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Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface

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0. Introduction.

In [16] Narasimhan and Seshadri proved that every stable vector bundle on a compact Riemann surface comes from an irreducible projective unitary representation of the fundamental group. In other words there exists an irreducible Hermitian Einstein metric on every stable vector bundle. In [2] Atiyah and Bott observed that the moduli space of stable vector bundles is considered as a Kähler quotient of the space of all holomorphic structures by the gauge group. In [6] Donaldson gave a different proof of the theorem of Narasimhan and Seshadri in this context. This theorem was generalized to higher dimensional cases by Donaldson [7, 8] and Uhlenbeck and Yau [19].

In [11] Hitchin extended this theory in another direction. He introduced the notion of a Higgs bundle, which is a generalization of a holomorphic vector bundle. He also introduced the notion of stability for Higgs bundles and showed that there exist irreducible Hermitian Einstein metrics on stable Higgs bundles and that stable Higgs bundles correspond to irreducible projective (not necessary unitary) representations of the fundamental group. He also pointed out that the moduli space of stable Higgs bundles can be viewed as a hyperkähler quotient. In [17] Simpson generalized this result to higher dimensional cases.

It seems quite natural to generalize these results to the case when base spaces are noncompact. In [15] Mehta and Seshadri introduced the notion of a parabolic vector bundle, which is a pair of a vector bundle and flags of fibers over some points called parabolic points. They showed that every stable parabolic vector bundle comes from an irreducible projective unitary representations of the fundamental group of the complement of the parabolic points. In [3] Biquard gave a different proof of this theorem from a gauge theoretical point of view by choosing the appropriate Sobolev completion of the space of all parabolic holomorphic structures.

In the case of stable parabolic Higgs bundles Simpson [17] showed the existence of Hermitian Einstein metrics. To do this, he solved a PDE in some function space. However to construct the moduli space of stable parabolic Higgs bundles, we must solve this PDE in a smaller function space.

In this paper we give the appropriate Sobolev completion of the space of all parabolic Higgs structures and show the existence of Hermitian Einstein metrics on stable parabolic Higgs bundles in this function space. Then we construct the moduli space of stable parabolic Higgs bundles as a hyperkähler quotient by the gauge group.

This paper is organized as follows. In section 1 we fix our notation and state our results. In section 2 we discussed the Sobolev completion of the space of all parabolic Higgs structures and construct the moduli space of stable parabolic Higgs bundles. In section 3 we construct and study the moduli space of irreducible Hermitian Einstein parabolic Higgs bundles. In section 4 we show the existence of Hermitian Einstein metrics on stable parabolic Higgs bundles so that we identify the moduli space of stable parabolic Higgs bundles and that of irreducible Hermitian Einstein parabolic Higgs bundles.

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After completing this work, the referees informed the author that in [**20**] Nasatyr proved a construction of the parabolic Higgs bundle moduli space by using orbifold methods. The author would like to thank the referees for it.

1. Notation and the results.

Let Σ be a compact Riemann surface with a Kähler form ω. We normalize the volume of Σ to be 1. Let E be a smooth vector bundle on Σ. We fix a finite set of points $P_1, \cdots, P_n$ of Σ, which we call parabolic points. We set $\Sigma_0=\Sigma\setminus\{P_1, \cdots, P_n\}$. Moreover at each parabolic point P_i we fix a flag and an increasing sequence of real numbers called weights:

$$E_{P_i}=F_1E_{P_i}\supsetneqq F_2E_{P_i}\supsetneqq\cdots\supsetneqq F_{a_i}E_{P_i}\supsetneqq F_{a_i+1}E_{P_i}=\{0\},$$

$$w_1^{(i)}<w_2^{(i)}<\cdots<w_{a_i}^{(i)},$$

where we assume $w_{a_i}^{(i)}-w_1^{(i)}<1$. We define $\alpha_k^{(i)}$ $(1\leqq k\leqq r)$ by

$$\alpha_k^{(i)}=w_j^{(i)} \qquad \text{if } r-\dim F_jE_{P_i}<k\leqq r-\dim F_{j+1}E_{P_i},$$

where $r=\operatorname{rank} E$. We define the parabolic degree by

$$\operatorname{pardeg} E=\deg E+\sum_{i=1}^{n}\sum_{k=1}^{r}\alpha_k^{(i)},$$

where $\deg E$ is the degree of E in the usual sense. Moreover we set

$$\mu=\frac{\operatorname{pardeg} E}{\operatorname{rank} E}.$$

We fix a Hermitian metric K on E, which is smooth and non singular on Σ_0 but singular at P_i as follows. Let U_i be a neighbourhood of P_i. We fix a local coordinate z_i on U_i with $z_i(P_i)=0$. Then we take a smooth frame $\{e_k^{(i)}\}$ for $E|U_i$ such that $F_jE_{P_i}$ is spanned by $\{e_k^{(i)}(P_i)\,|\,\alpha_k^{(i)}\geqq\omega_j^{(i)}\}$ for $1\leqq j\leqq a_i$. Then define K so that $\{e_k^{(i)}/|z_i|^{\alpha_k^{(i)}}\}$ is a unitary frame for $E|\Sigma_0$.

We shall write $\mathscr{B}'$ for the space of smooth holomorphic structures on E. For $\bar\partial_B\in\mathscr{B}'$ let d_B, R_B denote the Hermitian connection on E with respect to K induced by $\bar\partial_B$ and its curvature, let d_B^Z, R_B^Z denote the induced connection on $\wedge^r E$ and its curvature. We fix $\bar\partial_{B_0}\in\mathscr{B}$ such that

$$\frac{\sqrt{-1}}{2\pi}\Lambda R_{B_0}^Z=\mu\,\mathrm{id}_{\wedge^r E},$$

where Λ is the contraction with the Kähler form ω. It is easy to see that there always exists a holomorphic structure that satisfies the above condition. We set

$$\mathscr{B}=\{\bar\partial_B\in\mathscr{B}'\,|\,d_B^Z=d_{B_0}^Z\}.$$

At each P_i we define

$$B_i=\{g\in\operatorname{End}E_{P_i}\,|\,g(F_jE_{P_i})\subset F_jE_{P_i}\quad\text{for any } j\}$$

$$N_i=\{g\in\operatorname{End}E_{P_i}\,|\,g(F_jE_{P_i})\subset F_{j+1}E_{P_i}\quad\text{for any } j\}.$$

We set

$$\Omega^0(\operatorname{ParEnd}E)=\{g\in\Omega^0(\operatorname{End}E)\,|\,g_{P_i}\in B_i\}$$

$$\mathscr{G}^C=\{g\in\Omega^0(\operatorname{ParEnd}E)\,|\det g_x=1\quad\text{for any } x\in\Sigma\}.$$

Let End^0E denote the vector bundle of trace free endomorphisms of E. Now we can define the space of parabolic Higgs structures $\mathscr{D}$ as follows.

DEFINITION 1.1. We say $D''=\bar\partial_B+\theta\in\mathscr{D}$ if the following conditions hold:

(1) $\bar\partial_B\in\mathscr{B}$.

(2) θ is an End^0E valued $\bar\partial_B$-meromorphic (1, 0) form on Σ and $\bar\partial_B$-holomorphic on Σ_0.

(3) θ has a pole of at most 1st order with the residue in N_i at each P_i.

For $D''\in\mathscr{D}$ we call a pair (E, D'') a parabolic Higgs bundle. We define the right action of $\mathscr{G}^C$ on $\mathscr{D}$ by

$$D''\longmapsto g^{-1}\circ D''\circ g\qquad\text{for any } g\in\mathscr{G}^C,\ D''\in\mathscr{D}.$$

Let $(E, D''=\bar\partial_B+\theta)$ be a parabolic Higgs bundle. Let V be a subbundle of E. We say V is a sub Higgs bundle if $D''\Omega^0(V)\subset\Omega^1(V)$. This condition is equivalent to that V is a $\bar{\partial}_B$-holomorphic subbundle and $\theta(V)\subset K\otimes V$, where K is the canonical bundle of Σ. Next we define the induced parabolic structure on V

$$V_{P_i}=F_1V_{P_i}\supsetneqq F_2V_{P_i}\supsetneqq\cdots\supsetneqq F_{b_i}V_{P_i}\supsetneqq F_{b_i+1}V_{P_i}=\{0\},$$

$$x_1^{(i)}<x_2^{(i)}<\cdots<x_{b_i}^{(i)}.$$

Taking the greatest k such that $V_{P_i}\subset F_kE_{P_i}$, then we define $x_1^{(i)}=w_k^{(i)}$. To define $F_jV_{P_i}$ and $x_j^{(i)}$ inductively, assume $x_{j-1}^{(i)}=w_k^{(i)}$ and $F_{j-1}V_{P_i}=V_{P_i}\cap F_kE_{P_i}$. Then we define $F_jV_{P_i}=V_{P_i}\cap F_{k+1}E_{P_i}$ and, taking the greatest l such that $F_jV_{P_i}\subset F_lE_{P_i}$, we set $x_j^{(i)}=w_l^{(i)}$.

