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The Neumann and Dirichlet problems for elliptic operators

By Hisako WATANABE

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1. Introduction.

Let D be a bounded C^1-domain in $\boldsymbol{R}^d$. In [3] E.B. Fabes, M. Jodeit JR. and N.M. Rivière proved that, for every $f \in L^p(\partial D)$ satisfying $\int f d\sigma = 0$, there exists a function u which is harmonic in D, and $\langle \nabla u(X), N_P \rangle$ converges to $f(P)$ with an exception of a set of surface measure zero as X tends to P nontangentially. The corresponding results have been obtained even for a Lipschitz domain D in the case $1 < p < 2+\varepsilon$ (cf. [4], [2]).

On the other hand it is well-known that in $\boldsymbol{R}_+^{d+1}$ the Poisson integral of the Bessel potential $G_\alpha * f$ of each $f \in L^p(\boldsymbol{R}^d)$ converges not only nontangentially but also tangentially except for a set of an appropriately dimensional Hausdorff measure zero (cf. [1]).

In [7], for a bounded $C^{1,\alpha}$-domain D, we have studied the boundary behavior of the derivatives of solutions for the above Neumann problem, not up to an exception with a set of surface measure zero, but up to an exception with a set of β-dimensional Hausdorff measure zero for β satisfying $0 < \beta < d-1$.

In this paper we will consider the corresponding boundary behaviors of solutions of the Dirichlet and Neumann problems for uniformly elliptic differential operators.

Let L be a differential operator on $\boldsymbol{R}^d$ ($d \geqq 3$) defined by

$$L = \sum_{j,k=1}^d D_j(a_{jk} D_k), \tag{1.1}$$

where $D_j = \partial/\partial x_j$ and a_{jk} are of class $C^{1,\alpha}$ with $a_{jk} = a_{kj}$. Moreover L is assumed to be uniformly elliptic. This means that there exists a positive real number $\lambda > 1$ such that

$$\lambda^{-1}|\xi|^2 \leqq \sum_{j,k=1}^d a_{jk}(X)\xi_j\xi_k \leqq \lambda|\xi|^2$$

for all X, $\xi = (\xi_1, \cdots, \xi_d) \in \boldsymbol{R}^d$.

Let D be a bounded $C^{1,\alpha}$-domain in $\boldsymbol{R}^d$ and $0 < \beta < d-1$. To classify functions defined on ∂D, we use, as in [7], a countably sublinear functional γ_β and a function space $\mathcal{L}(\gamma_\beta, C(\partial D))$, instead of the L^p-norm and $L^p(\partial D)$, respectively.

More precisely, let $J(\partial D)$ be the class of the extended real-valued functions on ∂D and define, for $f \in J(\partial D)$,

$$\gamma_\beta(f) := \inf\{\sum_{j=1}^\infty b_j r_j^\beta ;\ b_j \in \boldsymbol{R}^+,\ \sum_{j=1}^\infty b_j \chi_{A(P_j, r_j)} \geqq |f| \text{ on } \partial D\},$$

where $A(P, r) = B(P, r) \cap \partial D$ and $B(P, r)$ stands for the open ball in $\boldsymbol{R}^d$ with center P and radius r.

The functional γ_β is countably sublinear, i.e., it is a mapping from $J(\partial D)$ to $\boldsymbol{R}^+ \cup \{+\infty\}$ with the following properties:

(i) $\gamma_\beta(f) = \gamma_\beta(|f|)$,

(ii) $\gamma_\beta(bf) = b\gamma_\beta(f)$ for each $b \in \boldsymbol{R}^+$,

(iii) $f, f_n \geqq 0,\quad f \leqq \sum_{n=1}^\infty f_n \Rightarrow \gamma_\beta(f) \leqq \sum_{n=1}^\infty \gamma_\beta(f_n)$.

To simplify the notations, we use $\gamma_\beta(E)$ instead of $\gamma_\beta(\chi_E)$ for a subset E of ∂D. A subset E of ∂D is called γ_β-polar if $\gamma_\beta(E) = 0$. We has shown in [7] that, a Borel set E is γ_β-polar if and only if it is of β-dimensional Hausdorff measure zero.

We say that a property holds γ_β-q.e. on ∂D if it holds on ∂D except for a γ_β-polar set. Note that, if $\gamma_\beta(f) < +\infty$, then $|f| < +\infty$ γ_β-q.e. on ∂D.

Let us denote by $\mathcal{L}(\gamma_\beta, C(\partial D))$ the class of all Borel measurable functions f such that $\gamma_\beta(f - f_n) \to 0$ for some sequence $\{f_n\} \subset C(\partial D)$, where $C(\partial D)$ stands for the class of all continuous real-valued functions on ∂D.

Furthermore we denote by $L(\gamma_\beta, C(\partial D))$ the family of the equivalent classes relative to the equivalent relation defined by $f = g$ γ_β-q.e. on ∂D. The space $L(\gamma_\beta, C(\partial D))$ is a Banach space with norm $\|f\| = \gamma_\beta(f)$ and it enables us to use the method of layer potentials.

