NM/T<n\leqq x+NM/T}\operatorname{sgn}(x-n)\int_{\tau|\log(x/n)|}^{M^2}\frac{\sin u}{u}du\Big)d\tau+O\Big(\frac{N}{TM}\Big)\\&=\frac{2}{T}\int_{T/2}^{T}\Big(\int_0^{M^2}\Sigma_1\frac{\sin u}{u}du\Big)d\tau+O\Big(\frac{N}{TM}\Big),\end{aligned}\tag{6}$$

where

$$\Sigma_1=\sum_{\substack{x-NM/T<n\leqq x+NM/T\\|\log(x/n)|\leqq u/\tau}}\operatorname{sgn}(x-n).$$

For $|n-x|\leqq NM/T$ we have

$$\left|\log\frac{x}{n}\right|=\frac{|n-x|}{x}+O\Big(\frac{M^2}{T^2}\Big),$$

hence for $T/2\leqq\tau\leqq T$ and $0\leqq u\leqq M^2$

$$\left|\Big\{n:\ |n-x|\leqq\frac{NM}{T},\ \left|\log\frac{x}{n}\right|\leqq\frac{u}{\tau}\Big\}\right|=\left|\Big\{n:\ |n-x|\leqq\min\Big(\frac{NM}{T},\frac{ux}{\tau}\Big)\Big\}\right|+O\Big(\frac{NM^2}{T^2}\Big).\tag{7}$$

The sum of $\operatorname{sgn}(x-n)$ over an interval symmetrical upon x is $O(1)$. Hence replacing the range of summation in Σ_1 by the set which appears in the expression on the right hand side of (7), from (7) we get

$$\Sigma_1 \ll 1+\frac{NM^2}{T^2} \ll 1+\frac{N}{TM^2}. \tag{8}$$

From (6) and (8) we get

$$\Sigma \ll \log M+\frac{N\log M}{TM^2}+\frac{N}{TM} \ll \frac{N}{TM}.$$

The proof of Theorem 1 is now as follows. We subdivide the interval $(x-NM/T, x+NM/T]$ into $P\ll M^2$ intervals of the form

$$I_j=(n_j, n_j+K], \ K=\frac{N}{TM}, \ n_j=x\pm jK, \ j=1, \cdots, P$$

(the two extreme intervals may be smaller). We may suppose that either $I_j\subset (0, x]$ or $I_j\subset [x, +\infty)$ for every j, hence $\mathrm{sgn}(x-n)$ is constant on each I_j.

If $N^{\varepsilon}\leqq T\leqq N^{1-\varepsilon}$ and $1\leqq M\leqq N^{\varepsilon/6}$, then the conditions of Lemma 1 and (4) and (5) are satisfied, hence by (5) and Lemma 1 we have

$$\begin{aligned} R(x, T) &= \frac{1}{\pi}\sum_{|n-x|\leq NM/T}(\Lambda(n)-1)\,\mathrm{sgn}(x-n)+O_{\varepsilon}\Big(\frac{N}{TM}\Big) \\ &\ll \sum_{j=1}^{P}|G(x, T, n_j)|\,\Big|\sum_{n\in I_j}(\Lambda(n)-1)\Big| \\ &\quad +\sum_{j=1}^{P}\sum_{n\in I_j}\Lambda(n)|G(x, T, n)-G(x, T, n_j)|+\frac{N}{TM}. \end{aligned} \tag{9}$$

By Lemma 1 of [**K-P**] we have

$$G(x, T, n_j) \ll \begin{cases} 1 & 1\leqq j\leqq M \\ \Big(\dfrac{M}{j}\Big)^2 & M\leqq j\leqq P \end{cases} \tag{10}$$

and, for $n\in I_j$,

$$G(x, T, n)-G(x, T, n_j) \ll \begin{cases} M^{-1} & 1\leqq j\leqq M \\ j^{-1} & M\leqq j\leqq P. \end{cases} \tag{11}$$

