J. Math. Soc. Japan
Vol. 44, No. 3, 1992

# Remarks on metaplectic representations of $SL(2)$*

By Hiroyuki YOSHIDA

(Received June 7, 1991)

## Introduction.

In a fundamental paper [9], Weil constructed oscillator representations of metaplectic groups. When specialized to the case $G=SL(2, k)$ where $k$ is a local field whose characteristic is not 2, the construction gives a projective representation $\pi$ of $G$ realized on $L^2(k)$ such that

$$(1)\qquad (\pi(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})F)(x)=\psi(bx^2)F(x),$$

$$(2)\qquad (\pi(w)F)(x)=\gamma F^*(x),$$

for $F\in L^2(k)$. Here $w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\psi$ is a non-trivial additive character of $k$, $F^*$ is the Fourier transformation of $F$ with respect to $\psi$ and $\gamma$ is a constant independent of $F$. When lifted to the 2-fold covering group of $G$ (if $k\neq \boldsymbol{C}$), $\pi$ becomes an ordinary representation. An important problem, already suggested in [9], is to construct analogous representations for an $n$-fold covering group of $G$, $n\geqq 3$. A natural candidate is to replace $x^2$ by $x^n$ in (1) and $F^*$ by a suitably generalized Fourier transformation. This problem was solved by Kubota [3], [4] for $k=\boldsymbol{C}$ and by Yamazaki [10] for $k=\boldsymbol{R}$ and $n$ is even.

In this paper, we shall give a conceptually simpler and unified treatment of these representations including the case $k=\boldsymbol{R}$, $n$ is odd. We are going to sketch our idea intuitively. First observation is that we should start from a representation $\pi$ (we choose it as a principal series representation corresponding to the parameter $s$, see the text) of an $n$-fold covering group $\tilde{G}$ of $G$ and then should examine its Kirillov realization. Thus we realize $\pi$ on a suitable (pre-Hilbert) space $V$ of functions $f$ on $k$ such that the action of $\pi\,\widetilde{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}}$ is given by $f(x)\rightarrow\psi(bx)f(x)$. Here, for $g\in G$, $\tilde{g}$ denotes some naturally defined element of $\tilde{G}$ which projects to $g$ (see § 1). Let $\tilde{V}$ be the vector space of all functions $F$ on $k$ defined by $F(x)=f(x^n)$, $x\in k$, $f\in V$. Set $F=\iota(f)$ and put

$$(3)\qquad \tilde{\pi}(g)F=\iota(\pi(g)f) \qquad \text{for } g\in\tilde{G}.$$

* This work was partially supported by the Inamori Foundation.

If (3) is well defined, we see that $\tilde{\pi}$ is a representation of $\tilde{G}$ on $\tilde{V}$ which automatically satisfies (1) with $x^n$ (resp. $\widetilde{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}}$) in the place of $x^2$ (resp. $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$); hence the only remaining task would be the explicit computation of the action of $\tilde{\pi}(\tilde{w})$.[1]

Obviously the well-definedness of (3) is equivalent to:

(4) If $f(x)=0$ for all $x \in k^n$, then $(\pi(g)f)(x)=0$ for all $x \in k^n$, $g \in \tilde{G}$.

When $k=\boldsymbol{C}$, the condition (4) is trivially satisfied so that we can construct Weil type representation of $SL(2, \boldsymbol{C})$ corresponding to any parameter $s$, $0<s<3/2$ (Theorem 5.3). Kubota's representation is a special instance for $s=2/n$, $n \geqq 2$. When $k=\boldsymbol{R}$ and $n$ is even, the condition (4) becomes non-trivial so that we are forced to choose the parameter $s$ of the representation $\pi$ as $s=1/n$. Then $\tilde{\pi}$ turns out to be the representation constructed by Yamazaki (Theorem 4.1, which holds also for odd $n$).

It seems that attempts to construct Weil type representation of $G=SL(2)$ for non-archimedean local fields are so far unsuccessful beyond Weil's original case (cf. Moen [**5**] for example). This fact could be interpreted, though we have no rigorous proof, that the condition (4) can never be met by any representation $\pi$ of $\tilde{G}$ if $n \geqq 3$.

The reader would notice that the idea sketched above is realized in the text in a straightforward manner, if some analytical details are disregarded. Concerning this technical part, we tried to be as precise as we could manage in compatibility of conciseness.

The author would like to express his sincere thanks to the Institute for advanced study for generous hospitality.

NOTATION. We denote the set of positive real numbers by $\boldsymbol{R}_+$. For $z \in \boldsymbol{C}$, $\Re(z)$ and $\Im(z)$ stand for the real part and the imaginary part of $z$ respectively. Let $z \in \boldsymbol{C}^\times$. We denote by $\arg(z)$ (resp. $\operatorname{Arg}(z)$) the argument of $z$ for which we take the branch so that $0 \leqq \arg(z) < 2\pi$ (resp. $-\pi < \operatorname{Arg}(z) \leqq \pi$).

## §1. The $n$-fold covering group of $SL(2, \boldsymbol{R})$.

We set $G=SL(2, \boldsymbol{R})$ and fix a positive integer $n \geqq 2$. For a positive integer $m$, put $\zeta_m = \exp(2\pi\sqrt{-1}/m)$. Let $\tilde{G}$ be the $n$-fold covering group of $G$. We can construct $\tilde{G}$ explicitly following Shimura's method (cf. [**6**], p. 443). For $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathfrak{H}$, set

$$j(g, z) = cz+d, \qquad x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c=0, \end{cases}$$

where $\mathfrak{H}$ denotes the complex upper half plane. Let $\mu_1(g, z), \cdots, \mu_n(g, z)$ be all holomorphic functions on $\mathfrak{H}$ which satisfy $\mu_l(g, z)^n = j(g, z)$, $1 \leqq l \leqq n$. Clearly we can choose $\mu_l(g, z)$ so that

$$(1.1) \qquad \frac{2\pi}{n}(l-1) \leqq \arg \mu_l(g, z) < \frac{2\pi}{n} l \qquad \text{for all } z \in \mathfrak{H},\ 1 \leqq l \leqq n.$$