Let $0 < \eta < 1$. The approach region at P is a nontangential region defined by

$$\Gamma_\eta(P) := \{X \in D;\ \langle P - X, N_P \rangle > \eta |X - P|\},$$

where $\langle\, ,\rangle$ is the inner product and N_P is the unit outer normal to the boundary at P.

Using the countably sublinear functional γ_β, we can estimate the nontangential maximal functions of 'double layer potentials' and the gradients of 'single layer potentials' by the same method as in the L^p theory, without technical skills.

In §6 the following Neumann problem with boundary data $\mathcal{L}(\gamma_\beta, C(\partial D))$ will be proved.

THEOREM 1. *Let $0 < \alpha < 1$ and D be a bounded $C^{1,\alpha}$-domain in $\boldsymbol{R}^d$. Furthermore, assume that $0 < \beta < d-1$ and $0 < \eta < 1$. Then for each function $f \in$*

$\mathcal{L}(\gamma_\beta, C(\partial D))$ *such that* $\int f d\sigma=0$ *there exists a function* u *in* D *and a subset* E *of* ∂D *having the following properties:*

(i) E *is a set of* β*-dimensional Hausdorff measure zero,*

(ii) $Lu=0$ *in* D.

(iii) $\lim_{X\to P, X\in\Gamma_\eta(P)}\langle A(P)N_P, \nabla u(X)\rangle=f(P)$ *for every* $P\in\partial D\setminus E$,

where $A(P)$ *stands for the matrix* $(a_{jk}(P))$.

In §7 the following Dirichlet problem with boundary data $\mathcal{L}(\gamma_\beta, C(\partial D))$ will be proved.

THEOREM 2. *Let* $0<\alpha<1$ *and* D *be a bounded* $C^{1,\alpha}$*-domain in* $\boldsymbol{R}^d$ *such that* $\boldsymbol{R}^d\setminus\bar{D}$ *is connected. Furthermore, assume that* $0<\beta<d-1$ *and* $0<\eta<1$. *Then for each function* $f\in\mathcal{L}(\gamma_\beta, C(\partial D))$ *there exists a function* v *in* D *and a subset* E *of* ∂D *having the following properties:*

(i) E *is a set of* β*-dimensional Hausdorff measure zero,*

(ii) $Lv=0$ *in* D,

(iii) $\lim_{X\to P, X\in\Gamma_\eta(P)}v(X)=f(P)$ *for every* $P\in\partial D\setminus E$.

We note that, if $\lambda>d-1-\beta>0$, then $\mathcal{L}(\gamma_\beta, C(\partial D))$ contains all functions of the form:

$$P\longmapsto\int|P-Q|^{\lambda+1-d}g(Q)d\sigma(Q)$$

for $g\in L^1(\partial D)$ (cf. [7]). Furthermore if $0<\alpha<d<\alpha+\beta$ and G_α be the Bessel kernel with order α, then the restriction of $G_\alpha*f$ to ∂D belongs to $\mathcal{L}(\gamma_\beta, C(\partial D))$ for every $f\in L^1(\boldsymbol{R}^d)$ (cf. [8]).

2. The fundamental solution.

In this paper, let D be a bounded $C^{1,\alpha}$-domain for $0<\alpha<1$. Recall that a domain D in $\boldsymbol{R}^d$ is called a $C^{1,\alpha}$-domain if to each point $Q\in\partial D$ there correspond a system of coordinates of $\boldsymbol{R}^d$ with origin Q and an open ball $B(Q, \rho)$ with center Q and radius ρ such that with respect to this coordinate system

$$D\cap B(Q, \rho)=\{(x, t);\ x\in\boldsymbol{R}^{d-1},\ t>\phi(x)\}\cap B(Q, \rho),$$

where $\phi\in C_0^{1,\alpha}(\boldsymbol{R}^{d-1})$ and $\phi(0)=D_j\phi(0)=0$. Note that $C_0^{1,\alpha}(\boldsymbol{R}^{d-1})$ stands for the space of all functions g in $C_0^1(\boldsymbol{R}^{d-1})$ with compact support satisfying

$$|D_jg(x)-D_jg(y)|\leqq M|x-y|^\alpha$$

for all $x, y\in\boldsymbol{R}^{d-1}$ and $1\leqq j\leqq d-1$.

We take a sufficient large number R such that $B(0, R)\supset\bar{D}$. To find a fundamental solution of the uniformly elliptic operator L defined by (1.1), we consider the differential operator

$$L_0 = L - b, \tag{2.1}$$

where b is a nonnegative function of class $C^{1,\alpha}$ such that

$$b = 0 \text{ on } B(0, 2R), \quad b = 1 \text{ on } \boldsymbol{R}^d \setminus B(0, 3R) \text{ and } 0 \leqq b \leqq 1.$$

Denote by $A(X)$ the matrix $(a_{jk}(X))$, by $A^{-1}(X) = (a^{jk}(X))$ the inverse matrix of $A(X)$ and by $\det A(X)$ the determinant of $A(X)$. The following function H defined on $\boldsymbol{R}^d \times \boldsymbol{R}^d$ is fundamental:

$$H(X, Y) := (d-2)^{-1}\omega_d^{-1}(\det A(Y))^{-1/2}\langle A^{-1}(Y)(X-Y), X-Y\rangle^{(2-d)/2}.$$

The following theorem is well-known (cf. [5, Theorem 20.1]).