Hence from (9), (11) and the Brun-Titchmarsh inequality we get

$$R(x, T) \ll \sum_{j=1}^{P}|G(x, T, n_j)|\,\Big|\sum_{n\in I_j}(\Lambda(n)-1)\Big|+\frac{N}{TM}\log(M+1),$$

so that by the Cauchy-Schwarz inequality and (10) we obtain that

$$\begin{aligned} I(N, T) &\ll M\int_N^{2N}\Big(\sum_{j=1}^{P}|G(x, T, n_j)|\,\Big|\sum_{n\in I_j}(\Lambda(n)-1)\Big|^2\Big)dx+\frac{N^3}{T^2}\Big(\frac{\log(M+1)}{M}\Big)^2 \\ &\ll M\sum_{j=1}^{P}a_j\int_N^{2N}\Big|\sum_{n\in I_j}(\Lambda(n)-1)\Big|^2dx+\frac{N^3}{T^2}\Big(\frac{\log(M+1)}{M}\Big)^2, \end{aligned} \tag{12}$$

where

$$a_j = \begin{cases} 1 & 1 \leqq j \leqq M \\ \left(\frac{M}{j}\right)^2 & M \leqq j \leqq P. \end{cases}$$

Using the substitution $x \pm jK = y$ in the integral (12) we finally obtain that

$$\begin{aligned} I(N, T) &\ll M \sum_{j=1}^{P} a_j \int_{N \pm jK}^{2N \pm jK} \left| \sum_{n=y}^{y+K} (\Lambda(n)-1) \right|^2 dy + \frac{N^3}{T^2}\left(\frac{\log(M+1)}{M}\right)^2 \\ &\ll M\left(\sum_{j=1}^{P} a_j\right) \int_{N}^{2N} \left| \sum_{n=x}^{x+K} (\Lambda(n)-1) \right|^2 dx + \frac{N^3}{T^2}\left(\frac{\log(M+1)}{M}\right)^2 \\ &\ll M^2 J_1(N, K) + \frac{N^3}{T^2}\left(\frac{\log(M+1)}{M}\right)^2 \end{aligned}$$

and Theorem 1 follows.

3. Proof of Theorem 2 and Corollary.

We proceed on the lines of [**P**]. Let $e(x) = \exp(2\pi i x)$,

$$S(\alpha) = \sum_{n=N}^{2N} \Lambda(n) e(n\alpha), \qquad L(\alpha) = \sum_{m=1}^{H} e(-m\alpha),$$

$$T(\alpha) = \sum_{n=N}^{2N} e(n\alpha) \quad \text{and} \quad S(N, H) = \int_{-1/2}^{1/2} |(S(\alpha)-T(\alpha))L(\alpha)|^2 d\alpha\,.$$

We may assume that $H \in \boldsymbol{N}$. Using the Parseval identity and the Brun-Titchmarsh inequality we get

$$J_1(N, H) = S(N, H) + O(H^3)\,. \tag{13}$$

Hence we study $S(N, H)$. Let $0 < \xi < 1/2$ to be chosen later on. Then by the Parseval identity we have

$$\left(\int_{-1/2}^{-\xi} + \int_{\xi}^{1/2}\right) |(S(\alpha)-T(\alpha))L(\alpha)|^2 d\alpha \ll \frac{NL}{\xi^2} \tag{14}$$

and

$$\int_{-\xi}^{\xi} |(S(\alpha)-T(\alpha))L(\alpha)|^2 d\alpha \ll H^2 \int_{-\xi}^{\xi} |S(\alpha)-T(\alpha)|^2 d\alpha\,, \tag{15}$$

since

$$L(\alpha) \ll \min\left(H, \frac{1}{|\alpha|}\right).$$

Now we dissect the interval $(-\xi, \xi)$ into $2J+1 = O(L)$ subintervals of the form

$$A_0 = \left(-\frac{1}{N^{1-\varepsilon/2}}, \frac{1}{N^{1-\varepsilon/2}}\right) \quad \text{and} \quad A_j = \left(\pm\frac{\xi}{2^j}, \pm\frac{\xi}{2^{j-1}}\right), \quad j = 1, \cdots, J$$

where $J=[\log_2 \xi N^{1-\varepsilon/2}]$ and $\varepsilon>0$ sufficiently small. Due to the symmetry of $|S(\alpha)-T(\alpha)|$ we will consider only $A_j=(\xi/2^j, \xi/2^{j-1})$. By Gallagher's lemma, see e.g. ch. 1 of [**Mo**], we have