Let $\tilde{G}$ be the group consisting of all couples $(g, \mu_l(g, z))$ with $g \in G$, $1 \leqq l \leqq n$ on which the multiplication is defined by

$$(1.2) \qquad (g_1, \mu_{l_1}(g_1, z))(g_2, \mu_{l_2}(g_2, z)) = (g_1 g_2, \mu_{l_1}(g_1, g_2(z))\mu_{l_2}(g_2, z)).$$

Let

$$p : \tilde{G} \ni (g, \mu_l(g, z)) \longrightarrow g \in G$$

be the projection homomorphism. We obtain a central extension

$$(1.3) \qquad 1 \longrightarrow \mu_n \longrightarrow \tilde{G} \stackrel{p}{\longrightarrow} G \longrightarrow 1$$

where $\mu_n = \mathrm{Ker}(p) = \{(1, \zeta) \mid \zeta^n = 1\}$. We identify $\mu_n$ with the cyclic group generated by $\zeta_n$. For $g \in G$, take the section $s_g \in \tilde{G}$ so that $s_g = (g, \mu_1(g, z))$. Then

$$\xi_1(g_1, g_2) = s_{g_1} s_{g_2} s_{g_1 g_2}^{-1}, \qquad g_1, g_2 \in G$$

is a 2-cocycle determined by (1.3). For $\theta \in \boldsymbol{R}$, put $r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and set

$$K = \{r(\theta) \mid \theta \in \boldsymbol{R}\} \cong SO(2, \boldsymbol{R}).$$

Since the restriction of $\xi_1$ to $K$ coincides with a cocycle defined by the $n$-fold covering map of $SO(2, \boldsymbol{R})$ to itself given by the $n$-th power, we find that the order of the cohomology class of $\xi_1$ in $H^2(G, \mu_n)$ is precisely $n$. By (1.2), we obtain

$$\xi_1(g_1, g_2) = \mu_1(g_1, g_2(z))\mu_1(g_2, z)\mu_1(g_1 g_2, z)^{-1}$$

which is independent of $z$. Hence we have

$$(1.4) \qquad \xi_1(g_1, g_2) = \begin{cases} 1 & \text{if } \arg \mu_1(g_1, g_2\sqrt{-1}) + \arg \mu_1(g_2, \sqrt{-1}) < 2\pi/n, \\ \zeta_n & \text{if } \arg \mu_1(g_1, g_2\sqrt{-1}) + \arg \mu_1(g_2, \sqrt{-1}) \geqq 2\pi/n. \end{cases}$$

By (1.4), we immediately obtain

$$(1.5) \qquad \xi_1(g_1, g_2) = \begin{cases} 1 & \text{if } \arg(j(g_1 g_2, \sqrt{-1})) \geqq \arg(j(g_2, \sqrt{-1})), \\ \zeta_n & \text{if } \arg(j(g_1 g_2, \sqrt{-1})) < \arg(j(g_2, \sqrt{-1})), \end{cases}$$

for $g_1, g_2 \in G$. For $g \in G$, put

$$y(g)=\begin{cases}1 & \text{if } x(g)>0,\\ \zeta_n & \text{if } x(g)<0,\end{cases}$$

and let

(1.6) $$\xi(g_1, g_2)=\xi_1(g_1, g_2)y(g_1g_2)y(g_1)^{-1}y(g_2)^{-1}, \qquad g_1, g_2\in G$$

be the 2-cocycle cohomologous to $\xi_1$. If $n=2$, we can show by a straightforward computation that

$$\xi(g_1, g_2)=(x(g_1), x(g_2))_{\boldsymbol{R}}(-x(g_1)^{-1}x(g_2), x(g_1g_2))_{\boldsymbol{R}}$$

where $(\ ,\ )_{\boldsymbol{R}}$ denotes the Hilbert symbol of $\boldsymbol{R}$. Thus $\xi$ coincides with the cocycle defined by Kubota [2] in this case.(2)

On the product set $G\times\mu_n$, we define the multiplication by

(1.7) $$(g, \zeta)(g', \zeta')=(gg', \zeta\zeta'\xi(g, g'))\,.$$

Then $G\times\mu_n$ has the group structure isomorphic to $\tilde{G}$. We take this model of $\tilde{G}$ for the convenience of calculation. For $g\in G$, set $\tilde{g}=(g, 1)\in\tilde{G}$. For a subgroup $H$ of $G$, we denote by $\tilde{H}$ the subgroup of $\tilde{G}$ defined by

$$\tilde{H}=\{(h, \zeta)\mid h\in H, \zeta\in\mu_n\}\,.$$

We define subgroups $T$, $B$, $B_+$ and $N$ of $G$ by

$$T=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\middle| a\in\boldsymbol{R}^\times\right\}, \qquad B=\left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}\middle| a\in\boldsymbol{R}^\times, b\in\boldsymbol{R}\right\},$$

$$B_+=\left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}\middle| a\in\boldsymbol{R}_+, b\in\boldsymbol{R}\right\}, \quad N=\left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\middle| b\in\boldsymbol{R}\right\}.$$

By (1.5) and (1.6), we obtain the following values of the cocycle.