THEOREM A. *Let L_0 be the differential operator defined by* (2.1). *Then L_0 has the fundamental solution F in $\boldsymbol{R}^d$ with the following properties:*

(a) *F is continuous outside of the diagonal set $\{(X, X); X \in \boldsymbol{R}^d\}$, together with first and second derivatives,*

(b) $|F(X, Y) - H(X, Y)| \leqq c|X-Y|^{\alpha+2-d}$,

$$\left|\frac{\partial(F-H)}{\partial x_j}(X, Y)\right| \leqq c|X-Y|^{\alpha+1-d} \quad \text{and} \quad \left|\frac{\partial^2(F-H)}{\partial x_j \partial x_k}(X, Y)\right| \leqq c|X-Y|^{\alpha-d}$$

for all $X, Y \in B(0, 3R)$.

(c) *For each $Y \in \boldsymbol{R}^d$*

$$L_0 F(\cdot, Y) = 0 \quad \text{in } \boldsymbol{R}^d \setminus \{Y\}.$$

3. The operators K and K^*.

We begin with the following lemma.

LEMMA A ([7, Lemma 2.6]). *If $0 < \beta < d-1$, then*

$$\gamma_\beta(M_\sigma f) \leqq c\gamma_\beta(f) \quad \text{for all } f \in \mathcal{L}(\gamma_\beta, C(\partial D)),$$

where

$$M_\sigma f(P) = \sup\left\{r^{1-d}\int_{A(P,r)} |f| d\sigma ; r > 0\right\}.$$

Now, let us define, for a Borel function $f \in J(\partial D)$ and $P \in \partial D$,

$$Kf(P) = -\int \langle A(Q)N_Q, \nabla_Q F(Q, P)\rangle f(Q) d\sigma(Q)$$

and

$$K^*f(P) = -\int\langle A(P)N_P, \nabla_P F(P, Q)\rangle f(Q)d\sigma(Q)$$

if they are well-defined, and $Kf(P)=0$, $K^*f(P)=0$ otherwise.

The operator K^* has the following properties.

LEMMA 3.1. *Let* $p>1$ *and* $0<\beta<d-1$. *Then*
(a) $|K^*f|\leqq cM_\sigma f$ *for all Borel measurable functions* f *in* $L^1(\sigma)$,
(b) K^* *is a compact operator on* $L^p(\sigma)$,
(c) K^* *is a compact operator on* $L(\gamma_\beta, C(\partial D))$.

PROOF. (a): Note that

$$(3.1)\qquad -\langle A(P)N_P, \nabla_P F(P, Q)\rangle = \langle (A(Q)-A(P))N_P, \nabla_P F(P, Q)\rangle + \langle (A(Q)N_P, \nabla_P(H(P, Q)-F(P, Q))\rangle - \langle (A(Q)N_P, \nabla_P H(P, Q)\rangle.$$

Since a_{jk} are of class $C^{1,\alpha}$, it follows from Theorem A that the absolute values of the first and second terms on the right-hand side of (3.1) are dominated by $c_1|P-Q|^{\alpha+1-d}$. Noting that

$$\langle A(Q)N_P, \nabla_P H(P, Q)\rangle = \omega_d^{-1}(\det A(Q))^{-1/2}\langle A^{-1}(Q)(P-Q), P-Q\rangle^{-d/2}\langle N_P, Q-P\rangle$$

and, both of $A(Q)$ and $A^{-1}(Q)$ are uniformly elliptic and that D is a $C^{1,\alpha}$-domain, we see that the absolute value of the last term is also dominated by $c_2|P-Q|^{\alpha+1-d}$. Therefore we obtain

$$(3.2)\qquad |K^*f(P)| \leqq c_3\int|P-Q|^{\alpha+1-d}|f(Q)|d\sigma(Q) \leqq c_4 M_\sigma f(P),$$

which shows (a).

(b) and (c): By virtue of (3.2) and Lemma A we see that

$$\|K^*f\|_p \leqq c_5\|f\|_p \qquad \text{for all } f\in L^p(\sigma)$$

and

$$\gamma_\beta(K^*f) \leqq c_6\gamma_\beta(f) \qquad \text{for all } f\in \mathcal{L}(\gamma_\beta, C(\partial D)).$$

Moreover the function $(P, Q)\mapsto\langle A(P)N_P, \nabla_P F(P, Q)\rangle$ is continuous at (P_0, Q_0) if $P_0\neq Q_0$, and $|\langle A(P)N_P, \nabla_P F(P, Q)\rangle|$ tends to $+\infty$ as $Q\to P$. Therefore, by the same methods as in Theorem 2 in [7], we can prove that K^* is a compact operator on $L^p(\sigma)$ and $L(\gamma_\beta, C(\partial D))$. □

LEMMA 3.2. *Let* $p>1$ *and* $0<\beta<d-1$. *Then*
(a) $|Kf|\leqq cM_\sigma f$ *for all Borel measurable functions* f *in* $L^1(\sigma)$,