Now we can introduce the notion of stability for parabolic Higgs bundles.

DEFINITION 1.2. We say $D''\in\mathcal{D}$ is stable if for any sub Higgs bundle V of (E, D''),

$$\frac{\operatorname{pardeg} V}{\operatorname{rank} V}<\mu.$$

We set $\mathcal{D}^{st}=\{D''\in\mathcal{D}\mid D''$ is stable$\}$. Since $\mathcal{G}^C$ preserves $\mathcal{D}^{st}$, we can construct the moduli space of stable parabolic Higgs bundles with fixed determinant and parabolic structures as $\mathcal{D}^{st}/\mathcal{G}^C$. In this paper we study this moduli space.

First we give the appropriate Sobolev completion of $\mathcal{B}$ and $\mathcal{D}$. To do this, we use the weighted Sobolev norm $\|\ \|_{D_k^p}$. See Section 2 for the precise definition. Let $\mathcal{D}_1^p$, $\mathcal{G}^C{}_2^p$ be the completions of $\mathcal{D}$ and $\mathcal{G}^C$ with respect to the norms $\|\ \|_{D_1^p}$ and $\|\ \|_{D_2^p}$ respectively. Then we show the following.

PROPOSITION 1.3. (2.7). *There exists $p>1$ such that the natural map*

$$i:\mathcal{D}/\mathcal{G}^C\longrightarrow\mathcal{D}_1^p/\mathcal{G}^C{}_2^p$$

is bijective.

Now we fix $p>1$ in Proposition 1.3. So we can define $\mathcal{D}^{st}{}_1^p$ naturally and the quotient space $\mathcal{D}^{st}{}_1^p/\mathcal{G}^C{}_2^p$ is the moduli space of stable parabolic Higgs bundles.

Let $\mathcal{E}_1^p$ be the Sobolev completion of $\mathcal{B}\times\Omega^{1,0}(\operatorname{End}^0E)$. As in the usual Higgs bundle case, there exists a hyperkähler structure $(g; I, J, K)$ on $\mathcal{E}_1^p$, which is preserved under the action of the gauge group $\mathcal{G}_2^p$. There exist the moment maps μ_1, μ_2, μ_3 corresponding to I, J, K respectively. Then we have $\mathcal{D}_1^p=\mu_2^{-1}(0)\cap\mu_3^{-1}(0)\subset\mathcal{E}_1^p$. For $D''\in\mathcal{D}_1^p$, $\mu_1(D'')=R_D^\perp$, where $R_D^\perp$ is a trace free part of the curvature of the connection on E corresponding to D''. See Section 3.1 for detail. So we define $\mathcal{D}_{HE}{}_1^p=\bigcap_{i=1}^3\mu_i^{-1}(0)$. Define $\mathcal{D}_{HE}^{irr}{}_1^p=\mathcal{D}_{HE}{}_1^p\cap\mathcal{D}^{irr}{}_1^p$, where $\mathcal{D}^{irr}{}_1^p$ is the space of irreducible parabolic Higgs structures on E. So we can construct the moduli space of irreducible Hermitian Einstein parabolic Higgs bundles as $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$, which is a hyperkähler quotient of $\mathcal{E}_1^p$ by $\mathcal{G}_2^p$ in the sense of [13]. Then we have the following.

PROPOSITION 1.4 (4.2). $\mathcal{D}_{HE}^{irr}{}_1^p \subset \mathcal{D}^{st}{}_1^p$.

Now we can identify the moduli space of stable parabolic Higgs bundles with that of irreducible Hermitian Einstein parabolic Higgs bundles as follows.

THEOREM 1.5 (4.3). *The natural map*

$$j: \mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p \longrightarrow \mathcal{D}^{st}{}_1^p/\mathcal{G}^C{}_2^p$$

is bijective.

Since Hermitian Einstein metrics is not smooth at parabolic points, the Sobolev completion is essential for the above bijection. The above theorem implies that there exists a unique Hermitian Einstein metric on every stable parabolic Higgs bundle. By studying $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$, we can show the following theorem.

THEOREM 1.6 (3.9, 3.10, 3.11). *The moduli space $\mathcal{D}^{st}/\mathcal{G}^C$ is a smooth hyperkähler manifold, whose complex dimension is*

$$2\{(g-1)(r^2-1)+\sum_{i=1}^{n}\dim_C N_i\},$$

If $\mathcal{D}_{HE}^{irr}{}_1^p=\mathcal{D}_{HE}{}_1^p$, the Riemannian metric on the moduli space is complete.

2. Sobolev completion of the space of parabolic Higgs structures.

In [3] Biquard introduced the appropriate Sobolev completion of the space of parabolic holomorphic structures. In this section we observe this completion also gives suitable settings in the case of parabolic Higgs bundles.

2.1. Weighted Sobolev spaces. In this subsection we review weighted Sobolev spaces. We set

$$U=\{z=x+\sqrt{-1}y=\rho\exp\sqrt{-1}\theta\in C\,|\,\rho\leqq 1\}.$$

Let L_k^p denote the usual Sobolev space of functions on U with k derivatives in L^p. For $f\in C^\infty(U)$ we define

$$\|f\|_{W_{k,\delta}^p}=\left\{\int_U\sum_{i+j\leq k}\left|\rho^{i+j-\delta}\frac{d^i}{dx^i}\frac{d^j}{dy^j}f\right|^p\frac{dxdy}{\rho^2}\right\}^{1/p}.$$

We set $\|f\|_{W_k^p}=\|f\|_{W^p_{k,k-2/p}}$, that is,

$$\|f\|_{W_k^p}=\left\{\int_U\sum_{i+j\leq k}\left|\rho^{i+j-k}\frac{d^i}{dx^i}\frac{d^j}{dy^j}f\right|^p dxdy\right\}^{1/p}.$$

Let W_k^p be the completion of $C^\infty(U)$ by the norm $\| \|_{W_k^p}$. If we write this norm in the coordinate (θ, ζ) on $U\setminus\{0\}$, where $\zeta=-\log\rho$, then this norm equivalent to

$$\|f\|'_{W_{k,\delta}^p}=\left\{\int_U \sum_{i+j\leq k}\left|e^{\delta\zeta}\frac{d^i}{d\theta^i}\frac{d^j}{d\zeta^j}f\right|^p d\theta d\zeta\right\}^{1/p}.$$

Therefore this norm is essentially the same as the one treated in Lockhart and McOwen [**14**]. Clearly we have the following lemma.

LEMMA 2.1. *Assume* $k>0$. *If* $f\in W_k^p$ *then* $f/\rho\in W_{k-1}^p$.

Biquard [**3**] showed the following lemma.

LEMMA 2.2. *Assume that for nonnegative integer* l, $l-1<k-2/p<l$ *holds. Then we have*

$$W_k^p=\{f\in L_k^p \mid f(0)=0, \cdots, (\nabla^{l-1}f)(0)=0\}$$

Moreover $\| \|_{W_k^p}$ *and* $\| \|_{L_k^p}$ *are equivalent in* W_k^p.

By this lemma we have for $1<p<2$

$$W_0^p=L^p, \qquad W_1^p=L_1^p, \qquad W_2^p=\{f\in L_2^p \mid f(0)=0\}.$$

We need the following lemma later.

LEMMA 2.3. *Assume* $0<\varepsilon<1$, $1<p<2/(2-\varepsilon)$, $2/(1+\varepsilon)$. *Then the following holds.*

(1) ρ^ε, $\rho^{-\varepsilon+1}\in W_2^p$.

(2) ρ^{-1} *does not belong to* L_1^p.

(3) $\rho^\varepsilon\times C^\infty(U)$ *is dense in* L_1^p.

(4) $\rho^{-\varepsilon}\times C^\infty(U)$ *is dense in* L_1^p.

PROOF. (1), (2) are clear. For (3) we fix $g\in L_1^p$. We can find $h\in C^\infty(U)$, which is near to g in L_1^p. Since $\rho^{-\varepsilon}h\in L_1^p$, we can find $k\in C^\infty$, which is near to $\rho^{-\varepsilon}h$ in L_1^p. Then $\rho^\varepsilon k$ is near to g in L_1^p because the map from L_1^p to L_1^p defined by

$$f\longmapsto \rho^\varepsilon f$$

is continuous. By a similar argument we can show (4). ■

2.2. Singular Hermitian connections. Recall that the Hermitian metric K on E is singular at parabolic points. So the Hermitian connection corresponding to a holomorphic structure on E is singular at parabolic points. In this subsection we describe this singular Hermitian connection around parabolic points and gives the definition the appropriate Sobolev norm.