(1.8) $$\xi(g_1, g_2)=1 \qquad \text{for } g_1, g_2\in G \text{ if } g_1\in B_+ \text{ or } g_2\in B_+.$$

(1.9) $$\xi\left(\begin{pmatrix}a_1 & 0\\ 0 & a_1^{-1}\end{pmatrix}, \begin{pmatrix}a_2 & 0\\ 0 & a_2^{-1}\end{pmatrix}\right)=\begin{cases}1 & \text{if } a_1>0 \text{ or } a_2>0,\\ \zeta_n^{-1} & \text{if } a_1<0 \text{ and } a_2<0.\end{cases}$$

By (1.9), we see that $\tilde{T}$ is commutative. We see also that $\tilde{K}$ is commutative. If $m$ is an integer such that $m\equiv 1 \bmod n$,

(1.10) $$\sigma_m((r(\theta), \zeta))=e^{\sqrt{-1}m\theta/n}y(r(\theta))^{-1}\zeta, \qquad 0\leqq\theta<2\pi,\ \zeta\in\mu_n$$

defines a one dimensional representation of $\tilde{K}$. Every genuine (i.e., $\sigma_m((1, \zeta))=\zeta$) continuous one dimensional representation $\sigma_m$ of $\tilde{K}$ is of this form. For the use of following sections, we note some relations among elements of $\tilde{G}$ which can be verified easily by (1.5) and (1.6). Put $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

(1.11) $$\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}=\widetilde{\begin{pmatrix}-u^{-1} & 1\\ 0 & -u\end{pmatrix}}\widetilde{\begin{pmatrix}1 & 0\\ u^{-1} & 1\end{pmatrix}},\qquad u\neq 0,$$

(1.12) $$\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}=\widetilde{\begin{pmatrix}1/\sqrt{u^2+1} & -u/\sqrt{u^2+1}\\ 0 & \sqrt{u^2+1}\end{pmatrix}}r(\widetilde{\theta}),\quad u\in\boldsymbol{R}$$

with $\cos\theta=-u/\sqrt{u^2+1}$, $\sin\theta=-1/\sqrt{u^2+1}$.

(1.13) $$\tilde{w}\widetilde{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}}=\widetilde{\begin{pmatrix}a^{-1} & 0\\ 0 & a\end{pmatrix}}\tilde{w},\quad a\in\boldsymbol{R}^{\times}.$$

(1.14) $$\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}\tilde{w}=\widetilde{\begin{pmatrix}-u^{-1} & 0\\ 0 & -u\end{pmatrix}}\widetilde{\begin{pmatrix}1 & -u\\ 0 & 1\end{pmatrix}}\tilde{w}\widetilde{\begin{pmatrix}1 & -u^{-1}\\ 0 & 1\end{pmatrix}},\quad u\neq 0.$$

(1.15) $$\widetilde{w^{-1}}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}=\widetilde{\begin{pmatrix}u^{-1} & -1\\ 0 & u\end{pmatrix}}\widetilde{\begin{pmatrix}1 & 0\\ u^{-1} & 1\end{pmatrix}}\times\begin{cases}1, & u>0,\\ \zeta_n, & u<0.\end{cases}$$

(1.16) $$\widetilde{w^{-1}}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}\tilde{w}=\widetilde{\begin{pmatrix}u^{-1} & -1\\ 0 & u\end{pmatrix}}\tilde{w}\widetilde{\begin{pmatrix}1 & -u^{-1}\\ 0 & 1\end{pmatrix}}\times\begin{cases}1, & u>0,\\ \zeta_n, & u<0.\end{cases}$$

## § 2. Principal series representations of $\tilde{G}$.

Let $\phi(x)=\exp(a\sqrt{-1}x)$ be an additive character of $\boldsymbol{R}$. We assume $a>0$. Let $du$ be the self-dual measure on $\boldsymbol{R}$ with respect to the self-duality $\langle x, y\rangle=\phi(xy)$. Then $du$ is $\sqrt{a/2\pi}$ times the usual Lebesgue measure. Let $\rho$ be a one dimensional representation of $\tilde{T}$. We assume that $\rho$ is genuine, i.e., $\rho((1, \zeta))=\zeta$, $\zeta\in\mu_n$. Set

(2.1) $$\rho\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}, \zeta\right)=\eta_a\chi(a)\zeta,\ \chi(a)=|a|^s,\qquad a\in\boldsymbol{R}^{\times},\ \zeta\in\mu_n$$

with $s\in\boldsymbol{C}$. Then $\rho$ is a homomorphism if and only if

$$\xi\left(\begin{pmatrix}a_1 & 0\\ 0 & a_1^{-1}\end{pmatrix}, \begin{pmatrix}a_2 & 0\\ 0 & a_2^{-1}\end{pmatrix}\right)=\eta_{a_1}\eta_{a_2}\eta_{a_1a_2}^{-1},\qquad a_1, a_2\in\boldsymbol{R}^{\times}.$$

Set

(2.2) $$\eta_a=\begin{cases}1 & \text{if } a>0,\\ \nu^{-1} & \text{if } a<0,\end{cases}$$

where $\nu^2=\zeta_n$. Then (2.1) is a general form of a continuous genuine one dimensional representation of $\tilde{T}$. We shall eventually take $\nu=-\zeta_{2n}$, the reason of which shall be clarified later. Set

$$\delta\left(\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}, \zeta\right)=|a|^2,\qquad a\in\boldsymbol{R}^{\times},\ b\in\boldsymbol{R},\ \zeta\in\mu_n,$$

which is the modular function of $\hat{B}$. Let $PS(\chi)$ denote the space of all $C^\infty$-functions $\varphi$ on $\tilde{G}$ which satisfy

$$\varphi(tng)=\delta(t)^{1/2}\rho(t)\varphi(g) \quad \text{for all } t\in\hat{T},\ n=\tilde{n}_1 \text{ with } n_1\in N,\ g\in\tilde{G}. \tag{2.3}$$

Let $\pi(\chi)$ denote the representation of $\tilde{G}$ realized on $PS(\chi)$ by right translations. For $\varphi\in PS(\chi)$, let $\Phi=R(\varphi)$ denote the $C^\infty$-function on $\boldsymbol{R}$ defined by

$$\Phi(u)=\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}), \qquad u\in\boldsymbol{R}. \tag{2.4}$$