(b) *K is a compact operator on $L^p(\sigma)$,*

(c) *K is a compact operator on $L(\gamma_\beta, C(\partial D))$.*

PROOF. From Lemma 3.1 and

$$\begin{aligned} &-\langle A(Q)N_Q, \nabla_Q F(Q, P)\rangle \\ &= \langle (A(P)-A(Q))N_Q, \nabla_Q F(Q, P)\rangle + \langle A(P)(N_P-N_Q), \nabla_Q F(Q, P)\rangle \\ &\quad -\langle A(P)N_P, \nabla_Q F(Q, P)\rangle \end{aligned}$$

we deduce

$$\begin{aligned} |Kf(P)| &\leqq c_1\left\{\int |P-Q|^{\alpha+1-d}|f(Q)|d\sigma(Q) + |K^*f(P)|\right\} \\ &\leqq c_2\int |P-Q|^{\alpha+1-d}|f(Q)|d\sigma(Q), \end{aligned} \tag{3.3}$$

which leads to (a). One can prove (b) and (c) by the same method as in Lemma 3.1. □

4. Single layer potentials.

Let us define the single layer potential u_f for a Borel measurable function $f \in L^1(\sigma)$ by

$$u_f(X) = -\int F(X, Q)f(Q)d\sigma(Q)$$

if it is well-defined, and by $u_f(X)=0$ if otherwise. Moreover, set

$$\Phi_f(X, P) := \langle A(P)N_P, \nabla_X u_f(X)\rangle = -\int \langle A(P)N_P, \nabla_X F(X, Q)\rangle f(Q)d\sigma(Q),$$

$$\Phi^*_{f,\delta}(P) := \sup\{|\Phi_f(X, P)|\,;\ X \in \Gamma_\eta(P),\ |X-P|<\delta\}$$

and

$$\Phi^{**}_{f,\delta}(P) := \sup\{|\Phi_f(X, P)|\,;\ X \in \Gamma^e_\eta(P),\ |X-P|<\delta\}$$

where

$$\Gamma^e_\eta(P) := \{X \in \boldsymbol{R}^d \setminus D\,; \langle X-P, N_P\rangle > \eta|X-P|\}.$$

LEMMA 4.1. *Assume that $p>1$, $0<\beta<d-1$ and $0<\eta<1$. Then there exist positive real numbers c, δ with the following properties:*

(a) *$\Phi^*_{f,\delta}(P) \leqq cM_\sigma f(P)$ and $\Phi^{**}_{f,\delta}(P) \leqq cM_\sigma f(P)$ for every Borel measurable function f in $L^1(\sigma)$.*

(b) *$\|\Phi^*_{f,\delta}\|_p \leqq c\|f\|_p$ and $\|\Phi^{**}_{f,\delta}\|_p \leqq \|f\|_p$ for every $f \in L^p(\sigma)$,*

(c) *$\gamma_\beta(\Phi^*_{f,\delta}) \leqq c\gamma_\beta(f)$ and $\gamma_\beta(\Phi^{**}_{f,\delta}) \leqq c\gamma_\beta(f)$ for every $f \in \mathcal{L}(\gamma_\beta, C(\partial D))$.*

PROOF. Recall that

$$F(X, Q)=H(X, Q)+G(X, Q),$$

where H is the function defined in § 2 and

$$|\nabla_X G(X, Q)| \leqq c_1 |X-Q|^{\alpha+1-d}.$$

Since

$$\begin{aligned}&-\langle A(P)N_P, \nabla_X H(X, Q)\rangle\\ &=\langle (A(Q)-A(P))N_P, \nabla_X H(X, Q)\rangle+\langle A(Q)(N_Q-N_P), \nabla_X H(X, Q)\rangle\\ &\quad -\langle A(Q)N_Q, \nabla_X H(X, Q)\rangle,\end{aligned}$$

we have

$$\begin{aligned}&|-\langle A(P)N_P, \nabla_X H(X, Q)\rangle|\\ &\leqq c_2\{|P-Q|^{\alpha}|X-Q|^{1-d}+|X-Q|^{-d}|\langle X-Q, N_Q\rangle|\}.\end{aligned}$$

Assume that $\phi\in C_0^{1,\alpha}(\boldsymbol{R}^{d-1})$, $|\nabla\phi|\leqq\eta/6$,

$$\partial D\cap B(P, r)=\{(z, \phi(z));\ z\in\boldsymbol{R}^{d-1}\}\cap B(P, r).$$

If $X=(x, t)\in\Gamma_\eta(P)\cap B(P, r)$ and $P=(y, \phi(y))$, then $t-\phi(y)>(5\eta/6)|x-y|$. Therefore, if $Q=(z, \phi(z))\in\partial D$ and $3|x-y|\geqq|y-z|$, then we have

$$\begin{aligned}|X-Q| &\geqq |t-\phi(z)| \geqq t-\phi(y)-|\phi(y)-\phi(z)|\\ &\geqq (5\eta/6)|x-y|-(\eta/6)|y-z| \geqq (\eta/9)|y-z| \geqq (\eta/18)|P-Q|.\end{aligned}$$

