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A Liouville theorem on an analytic space

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Introduction.

In [**TA2**], we showed Liouville theorems for harmonic maps and plurisubharmonic functions on a non-compact Kähler manifold by establishing a method to estimate the energy of those maps and functions. In this article we shall generalize those results to a reduced analytic space M of dimension $m\geqq 1$ which possesses a non-degenerate d-closed positive current ω of bidegree $(m-1, m-1)$ and an unbounded exhaustion function τ whose level hypersurface satisfies a slow volume growth condition relative to ω. Here it should be noted that any pseudoconvexity, pseudoconcavity nor subharmonicity for the exhaustion function τ is not required as in [**GW1**], [**KA1**], [**ST1**], [**TA1**] and [**WU1**]. This simplification of the assumptions for the triple (M, τ, ω) enables us to show a Casorati-Weierstrass theorem not only for holomorphic maps, but also for meromorphic maps (cf. [**TA2**]). Thus this theorem provides us a transparent understanding of Liouville property for those analytic objects and a generalization of the results obtained in [**KA1**], [**KA2**], [**SIW**], [**ST2**] and [**TA2**]. The reader should refer to [**NO**] or [**TU**] for a general reference about differential forms, positive currents, meromorphic maps and Stokes' theorem on analytic spaces.

Our main result is stated as follows.

THEOREM. *Let M be an irreducible reduced analytic space of pure dimension $m\geqq 1$. Suppose there exist*

(1) *a d-closed positive current ω of bidegree $(m-1, m-1)$ on M whose coefficients are Lipschitz continuous and non-degenerate at some non-singular point of M (we set $\omega\equiv 1$ if $m=1$), and*

(2) *an unbounded exhaustion function $\tau: M\to[\inf\tau, \infty)$ of class C^2 satisfying*

$$\limsup_{r\to\infty}\frac{\int_{M(r)}dd_c\tau\wedge\omega}{g(r)}<+\infty,$$

where $M(r):=\{\tau<r\}$, $d_c=\sqrt{-1}(\bar\partial-\partial)/2$ and $g:(-\infty, +\infty)\to(0, +\infty)$ is a positive

non-decreasing function with

$$\int_{\inf \tau}^{+\infty} \frac{dt}{g(t)} = +\infty .$$

Then we have the following:

(i) *Casorati-Weierstrass' theorem holds for meromorphic maps into the n-dimensional complex projective space* $\boldsymbol{P}_n$ *(cf. Remark 1); i.e., the image of any non-constant meromorphic map from M into* $\boldsymbol{P}_n$ *intersects almost all hyperplanes in* $\boldsymbol{P}_n$.

(ii) *Moreover, suppose* ω *is non-degenerate almost everywhere on M. If a smooth subharmonic function h relative to* ω *(i.e.,* $dd_c h \wedge \omega \geqq 0$ *on the set of non-singular points of M) satisfies*

$$\limsup_{r \to \infty} \frac{m(h, r)^2 \int_{M(r)} dd_c \tau \wedge \omega}{g(r)} < +\infty ,$$

where $m(h, r) := \sup_{z \in M(r)} |h(z)|$, *then h is constant. In particular, M admits no non-constant negative smooth subharmonic functions relative to* ω.

Before proving the above Theorem, we would like to mention several remarks and to state corollaries which follow from the above Theorem.

REMARK 1. Here we fix terminology for meromorphic maps. Let M and N be irreducible reduced analytic spaces of pure dimension m and n respectively. A meromorphic map $f: M \to N$ from M to N is defined as a reduced analytic subspace Γ_f of $M \times N$ such that the natural projection $p: \Gamma_f \to M$ is proper and there are open dense subsets V of Γ_f and W of M so that p induces a biholomorphic map from V onto W. For any subset A of M the image $f(A)$ of A is defined by $q \circ p^{-1}(A)$, where $q: \Gamma_f \to N$ is the natural projection. We denote by E the degeneracy set of the natural projection $p: \Gamma_f \to M$ and put $W := M \setminus p(E)$. Then

(i) Γ_f is an irreducible reduced analytic space of pure dimension m;

(ii) p induces a biholomorphic map from $\Gamma_f \setminus E$ onto W and $p(E)$ is a lower dimensional closed analytic subspace of M;

(iii) $f_W := f|_W: W \to N$ is a holomorphic map so that the closure of the graph of f_W coinsides with Γ_f (cf. [**NO**], 4.4).