$$\int_{A_0}|S(\alpha)-T(\alpha)|^2 d\alpha \ll \frac{1}{N^{2-\varepsilon}}\int_0^{3N}|\psi(x+N^{1-\varepsilon/2})-\psi(x)-N^{1-\varepsilon/2}|^2 dx$$

and hence by standard techniques, see e.g. [**H-B**], we obtain that

$$\int_{A_0}|S(\alpha)-T(\alpha)|^2 d\alpha \ll N\exp(-cL^{1/4}). \tag{16}$$

For every non-trivial zero ρ of $\zeta(s)$ we define

$$T_\rho(\alpha)=\sum_{n=N}^{2N} a_{n,\rho}e(n\alpha), \qquad a_{n,\rho}=\int_n^{n+1} t^{\rho-1}dt$$

and let $T_j\in[N^{\varepsilon/4}, N^{1-\varepsilon/4}]$, $j=1, \cdots, J$, be parameters to be chosen later on. For $\alpha\in A_j$ we write

$$S(\alpha)-T(\alpha)=-\sum_{|\gamma|\leq T_j} w\Big(\frac{|\gamma|}{T_j}\Big)T_\rho(\alpha)+R_j(\alpha)$$

where the remainder $R_j(\alpha)$ is defined by

$$R_j(\alpha)=\sum_{n=N}^{2N} a_j(n)e(n\alpha), \quad a_j(n)=\Lambda(n)-1+\sum_{|\gamma|\leq T_j} w\Big(\frac{|\gamma|}{T_j}\Big)a_{n,\rho}.$$

Hence

$$\begin{aligned}\int_{A_j}|S(\alpha)-T(\alpha)|^2 d\alpha &\ll \Big(\sum_{|\gamma|\leq T_j}\Big(\int_{A_j}|T_\rho(\alpha)|^2 d\alpha\Big)^{1/2}\Big)^2+\int_{A_j}|R_j(\alpha)|^2 d\alpha\\ &= E_1(j)+E_2(j),\end{aligned}$$

say, so that by (16) we get

$$\int_{-\xi}^{\xi}|S(\alpha)-T(\alpha)|^2 d\alpha \ll \sum_{j=1}^{J}(E_1(j)+E_2(j))+N\exp(-cL^{1/4}). \tag{17}$$

In order to estimate $E_2(j)$ we need the following

LEMMA 2. *Let* $0<\varepsilon<1/4$, $N^{\varepsilon/4}\leqq T\leqq N^{1-\varepsilon/4}$ *and*

$$\tilde{I}(N, T)=\sum_{m=N}^{2N}|R(m, T)|^2.$$

Then

$$\tilde{I}(N, T)\ll_\varepsilon I(N, T)+\frac{N^3}{T^2}\exp(-cL^{1/4}).$$

PROOF. Let $N\leqq x\leqq 2N$. For $n\neq[x]$ we have $\mathrm{sgn}(x-n)=\mathrm{sgn}([x]-n)$, and the intervals $|x-n|\leqq MN/T$ and $|[x]-n|\leqq MN/T$ differ at most for the two endpoints. By (5) we have

$$\begin{aligned}
&R(x,T)-R([x],T)\\
&=\frac{1}{\pi}\sum_{2<|n-x|<MN/T-2}\Lambda(n)\operatorname{sgn}(x-n)(G(x,T,n)-G([x],T,n))+O\Big(\frac{N}{TM}\Big)\\
&\ll\frac{1}{T}\sum_{|n-x|\leq MN/T}\Lambda(n)\int_{T/2}^{T}\Big(\int_{\tau|\log([x]/n)|}^{\tau|\log(x/n)|}\frac{\sin u}{u}du\Big)d\tau+\frac{N}{TM}\\
&\ll T\sum_{|n-x|\leq MN/T}\Lambda(n)\Big|\Big|\log\frac{x}{n}\Big|-\Big|\log\frac{[x]}{n}\Big|\Big|+\frac{N}{TM}\\
&\ll\frac{T}{N}\sum_{|n-x|\leq MN/T}\Lambda(n)+\frac{N}{TM}\ll M+\frac{N}{TM}.
\end{aligned}$$

Hence, choosing $M=\exp(cL^{1/4})$, for any $m\in[N,2N]$ we have that

$$|R(m,T)|^2\ll\int_{m-1}^{m}|R(x,T)|^2dx+\frac{N^2}{T^2}\exp(-cL^{1/4})$$

and Lemma 2 follows summing over m.