We choose $s$ in (2.1) so that $0<\sigma=\Re(s)<1$. By (1.11), we have

$$\Phi(u)=\eta_{-u}|u|^{-s-1}\varphi(\widetilde{\begin{pmatrix}1 & 0\\ u^{-1} & 1\end{pmatrix}}), \qquad u\neq 0.$$

Hence we easily obtain

$$\Phi^{(k)}(u)=O(|u|^{-\sigma-1}) \qquad \text{for } |u|\to\infty,\ k\geqq 0 \tag{2.5}$$

where $\Phi^{(k)}$ denotes the $k$-th derivative of $\Phi$. Let $f=\mathcal{F}(\Phi)$ be the Fourier transform of $\Phi$, i.e.,

$$f(x)=\int_{\boldsymbol{R}}\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}})\overline{\psi(ux)}dx, \qquad x\in\boldsymbol{R}. \tag{2.6}$$

By (2.5), this integral converges absolutely and $f$ is a continuous function. Furthermore we see

$$f(x)=O(|x|^{-N}), \qquad |x|\longrightarrow\infty \quad \text{for every } N>0 \tag{2.7}$$

using integration by parts. By Fourier inversion, we have

$$\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}})=\int_{\boldsymbol{R}}f(x)\psi(ux)dx. \tag{2.8}$$

Let $V_n$ denote the vector space

$$\{f \mid f=\mathcal{F}(R(\varphi)) \qquad \text{for some } \varphi\in PS(\chi)\}.$$

Since $R$ is injective, we can transport the representation $\pi(\chi)$ to the representation $\pi_0$ of $\tilde{G}$ on $V_n$. By (2.6) and (1.8), we obtain

$$(\pi_0(\widetilde{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}})f)(x)=\psi(bx)f(x), \qquad f\in V_n,\ b,\ x\in\boldsymbol{R}. \tag{2.9}$$

By (2.6), (1.13) and (2.1), we obtain

$$(\pi_0(\widetilde{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}})f)(x)=\eta_a|a|^{1-s}f(a^2x), \qquad f\in V_n,\ a\in\boldsymbol{R}^\times,\ x\in\boldsymbol{R}. \tag{2.10}$$

We are going to compute the action of $\pi_0(\tilde{w})$ on $V_n$. Take $f=\mathcal{F}(R(\varphi))\in V_n$, $\varphi\in PS(\chi)$. By definition, we have

$$(\pi_0(\tilde{w})f)(x)=\int_R \varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}\tilde{w})\overline{\psi(ux)}du.$$

Put $\rho_0(a)=\rho(\widetilde{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}})$, $a\in \boldsymbol{R}^{\times}$. By (1.14), we get

$$(\pi_0(\tilde{w})f)(x)=\int_R \rho_0(-u^{-1})|u|^{-1}\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & -u^{-1}\\ 0 & 1\end{pmatrix}})\overline{\psi(ux)}du$$

$$=\int_R \rho_0(v)|v|^{-1}\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}})\overline{\psi(-v^{-1}x)}dv=\int_R \eta_v|v|^{s-1}\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}})\psi(v^{-1}x)dv.$$

Up to this point, the integrals are absolutely convergent. By (2.8), we obtain

$$(\pi_0(\tilde{w})f)(x)=\int_R\Big(\int_R f(y)\psi(vy)dy\Big)\eta_v|v|^{s-1}\psi(v^{-1}x)dv. \tag{2.11}$$

This double integral does not converge absolutely hence some care is called for to interchange two integrals. We shall show that this is permissible so that

$$(\pi_0(\tilde{w})f)(x)=\int_R\Big(\int_R \eta_v|v|^{s-1}\psi(vy+v^{-1}x)dv\Big)f(y)dy, \tag{2.12}$$

where the inner integral is understood in the sense $\lim_{T\to+\infty}\int_{|v|\leq T}$. First, by (2.7) and (2.11), we have

$$(\pi_0(\tilde{w})f)(x)=\lim_{T\to+\infty}\int_R\Big(\int_{-T}^{T}\eta_v|v|^{s-1}\psi(vy+v^{-1}x)dv)f(y)dy\,.$$

Assume $y\neq 0$. The convergence of $\lim_{T\to+\infty}\int_{-T}^{T}\eta_v|v|^{s-1}\psi(vy)dv$ is well known and can easily be verified. The integral

$$\int_{-\infty}^{\infty}\eta_v|v|^{s-1}\psi(vy)(\psi(v^{-1}x)-1)dv \tag{2.13}$$

is absolutely convergent. Hence

$$\lim_{T\to+\infty}\int_{-T}^{T}\eta_v|v|^{s-1}\psi(vy+v^{-1}x)dv$$

exists. By the Lebesgue dominated convergence theorem, it suffices to show

$$\Big|\Big(\int_{-T}^{T}\eta_v|v|^{s-1}\psi(vy+v^{-1}x)dv\Big)f(y)\Big|\leqq |H(y)|, \qquad y\neq 0$$

with $H\in L^1(\boldsymbol{R})$ which is independent of $T$. In view of the absolutely convergence of (2.13) and $f\in L^1(\boldsymbol{R})$, it suffices to show

$$(2.14)\qquad \left|\left(\int_{-T}^{T}\eta_v|v|^{s-1}\phi(vy)dv\right)f(y)\right| \leqq |H_1(y)|, \qquad y\neq 0$$

with $H_1\in L^1(\boldsymbol{R})$ independent of $T$. We recall a well known integration formula

$$(2.15)\qquad \int_0^\infty v^{s-1}\exp(c\sqrt{-1}v)dv = \sqrt{\frac{a}{2\pi}}\frac{\Gamma(s)}{c^s}e^{s\pi\sqrt{-1}/2}, \qquad c>0.$$