By the same method as in the proof of Theorem 1.3 in [3] we can choose positive real numbers δ, c_4, c_6, independent of f, such that

$$\begin{aligned}&\sup\left\{\int |X-Q|^{-d}|\langle X-Q, N_Q\rangle||f(Q)|d\sigma(Q);\ X\in\Gamma_\eta(P),\ |X-P|<\delta\right\}\\ &\leqq c_3\left\{M_\sigma f(P)+\int |P-Q|^{\alpha+1-d}|f(Q)|d\sigma(Q)\right\}\leqq c_4 M_\sigma f(P)\end{aligned}$$

and

$$\begin{aligned}&\sup\left\{\int (|X-Q|^{\alpha+1-d}+|P-Q|^{\alpha}|X-Q|^{1-d})|f(Q)|d\sigma(Q);\ X\in\Gamma_\eta(P),\ |X-P|<\delta\right\}\\ &\leqq c_5\int |P-Q|^{\alpha+1-d}|f(Q)|d\sigma(Q)\leqq c_6 M_\sigma f(P).\end{aligned}$$

Thus we have the estimate of $\Phi^*_{f,\delta}$. Similarly the estimate of $\Phi^{**}_{f,\delta}$ is also obtained.

The estimates of (b) are easy consequences of (a). The estimates of (c) are deduced from (a) and Lemma A. □

Using Green's formula, we can easily show the following properties of H.

LEMMA 4.2.

(a) *For* $X \in D$

$$\int \langle A(X)N_Q,\ \nabla_Q H(Q,\ X)\rangle d\sigma(Q) = -1,$$

(b) *For* $X \in B(0,\ R) \setminus \bar{D}$

$$\int \langle A(X)N_Q,\ \nabla_Q H(Q,\ X)\rangle d\sigma(Q) = 0,$$

(c) *For* $P \in \partial D$

$$\int \langle A(P)N_Q,\ \nabla_Q H(Q,\ P)\rangle d\sigma(Q) = -1/2.$$

LEMMA 4.3. *Let* $P \in \partial D$. *Then*

$$\lim_{X \to P,\ X \in \Gamma_\eta(P)} \Phi_1(X,\ P) = K^*(1) - 1/2 \tag{4.1}$$

and

$$\lim_{X \to P,\ X \in \Gamma_\eta^e(P)} \Phi_1(X,\ P) = K^*(1) + 1/2. \tag{4.2}$$

PROOF. Note that

$$\begin{aligned} &-\langle A(P)N_P,\ \nabla_X F(X,\ Q)\rangle \\ &= -\langle (A(P)-A(Q))N_P,\ \nabla_X F(X,\ Q)\rangle - \langle A(Q)(N_P - N_Q),\ \nabla_X F(X,\ Q)\rangle \\ &\quad -\langle A(Q)N_Q,\ \nabla_X (F(X,\ Q) - H(X,\ Q))\rangle \\ &\quad -\{\langle A(Q)N_Q,\ \nabla_X H(X,\ Q)\rangle + \langle A(X)N_Q,\ \nabla_Q H(Q,\ X)\rangle\} \\ &\quad +\langle A(X)N_Q,\ \nabla_Q H(Q,\ X)\rangle. \end{aligned}$$

The absolute value of each term, except for the last term, on the right-hand side is dominated by $c|X-Q|^{\alpha+1-d}$ and the integral of the last term over ∂D takes the value -1 by Lemma 4.2. Therefore we have

$$\begin{aligned} &\lim_{X \to P,\ X \in \Gamma_\eta(P)} \Phi_1(X,\ P) \\ &= -\int \langle (A(P)-A(Q))N_P,\ \nabla_P F(P,\ Q)\rangle d\sigma(Q) \\ &\quad -\int \langle A(Q)(N_P - N_Q),\ \nabla_P F(P,\ Q)\rangle d\sigma(Q) \\ &\quad -\int \langle A(Q)N_Q,\ \nabla_P (F(P,\ Q) - H(P,\ Q))\rangle d\sigma(Q) \\ &\quad -\int \{\langle A(Q)N_Q,\ \nabla_P H(P,\ Q)\rangle + \langle A(P)N_Q,\ \nabla_Q H(Q,\ P)\rangle\} d\sigma(Q) - 1 \end{aligned}$$

$$= -\int\langle A(P)N_P, \nabla_P F(P, Q)\rangle d\sigma(Q) - \int\langle A(P)N_Q, \nabla_Q H(Q, P)\rangle d\sigma(Q) - 1$$

$$= K^*(1) - 1/2.$$

Similarly the relation (4.2) is also obtained.

LEMMA 4.4. *Let* $0<\beta<d-1$, $0<\eta<1$. *If* $f\in\mathcal{L}(\gamma_\beta, C(\partial D))$, *then there exists a* γ_β*-polar set* E *such that*

$$\lim_{X\to P,\, X\in\Gamma_\eta(P)}\Phi_f(X, P) = (K^* - (1/2)I)f(P) \tag{4.3}$$

and

$$\lim_{X\to P,\, X\in\Gamma_\eta^e(P)}\Phi_f(X, P) = (K^* + (1/2)I)f(P) \tag{4.4}$$

for every $P\in\partial D\setminus E$.