Let

$$\xi_j=\frac{\xi}{2^{j-1}},\quad H_j=(2\xi_j)^{-1}\quad\text{and}\quad b_j(n)=\begin{cases}a_j(n) & N\leqq n\leqq 2N\\ 0 & \text{otherwise.}\end{cases}$$

By Gallagher's lemma we have

$$E_2(j)\ll\int_{-\xi_j}^{\xi_j}|R_j(\alpha)|^2d\alpha\ll H_j^{-2}\int_{N-H_j}^{2N}\Big|\sum_{x<n\leqq x+H_j}b_j(n)\Big|^2dx. \tag{18}$$

If $0\leqq h\leqq H_j$ and $N\leqq y,\ y+h\leqq 2N$, from (4) we get

$$\begin{aligned}
\sum_{y<n\leqq y+h}b_j(n)&=\psi(y+h)-\psi(y)-h+O(1)\\
&\quad+\sum_{|\gamma|\leqq T_j}w\Big(\frac{|\gamma|}{T_j}\Big)\frac{([y+h]+1)^{\rho}-([y]+1)^{\rho}}{\rho}\\
&\ll L+|R([y+h]+1,T_j)|+|R([y]+1,T_j)|.
\end{aligned}\tag{19}$$

From (19), Lemma 2 and the estimate $R(x,T)\ll N/T$, where $N/2\leqq x\leqq 3N$ and $N^{\varepsilon/4}\leqq T\leqq N^{1-\varepsilon/4}$ (see the Corollary in [**K-P**]), we obtain that

$$\begin{aligned}
\int_{N-H_j}^{2N}\Big|\sum_{x<n\leqq x+H_j}b_j(n)\Big|^2dx&\ll I(N,T_j)+\frac{N^3}{T_j^2}\exp(-cL^{1/4})+NL^2+H_j\frac{N^2}{T_j^2}\\
&\ll I(N,T_j)+\frac{N^3}{T_j^2}\exp(-cL^{1/4}).
\end{aligned}\tag{20}$$

Hence from (18) and (20) we get

$$E_2(j)\ll H_j^{-2}I(N,T_j)+\frac{N^3}{(H_jT_j)^2}\exp(-cL^{1/4}). \tag{21}$$

Now we estimate $E_1(j)$. Let P be a sufficiently large but fixed integer.

Then, as in [**P**], we have

$$T_\rho(\alpha) = \sum_{n=N}^{2N} n^{\rho-1}e(n\alpha)+\sum_{k=2}^{P}\frac{1}{k\,!}\sum_{n=N}^{2N}(\rho-1)\cdots(\rho-k+1)n^{\rho-k}e(n\alpha)+O(N^{\beta-10})$$

$$= T_{\rho,1}(\alpha)+\sum_{k=2}^{P}\frac{(\rho-1)\cdots(\rho-k+1)}{k\,!}T_{\rho,k}(\alpha)+O(N^{\beta-10}),$$

say. By Abel's inequality

$$T_{\rho,k}(\alpha) \ll N^{\beta-k}\max_{N\leq Y\leq 2N}\Big|\sum_{N\leq n\leq Y}e(f_\rho(n))\Big|,$$

where

$$f_\rho(n) = \frac{\gamma}{2\pi}\log n+\alpha n.$$

Since $T_j \leqq N$ we have that

$$T_\rho(\alpha) \ll N^{\beta-1}\max_{N\leq Y\leq 2N}|L_\rho(\alpha)|+N^{\beta-10}, \tag{22}$$

where

$$L_\rho(\alpha) = \sum_{n=N}^{Y}e(f_\rho(n)).$$

The term $N^{\beta-10}$ in (22) contributes to $E_1(j)$ at most $O(N\exp(-cL^{1/4}))$. Since