(cf. [1], p. 420-421.) By (2.15), we see that

$$\left(\int_{-\infty}^{\infty}\eta_v|v|^{s-1}\phi(vy)dv\right)f(y)\in L^1(\boldsymbol{R}),$$

since $|y|^{-s}$ is locally integrable at $y=0$. If $T\leqq|1/y|$, we have

$$\left|\int_{-T}^{T}\eta_v|v|^{s-1}\phi(vy)dv\right| \leqq \frac{2}{\sigma}|y|^{-\sigma}.$$

If $T\geqq|1/y|$, we have

$$\left|\int_{|v|\geq T}\eta_v|v|^{s-1}\phi(vy)dv\right| \leqq CT^{\sigma-1}|y|^{-1}\leqq C|y|^{-\sigma}$$

with a constant $C$ which does not depend on $T$ and $y$ using integration by parts. This proves (2.14). Hence (2.12) is justified. Changing variables, we obtain

$$(2.16)\qquad (\pi_0(\tilde{w})f)(x) = \int_{\boldsymbol{R}}\left(\int_{\boldsymbol{R}}\eta_{vy}|v|^{s-1}\phi(v+v^{-1}xy)dv\right)f(y)|y|^{-s}dy.$$

Put

$$(2.17)\qquad G(x, y) = \int_{\boldsymbol{R}}\eta_{vy}|v|^{s-1}\phi(v+v^{-1}xy)dv, \qquad x, y\in\boldsymbol{R}.$$

LEMMA 2.1. *Let* $s\in\boldsymbol{R}$, $0<s<1$. *Put* $\nu=-\nu_1^2$, $\nu_1=\exp(\alpha\sqrt{-1}\pi)$, $0<\alpha<1$, *i.e.,* $\alpha=1/2n$ *(resp.* $\alpha=1/2n+1/2$*) if* $\nu=-\zeta_{2n}$ *(resp.* $\nu=\zeta_{2n}$*). Then* $G(x, y)$ *equals*

$$\sqrt{2\pi a}\nu_1^{-1}\sqrt{-1}(xy)^{s/2}\left\{\cos\left(\left(\frac{s}{2}+\alpha\right)\pi\right)J_s(2a(xy)^{1/2})-\sin\left(\left(\frac{s}{2}+\alpha\right)\pi\right)N_s(2a(xy)^{1/2})\right\},$$

$$-\sqrt{2\pi a}\nu_1^{-1}\sqrt{-1}(xy)^{s/2}\left\{\cos\left(\left(\frac{s}{2}-\alpha\right)\pi\right)J_s(2a(xy)^{1/2})-\sin\left(\left(\frac{s}{2}-\alpha\right)\pi\right)N_s(2a(xy)^{1/2})\right\},$$

$$-\sqrt{\frac{a}{2\pi}}4\nu_1^{-1}\sqrt{-1}|xy|^{s/2}\sin\left(\left(\frac{s}{2}-\alpha\right)\pi\right)K_s(2a|xy|^{1/2}),$$

$$\sqrt{\frac{a}{2\pi}}4\nu_1^{-1}\sqrt{-1}|xy|^{s/2}\sin\left(\left(\frac{s}{2}+\alpha\right)\pi\right)K_s(2a|xy|^{1/2}),$$

*according as the cases* $x>0$, $y>0$; $x<0$, $y<0$; $x>0$, $y<0$; $x<0$, $y>0$ *respec-*

*tively.*

PROOF. We use the following integration formulas.[3]

$$\int_0^\infty v^{s-1}\exp\Big(a\sqrt{-1}\Big(v-\frac{b^2}{v}\Big)\Big)dv=\sqrt{\frac{a}{2\pi}}2e^{s\pi\sqrt{-1}/2}b^sK_s(2ab),\quad a>0,\ b>0, \tag{2.18}$$

$$\begin{aligned}&\int_0^\infty v^{s-1}\exp\Big(a\sqrt{-1}\Big(v+\frac{b^2}{v}\Big)\Big)dv\\ &=\sqrt{\frac{a}{2\pi}}\pi e^{s\pi\sqrt{-1}/2}b^s[\sqrt{-1}J_s(2ab)-N_s(2ab)],\qquad a>0,\ b>0,\end{aligned} \tag{2.19}$$

in the standard notation of Bessel functions (cf. [**1**], p. 470). Since

$$G(x,y)=\eta_y\int_0^\infty v^{s-1}\exp\Big(a\sqrt{-1}\Big(v+\frac{xy}{v}\Big)\Big)dv+\eta_{-y}\overline{\int_0^\infty v^{s-1}\exp\Big(a\sqrt{-1}\Big(v+\frac{xy}{v}\Big)\Big)dv},$$

the assertion follows from (2.18) and (2.19) by simple computations.

COROLLARY 2.2. *$G(x, y)=0$ whenever $x>0$, $y<0$ if and only if $s=1/n$, $\nu=-\zeta_{2n}$. In this choice of parameters, we have*

$$G(x,y)=\sqrt{2\pi a}\zeta_{4n}^{-1}\sqrt{-1}(xy)^{1/2n}J_{-1/n}(2a(xy)^{1/2})\qquad \textit{if } x>0,\ y>0.$$

## § 3. The intertwining operator and unitary structure.

For $\varphi\in PS(\chi)$, set

$$(T_w(\varphi))(g)=\int_{\boldsymbol{R}}\varphi(\tilde{w}^{-1}\widetilde{\begin{pmatrix}1&u\\0&1\end{pmatrix}}g)du,\qquad g\in\tilde{G}. \tag{3.1}$$