Fix a parabolic point P_i. Recall that U_i is a neighbourhood of P_i, z_i is a holomorphic local coordinate on U_i with $z_i(P_i)=0$ and $\{e_k^{(i)}\}_{k=1}^{r}$ is a smooth frame of $E|U_i$. From now on we omit the suffix i if there is no confusion. We set

$$S=\begin{pmatrix} |z|^{-\alpha_1} & & 0 \\ & \ddots & \\ 0 & & |z|^{-\alpha_r} \end{pmatrix}, \quad \alpha=\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_r \end{pmatrix}.$$

Let $\bar{\partial}_B$ be a holomorphic structure on $E|U$. We write

$$\bar{\partial}_B=\bar{\partial}+B \qquad \text{with respect to } \{e_j\},$$

where B is an $\mathrm{End}\,(\boldsymbol{C}^r)$ valued (0, 1) form. Then we have

$$\bar{\partial}_B=\bar{\partial}-\frac{\alpha}{2}\frac{d\bar{z}}{\bar{z}}+S^{-1}BS \qquad \text{with respect to } \left\{\frac{e_k}{|z|^{\alpha_k}}\right\}.$$

Therefore we have

$$\begin{aligned} d_B &= d+\frac{\alpha}{2}\Big(\frac{dz}{z}-\frac{d\bar{z}}{\bar{z}}\Big)+\{S^{-1}BS-(S^{-1}BS)^*\} \\ &= d+\sqrt{-1}\alpha d\theta+\{S^{-1}BS-(S^{-1}BS)^*\}, \end{aligned}$$

where we write $z=\rho\exp\sqrt{-1}\theta$. We set

$$d_0=d+\sqrt{-1}\alpha d\theta \qquad \text{with respect to } \left\{\frac{e_k}{|z|^{\alpha_k}}\right\}.$$

Note that $d\theta$ has a pole at the origin.

REMARK. d_0 is a unitary flat connection on $U\backslash\{P\}$. Let R_0 be the curvature of d_0. Then we have

$$\frac{\sqrt{-1}}{2\pi}\Lambda R_0=-\alpha\delta_0,$$

where δ_0 is Dirac's delta function with the support at the origin.

Decompose $E|U=\bigoplus_{j=1}^{a}E_j$ such that $E_j=\mathrm{span}\{e_k\,|\,\alpha_k=w_j\}$. Then we have $\mathrm{End}\,(E|U)=E_D\oplus E_H$, where

$$E_D=\bigoplus_{j=1}^{a}\mathrm{End}\,E_j, \qquad E_H=\bigoplus_{j\neq k}\mathrm{Hom}\,(E_j,E_k).$$

For $u\in\Omega^0(\mathrm{End}\,(E|U))$ we write

$$u=u_D+u_H$$

corresponding to the above decomposition. Since

$$d_0u=du+[\sqrt{-1}\alpha,\,u]d\theta \qquad \text{with respect to } \left\{\frac{e_k}{|z|^{\alpha_k}}\right\},$$

we have

$$(d_0 u)_D = d(u_D),$$

$$(d_0 u)_H = d(u_H) + [\sqrt{-1}\alpha,\ u_H]d\theta .$$

Therefore Biquard [3] introduced the following Sobolev norms

$$\|u\|_{D_k^p} = \|u_D\|_{L_k^p} + \|u_H\|_{W_k^p} .$$

where we use the Hermitian metric K on E to define this Sobolev norm. So we have continuous maps

$$d_0 : D_k^p \Omega^p(\mathrm{End}(E|U)) \longrightarrow D_{k-1}^p \Omega^1(\mathrm{End}(E|U)),$$

where the function space $D_k^p\Omega^0(\mathrm{End}(E|U))$ is the Sobolev completion of $\Omega^0(\mathrm{End}(E|U))$ with respect to the norm $\| \ \|_{D_k^p}$.

2.3. Sobolev completion of the space of parabolic Higgs structures. In the last subsection we defined the Sobolev norms in a neighbourhood of each parabolic point. We patch them with the usual Sobolev norms on the complement to get the Sobolev norms on $\Omega^0(\mathrm{End}^0 E)$. Let $D_2^p\Omega^0(\mathrm{End}^0 E)$ denote the Sobolev completion of $\Omega^0(\mathrm{End}^0 E)$ with respect to the norm $\| \ \|_{D_2^p}$. Recall that we have fixed $\bar{\partial}_{B_0} \in \mathcal{B}$ in Section 1. Define

$$\mathcal{B}_1^p = \bar{\partial}_{B_0} + D_1^p \Omega^{0,1}(\mathrm{End}^0 E) .$$

So we have the continuous map

$$\bar{\partial}_B : D_2^p \Omega^0(\mathrm{End}^0 E) \longrightarrow D_1^p \Omega^{0,1}(\mathrm{End}^0 E),$$

where $\bar{\partial}_B \in \mathcal{B}_1^p$. Biquard [3] showed the following lemma using the theory of Lockhart and McOwen [14].

LEMMA 2.4. *If $p>1$ satisfies the following condition*

$$1 < p < \frac{2}{2+\alpha_k^{(i)}-\alpha_j^{(i)}} \qquad \textit{if } \alpha_j^{(i)} > \alpha_k^{(i)} ,$$

$$1 < p < \frac{2}{1+\alpha_k^{(i)}-\alpha_j^{(i)}} \qquad \textit{if } \alpha_j^{(i)} < \alpha_k^{(i)}$$

for each parabolic point P_i, then the maps

$$\bar{\partial}_B : D_2^p \Omega^0(\mathrm{End}^0 E) \longrightarrow D_1^p \Omega^{0,1}(\mathrm{End}^0 E) ,$$

$$\bar{\partial}_B : D_1^p \Omega^{1,0}(\mathrm{End}^0 E) \longrightarrow D_0^p \Omega^{1,1}(\mathrm{End}^0 E) ,$$

are Fredholm operators.

So it is natural to define the following.

DEFINITION 2.5. We say $p>1$ is compatible with the parabolic structure of E if the assumption in Lemma 2.4 holds.

We define

$$\mathcal{G}^{C}{}_{2}^{p} = \{g \in D_{2}^{p}\Omega^{0}(\mathrm{End}\, E) \mid \det g_{x}=1 \quad \text{for any } x \in \Sigma\}$$

$$\mathcal{G}_{2}^{p} = \{g \in \mathcal{G}^{C}{}_{2}^{p} \mid g_{x} g_{x}^{*}=1 \quad \text{for any } x \in \Sigma_{0}\}$$

$$\mathcal{D}_{1}^{p} = \{D''=\bar{\partial}_{B}+\theta \in \mathcal{B}_{1}^{p} \times D_{1}^{p}\Omega^{1,0}(\mathrm{End}^{0} E) \mid \bar{\partial}_{B}\theta=0\}.$$

By the Sobolev embedding theorem, it is easy to see that for $p>1$, $\mathcal{G}^{C}{}_{2}^{p}$ forms a group and that $\mathcal{G}^{C}{}_{2}^{p}$ acts on $\mathcal{D}_{1}^{p}$ from the right. We can show the following lemma.

LEMMA 2.6. *If $p>1$ is compatible with the parabolic structure of E, then the following holds.*

(1) *$\mathcal{B}$ is dense in $\mathcal{B}_{1}^{p}$.*

(2) *$\mathcal{G}^{C}$ is dense in $\mathcal{G}^{C}{}_{2}^{p}$.*

(3) *$\mathcal{D} \subset \mathcal{D}_{1}^{p}$.*

(4) *Assume $\bar{\partial}_{B} \in \mathcal{B}$ and s is a section of* $\mathrm{End}^{0} E$ *and $\bar{\partial}_{B}s=0$ on Σ.*

(a) *If $s \in D_{2}^{p}\Omega^{0}(\mathrm{End}\, E)$ then s is $\bar{\partial}_{B}$-holomorphic on Σ and $s(P_{i}) \in B_{i}$.*

(b) *If $s \in D_{1}^{p}\Omega^{0}(\mathrm{End}\, E)$ then s is ∂_{B}-holomorphic on Σ_{0} and has a pole of at most 1st order with the residue in N_{i} at P_{i}.*

PROOF. First we prove (1). If we write

$$s e_{k}^{(i)} = \sum_{j=1}^{r} s_{jk}^{(i)} e_{j}^{(i)} \qquad \text{on } U_{i},$$

then we have

$$s \frac{e_{k}^{(i)}}{|z_{i}|^{\alpha_{k}^{(i)}}} = \sum_{j=1}^{r} |z_{i}|^{\alpha_{j}^{(i)}-\alpha_{k}^{(i)}} s_{jk}^{(i)} \frac{e_{j}^{(i)}}{|z_{i}|^{\alpha_{j}^{(i)}}}.$$

So by Lemma 2.3 we can conclude that (1) holds. Since $s(P_{i}) \in B_{i}$ if and only if $s_{jk}^{(i)}(P_{i})=0$ for $\alpha_{j}^{(i)}<\alpha_{k}^{(i)}$, (2) follows. Since $s(P_{i}) \in N_{i}$ if and only if $s_{jk}^{(i)}(P_{i})=0$ for $\alpha_{j}^{(i)} \leqq \alpha_{k}^{(i)}$, (3) follows. We can prove (4) similarly. ∎

Since $\mathcal{D} \subset \mathcal{D}_{1}^{p}$ and $\mathcal{G}^{C} \subset \mathcal{G}^{C}{}_{2}^{p}$, we can define the map

$$i : \mathcal{D}/\mathcal{G}^{C} \longrightarrow \mathcal{D}_{1}^{p}/\mathcal{G}^{C}{}_{2}^{p}.$$

Now we state main result of this section.