Γ_f is called the graph of f. Let $g: L \to M$ be a surjective holomorphic map from an analytic space L onto M. Then the composition of the maps f and g can be defined as a meromorphic map and is written as $h = f \circ g: L \to N$ (cf. [**NO**], 4.4).

REMARK 2. We point out an intrinsic relation between the assumptions for the triple (M, τ, ω) in Theorem and meromorphic maps from M to analytic spaces. Let $f: M \to N$ be a meromorphic map from M onto a reduced analytic space N, and G_f the graph of f. Using the natural projection $p: G_f \to M$, we lift the exhaustion function τ and the current ω on G_f. We set $\tau^* := \tau \circ p$ and $\omega^* := p^* \omega$. Since p is proper, in view of Remark 1 the triple (G_f, τ^*, ω^*) also satisfies the same properties as (M, τ, ω). In particular, if M satisfies Liouville properties for meromorphic maps and plurisubharmonic functions on M, then so does N. This argument is used to show Casorati-Weierstrass' theorem for meromorphic maps in the proof of Theorem.

REMARK 3. Let M be an irreducible reduced analytic space of pure dimension $m \geqq 1$, and let Ψ be a smooth unbounded plurisubharmonic exhaustion function on M (cf. [**RI**]). We set $M_o := M \setminus S_M$ with the singular locus S_M of M. The following condition was introduced in [**KA1**], p. 293, (C, 1):

There exists a non-increasing continuous function $\eta \geqq 0$ such that

$$\overset{m}{\wedge} dd_c \Psi \leqq \eta(\Psi) d\Psi \wedge d_c \Psi \wedge \overset{m-1}{\wedge} dd_c \Psi \quad \text{on} \quad M_o \setminus K$$

and

$$\int_{\delta_1}^{+\infty} \exp\left(-\int_{\delta_2}^{t} \eta(s) ds\right) dt = +\infty$$

for a compact subset K of M and $\delta_1 > \delta_2$.

It easily follows from this condition that $\int_{M(r)} \overset{m}{\wedge} dd_c \Psi \leqq C g(r)$ with a constant $C > 0$ not depending on r, where $g(r) := \exp\left(\int_{\delta_2}^{r} \eta(t) dt\right)$ (cf. Section 1). Hence by assuming a non-degenerate condition for Ψ, we can verify the same Liouville properties as in Theorem for the triple $(M, \Psi, \overset{m-1}{\wedge} dd_c \Psi)$ (cf. [**KA1**], $(C, 2)$).

For instance we consider a parabolic space (M, Φ) of pure dimension $m \geqq 1$ in the sense of Stoll (cf. [**ST2**]); i.e., Φ is a smooth unbounded exhaustion function which satisfies the following conditions:

(i) $dd_c \Phi \geqq 0$ and $\overset{m}{\wedge} dd_c \Phi \equiv 0$ on $M_o \setminus K$ for a compact subset K of M;

(ii) $\overset{m}{\wedge} dd_c \rho \not\equiv 0$ (or $\neq 0$) on M_o, where $\rho = \exp \Phi$. Then $\Psi = \log(1 + \rho)$ and $\eta(r) = 1/r$ satisfy the condition stated above.

From Theorem we obtain the following corollary which is a generalization of our previous result [**TA2**], Theorem 2 (cf. [**KA2**], [**SIW**]).