By (1.15), we get

$$\varphi(\widetilde{w^{-1}}\widetilde{\begin{pmatrix}1&u\\0&1\end{pmatrix}}g)=\eta_u|u|^{-s-1}\varphi(\widetilde{\begin{pmatrix}1&0\\u^{-1}&1\end{pmatrix}}g)\times\begin{cases}1, & u>0,\\ \zeta_n, & u<0,\end{cases} \tag{3.2}$$

$$\varphi(\widetilde{w^{-1}}\widetilde{\begin{pmatrix}1&u\\0&1\end{pmatrix}}g)=O(|u|^{-\sigma-1}),\qquad |u|\to\infty. \tag{3.3}$$

Therefore the integral (3.1) is absolutely convergent. Furthermore by (3.2), we see that differentiations under the integral is legitimate so that $T_w(\varphi)$ defines a $C^\infty$-function on $\tilde{G}$. By a direct computation, we find that $T_w(\varphi)$ obeys the transformation rule (2.3) with $\chi^{-1}$ in the place of $\chi$. Hence we have $T_w(\varphi)\in PS(\chi^{-1})$. Thus we obtain an intertwining operator $T_w$ from $PS(\chi)$ to $PS(\chi^{-1})$. Assume $s\in\boldsymbol{R}$, i.e., $0<s<1$. Then for $\varphi_1$, $\varphi_2\in PS(\chi)$, we have

$$(T_w(\varphi_1))(bg)\overline{\varphi_2(bg)}=\delta(b)T_w(\varphi_1)(g)\overline{\varphi_2(g)}\qquad\text{for every } b\in\tilde{B},\ g\in\tilde{G}.$$

Therefore

(3.4) $$\langle \varphi_1, \varphi_2 \rangle = \int_{\tilde{B}\backslash\tilde{G}} (T_w(\varphi_1))(g)\overline{\varphi_2(g)}dg, \qquad \varphi_1, \varphi_2 \in PS(\chi)$$

defines an invariant sesqui-linear form on $PS(\chi)$. Here we choose the invariant measure $dg$ on $\tilde{B}\backslash\tilde{G}$ so that

(3.5) $$\langle \varphi_1, \varphi_2 \rangle = \int_{\boldsymbol{R}} (T_w(\varphi_1))(\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}})\overline{\varphi_2(\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}})}du$$

holds.

We are going to calculate the action of $T_w$ on $\Phi = R(\varphi)$. Let $\varphi \in PS(\chi)$ and put $\Phi = R(\varphi)$, $\Psi = R(T_w(\varphi))$. By (1.16), we have

$$\Psi(u) = (T_w(\varphi))(\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}}) = \int_{\boldsymbol{R}} \varphi(\tilde{w}^{-1}\widetilde{\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}}\tilde{w}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}})dv$$

$$= \int_{\boldsymbol{R}} \eta_v^{-2}\rho_0(v^{-1})|v|^{-1}\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & -v^{-1}\\ 0 & 1\end{pmatrix}}\widetilde{\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}})dv$$

$$= \int_{\boldsymbol{R}} \eta_{-v}^{-1}|v|^{s-1}\varphi(\tilde{w}\widetilde{\begin{pmatrix}1 & u+v\\ 0 & 1\end{pmatrix}})dv.$$

Define a locally integrable function $T$ on $\boldsymbol{R}$ by

(3.6) $$T(x) = \eta_{-x}^{-1}|x|^{s-1}.$$

The calculation above shows that

(3.7) $$\Psi = \check{T} * \Phi\,,$$

where $\check{T}(x) = T(-x)$. The integral defining $\check{T}*\Phi$ is absolutely convergent by (2.5). Let $\varphi_i \in PS(\chi)$, $\Phi_i = R(\varphi_i)$, $i=1, 2$. By (3.5) and (3.7), we have

$$\langle \varphi_1, \varphi_2 \rangle = (\check{T}*\Phi_1)(\bar{\Phi}_2)\,.$$

Put $\tilde{\Phi}_2(x) = \overline{\Phi_2(-x)}$, $x \in \boldsymbol{R}$ and regard $T$ as a distribution on $\boldsymbol{R}$. Since $\Phi_1 * \tilde{\Phi}_2 \in L^1(\boldsymbol{R})$, the double integral defining $T(\Phi_1 * \tilde{\Phi}_2)$ is absolutely convergent. Hence we obtain[(4)]

(3.8) $$\langle \varphi_1, \varphi_2 \rangle = T(\Phi_1 * \tilde{\Phi}_2) \qquad \text{for } \varphi_1, \varphi_2 \in PS(\chi).$$

The inverse Fourier transformation

$$(\mathcal{F}'T)(x) = \lim_{A\to+\infty}\int_{-A}^{A} T(y)\psi(xy)dy$$

of $T$ can be calculated by (2.15) and we obtain

$$(3.9)\qquad (\mathcal{F}'T)(x)=-2\sqrt{-1}\nu_1\sqrt{\frac{a}{2\pi}}\Gamma(s)a^{-s}|x|^{-s}\times\begin{cases}\sin\left(\left(\frac{s}{2}+\alpha\right)\pi\right), & x>0,\\ -\sin\left(\left(\frac{s}{2}-\alpha\right)\pi\right), & x<0,\end{cases}$$

in the notation of Lemma 2.1. Put

$$f_i=\mathcal{F}(\Phi_i),\quad i=1, 2,\quad f=f_1\bar{f}_2,\quad \Phi=\Phi_1*\tilde{\Phi}_2\,.$$

We have

$$\mathcal{F}(\Phi)=\mathcal{F}(\Phi_1)\overline{\mathcal{F}(\Phi_2)}=f\,.$$

Since $f$ is rapidly decreasing (cf. (2.7)), we have $\mathcal{F}'(f)=\Phi$ by Fourier inversion. Hence we have

$$T(\mathcal{F}'f)=\lim_{A\to+\infty}\int_{-A}^{A}\left(\int_{R}f(y)\psi(xy)dy\right)T(x)dx$$

$$=\lim_{A\to+\infty}\int_{R}\left(\int_{-A}^{A}T(x)\psi(xy)dx\right)f(y)dy.$$

As in §2, we can apply the Lebesgue dominated convergence theorem and obtain $T(\mathcal{F}'f)=(\mathcal{F}'T)(f)$. Therefore we have shown

$$(3.10)\qquad \langle\varphi_1,\varphi_2\rangle=\int_{-\infty}^{\infty}(\mathcal{F}'T)(x)f_1(x)\overline{f_2(x)}dx.$$