PROPOSITION 2.7. *If $p>1$ is compatible with the parabolic structure of E, the map i is bijective.*

PROOF. First we show the injectivity of the map i. We set

$$D_i'' = \bar{\partial}_{B_i} + \theta_i \in \mathcal{D} \qquad (i=1, 2)$$

$$D_1'' = g^{-1} \circ D_2'' \circ g \qquad \text{for some } g \in \mathcal{G}^{C}{}_2^p .$$

Then we have $\bar{\partial}_{B_1} = g^{-1} \circ \bar{\partial}_{B_2} \circ g$. By Lemma 2.6 (4) g is smooth on whole Σ. So $g \in \mathcal{G}^C$.

To prove the surjectivity of the map i, we fix $D'' = \bar{\partial}_B + \theta \in \mathcal{D}_1^p$. Lemma 2.4 and 2.6 says that

$$\bar{\partial}_B : D_2^p \Omega^0(\mathrm{End}^0 E) \longrightarrow D_1^p \Omega^{0,1}(\mathrm{End}^0 E)$$

is Fredholm and that $\mathcal{B}$ is dense in $\mathcal{B}_1^p$. So by the argument of Atiyah and Bott [1] (Lemma 14.8) we conclude that there exists $g \in \mathcal{G}^C{}_2^p$ such that $g^{-1} \circ \bar{\partial}_B \circ g \in \mathcal{B}$. By Lemma 2.6 (4) we have $g^{-1} \circ D'' \circ g \in \mathcal{D}$. ∎

By this proposition we can define $\mathcal{D}^{st}{}_1^p$ naturally. The quotient space $\mathcal{D}^{st}{}_1^p / \mathcal{G}^C{}_2^p$ is the moduli space of stable parabolic Higgs bundles. From now on we always assume that $p > 1$ is compatible with the parabolic structure of E.

3. Irreducible Hermitian Einstein parabolic Higgs bundles.

We set $\mathcal{E}_1^p = \mathcal{B}_1^p \times D_1^p \Omega^{1,0}(\mathrm{End}^0 E)$. In this section we construct and study the moduli space of irreducible Hermitian Einstein parabolic Higgs bundles as a hyperkähler quotient of $\mathcal{E}_1^p$.

3.1. Construction of the moduli space. In this subsection we construct the moduli space of irreducible Hermitian Einstein parabolic Higgs bundles. The tangent space of $\mathcal{E}_1^p$ at any $(\bar{\partial}_B, \theta) \in \mathcal{E}_1^p$ is naturally isomorphic to

$$\mathcal{T}_1^p = D_1^p \Omega^{0,1}(\mathrm{End}^0 E) \times D_1^p \Omega^{1,0}(\mathrm{End}^0 E) .$$

We define a Riemannian metric g on $\mathcal{E}_1^p$ by

$$g((\xi, \phi), (\eta, \psi)) = -\frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\{(\xi - \xi^*) \wedge \sqrt{-1}(\eta + \eta^*)\}$$
$$+ \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\{(\phi - \phi^*) \wedge \sqrt{-1}(\psi + \psi^*)\},$$

for $(\xi, \phi), (\eta, \psi) \in \mathcal{T}_1^p$. This is well defined thanks to the Sobolev embedding theorem. Moreover we define three complex structures $I, J, K : \mathcal{T}_1^p \to \mathcal{T}_1^p$ on $\mathcal{E}_1^p$ by

$$I(\xi, \phi) = (\sqrt{-1}\xi, \sqrt{-1}\phi), \qquad J(\xi, \phi) = (\sqrt{-1}\phi^*, -\sqrt{-1}\xi^*),$$
$$K(\xi, \phi) = (-\phi^*, \xi^*) .$$

It is easy to see that g is a Kähler metric for I, J, K respectively and that I, J, K satisfy the relation $IJ=-JI=K$. Therefore $(g; I, J, K)$ is a hyperkähler structure on $\mathcal{E}_1^p$ by definition. We define the Kähler form ω_1 on $\mathcal{E}_1^p$ by

$$\omega_1((\xi, \phi), (\eta, \psi)) = g(I(\xi, \phi), (\eta, \psi)),$$

for $(\xi, \phi), (\eta, \psi) \in \mathcal{T}_1^p$. Similarly we define the Kähler forms ω_2, ω_3 corresponding to J, K respectively. The natural right action of $\mathcal{G}_2^p$ on $\mathcal{E}_1^p$ is given by

$$(\bar\partial_B, \theta) \longmapsto (g^{-1}\circ\bar\partial_B\circ g,\ g^{-1}\circ\theta\circ g),$$

for $g\in\mathcal{G}_2^p$, $(\bar\partial_B, \theta)\in\mathcal{E}_1^p$. This action preserves the hyperkähler structure. We write $\mathrm{End}_{sk}^0 E$ for the vector bundle of trace free skew endomorphisms of E. Then there exists the moment map

$$\mu_i : \mathcal{E}_1^p \longrightarrow D_0^p\Omega^2(\mathrm{End}_{sk}^0 E)$$

for each action of $\mathcal{G}_2^p$ on $(\mathcal{E}_1^p, \omega_i)$. Since the Lie algebra of $\mathcal{G}_2^p$ is $D_2^p\Omega^0(\mathrm{End}_{sk}^0 E)$, we consider $D_0^p\Omega^2(\mathrm{End}_{sk}^0 E)$ as a subspace of the dual space of the Lie algebra of $\mathcal{G}_2^p$ by the natural pairing

$$\langle\alpha, \xi\rangle = -\frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}(\alpha\xi),$$

for $\alpha\in D_2^p\Omega^0(\mathrm{End}_{sk}^0 E)$, $\xi\in D_0^p\Omega^2(\mathrm{End}_{sk}^0 E)$. This is well defined thanks to the Sobolev embedding theorem. We can write down the moment maps explicitly.

$$\mu_1(\bar\partial_B, \theta) = R_B^\perp + [\theta, \theta^*], \qquad \{\mu_2+\sqrt{-1}\mu_3\}(\bar\partial_B, \theta) = -2\bar\partial_B\theta,$$

where $R_B^\perp$ is the trace free part of R_B. Therefore we conclude

$$\mathcal{D}_1^p = \mu_2^{-1}(0)\cap\mu_3^{-1}(0).$$

For $D''=\bar\partial_B+\theta\in\mathcal{D}_1^p\subset\mathcal{E}_1^p$, we set

$$D' = \partial_B+\theta^*, \qquad D = D''+D', \qquad R_D = D^2 = R_B+[\theta, \theta^*].$$

DEFINITION 3.1. We say $D''\in\mathcal{D}_1^p\subset\mathcal{E}_1^p$ is Hermitian Einstein if

$$R_D^\perp = 0.$$

We set

$$\mathcal{D}_{HE}{}_1^p = \{D''\in\mathcal{D}_1^p \mid D'' \text{ is Hermitian Einstein}\}.$$

So we have

$$\mathcal{D}_{HE}{}_1^p = \bigcap_{i=1}^3 \mu_i^{-1}(0).$$

DEFINITION 3.2. We say $D''\in\mathcal{D}_1^p$ is reducible if there exist sub Higgs bundles V, $W\subset(E, D'')$ such that $E=V\oplus W$. We say irreducible otherwise.

We set

$$\mathcal{D}^{irr}{}^{p}_{1} = \{D'' \in \mathcal{D}^{p}_{1} \mid D'' \text{ is irreducible}\},$$

and

$$\mathcal{D}^{irr}_{HE}{}^{p}_{1} = \mathcal{D}_{HE}{}^{p}_{1} \cap \mathcal{D}^{irr}{}^{p}_{1}.$$

Thus we can construct the moduli space of irreducible Hermitian Einstein parabolic Higgs bundles as $\mathcal{D}^{irr}_{HE}{}^{p}_{1}/\mathcal{G}^{p}_{2}$. This is a hyperkähler quotient of $\mathcal{E}^{p}_{1}$ by $\mathcal{G}^{p}_{2}$ in the sense of [13].