PROOF. Let δ be a positive real number satisfying (a) and (c) in Lemma 4.1. For $f\in\mathcal{L}(\gamma_\beta, C(\partial D))$ and a positive real number b we put

$$E_{f,b} = \{P\in\partial D;\ \Phi^*_{f,\delta}(P) > b\}.$$

By the aid of Lemma 4.1 we have

$$\gamma_\beta(E_{f,b}) \leqq b^{-1}\gamma_\beta(\Phi^*_{f,\delta}) \leqq cb^{-1}\gamma_\beta(f).$$

Especially, let f be a function of C^1 class on ∂D. From Lemma 4.3 we deduce

$$\lim_{X\to P,\, X\in\Gamma_\eta(P)}\Phi_f(X, P)$$

$$= -\lim_{X\to P,\, X\in\Gamma_\eta(P)}\int\langle A(P)N_P, \nabla_X F(X, Q)\rangle(f(Q)-f(P))d\sigma(Q)$$

$$+\lim_{X\to P,\, X\in\Gamma_\eta(P)}f(P)\Phi_1(X, P)$$

$$= -\int\langle A(P)N_P, \nabla_P F(P, Q)\rangle(f(Q)-f(P))d\sigma(Q) + K^*(1)f(P) - (1/2)f(P)$$

$$= K^*f(P) - (1/2)f(P).$$

On the other hand the space $C^1(\partial D)$ is uniformly dense in $C(\partial D)$ and hence it is dense in $L(\gamma_\beta, C(\partial D))$. Therefore Theorem A in [7], which is a generalized Fatou type theorem with respect to a countably linear functional, leads to (4.3). Similarly one can also show (4.4). □

LEMMA 4.5. *Let* $p>1$ *and* $0<\eta<1$. *Then for every* $f\in L^p(\sigma)$ *there exists a set* $E\subset\partial D$ *such that* $\sigma(E)=0$ *and,* (4.3) *and* (4.4) *hold for every* $P\in\partial D\setminus E$.

PROOF. The operator K^* is bounded in $L^p(\sigma)$ and (4.3) holds for every $f\in C^1(\partial D)$. On account of (b) in Lemma 4.1 we conclude that (4.3) holds at every point $P\in\partial D$ except for a set of surface measure 0. □

5. Double layer potentials.

In this section we prepare some lemmas corresponding to Lemmas in § 4 to solve the Dirichlet problem. Let us define, for Borel measurable function f in $L^1(\sigma)$, the double layer potential Ψ_f defined by

$$\Psi_f(X):=-\int\langle A(Q)N_Q, \nabla_Q F(Q, X)\rangle f(Q)d\sigma(Q)$$

at $X\in \boldsymbol{R}^d\setminus\partial D$. We also define, for $P\in\partial D$,

$$\Psi^*_{f,\delta}(P):=\sup\{|\Psi_f(X)|\,;\ X\in\Gamma_\eta(P),\ |X-P|\leqq\delta\},$$

$$\Psi^{**}_{f,\delta}(P):=\sup\{|\Psi_f(X)|\,;\ X\in\Gamma^e_\eta(P),\ |X-P|\leqq\delta\}.$$

Then we have the corresponding lemma to Lemma 4.1.

LEMMA 5.1. *Assume that $p>1$, $0<\beta<d-1$ and $0<\eta<1$. Then there exist positive real numbers c, δ having the following properties:*

(a) $\Psi^*_{f,\delta}(P)\leqq cM_\sigma f(P)$, $\quad\Psi^{**}_{f,\delta}(P)\leqq cM_\sigma f(P)$

for every Borel measurable function f in $L^1(\sigma)$ and for every $P\in\partial D$,

(b) $\|\Psi^*_{f,\delta}\|_p\leqq c\|f\|_p$ *and* $\|\Psi^{**}_{f,\delta}\|_p\leqq c\|f\|_p$ *for every $f\in L^p(\sigma)$,*

(c) $\gamma_\beta(\Psi^*_{f,\delta})\leqq c\gamma_\beta(f)$ *and* $\gamma_\beta(\Psi^{**}_{f,\delta})\leqq c\gamma_\beta(f)$.

PROOF. Noting that

$$\begin{aligned}&-\langle A(Q)N_Q, \nabla_Q F(Q, X)\rangle\\&=\langle (A(X)-A(Q))N_Q, \nabla_Q F(Q, X)\rangle-\langle A(X)N_Q, \nabla_Q(F(Q, X)-H(Q, X))\rangle\\&\quad-\langle A(X)N_Q, \nabla_Q H(Q, X)\rangle,\end{aligned}$$

we obtain

$$\begin{aligned}&|\Psi_f(X)|\\&\leqq c_1\left\{\int|X-Q|^{\alpha+1-d}|f(Q)|d\sigma(Q)+\int|X-Q|^{-d}|\langle X-Q, N_Q\rangle||f(Q)|d\sigma(Q)\right\}.\end{aligned}$$

By the same method as in the proof of Lemma 4.1 we have

$$\Psi^*_{f,\delta}(P)\leqq c_2 M_\sigma f(P).$$

Similarly we have also the estimate of $\Psi^{**}_{f,\delta}$. The estimates of (b) and (c) follow from (a) and Lemma A. □

The following lemma can be proved by the same method as in Lemma 4.4.