$$|f_\rho'(n)| = \left|\frac{\gamma}{2\pi n}+\alpha\right| \leqq \frac{1}{2},$$

by Lemma 4.8 of [**T**] we have

$$L_\rho(\alpha) = \int_N^Y e(f_\rho(x))dx+O(1).$$

Choosing $T_j=N/(100H_j)$, we have that

$$\left|\frac{\gamma}{2\pi n}\right| \leqq \frac{\xi_j}{100\pi} \quad\text{and}\quad \alpha \geqq \frac{\xi_j}{2}$$

for any ρ with $|\gamma|\leqq T_j$, $n\in[N, 2N]$ and $\alpha\in A_j$. Hence $|f_\rho'(n)|\gg|\alpha|$, and by Lemma 4.2 of [**T**] and (22) we get

$$T_\rho(\alpha) \ll \frac{N^{\beta-1}}{|\alpha|}, \qquad \alpha\in A_j.$$

Using Ingham's density estimate (see ch. 12 of [**Mo**]), Vinogradov's zero-free region and writing

$$N(\sigma, T) = |\{\rho = \beta+i\gamma : \zeta(\rho) = 0,\ \beta \geqq \sigma \text{ and } |\gamma| \leqq T\}|$$

we obtain

$$E_1(j) \ll \Big(\sum_{|\gamma|\leq T_j} H_j^{1/2} N^{\beta-1}\Big)^2 \ll L^c H_j N^{-2}(\sup_{0<\sigma<1} N^{\sigma} N(\sigma, T_j))^2$$
$$\ll N\exp(-cL^{1/4}). \tag{23}$$

From (14), (15), (17), (21) and (23) we obtain that

$$S(N, H) \ll H^2 \sum_{j=1}^{J} H_j^{-2} I(N, T_j) + \frac{NL}{\xi^2} + NH^2 \exp(-cL^{1/4}). \tag{24}$$

Theorem 2 follows now from (13) and (24), choosing $\xi = L^{1/2}K/H$, where $1 \leqq K \leqq \exp(cL^{1/4})$.

REMARK. We point out that in order to obtain the estimate (23) it is sufficient to use a density estimate of the form

$$N(\sigma, T) \ll T^{3/2-\sigma} L^c.$$

The proof of the Corollary is very simple. Let $\delta>0$ be a sufficiently small constant and $A>0$ be arbitrary. Choose $M=L^A$ in Theorem 1. Then

$$I(N, T) \ll M^2 J_1\Big(N, \frac{N}{TM}\Big) + \frac{N^3}{T^2} L^{-A} \ll \frac{N^3}{T^2} L^{-A}$$

provided $T \leqq N^{1-\theta_1-\delta} L^A$. Hence we have $\theta_2 \geqq 1-\theta_1-\delta$. On the other hand, choose $K=L^A$ in Theorem 2. Then

$$J_1(N, H) \ll H^2 \sum_{j=1}^{J} H_j^{-2} I(N, T_j) + NH^2 L^{-A} \ll NH^2 L^{-A+1}$$

provided $H \geqq N^{1-\theta_2+\delta} L^{A+1/2}$, which gives $\theta_1 \leqq 1-\theta_2+\delta$. Hence $1-\delta \leqq \theta_1+\theta_2 \leqq 1+\delta$, and the Corollary follows since $\delta>0$ is arbitrarily small.

4. An example.

In this section we construct a sequence of "prime-like" numbers having the following properties

i) the global behaviour reflects the global behaviour of primes

ii) the mean-square of the first and the second differences behave in qualitatively different ways.

For sake of simplicity we define such numbers only in the interval $[N, 2N]$, but it is of course possible to define them in $[1, +\infty)$.

Let $N^{\varepsilon} \leqq H \leqq N^{1-\varepsilon}$ be fixed and define

$$f(t) = \begin{cases} \frac{1}{N}\Big(1+\frac{1}{2}\sin\Big(\frac{2\pi t}{H}\Big)\Big) & N \leqq t \leqq 2N \\ 0 & \text{otherwise,} \end{cases}$$

$$F(x)=\int_{-\infty}^{x} f(t)dt.$$

For $k \geqq 1$ define

$$p_k = \min\left\{m \leqq 2N:\ \frac{N}{L}F(m) \geqq k\right\},$$

$$\tilde{\pi}(x) = \sum_{p_k \leqq x} 1$$

$$\tilde{\Delta}_1(x,\ K) = \tilde{\pi}(x+K) - \tilde{\pi}(x) - \frac{K}{L}$$

$$\tilde{\Delta}_2(x,\ K) = \tilde{\Delta}_1(x,\ K) - \tilde{\Delta}_1(x-K,\ K).$$