3.2. Vanishing theorems. In principle Higgs bundles share many properties with vector bundles. One of them is the following Kähler identity.

LEMMA 3.3. *For $D'' \in \mathcal{D}^{p}_{1}$ we have*
(1) $D''^{*} = -\sqrt{-1}[\Lambda, D']$
(2) $D'^{*} = \sqrt{-1}[\Lambda, D'']$
(3) $D^{*} = D''^{*} + D'^{*}$.

Now we introduce the Laplacian for Hermitian Higgs bundles as follows.

$$\Delta'' = D''D''^{*} + D''^{*}D'', \qquad \Delta' = D'D'^{*} + D'^{*}D', \qquad \Delta = DD^{*} + D^{*}D.$$

Then we have

LEMMA 3.4.
(1) $\Delta = \Delta'' + \Delta'$
(2) $\Delta' - \Delta'' = \sqrt{-1}[\Lambda, R_D]$.

We need the following vanishing theorem for irreducible Hermitian Einstein parabolic Higgs bundles.

PROPOSITION 3.5. *If $D'' \in \mathcal{D}^{irr}_{HE}{}^{p}_{1}$, then the map*

$$D'' : D^{p}_{2}\Omega^{0}(\mathrm{End}^{0}E) \longrightarrow D^{p}_{1}\Omega^{1}(\mathrm{End}^{0}E)$$

has a trivial kernel.

PROOF. If $D''s = 0$, and $s \in D^{p}_{2}\Omega^{0}(\mathrm{End}^{0}E)$, then

$$0 = \int_{\Sigma} \langle [\sqrt{-1}\Lambda R_D, s], s\rangle \omega = \int_{\Sigma} \langle (\Delta' - \Delta'')s, s\rangle \omega$$

$$= \int_{\Sigma} \langle D'^{*}D's, s\rangle \omega = \int_{\Sigma} \|D's\|^{2} \omega .$$

Therefore $D's = 0$. So we have

$$d_B s = 0, \qquad \theta s = s\theta, \qquad \theta^{*} s = s\theta^{*},$$

where $D'' = \bar{\partial}_B + \theta$. If we set $t = s + s^{*}$, then $d_B t = 0$, $\theta t = t\theta$. So we conclude that the eigenvalue of t is constant on $\sum_0$. We assume t is non trivial. Let $\lambda_1, \cdots, \lambda_l$ be the eigenvalues of t. Since $\mathrm{Tr}\, t \equiv 0$, we have $l \geqq 2$. Let $E_m (m=1, \cdots, l)$ be the vector bundle on $\sum_0$, which consist of the eigenspace of t with the eigenvalue λ_m. Then we have

$$(E, D'') = (E_1, D''|E_1) \oplus \cdots \oplus (E_l, D''|E_l) \qquad \text{on } \textstyle\sum_0 .$$

We have to show that this splitting extends to whole $\sum$. Let π_m be the orthogonal projection of E to E_m. Then we have

$$\bar{\partial}_B \pi_m = 0 \qquad \text{on } \textstyle\sum_0 .$$

If we represent

$$\pi_m \frac{e_k^{(i)}}{|z_i|^{\alpha_k^{(i)}}} = \sum_j a_{jk}^{(i)} \frac{e_j^{(i)}}{|z_i|^{\alpha_j^{(i)}}} \qquad \text{on } U_i ,$$

then $a_{jk}^{(i)}$ is bounded on U_i, because π_m is an orthogonal projection. We note $\alpha_k^{(i)} - \alpha_j^{(i)} > -1$ and rewrite

$$\pi_m e_k^{(i)} = \sum_j |z_i|^{\alpha_k^{(i)} - \alpha_j^{(i)}} a_{jk}^{(i)} e_j^{(i)} ,$$

then we conclude that π_m is $\bar{\partial}_B$-holomorphic on whole $\sum$.

If we note at P_i

$$\operatorname{rank} E = \operatorname{rank} \Big(\sum_{m=1}^{l} \pi_m\Big) \leqq \sum_{m=1}^{l} \operatorname{rank} \pi_m \leqq \sum_{m=1}^{l} \operatorname{rank} E_m = \operatorname{rank} E ,$$

we have at P_i

$$\operatorname{rank} \pi_m = \operatorname{rank} E_m, \qquad E_{P_i} = \operatorname{Im} \pi_1 \oplus \cdots \oplus \operatorname{Im} \pi_l .$$

If we set $E_m = \operatorname{Im} \pi_m$, then we have

$$(E, D'') = (E_1, D''|E_1) \oplus \cdots \oplus (E_l, D''|E_l) \qquad \text{on } \textstyle\sum .$$

Since $D'' \in \mathcal{D}^{irr}{}_1^p$, this is a contradiction. Therefore $t = s + s^* = 0$. By the same argument $s - s^* = 0$. So we have $s = 0$. ■

Next we show the following Hodge decomposition theorem for parabolic Higgs bundles.

PROPOSITION 3.6.

(1) *If* $D'' \in \mathcal{D}_1^p$, *we have*

$$D_1^p \Omega^1(\mathrm{End}^0 E) = D''(D_2^p \Omega^0) \oplus D'(D_2^p \Omega^0) \oplus \boldsymbol{H}^1 ,$$

where $\boldsymbol{H}^1 = \{a \in D_1^p \Omega^1(\mathrm{End}^0 E) \,|\, D'' a = 0,\ D' a = 0\}$.

(2) *Let* $\mathrm{Ker}\, D''$, $\mathrm{Ker}\, D'$ *be the kernel of the map*

$$D'', D' : D_1^p \Omega^1(\mathrm{End}^0 E) \longrightarrow D_0^p \Omega^2(\mathrm{End}^0 E) .$$

Then we have

$$\operatorname{Ker} D'' = D''(D_2^p\Omega^0)\oplus \boldsymbol{H}^1,$$

$$\operatorname{Ker} D' = D'(D_2^p\Omega^0)\oplus \boldsymbol{H}^1.$$

PROOF. By Lemma 2.4 $D''(D_2^p\Omega^0)$ and $D'(D_2^p\Omega^0)$ is closed in $D_1^p\Omega^1(\mathrm{End}^0E)$. So we can prove (1), (2) by the standard argument. ■

We note that there is a following real structure of End^0E.

$$\mathrm{End}^0E = \mathrm{End}_{sk}^0E\oplus\sqrt{-1}\,\mathrm{End}_{sk}^0E.$$

Then we have the following proposition.

PROPOSITION 3.7. *If $D''\in\mathcal{D}_{HE}^{irr}{}_1^p$, then the map*

$$\Delta: D_2^p\Omega^0(\mathrm{End}^0E)\longrightarrow D_0^p\Omega^0(\mathrm{End}^0E)$$

is an isomorphism, which preserves the real structure.

PROOF. Thanks to Proposition 3.5 we have only to show that the map

$$\Delta: D_2^p\Omega^0(\mathrm{End}^0E)\longrightarrow D_0^p\Omega^0(\mathrm{End}^0E)$$

is a Fredholm operator with index zero. However we can prove this by the similar argument in the proof of Proposition 2.13 in [3]. ■

3.3. Properties of the moduli space. In subsection 3.1 we have constructed the moduli space of irreducible Hermitian Einstein parabolic Higgs bundles $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$ by a hyperkähler quotient method. In this subsection we study this moduli space.

First we introduce a manifold structure on the moduli space. We fix $D''=\bar{\partial}_B+\theta\in\mathcal{D}_{HE}^{irr}{}_1^p$. We define an elliptic complex which describes deformations of a Hermitian Einstein parabolic Higgs bundle. We define

$$C^0 = D_2^p\Omega^0(\mathrm{End}_{sk}^0E),\qquad C^1 = D_1^p\Omega^1(\mathrm{End}^0E).$$

Note that C^0 is the Lie algebra of $\mathcal{G}_2^p$ and that C^1 is the tangent space of $\mathcal{E}_1^p$ at D''. Define $d^0: C^0\to C^1$ by

$$d^0X = D''X \qquad \text{for } X\in C^0.$$

Note that d^0 is the differential of the map $\mathcal{G}_2^p\to\mathcal{D}_1^p$ defined by

$$g\longmapsto g^{-1}\circ D''\circ g \qquad \text{for } g\in\mathcal{G}_2^p.$$

Define $C^2 = D_0^p\Omega^2(\mathrm{End}_{sk}^0E)\oplus D_0^p\Omega^2(\mathrm{End}^0E)$. For $\xi\in D_1^p\Omega^{0,1}(\mathrm{End}^0E)$, $\phi\in D_1^p\Omega^{1,0}(\mathrm{End}^0E)$,

$$(\bar{\partial}_B+\xi)+(\theta+\phi) \in \mathcal{D}_{HE}{}^{p}_{1}$$

if and only if

$$d^1(\xi+\phi)+Q(\xi+\phi)=0\,,$$

where d^1, $Q: C^1 \to C^2$ are defined by

$$d^1(\xi+\phi)=(D'(\xi+\phi)-\{D'(\xi+\phi)\}^*,\ D''(\xi+\phi))$$

$$Q(\xi+\phi)=(-[\xi,\ \xi^*]+[\phi,\ \phi^*],\ [\xi,\ \phi]).$$

So we have defined the fundamental elliptic complex

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \longrightarrow 0\,.$$

The next lemma admits us to calculate the cohomology groups associated to this complex.