Hereafter we choose $s=1/n$, $\nu=-\zeta_{2n}$. We have

$$(3.11)\qquad \langle\varphi_1,\varphi_2\rangle=c_n\int_{0}^{\infty}f_1(x)\overline{f_2(x)}x^{-1/n}dx,$$

$$c_n=-2\sqrt{-1}\zeta_{4n}\sin\frac{\pi}{n}\sqrt{\frac{a}{2\pi}}\Gamma\left(\frac{1}{n}\right)a^{-1/n}.$$

Dropping the constant $c_n$, put

$$(3.12)\qquad (\varphi_1,\varphi_2)=\int_{0}^{\infty}f_1(x)\overline{f_2(x)}x^{-1/n}dx.$$

Then (3.12) defines an invariant sesqui-linear form on $PS(\chi)$ such that $(\varphi,\varphi)\geqq 0$ for every $\varphi\in PS(\chi)$. Therefore (3.12) is a positive semi-definite invariant hermitian form on $PS(\chi)$. Let $V_n^+$ be the space of all functions $f(x)$ on $\boldsymbol{R}_+$ such that $f(x)=f_1(x)$, $x>0$ for some $f_1\in V_n$. On $V_n^+$, we introduce the norm

$$(3.13)\qquad \|f\|_n=\left(\int_{0}^{\infty}|f(x)|^2x^{-1/n}dx\right)^{1/2}.$$

Then $V_n^+$ is a pre-Hilbert space with respect to $\|\ \|_n$.

PROPOSITION 3.1. *Let $f\in V_n^+$ and $g\in\tilde{G}$. Take any $f_1\in V_n$ such that $f(x)=f_1(x)$, $x>0$. Set*

$$(\pi_1(g)f)(x)=(\pi_0(g)f_1)(x), \qquad x>0.$$

*Then $\pi_1(g)$ is a well defined unitary operator on $V_n^+$.*

PROOF. Let $f_1 \in V_n$ and assume $f_1(x)=0$ for all $x>0$. The well-definedness follows if we can show

$$(\pi_0(g)f_1)(x)=0 \qquad \text{for all } x>0.$$

This is clear for $g\in\tilde{N}$ by (2.9); for $g=\tilde{w}$, this follows from (2.16) and Corollary 2.2. Since $\tilde{G}$ is generated by $\tilde{N}$ and $\tilde{w}$, the general case follows. The unitarity of $\pi_1(g)$ is an obvious consequence of the invariance of the positive definite hermitian form (3.12) on $V_n^+$. This completes the proof.

Let $H_n$ be the Hilbert space of all measurable functions $f(x)$ on $\boldsymbol{R}_+$ such that

$$\|f\|_n^2=\int_0^\infty |f(x)|^2 x^{-1/n}dx<\infty.$$

PROPOSITION 3.2. *$V_n^+$ is a dense subspace of $H_n$.*

PROOF. First we shall show $V_n^+\neq\{0\}$. For an integer $m$ such that $m\equiv n+1 \bmod 2n$, define a function $\varphi_m$ on $\tilde{G}$ by

$$\varphi_m(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}r(\theta),\zeta)=a^{s+1}e^{\sqrt{-1}m\theta/n}y(r(\theta))^{-1}\zeta$$

for $a\in\boldsymbol{R}_+$, $b\in\boldsymbol{R}$, $0\leqq\theta<2\pi$, $\zeta\in\mu_n$. Then we can verify that $\varphi_m$ transforms according to $\sigma_m$ (cf. (1.10)) under the right action of $\tilde{K}$ and that $\varphi_m\in PS(\chi)$. Put $\Phi_m=R(\varphi_m)$, $f_m=\mathcal{F}(\Phi_m)$. By (1.12), we have

$$\Phi_m(u)=\zeta_n^{-1}(\sqrt{u^2+1})^{-s-1}e^{\sqrt{-1}\,m\theta/n}$$

with $\cos\theta=-u/\sqrt{u^2+1}$, $\sin\theta=-1/\sqrt{u^2+1}$, $0\leqq\theta<2\pi$. We have

$$e^{\sqrt{-1}\theta}=-(u+\sqrt{-1})^{1/2}/(u-\sqrt{-1})^{1/2}, \qquad \sqrt{u^2+1}=(u+\sqrt{-1})^{1/2}(u-\sqrt{-1})^{1/2}$$

when we choose the branches so that

$$0<\operatorname{Arg}(\log(u+\sqrt{-1}))<\pi, \qquad -\pi<\operatorname{Arg}(\log(u-\sqrt{-1}))<0$$

for $u\in\boldsymbol{R}$. Put $m=n+1+2nt$ with $t\in\boldsymbol{Z}$. Then we get

$$\Phi_m(u)=-\zeta_{2n}^{-1}(u+\sqrt{-1})^t(u-\sqrt{-1})^{-t-(n+1)/n}, \qquad u\in\boldsymbol{R},$$

$$f_m(x)=-\zeta_{2n}^{-1}\int_{-\infty}^{\infty}(u+\sqrt{-1})^t(u-\sqrt{-1})^{-t-(n+1)/n}\exp(-\sqrt{-1}axu)du, \quad x\in\boldsymbol{R}.$$

For $R>1$, consider the integration of $\Phi_m(u)\exp(-\sqrt{-1}axu)$ along the contour