COROLLARY 1. *Let $A \hookrightarrow \boldsymbol{C}^n$ be an irreducible (reduced) analytic subset of pure dimension $m \geqq 1$. Suppose*

$$\int_{\delta}^{+\infty} \frac{dt}{t n(A, t)} = +\infty,$$

where $n(A, r) := \int_{A \cap \{\|z\| < r\}} \overset{m}{\wedge} dd_c \|z\|^2 / r^{2m}$ *and* $\delta > 0$. *Then Casorati-Weierstrass'* *theorem holds for meromorphic maps from A into $\boldsymbol{P}_n$ and A admits no non-constant smooth negative plurisubharmonic functions.*

Set $\tau(z)=\log(1+\|z\|^2)$ and $\omega=\overset{m-1}{\wedge} dd_c\tau$. Then the above corollary immediately follows from Theorem because $n(A, r)$ is a non-decreasing function in r. The class of analytic subsets of $\boldsymbol{C}^n$ satisfying the condition $\int_{\delta}^{+\infty}(tn)A, t))^{-1}dt=+\infty$ is strictly larger than that of affine algebraic varieties (cf. [**ST1**]).

Next we obtain the following corollary (cf. [**KA1**], Corollary and [**TA2**], Theorem 3).

COROLLARY 2. *Let (M, ds_M^2) be an m $(\geqq 1)$-dimensional complete Kähler manifold with a pole $0\in M$, and let Φ be the distance function from $0\in M$ relative to ds_M^2. Suppose that*

$$|\text{radial curvature at } x| \leqq \varepsilon/((\Phi(x)+\gamma)^2\log(\Phi(x)+\gamma))$$

for all $x\in M$, where $\gamma>e$ is a constant and $\varepsilon=1/((4m-1)(\gamma+1))$.

Then Casorati-Weierstrass' theorem holds for meromorphic maps into $\boldsymbol{P}_n$. Moreover, if a non-negative plurisubharmonic function h on M satisfies

$$\limsup_{r\to\infty}\frac{m(h, r)}{(\log r)^{\delta}}<+\infty,$$

where $m(h, r):=\sup_{w\in M(r)}h(w)$ with $M(r)=\{\Phi<r\}$, and $\delta=(1+2\varepsilon-4m\varepsilon)/2$, then h is constant.

Corollary 2 will be proved later.

REMARK 4. In the above corollaries the analytic set A and the manifold M are Stein. However there exists an example which satisfies the conditions of Theorem and is neither holomorphically convex nor holomorphically separable but is meromorphically separable. It is constructed as follows:

Let X be an m-dimensional non-singular projective algebraic variety equipped with a Kähler metric ω_X. Let F be a flat line bundle of infinite order on X. We can choose a system of transition functions $\{f_{ij}\}$ of F such that $|f_{ij}|=1$ on $U_i\cap U_j$ for a suitable locally finite covering $\{U_i\}$ of X. Let ζ_i be a fibre coordinate function of F restricted over U_i. Then $\rho(\zeta):=|\zeta_i|^2$ defines a smooth exhaustion function on F. Putting $\tau=\log(1+\rho)$ (or $\log\rho$) and a Kähler metric $\omega_F=dd_c\rho+\omega_X$, we have $\int_F dd_c\tau\wedge\overset{m}{\wedge}\omega_F<+\infty(\dim_{\boldsymbol{C}}F=m+1)$. Since any global holomorphic function on F is constant unless F is of finite order, F is neither holomorphically convex nor holomorphically separable. However F is meromorphically separable because F admits a projective algebraic compactification. The reader should refer to [**GR**] about the detail of the above discussion.