LEMMA 3.8.

(1) $\operatorname{Ker} d^0=0$.

(2) $C^1=\operatorname{Im} d^0 \oplus \boldsymbol{H}^1 \oplus \sqrt{-1}\operatorname{Im} d^0 \oplus \operatorname{Im} D'$ *as a vector space over* $\boldsymbol{R}$.

(3) $\operatorname{Ker} d^1=\operatorname{Im} d^0 \oplus \boldsymbol{H}^1$.

(4) *If we set* $W=\sqrt{-1}\operatorname{Im} d^0 \oplus \operatorname{Im} D'$, *then*

$$d^1|W: W \longrightarrow C^2$$

is an isomorphism.

PROOF. By Proposition 3.5 we have (1). By Proposition 3.5 and 3.6 we have (2). To prove (3) and (4) we write $d^1(\xi+\phi)=(d^1_r(\xi+\phi),\ d^1_c(\xi+\phi))$. First by Proposition 3.7 we have

$$\operatorname{Ker} d^1_r=\operatorname{Im} d^0 \oplus \boldsymbol{H}^1 \oplus \operatorname{Im} D'\,.$$

Moreover we conclude that the map

$$d^1_r|\sqrt{-1}\operatorname{Im} d^0: \sqrt{-1}\operatorname{Im} d^0 \longrightarrow D^p_0\Omega^2(\operatorname{End}^0_{sk} E)$$

is an isomorphism. Again by Proposition 3.7 we have

$$\operatorname{Ker} d^1_c=\operatorname{Im} d^0 \oplus \boldsymbol{H}^1 \oplus \sqrt{-1}\operatorname{Im} d^0\,.$$

Moreover we conclude that the map

$$d^1_c|\operatorname{Im} D': \operatorname{Im} D' \longrightarrow D^p_0\Omega^2(\operatorname{End}^0 E)$$

is an isomorphism. So we have (3), (4). ■

Now we define Kuranishi map $F: C^1 \to C^1$ by

$$F(\xi+\phi)=(\xi+\phi)+(d^1|W)^{-1}Q(\xi+\phi)\,.$$

Since Q is quadratic with respect to ξ, ϕ, there exists a neighbourhood of 0 in C^1 such that

$$F|U : U \longrightarrow F(U)$$

is a diffeomorphism. If we set

$$V = \{\xi+\phi \in \boldsymbol{H}^1 \oplus \sqrt{-1}\,\mathrm{Im}\, d^0 \oplus \mathrm{Im}\, D' \mid d^1(\xi+\phi)+Q(\xi+\phi)=0\},$$

then

$$F|V\cap U : V\cap U \longrightarrow \boldsymbol{H}^1 \cap U$$

is a diffeomorphism. By the same argument in [2] we can show $V\cap U$ is a slice of the action of $\mathcal{G}_2^p$ on $\mathcal{D}_{HE}^{irr}{}_1^p$ if we choose U small enough. So we have the following theorem.

THEOREM 3.9. *$\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$ is a smooth hyperkähler manifold.*

PROOF. By the above argument we have introduced a smooth manifold structure on $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$. Since $\mathcal{D}_{HE}{}_1^p=\bigcap_{i=1}^3 \mu_i^{-1}(0)$ as we saw in Section 3.1, this manifold has a natural hyperkähler structure by the same argument in the proof of Theorem 6.7 in [11]. ■

Now we can show the following theorem by the same argument in the proof of Theorem 6.1 in [11].

THEOREM 3.10. *If $\mathcal{D}_{HE}^{irr}{}_1^p=\mathcal{D}_{HE}{}_1^p$, the natural Riemannian metric on the moduli space is complete.*

Now we can calculate the dimension of $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$.

THEOREM 3.11. *If $\mathcal{D}_{HE}^{irr}{}_1^p \neq \emptyset$, then the complex dimension of $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$ is*

$$2\{(g-1)(r^2-1)+\sum_{i=1}^{n} \dim_C N_i\},$$

where g is the genus of Σ.

PROOF. Suppose $D''=\bar{\partial}_B+\theta \in \mathcal{D}_{HE}^{irr}{}_1^p$. By the above argument we saw that the tangent space of $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$ at $[D'']$ is isomorphic to $\boldsymbol{H}^1$. So we have only to calculate the dimension of $\boldsymbol{H}^1$. If we define the map

$$S : D_1^p\Omega^1(\mathrm{End}^0 E) \longrightarrow D_0^p\Omega^2(\mathrm{End}^0 E) \oplus D_0^p\Omega^2(\mathrm{End}^0 E)$$

by

$$S(a) = (D''a,\, D'a),$$

then S is a Fredholm operator. We write $\mathrm{ind}(S)$ for the Fredholm index of S. By Proposition 3.7 we have

$$\dim_C \boldsymbol{H}^1 = \mathrm{ind}(S).$$

Since the Fredholm index is invariant under the deformation by compact operators, we have

$$\begin{aligned}\operatorname{ind} S &= \operatorname{ind}\{\bar{\partial}_B : D_1^p\Omega^{1,0}(\mathrm{End}^0 E) \longrightarrow D_0^p\Omega^2(\mathrm{End}^0 E)\}\\ &\quad+\operatorname{ind}\{\partial_B : D_1^p\Omega^{0,1}(\mathrm{End}^0 E) \longrightarrow D_0^p\Omega^2(\mathrm{End}^0 E)\}\\ &= 2\operatorname{ind}\{\partial_B : D_1^p\Omega^{0,1}(\mathrm{End}^0 E) \longrightarrow D_0^p\Omega^2(\mathrm{End}^0 E)\}.\end{aligned}$$

So by Proposition 3.7 we have

$$\operatorname{ind} S = -2\operatorname{ind}\{\bar{\partial}_B : D_2^p\Omega^0(\mathrm{End}^0 E) \longrightarrow D_1^p\Omega^{0,1}(\mathrm{End}^0 E)\}.$$

By Lemma 2.4 and 2.6 we have

$$\begin{aligned}&\operatorname{ind}\{\bar{\partial}_B : D_2^p\Omega^0(\mathrm{End}^0 E) \longrightarrow D_1^p\Omega^{0,1}(\mathrm{End}^0 E)\}\\ &\quad= \operatorname{ind}\{\bar{\partial}_B : \Omega^0(\mathrm{ParEnd}^0 E) \longrightarrow \Omega^{0,1}(\mathrm{End}^0 E)\}.\end{aligned}$$

By the standard argument we see that the Dolbeault cohomology group of

$$0 \longrightarrow \Omega^0(\mathrm{ParEnd}^0 E) \xrightarrow{\bar{\partial}_B} \Omega^{0,1}(\mathrm{End}^0 E) \longrightarrow 0$$

is isomorphic to the sheaf cohomology group of $\mathcal{O}(\mathrm{ParEnd}^0 E)$, which is a sheaf of $\bar{\partial}_B$-holomorphic sections of $\mathrm{End}^0 E$ preserving the flag at each parabolic point P_i. By considering the exact sequence

$$0 \longrightarrow \mathcal{O}(\mathrm{ParEnd}^0 E) \longrightarrow \mathcal{O}(\mathrm{End}^0 E) \longrightarrow \mathcal{O}(\mathrm{End}^0 E)/\mathcal{O}(\mathrm{ParEnd}^0 E) \longrightarrow 0,$$

we have the theorem. ■

4. Identification of the two moduli spaces.

In Section 2 we have constructed the moduli space of stable parabolic Higgs bundles as $\mathcal{D}^{st}{}_1^p/\mathcal{G}^{c}{}_2^p$. In the last section we have constructed the moduli space of irreducible Hermitian Einstein parabolic Higgs bundles as $\mathcal{D}_{HE}^{irr}{}_1^p/\mathcal{G}_2^p$. In this section we identify these two moduli spaces. As stated at the end of Section 2, we assume that $p>1$ is compatible with the parabolic structure of E.

4.1. Hermitian Einstein metrics. First we review the Chern-Weil formula for parabolic Higgs bundles. (See [17] Lemma 3.2, Lemma 10.5.)