LEMMA 5.2. *Let $0<\beta<d-1$ and $0<\eta<1$. Then for every $f\in\mathcal{L}(\gamma_\beta, C(\partial D))$*

there exists a γ_β-polar set E such that

(5.1) $$\lim_{X\to P,\, X\in\Gamma_\eta(P)}\phi_f(X)=(K+(1/2)I)f(P)$$

and

(5.2) $$\lim_{X\to P,\, X\in\Gamma^e_\eta(P)}\phi_f(X)=(K-(1/2)I)f(P)$$

for each $P\in\partial D\setminus E$.

Using (b) of Lemma 5.1, we can prove the following lemma.

LEMMA 5.3. *Let $p>1$ and $0<\eta<1$. Then for each $f\in L^p(\sigma)$ there exists a subset of ∂D such that $\sigma(E)=0$ and* (5.1), (5.2) *hold at every point $P\in\partial D\setminus E$.*

6. The Neumann problem.

Before the proof of Theorem 1 we prepare a lemma.

LEMMA 6.1. *Let $p>1$ and set*

$$S_p:=\left\{f\in L^p(\sigma);\ \int f\,d\sigma=0\right\}.$$

Then $K^*-(1/2)I$ is invertible on S_p.

PROOF. Let $f\in S_p$. Noting that

$$|K^*f(P)|\leqq c\int|P-Q|^{\alpha+1-d}|f(Q)|\,d\sigma(Q),$$

we see that K^*f is continuous or it belongs to $L^s(\sigma)$ for the positive real number s such that $1/p-\alpha/(d-1)=1/s$. By repeating this, we conclude that f belongs to $L^t(\sigma)$ for every $t>1$.

Let $(K^*-(1/2)I)f=0$. Set

$$u(X)=-\int F(X,Q)f(Q)\,d\sigma(Q).$$

Then u is continuous everywhere. On account of the uniform ellipticity and Lemma 4.5 we obtain

$$\int_D|\nabla u(X)|^2dX\leqq\lambda\int_D\sum_{j,k=1}^d a_{jk}(X)\frac{\partial u(X)}{\partial x_j}\frac{\partial u(X)}{\partial x_k}dX$$
$$=\lambda\int_{\partial D}(K^*-(1/2)I)f(Q)u(Q)\,d\sigma(Q)=0.$$

Therefore u is a constant c on D and hence on $\bar{D}$. Assume that $c\geqq0$. Noting that $L_0u=0$ in $\boldsymbol{R}^d\setminus\bar{D}$ and $\lim_{|X|\to\infty}u(X)=0$, we see by the maximum principle that u takes the maximum at every point $P\in\partial D$. Therefore, as X converges to P along the nontangential region $\Gamma^e_\eta(P)$, the function: $X\mapsto\langle A(P)N_P,\nabla u(X)\rangle$

is nonnegative. But this is equal to $(K^{*}+(1/2))f=f$ σ-a.e., whence f is nonnegative σ-a.e, on ∂D. Noting that $\int f d\sigma=0$, we see that $f=0$ σ-a.e.. Similarly we can also show that $f=0$ σ-a.e. in the case $c<0$. Thus we see that the operator $K^{*}-(1/2)I$ is injective on the closed subspace S_{p} of $L^{p}(\sigma)$. Since K^{*} is compact on S_{p} by Lemma 3.1, $K^{*}-(1/2)I$ is invertible on S_{p}.

LEMMA 6.2. *Set*

$$S_{\beta}:=\left\{f\in L(\gamma_{\beta},\, C(\partial D));\ \int f d\sigma=0\right\}.$$

Then $K^{*}-(1/2)I$ *is invertible on* S_{β}.

PROOF. Let f be a function in $\mathcal{L}(\gamma_{\beta}, C(\partial D))$ such that $\int f d\sigma=0$ and $(K^{*}-(1/2)I)f=0$ γ_{β}-q.e.. Noting that $f\in L^{p}(\sigma)$ for $p=(d-1)/\beta$, we see by Lemma 6.1 that $K^{*}f=(1/2)f$ σ-a.e. and hence $f=0$ σ-a.e.. Therefore $K^{*}f=0$ and hence $f=0$ γ_{β}-q.e.. Thus $K^{*}-(1/2)I$ is injective on the closed subspace S_{β} of $L(\gamma_{\beta}, C(\partial D))$. Since K^{*} is compact on S_{β} by Lemma 3.1, $K^{*}-(1/2)I$ is invertible on S_{β}. □

Next, we prove Theorem 1.

PROOF OF THEOREM 1. Let f be a function in $\mathcal{L}(\gamma_{\beta}, C(\partial D))$ such that $\int f d\sigma=0$. By the aid of Lemma 6.2 we can choose a function $g\in\mathcal{L}(\gamma_{\beta}, C(\partial D))$ such that

$$(K^{*}-(1/2)I)g=f \ \ \gamma_{\beta}\text{-q.e..}$$

By Lemma 4.4 we see that the single layer potential u_{g} of g is the desired function. □

Similarly, using Lemmas 4.5 and 6.1 we can prove the following theorem.