REMARK 5. The growth condition for the integral in Theorem is optimal in the following sense:

Let (M, p, ds_M^2) be a real two dimensional model space (i.e., the metric ds_M^2 is rotationally symmetric relative to the point p of M) whose metric in polar coordinates centered at p is written as $ds_M^2=dr^2+f(r)^2d\theta^2$. Then the function $f: [0, +\infty)\to[0, +\infty)$ satisfies that $f(0)=0$, $f'(0)=1$, $f(r)>0$ for $r>0$ and $f''(r)=K(r)f(r)$ ($K(r)$ is called the radial curvature function). For a given $\varepsilon>1$, we assume that f is convex and $f(r)=r(\log r)^{\varepsilon}$ outside a compact subset $[0, b_\varepsilon]$, where $b_\varepsilon>0$ is a constant depending on ε. The convexity of f assures nonpositive radial curvature. Setting $\Phi(x):=\mathrm{dist}_M(p, x)^2$, we obtain a smooth exhaustion function on M. Then $\log\Phi$ is subharmonic on $M\backslash(p)$ with respect to the complex structure determined by ds_M^2 (cf. [**GW2**], Proposition 2.24). Hence $\tau:=\log(1+\Phi)$ is a strictly subharmonic exhaustion function on M and it is easily verified that $\int_{M(r)} dd_c\tau\sim r^{2\varepsilon}$ for any $r>b_\varepsilon$ and $M(r)=\{\tau<r\}$ (cf. [**WU2**]). Since $\varepsilon>1$, $\int_1^\infty dr/r^{2\varepsilon}<+\infty$. On the other hand, M has the conformal type of the unit disk since $\int_1^\infty dr/f(r)<+\infty$ (cf. [**GW2**], Proposition 5.13). Therefore M admits many non-constant bounded holomorphic functions.

However in the case of Corollary 1 it is not clear whether the growth condition for the function $n(A, r)$ is optimal or not. Nevertheless, it seems natural to expect that the analyticity of A in $\boldsymbol{C}^n$ may imply a certain stronger result than ours. Here we propose the following problem:

PROBLEM. *Let $A\hookrightarrow\boldsymbol{C}^n$ be an irreducible (reduced) analytic subset of pure dimension $m\geqq 1$. Suppose*

$$\lim_{r\to\infty}\frac{\log n(A, r)}{\log r}=0$$

Then, does A admit no non-constant bounded holomorphic functions?

1. Proof of Theorem.

We denote by $\boldsymbol{P}_n^*$ the dual projective space of $\boldsymbol{P}_n$. We denote the homogeneous coordinates of $\boldsymbol{P}_n$ (resp. $\boldsymbol{P}_n^*$) by $\sigma=(\sigma_0:\sigma_1:\cdots:\sigma_n)$ (resp. $\xi=(\xi^0:\xi^1:\cdots:\xi^n)$). Then it is easily verified that a function $\Lambda(\sigma, \xi):=\|\sigma\|\|\xi\|/|\langle\sigma, \bar{\xi}\rangle|\geqq 1$ on $\boldsymbol{P}_n\times\boldsymbol{P}_n^*$ ($\langle a, b\rangle=\sum_{i=0}^n a_i\bar{b}^i$ and $\|a\|=\sqrt{\langle a, a\rangle}$) satisfies the following properties:

(i) $$\chi=dd_c\log\Lambda(\sigma, \xi)\quad\text{on}\quad\boldsymbol{P}_n\diagdown\mathrm{Supp}(\xi)$$

for any $\xi\in\boldsymbol{P}_n^*$, where χ is the Fubini-Study form on $\boldsymbol{P}_n$;

(ii) $$A := \int_{\xi \in P_n^*} \Lambda(\sigma, \xi) \overset{n}{\wedge} \chi^*$$

is a positive constant not depending on $\sigma \in \boldsymbol{P}_n$, where χ^* is the Fubini-Study form on $\boldsymbol{P}_n^*$;

(iii) There exists a positive constant C_* not depending on $(\sigma, \xi) \in \boldsymbol{P}_n \times \boldsymbol{P}_n^*$ such that

$$|\partial_\sigma \log \Lambda(\sigma, \xi)|_\chi \leqq C_* \Lambda(\sigma, \xi)$$

for any $\xi \in \boldsymbol{P}_n^*$ and any $\sigma \in \boldsymbol{P}_n \setminus \mathrm{Supp}(\xi)$.