LEMMA 4.1. *Assume that $D''\in\mathcal{D}_1^p$ and $V\subset(E, D'')$ is a sub Higgs bundle. Let $\pi: E|\Sigma_0\to V|\Sigma_0$ be an orthogonal projection. Then the following holds.*

$$\operatorname{pardeg} V = \frac{\sqrt{-1}}{2\pi}\int_\Sigma \mathrm{Tr}(\pi R_D) - \frac{1}{2\pi}\int_\Sigma |D''\pi|^2.$$

By this lemma we can show the following.

PROPOSITION 4.2. $\mathcal{D}_{HE}^{irr}{}_1^p \subset \mathcal{D}^{st}{}_1^p$.

PROOF. We fix $D''=\bar{\partial}_B+\theta \in \mathcal{D}_{HE}^{irr}{}_1^p$. Let $V \subset (E, D'')$ be a sub Higgs bundle with $0<\operatorname{rank} V<r$. By Lemma 4.1 we have

$$\frac{\operatorname{pardeg} V}{\operatorname{rank} V} \leqq \frac{\operatorname{pardeg} E}{\operatorname{rank} E}=\mu .$$

We have to show that the equality does not occur. Assume the equality holds. Then we have $D''\pi=0$ on Σ_0 by Lemma 4.1. Since $\pi=\pi^*$, we have $d_B\pi=0$, $\theta\pi=\pi\theta$. By the same argument in the proof of Proposition 3.5, we can show that there exists a sub Higgs bundle $W \subset (E, D'')$ such that $E=V \oplus W$ on Σ, and $W=V^{\perp}$ on Σ_0. Since $D'' \in \mathcal{D}^{irr}{}_1^p$ this is a contradiction. Therefore we have

$$\frac{\operatorname{pardeg} V}{\operatorname{rank} V}<\mu . \quad \blacksquare$$

So we can define the map

$$j: \mathcal{D}_{HE}^{irr}{}_1^p / \mathcal{G}_2^p \longrightarrow \mathcal{D}^{st}{}_1^p / \mathcal{G}^{C}{}_2^p .$$

Now we state main result of this section.

THEOREM 4.3. *j is bijective.*

To prove this theorem, we reformulate Theorem 4.3. We have fixed the Hermitian metric K on E and varied Higgs structures on E. But from now on we fix a Higgs structure $D''=\bar{\partial}_B+\theta \in \mathcal{D}_1^p$ and vary Hermitian metrics on E. So we have to write explicitly which Hermitian metric we use. For example

$$D_K'=\partial_K+\theta^{*K}, \qquad D_K=D''+D_K', \qquad R_K=D_K^2 .$$

We define the space of Hermitian metrics compatible with the parabolic structure on E as follows.

$$S(K)_2^p=\{s \in D_2^p \Omega^0(\operatorname{End}^0 E) \mid s^{*K}=s\},$$

$$MET_2^p=\{H=Ke^s \mid s \in S(K)_2^p\}.$$

For $H=Kh \in MET_2^p$, a Hermitian metric is defined by

$$\langle v, w\rangle_H=\langle hv, w\rangle_K \qquad \text{for } v, w \in E_x \quad \text{for some } x \in \Sigma_0 .$$

Then we have the following lemma.

LEMMA 4.4. *For $H=Kh \in MET_2^p$,*

(1) $D_H'=h^{-1} \circ D_K' \circ h$,

(2) $R_H = R_K + D''(h^{-1}D'_K h)$.

On the other hand, if we fix a Hermitian metric K on E, the corresponding connection to $g^{-1}\circ D''\circ g$ for $g\in\mathcal{G}C_2^p$ is

$$g^{-1}\circ(D''+h^{-1}\circ D'_K\circ h)\circ g\,,$$

where $h=(gg^{*K})^{-1}$. Therefore if there is a Hermitian metric $H=Ke^s\in MET_2^p$ such that $R_H^\perp=0$, then $g=e^{-s/2}\in\mathcal{G}C_2^p$ and $g^{-1}\circ D''\circ g\in\mathcal{D}_{HE}{}_1^p$. So the following theorem is equivalent to Theorem 4.3.

THEOREM 4.5. *Assume $p>1$ is compatible with the parabolic structure of E. Fix $D''\in\mathcal{D}^{st}$. Then there exists a unique Hermitian metric $H=Ke^s\in MET_2^p$ such that $R_H^\perp=0$.*

4.2. The Donaldson functional. In this subsection we define the Donaldson functional (See [7, 8]) in the formulation of Simpson [**17**].

For a smooth function $f: \boldsymbol{R}\to\boldsymbol{R}$, we define $f: S(K)\to S(K)$ as follows. Let $s\in S(K)$. For any $x\in\sum$ there exists an orthonormal basis $\{e_j\}$ for E_x such that $se_j=\lambda_j e_j$ for all j. Then we set $f(s)e_j=f(\lambda_j)e_j$.

For a smooth function $F: \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$, we define $F: S(K)\to S_K(\mathrm{End}\,E)$, where $S_K(\mathrm{End}\,E)=\{T\in\mathrm{End}(\mathrm{End}\,E)\mid T^{*K}=T\}$. For $A\in\mathrm{End}\,E_x$, we write $Ae_k=\sum_j a_{jk}e_j$. Then we define

$$\{F(s)A\}e_k = \sum_j F(\lambda_k, \lambda_j)a_{jk}e_j\,.$$

Simpson [**17**] showed how these constructions behave on L_k^p spaces as follows.

LEMMA 4.6. *Let $S(K)_{k,b}^p=\{s\in D_k^p\Omega^0(\mathrm{End}\,E)\mid s^{*K}=s,\ \sup|s|_K\leqq b\}$. Let $f: S(K)\to S(K)$ and $F: S(K)\to S_K(\mathrm{End}\,E)$ as above. Then we have the following.*

(1) *The map F extends to a map*

$$F: S(K)_{0,b}^p \longrightarrow \mathrm{Hom}(D_0^p\Omega^0(\mathrm{End}\,E),\ D_0^q\Omega^0(\mathrm{End}\,E))$$

for $q\leqq p$. For $q<p$ the map is continuous in the operator norm topology.

(2) *The map f extends to a map*

$$f: S(K)_{1,b}^p \longrightarrow S(K)_{1,b'}^q\,,$$

for $q\leqq p$. For $q<p$ the map is continuous.

(3) *Define $df: \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$ by*

$$df(x, y) = \frac{f(x)-f(y)}{x-y} \qquad \text{if } x\neq y,$$

$$df(x, x) = f'(x)\,.$$

Then for $s\in S(K)_{1,b}^p$ and $D''\in\mathcal{D}_1^p$,

$$D''f(s) = df(s)(D''s).$$

We define $\Psi : \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}_{>0}$ by

$$\Psi(x, y) = \frac{e^{y-x}-(y-x)-1}{(y-x)^2}.$$

Now we can define the Donaldson functional.

DEFINITION 4.7. The Donaldson functional $M: MET_2^p\times MET_2^p\to\boldsymbol{R}$ is defined by

$$M(K, Ke^s) = 2\sqrt{-1}\int_\Sigma \mathrm{Tr}(sR_K)+2\int_\Sigma \langle\Psi(s)(D''s), D''s\rangle_K .$$

Following properties of this functional are proved by Donaldson [7, 8] for vector bundles and by Simpson [17] for Higgs bundles.

LEMMA 4.8. *Suppose* $K, H, J\in MET_2^p$. *Then*

(1) $M(K, H)+M(H, J)=M(K, J)$.

(2) $d/dtM(K, He^{ts}) = 2\sqrt{-1}\int_\Sigma \mathrm{Tr}(sR_{He^{ts}})$ *for* $t\geqq 0$ *and* $s\in S(H)_2^p$.

(3) $d^2/dt^2M(K, He^{ts})|_{t=0} = 2\int_\Sigma \|D''s\|_H^2$.

4.3. Proof of Theorem 4.5. To prove Theorem 4.5 we have to solve the variational problem with respect to the Donaldson functional. In [17] Simpson solve this problem in a certain function space. However, to construct the moduli space of stable parabolic Higgs bundles, we have to solve this variational problem in MET_2^p, which is different from Simpson's function space.

For positive number $B>0$, we set

$$MET_2^p(B) = \{H\in MET_2^p \| \Lambda R_H\|_{L^p, H}\leqq B\},$$

where $\| \ \|_{L^p, H}$ is the norm with respect to H.

The following lemma is due to Simpson [17] (Proposition 5.3, Corollary 10.7 and 10.8).