THEOREM 3. *Let* $0<\alpha<1$ *and* D *be a bounded* $C^{1,\alpha}$*-domain in* $\boldsymbol{R}^{d}$. *Furthermore, assume that* $p>1$ *and* $0<\eta<1$. *Then for each function* $f\in L^{p}(\sigma)$ *satisfying* $\int f d\sigma=0$ *there exists a function* u *in* D *and a subset* E *of* ∂D *having the following properties:*

(i) $\sigma(E)=0$,

(ii) $Lu=0$ *in* D,

(iii) $\lim_{X\to P,\, X\in\Gamma_{\eta}(P)}\langle A(P)N_{P}, \nabla u(X)\rangle=f(P)$ *for every* $P\in\partial D\setminus E$.

7. The Dirichlet problem.

Let L be the differential operator in § 1. Let us find, for $f\in\mathcal{L}(\gamma_\beta, C(\partial D))$, a function v defined on D such that $Lv=0$ on D and v converges nontangentially to f γ_β-q. e. on ∂D.

We begin with the following lemma.

LEMMA 7.1. *Assume that $\boldsymbol{R}^d\setminus\bar{D}$ is connected. Then the operator $K^*+(1/2)I$ is injective on $L^q(\sigma)$ for every $q>1$ and $K+(1/2)I$ is also injective on $L^p(\sigma)$ for every $p>1$.*

PROOF. Suppose that $(K^*+(1/2)I)f=0$ σ-a. e. for $f\in L^q(\sigma)$. Set

$$u(X):=-\int F(X, Q)f(Q)d\sigma(Q).$$

As in the proof of Lemma 6.1 we see that u is continuous everywhere. Noting that

$$\langle A(P)N_P, \nabla u(X)\rangle=-\int\langle A(P)N_P, \nabla_Q F(Q, P)\rangle d\sigma(Q),$$

we deduce from Lemma 4.5

$$\int_{R^d\setminus\bar{D}}|\nabla u(X)|^2 dX\leqq\lambda\int_{R^d\setminus\bar{D}}\sum_{j,k=1}^d a_{jk}(X)\frac{\partial u(X)}{\partial x_j}\frac{\partial u(X)}{\partial x_k}dX$$

$$=\lambda\int_{\partial D}(K^*+(1/2)I)f(Q)u(Q)d\sigma(Q)=0,$$

which shows that u is constant on $\boldsymbol{R}^d\setminus\bar{D}$. Since $\lim_{|X|\to\infty}u(X)=0$, we see that $u=0$ on $\boldsymbol{R}^d\setminus\bar{D}$ and hence $u=0$ on ∂D. By the aid of the maximum principle u is also equal to 0 on D. Noting that

$$(K^*-(1/2)I)f(P)=\lim_{X\to P,\, X\in\Gamma_\eta(P)}\langle A(P)N_P, \nabla u(X)\rangle=0 \qquad \sigma\text{-a. e.}$$

and $(K^*+(1/2)I)f(P)=0$ σ-a. e., we conclude that $f=0$ σ-a. e. on ∂D and hence $K^*+(1/2)I$ is injective on $L^q(\sigma)$.

Let p be the positive real number such that $1/p+1/q=1$. Since K (resp. K^*) is compact on $L^p(\sigma)$ (resp. $L^q(\sigma)$) and $K^*+(1/2)I$ is an adjoint operator of $K+(1/2)I$, the operator $K+(1/2)I$ is also injective on $L^p(\sigma)$. □

We have also the following lemma in the space $L(\gamma_\beta, C(\partial D))$.

LEMMA 7.2. *Let $0<\beta<d-1$ and assume that $\boldsymbol{R}^d\setminus\bar{D}$ is connected. Then $K+(1/2)I$ is invertible on $L(\gamma_\beta, C(\partial D))$.*

PROOF. It suffices to show that $K+(1/2)I$ is injective on $L(\gamma_\beta, C(\partial D))$ because it is a compact operator by Lemma 3.1. Assume that $(K+(1/2)I)f=0$

γ_{β}-q.e. on ∂D. Since $f \in L^{p}(\sigma)$ for $p=(d-1)/\beta$, we see by Lemma 7.1 that $f=0$ σ-a.e. and hence $Kf=0$ on ∂D. Therefore it must be concluded that $f=0$ γ_{β}-q.e. on ∂D. □

PROOF OF THEOREM 2. Let f be a function in $\mathcal{L}(\gamma_{\beta}, C(\partial D))$. Using Lemma 7.1, we can choose a function $g \in \mathcal{L}(\gamma_{\beta}, C(\partial D))$ such that $(K+(1/2)I)g=f$ γ_{β}-q.e. on ∂D. By the aid of Lemma 5.2 we see that the function Ψ_{g} defined in §5 is the desired function. □

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Hisako WATANABE
Department of Mathematics
Ochanomizu University
Bunkyō-ku, Tokyo 112
Japan