Here (ξ) is the hyperplane on $\boldsymbol{P}_n$ defined by $\xi \in \boldsymbol{P}_n^*$.

First we consider a holomorphic map $f: M \to \boldsymbol{P}_n$ from M to $\boldsymbol{P}_n$. Suppose that f is non-constant and there exists a subset E of $\boldsymbol{P}_n^*$ such that E has positive measure and the image of f in $\boldsymbol{P}_n$ does not intersect any hyperplane contained in E.

Since we may assume that M is a paracompact Hausdorff space, M has a partition of unity of class C^∞. Therefore there exists a smooth real (1, 1) form ω_M such that coefficients of ω_M are locally bounded around S_M and that ω_M defines a smooth hermitian metric on $M_o = M \setminus S_M$.

We set $\psi := \log \Lambda$. Since τ is an exhaustion function of class C^2 and coefficients of $d_c f^* \psi \wedge \omega$ are continuous on M, there exists a measure zero set $I_1 \subset \tau(M)$ such that for any $r \in \tau(M) \setminus I_1$, $\partial M_r \cap M_0 (\partial M_r := \{\tau = r\})$ is a real hypersurface of class C^2 in M_0 and the following Stokes' Theorem holds

$$\int_{M(r)} d d_c f^* \psi \wedge \omega = \int_{\partial M_r \cap M_0} d_c f^* \psi \wedge \omega$$

(cf. [**TU**], VII. Integration Theorems).

Since $\partial M_r \cap M_0$ is a real hypersurface of class C^2, we define its volume element dS_r by $dS_r := *(d\tau / |d\tau|_{M_0})$ so that $\overset{m}{\wedge} \omega_{M|M_0} / m! = d\tau / |d\tau|_{M_0} \wedge dS_r$ at every point of $\partial M_r \cap M_0$. Here $*$ is the star operator relative to $\omega_{M|M_0}$. On M_0, we can represent the form ω locally

$$\omega = \sigma_{m-1}^* \sum \omega_{i\bar{j}} dz^{J_i} \wedge d\bar{z}^{J_j}$$

so that $\omega_{i\bar{j}}$ are locally bounded around S_M. Here $\sigma_k^* = (\sqrt{-1}/2)(-1)^{k-1} \sigma_{k-1}^* (k = 1, \cdots, m$ and $\sigma_0^* = 1)$, $J_i = (j_1, \cdots, j_{m-1})$, $j_1 < \cdots < j_{m-1}$ and $i \notin J_i$. For any $r \in \tau(M) \setminus I_1$, we define a function h_r on $\partial M_r \cap M_0$ by the following equation:

$$h_r \lambda_r = d_c f^* \psi \wedge \omega \quad \text{and} \quad \lambda_r = dS_r / |d\tau|_{M_0} \quad \text{on} \quad \partial M_r \cap M_0.$$

For any point $z \in \partial M_r \cap M_0$, we can choose local coordinate systems (z^i) around z and (w^α) around $w = f(z)$ so that $\omega_M(z) = (\sqrt{-1}/2) \sum_{i=1}^m dz^i \wedge d\bar{z}^i$, $\omega(z) = \sigma_{m-1}^* \sum_{i=1}^m \omega_{i\bar{i}}(z) dz^{J_i} \wedge d\bar{z}^{J_i}$, and $\chi(w) = \sqrt{-1} \sum_{\alpha=1}^n dw^\alpha \wedge d\bar{w}^\alpha$. Since $\omega_{i\bar{i}}(z) \geqq 0$ and $f(\partial M_r) \cap \mathrm{Supp}(\xi) = \emptyset$ for any $\xi \in E \subset \boldsymbol{P}_n^*$, we have by the property (i) and