LEMMA 4.9. *Fix* $D''\in\mathfrak{D}^{st}$ *and* $B>0$. *Then there are constants* $C_1, C_2>0$ *such that*

$$\sup|s|_K \leqq C_1M(K, Ke^s)+C_2 \qquad \textit{for any } Ke^s\in MET_2^p(B).$$

COROLLARY 4.10. $\{M(K, Ke^s)|Ke^s\in MET_2^p(B)\}$ *is bounded from below.*

The next lemma shows this variational problem is solvable in $MET_2^p(B)$.

LEMMA 4.11. *There exists* $H_\infty = Ke^{s_\infty}\in MET_2^p(B)$ *such that*

$$M(K, Ke^{s_\infty}) = \inf\{M(K, Ke^s) \mid Ke^s \in MET_2^p(B)\}.$$

PROOF. Let $\{H_i = Ke^{s_i} = Kh_i\} \subset MET_2^p(B)$ be a minimizing sequence of $M(K, \ )$ in $MET_2^p(B)$. By Lemma 4.9 we have the following estimate.

$$\sup|s_i|_K \leqq C_1, \tag{1}$$

where C_1 is independent of i.

Recall that the Donaldson functional is the following

$$M(K, Ke^{s_i}) = 2\sqrt{-1}\int_\Sigma \mathrm{Tr}(s_i R_K) + 2\int_\Sigma \langle \Psi(s_i)(D''s_i), D''s_i\rangle_K.$$

So we have

$$\int_\Sigma \langle \Psi(s_i)(D''s_i), D''s_i\rangle_K \leqq C_2,$$

where C_2 is independent of i. Therefore $\|D''s_i\|_{L^2} \leqq C_3$. So we have the following estimate.

$$\|s_i\|_{L_1^2} \leqq C_4. \tag{2}$$

On the other hand by Lemma 4.4 we have

$$R_{H_i} = R_K + D''(h_i^{-1}D'_K h_i).$$

Since $H_i \in MET_2^p(B)$, we have

$$\|D''(h_i^{-1}D'_K h_i)\|_{L^p} \leqq C_5. \tag{3}$$

Recall

$$D''(h_i^{-1}D'_K h_i) = \bar\partial_B(h_i^{-1}\partial_K h_i) + [\theta, h_i^{-1}[\theta^{*K}, h_i]], \tag{4}$$

where $D'' = \bar\partial_B + \theta$. By (1) we have

$$\|[\theta, h_i^{-1}[\theta^{*K}, h_i]]\|_{L^p} \leqq C_6. \tag{5}$$

By (3), (4) and (5) we have

$$\|\bar\partial_B(h_i^{-1}\partial_K h_i)\|_{L^p} \leqq C_7. \tag{6}$$

By (2), (6) we have

$$\|h_i^{-1}\partial_K h_i\|_{L_1^p} \leqq C_8. \tag{7}$$

We set $a_i = h_i^{-1}\partial_K h_i$. So we have

$$\partial_K h_i = h_i a_i. \tag{8}$$

By (7) we have $\|a_i\|_{L_1^p} \leqq C_8$. By (1) we can show $\|h_i a_i\|_{L^{2p}} \leqq C_9$. By (8) we have $\|h_i\|_{L_1^{2p}} \leqq C_{10}$. Then we have $\|h_i a_i\|_{L_1^p} \leqq C_{11}$. Again by (8) we have

$$\|h_i\|_{L_2^p} \leqq C_{12}\,.$$

By taking a subsequence, if necessary, we may assume that $\{s_i\}$ converges to s_∞ weakly in $D_2^p\Omega^0(\mathrm{End}^0 E)$ and strongly in $D_1^{2p}\Omega^0(\mathrm{End}^0 E)$. Then by Lemma 4.6 we have

$$\lim_{i\to\infty} M(K, Ke^{s_i}) = M(K, Ke^{s_\infty})\,.$$

So we have completed the proof of Lemma 4.11. ■

Next we have to show that $H_\infty = Ke^{s_\infty}$ is a Hermitian Einstein metric. To do this we need the following lemma.

LEMMA 4.12. *Assume $H \in MET_2^p$ and $s \in S(H)_2^p$. Then*
(1) *If $D''s=0$, then $s=0$.*
(2) *If $D'_H s=0$, then $s=0$.*

PROOF. First we prove (1). Since $D''s=0$ and $s^{*H}=s$, we have $D'_H s=0$. By the same argument in the proof of Proposition 3.5, we have $s=0$. The proof of (2) is the same. ■

Now we come to the final stage of the proof of Theorem 4.5.

LEMMA 4.13. *$H_\infty \in MET_2^p$ is a Hermitian Einstein metric, that is, $R_{H_\infty}^\perp=0$.*

PROOF. First we claim that there exists $s \in S(H_\infty)_2^p$ such that

$$D'' D'_{H_\infty} s = -R_{H_\infty}^\perp\,.$$

In fact, since by Proposition 3.7 the map

$$\sqrt{-1}\Lambda D'' D'_{H_\infty} : S(H_\infty)_2^p \longrightarrow S(H_\infty)_0^p$$

is a Fredholm operator with index zero, by Lemma 4.12 we conclude that this map is surjective. So the existence of s is clear.

Since

$$\frac{d}{dt}\|\Lambda R_{H_\infty e^{ts}}^\perp\|_{L^p, H_\infty e^{ts}}^p\big|_{t=0} = -p\|\Lambda R_{H_\infty}^\perp\|_{L^p, H_\infty}^p,$$

we have $H_\infty e^{ts} \in MET_2^p(B)$ for small $t \geqq 0$. By Lemma 4.8 we have

$$\frac{d}{dt}M(K, H_\infty e^{ts})\big|_{t=0} = 2\sqrt{-1}\int_\Sigma \mathrm{Tr}(sR_{H_\infty}) = -2\int_\Sigma \|D''s\|_{H_\infty}^2\,.$$

Since H_∞ attains the minimum of $M(K,\ )$ on $MET_2^p(B)$, we have $D''s=0$. By Lemma 4.12 we have $s=0$. Therefore

$$R_{H_\infty}^\perp = -D'' D'_{H_\infty} s = 0\,. \quad ■$$

We have to show the uniqueness of Hermitian Einstein metric. If we use Lemma 4.8 (3), the proof is easy. So we omit the proof. Thus we have completed the proof of Theorem 4.5.

References

[1] M.F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A, **308** (1982), 523-615.

[2] M.F. Atiyah, N.J, Hitchin and I.M. Singer, Self-duality in four dimensionl Riemannian geometry, Proc. Roy. Soc. London Ser. A, **362** (1978), 425-461.

[3] O. Biquard, Fibrés parabolique stables et connexions singulières plates, Bull. Soc. Math. France, **119** (1991), 231-257.

[4] H. Boden, Representations of orbifold groups and parabolic bundles, Comment. Math. Helve., **66** (1991), 389-447.

[5] S.B. Bradlow, Special metrics and stability for holomorphic bundles with global sections, J. Differential Geom., **33** (1991), 169-213.

[6] S.K. Donaldson, A new proof of a theorem of Narasimhan and Seshadri, J. Differential Geom., **18** (1983), 269-277.

[7] S.K. Donaldson, Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable bundles, Proc. London Math. Soc. (3), **50** (1985), 1-26.

[8] S.K. Donaldson, Infinite determinants, stable bundles and curvature, Duke Math. J., **54** (1987), 231-247.

[9] A. Fujiki, Hyperkähler structure on the moduli space of flat bundles, Lecture Notes in Math., **1468**, Springer, 1991, pp. 1-83.

[10] M. Furuta and B. Steer, Seifert fibred homology 3-spheres and Yang-Mills equations on Riemann surfaces with marked points, preprint.

[11] N.J. Hitchin, The self duality equations on a Riemann surface, Proc. London Math. Soc. (3), **55** (1987), 58-126.

[12] N.J. Hitchin, The symplectic geometry of moduli space of connections and geometric quantization, Progr. of Theoret. Phys., Supplement, **102** (1990), 159-174.

[13] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Hyperkähler metrics and supersymmetry, Comm. Math. Phys., **108** (1987), 535-589.

[14] R.B. Lockhart and R.C. McOwen, Elliptic differential operators on noncompact manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **12** (1985), 409-447.

[15] V. Mehta and C.S. Seshadri, Moduli of vector bundles on curves with parabolic structures, Math. Ann., **248** (1980), 205-239.

[16] M.S. Narasimhan and C.S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, Ann. of Math., **82** (1965), 540-564.

[17] C.T. Simpson, Constructing variation of Hodge structure using Yang-Mills theory and applications to uniformization, J. Amer. Math. Soc., **1** (1988), 867-918.

[18] C.T. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc., **3** (1990), 713-770.

[19] K.K. Uhlenbeck and S.T. Yau, On the existence of Hermitian Yang-Mills connections in stable bundles, Comm. Pure Appl. Math., **39-S** (1986), 257-293.

[20] B. Nasatyr, Oxford thesis.

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