We will briefly sketch a proof of the following estimates

$$\tilde{\pi}(x+K) - \tilde{\pi}(x) \sim \frac{K}{L} \quad \text{if} \quad H = o(K) \quad \text{and} \quad x \in [N,\ 2N] \tag{25}$$

$$\tilde{J}_1(N,\ K) = \int_N^{2N} |\tilde{\Delta}_1(x,\ K)|^1 dx \sim \frac{1}{8}\frac{NK^2}{L^2} \quad \text{if} \quad K = o(H) \tag{26}$$

$$\tilde{J}_2(N,\ K) = \int_N^{2N} |\tilde{\Delta}_2(x,\ K)|^2 dx \ll \left(\frac{K}{H}\right)^2 \frac{NK^2}{L^2} + N \quad \text{if} \quad K = o(H). \tag{27}$$

Choosing e.g. $H = N^{1/7}$ we see that the properties i) and ii) are satisfied for suitable values of K.

Let

$$p_{k-1} < x \leqq p_k < \cdots < p_{k'} \leqq x+K < p_{k'+1}.$$

Since

$$F(m+1) - F(m) \ll \frac{1}{N},$$

we have that

$$\frac{N}{L}F(p_j) = j + O\left(\frac{1}{L}\right) \tag{28}$$

and

$$p_{j+1} - p_j = O(L). \tag{29}$$

From (28) and (29) we get

$$\begin{aligned}
\tilde{\pi}(x+K) - \tilde{\pi}(x) &= k' - k + O(1) \\
&= \frac{N}{L}(F(x+K) - F(x)) + O(1) \\
&= \frac{1}{L}\int_x^{x+K}\left(1 + \frac{1}{2}\sin\left(\frac{2\pi t}{H}\right)\right)dt + O(1) \\
&= \frac{K}{L}\left(1 + O\left(\frac{H}{K}\right)\right)
\end{aligned}$$

which gives (25). In a similar way we get

$$\tilde{\Delta}_1(x, K) = \frac{1}{L}\int_x^{x+K}\left(1+\frac{1}{2}\sin\left(\frac{2\pi t}{H}\right)\right)dt - \frac{K}{L} + O(1)$$

$$= -\frac{H}{4\pi L}\left(\cos\left(\frac{2\pi(x+K)}{H}\right) - \cos\left(\frac{2\pi x}{H}\right)\right) + O(1). \tag{30}$$

Hence by well known trigonometric identities we obtain

$$\tilde{J}_1(N, K) = \frac{H^2}{4\pi^2 L^2}\sin^2\left(\frac{\pi K}{H}\right)\int_N^{2N}\sin^2\left(\frac{\pi(2x+K)}{H}\right)dx + O\left(\frac{NK}{L}\right)$$

$$\sim \frac{1}{8}\frac{NK^2}{L^2},$$

which gives (26). Finally, from (30) and well known trigonometric identities we obtain that

$$|\tilde{\Delta}_2(x, K)|^2 \ll \frac{H^2}{4\pi^2 L^2}\sin^2\left(\frac{\pi K}{H}\right)\left|\sin\left(\frac{\pi(2x+K)}{H}\right) - \sin\left(\frac{\pi(2x-K)}{H}\right)\right|^2 + 1$$

$$\ll \frac{K^2}{L^2}\left|\int_{\pi(2x-K)/H}^{\pi(2x+K)/H}\cos t\, dt\right|^2 + 1$$

$$\ll \frac{K^2}{L^2}\left(\frac{K}{H}\right)^2 + 1$$

and (27) follows by integration over $[N, 2N]$.

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Jerzy KACZOROWSKI
Institute of Mathematics
A. Mickiewicz University
Poznań
Poland

Alberto PERELLI
Dipartimento di Matematica
Universitá di Genova
Via L. B. Alberti 4
16132 Genova
Italy