Schwarz's inequality

$$h_r(z) = m!2\,\mathrm{Re}\{\textstyle\sum_{i=1}^m \sum_{\alpha=1}^n f_i^\alpha \partial_\alpha \phi(w,\xi)\tau_{\bar{i}}\omega_{i\bar{i}}\}(z)$$
$$\leqq f^*(|\partial_\sigma\phi(\sigma,\xi)|_\chi)(z)\Psi_r(z)$$

for $z\in\partial M_r\cap M_o$, where $\Psi_r(z):=\sqrt{*(d\tau\wedge d_c\tau\wedge\omega)*(dd_cf^*\phi\wedge\omega)(z)}$, defined on $\partial M_r\cap M_o$. Since there is a measure zero set I_2 such that

$$0\leqq \frac{\partial}{\partial r}\int_{M(r)} dd_cf^*\phi\wedge\omega = \int_{\partial M_r\cap M_o} *(dd_cf^*\phi\wedge\omega)\lambda_r < +\infty$$

and

$$0\leqq \int_{\partial M_r\cap M_o} *(d\tau\wedge d_c\tau\wedge\omega)\lambda_r = \int_{\partial M_r\cap M_o} d_c\tau\wedge\omega$$
$$= \int_{M(r)} dd_c\tau\wedge\omega < +\infty$$

for any $r\in\tau(M)\setminus(I_1\cup I_2)$, by Schwarz's inequality the function Ψ_r is integrable over $\partial M_r\cap M_o$ for every $r\in\tau(M)\setminus(I_1\cup I_2)$. Since $f^*(|\partial_\sigma\phi|_\chi)$ is integrable on $(\partial M_r\cap M_o)\times E$ and E has positive measure, the properties (ii) and (iii) and Fubini's theorem imply

$$\int_{M(r)} dd_cf^*\phi\wedge\omega \leqq C\int_{\partial M_r\cap M_o}\Psi_r\lambda_r \leqq C\sqrt{\int_{M(r)} dd_c\tau\wedge\omega\frac{\partial}{\partial r}\int_{M(r)} dd_cf^*\phi\wedge\omega},$$

where C is a positive constant not depending on r. Since $\omega\geqq 0$ on M_o and $\omega>0$ at least one point of M_o, by the unique continuation theorem for holomorphic maps we have $u(r):=\int_{M(r)} dd_cf^*\phi\wedge\omega>0$ for all sufficiently large r. Hence by the assumption and the above inequality we have

$$(*)\qquad u(r)^2 \leqq C'g(r)\frac{\partial}{\partial r}u(r)$$

for any $r(\gg 0)\in\tau(M)\setminus(I_1\cup I_2)$, where C' is a positive constant not depending on r.

The inequality (*) implies that

$$\int_{r_0}^{+\infty}\frac{dt}{g(t)} < +\infty$$

for some $r_0>0$. This contradicts the choice of $g(r)$.

Next we consider a meromorphic map $h: M\to \boldsymbol{P}_n$ from M to $\boldsymbol{P}_n$. Suppose that h is non-constant and there is a subset E of $\boldsymbol{P}_n^*$ such that E has positive measure and the image of h does not intersect any hyperplane contained in E.

Let $\Gamma_h\hookrightarrow M\times\boldsymbol{P}_n$ be the graph of h. Then the natural projection $q: \Gamma_h\to \boldsymbol{P}_n$ is holomorphic and satisfies the same property as h. However in view of the observation made in Remark 2 and the above discussion we have already seen that Casorati-Weierstrass' theorem holds for holomorphic maps from Γ_h to $\boldsymbol{P}_n$. Hence q is constant, and so is h. Therefore Casorati-Weierstrass' theorem holds for meromorphic maps from M to $\boldsymbol{P}_n$.

To show the latter assertion we have only to consider the integral $v(r):=\int_{M(r)} dh\wedge d_c h\wedge\omega$ for a smooth subharmonic function h relative to ω. If h is non-constant, then by the assumption we can show the inequality (*) for $v(r)$ similarly. The proof of Theorem is completed.

REMARK 6. Let $f: M\to\boldsymbol{P}_n$ be a meromorphic map from an irreducible reduced analytic space M to $\boldsymbol{P}_n$. If the image of f omits a hyperplane $H\subset\boldsymbol{P}_n$, then the image $f(x)$ of any point x of M consists only one point because $f(x)$ is a connected compact analytic subset of $\boldsymbol{P}_n\setminus H$ and $\boldsymbol{P}_n\setminus H$ is Stein. This fact implies that the meromorphic map f is already holomorphic if M is a normal analytic space. However this argument does not hold generally if M is not normal (cf. [**NO**], Theorems (4.4.1), (4.4.6), (4.4.8). The results there are for complex manifolds, but they can be applied to normal analytic spaces; see [**F1**], Appendix).

2. Proof of Corollary 2.

Let ϕ be the distance function from the point 0 of M relative to ds_M^2. Then by Hessian comparison theorem for Φ^2 we can verify the following facts:

(2.1) $\tau:=(\log(1+\Phi^2))^{1+2\varepsilon}$ is strictly plurisubharmonic on M (cf. [**TA2**], Proof of Theorem 3, (2.34));

(2.2) $\int_{\{\Phi<s\}}\overset{m}{\wedge} dd_c\Phi^2\leqq C_1 s^{2m}(\log(1+s))^{2m}$ for any $s>0$, where C_1 is a positive constant not depending on s (cf. [**TA2**], Proof of Theorem 2.4, (2.18) and (2.19)).

Hence we have the following estimate:

$$\int_{\{\tau<r\}}\overset{m}{\wedge} dd_c\tau\leqq C_{2,\varepsilon} r^{4m\varepsilon/(1+2\varepsilon)} \tag{2.3}$$

for any $r\gg 0$, where $C_{2,\varepsilon}$ is a positive constant not depending on r.

Since $0<4m\varepsilon<1$, Casorati-Weierstrass' theorem for the triple $(M, \tau, \overset{m-1}{\wedge} dd_c\tau)$ follows from (2.3).

Next we show the latter assertion. Let h be a non-constant, non-negative, plurisubharmonic function on M satisfying

$$\limsup_{r\to\infty}\frac{m(h, r)}{(\log s)^{\delta}}<+\infty, \tag{2.4}$$

where $m(h, s)=\sup_{z\in\{\Phi<s\}} h(z)$.

By (2.3) and (2.4), there exist $r_0>0$ and $C_3>0$ not depending on r such that

$$m(h, r)^2\int_{M(r)} \overset{m}{\wedge} dd_c\tau \leqq C_3 r \tag{2.5}$$

for $r>r_0$, where $M(r)=\{\tau<r\}$ and $m(h, r)=\sup_{w\in M(r)}h(w)$.

Since τ is a smooth strictly plurisubharmonic exhaustion function on M, M is a Stein manifold. Hence there exists a proper embedding $\zeta: M\hookrightarrow \boldsymbol{C}^{2m+1}$ of M into the $2m+1$-dimensional complex vector space $\boldsymbol{C}^{2m+1}$. By identifying M with the image $\zeta(M)$, we may consider that M is a closed submanifold of $\boldsymbol{C}^{2m+1}$. Then there exist a neighborhood V of M in $\boldsymbol{C}^{2m+1}$ and a holomorphic retraction $\pi: V\to M$ (cf. [**GUR**], Chap. VII, C and Chap. VIII, C). Since the pull back h^* of h by π is plurisubharmonic on $V\subset \boldsymbol{C}^{2m+1}$, we can apply a standard smoothing argument to h^*. Hence for a sequence $\{r_j\}_{j\geq 0}$ of positive numbers with $r_j=2^j r_0$ $(j\geqq 1)$ there is a decreasing sequence $\{h_j\}_{j\geq 1}$ of smooth functions on M such that

(2.6) h_j is plurisubharmonic on $M(r_j)$

(2.7) $\{h_j\}_{j\geq 1}$ converges to h on $M(r_i)$

(2.8) $m(h_j, r_i)\leqq 2m(h, r_i)$ for all $j\geqq i$.

We set

$$\omega := \overset{m-1}{\wedge} dd_c\tau, \qquad u(h_j, r) := \int_{M(r)} dd_c h_j^2\wedge\omega .$$

Since $h_j\geqq h\geqq 0$, by (2.6) h_j^2 is plurisubharmonic on $M(r_i)$ for all $j\geqq i$, so that $u(h_j, r)$ is non-negative for any $r\in(r_0, r_i]$. By Stokes' theorem, (2.5), (2.6) and (2.8) we have the following inequality similarly to (*) in the proof of Theorem:

$$u(h_j, r)^2 \leqq C_4 r\frac{\partial}{\partial r}u(h_j, r) \tag{2.9}$$

for any $r\in(r_0, r_i]$ and $j\geqq i$, where C_4 is a positive constant not depending on r and i. Now we may assume that for any fixed i, there is an infinite number of integers $j\geqq 2i$ so that $u(h_j, r_i)>0$. Otherwise, for every i there would be an integer $N(i)\geqq 2i$ such that $u(h_{N(i)}, r_i)=0$; i.e., $h_{N(i)}$ is constant on $M(r_i)$. By (2.7) this implies that h is constant on all $M(r_i)$ and hence on M. This is a contradiction. Thus, if necessary, taking a subsequence of $\{h_j\}$, we may assume that $u(h_j, r_i)>0$ for any i and j with $j\geqq 2i$. Since $u(h_j, r)\geqq u(h_j, r_i)$ for any $r\in[r_i, r_{2i}]$, integrating the inequality (2.9) we have

$$u(h_j, r_i)\leqq \frac{C_5}{i}$$

for any $j \geqq 2i$, where C_5 is a positive constant not depending on i and j. Hence by (2.6) we have

$$\lim_{i\to\infty} u(h_{2i}, r_k) = 0 \quad \text{for any} \quad k \geqq 1. \tag{2.10}$$

On the other hand, h^2 is also plurisubharmonic on M since h is a non-negative plurisubharmonic function on M. Hence h^2 is integrable on every $M(r)$ and h_{2j}^2 converges to h^2 on every $M(r_{2i})$. Moreover we may assume that $\{h_{2j}^2\}$ is uniformly bounded on $M(r_{2i})$ because h_{2j} decreasingly converges to h on $M(r_{2i})$. Therefore, if necessary, taking a subsequence of $\{h_{2j}\}$, we conclude that h_{2j}^2 converges to h^2 in $L^1(M(r_i), \overset{m}{\wedge} dd_c\tau)$. Therefore by (2.10) we have

$$\int_M h^2 dd_c\theta \wedge \omega = \lim_{i\to\infty}\int_M h_{2i}^2 dd_c\theta \wedge \omega$$

$$= \lim_{i\to\infty} -\int_M \theta dd_c h_{2i}^2 \wedge \omega = 0$$

for any smooth function θ with compact support on M.

Since $dd_c\tau$ gives a Kähler metric on M by (2.1), it follows that $\Delta_\tau h^2 \equiv 0$ in the sense of distribution, where Δ_τ is the Laplacian defined by $dd_c\tau$. Hence h^2 is smooth on M by Weyl's lemma. Since we may assume $\inf_{z\in M} h(z) > 0$ without loss of generality, it follows that h is a smooth plurisubharmonic function on M with $\Delta_\tau h^2 = 2h\Delta_\tau h + 2|dh|_\tau^2 \equiv 0$ on M. Therefore h is a constant. This is a contradiction.

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