$R$ to $-R$ on the real line, $-R$ to $R$ on the semi-circle $C$ lying in the lower half plane, of radius $R$, the center at the origin. If $x>0$, the integral on $C$ tends to 0 as $R\to+\infty$. Therefore we obtain

$$f_m(x)=\sqrt{\frac{a}{2\pi}}2\pi\sqrt{-1}\zeta_{2n}^{-1}\times\text{Residue of }(u+\sqrt{-1})^t(u-\sqrt{-1})^{-t-(n+1)/n} \tag{3.14}$$

$$\exp(-\sqrt{-1}axu)\quad\text{at } u=-\sqrt{-1},\qquad x>0.$$

From (3.14), we see immediately that

$$f_m(x)=0\qquad\text{for all } x>0\quad\text{if } t\geqq 0,$$

$$f_{-n+1}(x)=\sqrt{\frac{a}{2\pi}}2\pi\sqrt{-1}\zeta_{4n}^{-1}2^{-1/n}\exp(-ax),\qquad x>0, \tag{3.15}$$

$$f_m(x)=\sqrt{\frac{a}{2\pi}}\times\text{a polynomial of degree (precisely) } |t|-1 \text{ of } ax$$

$$\times\exp(-ax),\qquad x>0,\ t<0.$$

In particular, we obtain $V_n^+\neq\{0\}$.

Let $\bar{V}_n^+$ be the closure of $V_n^+$ in $H_n$. Let $f\in V_n^+$, $f\neq 0$ and take any function $h\in H_n$ from the orthogonal complement of $\bar{V}_n^+$ in $H_n$. Put $f_1(x)=f(x)x^{-s/2}$, $h_1(x)=\overline{h(x)}x^{-s/2}$. Then $f_1$, $h_1\in L^2(\boldsymbol{R}_+)$. Since $\psi(ux)f(ax)\in V_n^+$ for every $u\in\boldsymbol{R}$, $a\in\boldsymbol{R}_+^\times$ by (2.9) and (2.10), we have

$$\int_0^\infty\psi(ux)f_1(ax)h_1(x)dx=0\qquad\text{for every } u\in\boldsymbol{R},\ a\in\boldsymbol{R}_+^\times. \tag{3.16}$$

Fix $a>0$ and put $F(x)=f_1(ax)h_1(x)$ for $x>0$, $F(x)=0$ for $x\leqq 0$. Then $F\in L^1(\boldsymbol{R})$ and (3.16) implies $\mathcal{F}(F)=0$. Hence $F=0$ as a distribution which implies $F(x)=0$ for almost all $x$. Since $f_1$ is continuous, we can find $0<\alpha<\beta$ such that $f_1(x)\neq 0$ for all $x\in(\alpha,\beta)$. Then we have $h_1(x)=0$ for almost all $x\in(a^{-1}\alpha, a^{-1}\beta)$. Since $\bigcup_{a\in\boldsymbol{Q}_+}(a^{-1}\alpha, a^{-1}\beta)=\boldsymbol{R}_+$, we obtain $h_1(x)=0$ for almost all $x$. This implies $h=0$ in $H_n$. Hence $\bar{V}_n^+=H_n$ and this completes the proof.

REMARK 3.3. (1) Let $s=1/n$, $\nu=-\zeta_{2n}$. By (3.11) and (3.15), we have $\varphi_m\in\mathrm{Ker}(T_w)$, $m=n+1+2nt$ if and only if $t\geqq 0$. Hence $PS(\chi)$ is reducible. The $\tilde{K}$-type of $\pi_1$ is determined by (3.15).

(2) Let $\nu=-\zeta_{2n}$. We see by a similar argument as above that $PS(\chi)$ is irreducible for $0<s<1$, $s\neq 1/n$. We can also see by more argument using (3.9) and (3.10) that $PS(\chi)$ is unitarizable for $0<s<1/n$ but not unitarizable for $1/n<s<1$.

By Proposition 3.2, $\pi_1$ extends to a unitary representation of $\tilde{G}$ on $H_n$. We use the same letter $\pi_1$ for this representation. By (3.4), we see easily that $g\to\pi_1(g)f$ is a continuous map from $\tilde{G}$ to $H_n$ for fixed $f\in H_n$. Hence $\pi_1$ is a continuous unitary representation of $\tilde{G}$ on $H_n$ (cf. Warner [7], p. 219, p. 237, Proposition 4.2.2.1).

PROPOSITION 3.4. *There exists a unique irreducible unitary representation* $\pi_1$ *of* $\tilde{G}$ *on* $H_n$ *which satisfies the following conditions for* $f\in H_n$.

(1) $$(\pi_1(\widetilde{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}})f)(x)=\psi(bx)f(x), \qquad b\in\boldsymbol{R},\ x\in\boldsymbol{R}_+.$$

(2) $$(\pi_1(\widetilde{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}})f)(x)=\eta_a|a|^{1-1/n}f(a^2x), \qquad a\in\boldsymbol{R}^{\times},\ x\in\boldsymbol{R}_+.$$

(3) $$(\pi_1(\tilde{w})f)(x)=\lim_{T\to+\infty}\int_0^T k(xy)f(y)y^{-1/n}dy, \qquad x\in\boldsymbol{R}_+,$$

*where*

$$k(z)=\sqrt{2\pi a}\zeta_{4n}^{-1}\sqrt{-1}z^{1/2n}J_{-1/n}(2az^{1/2}), \qquad z>0.$$

PROOF. The formulas (1) and (2) follow from (2.9) and (2.10) respectively. By the well known asymptotic formula

(3.17) $$J_\nu(z)=\sqrt{\frac{2}{\pi z}}\cos\Big(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\Big)+O(|z|^{-3/2}), \qquad \Re(\nu)>-\frac{1}{2},$$

for $|z|\to\infty$ satisfying $-\pi+\delta\leqq\operatorname{Arg}(z)\leqq\pi$ for fixed $0<\delta<\pi/2$ (cf. [8], p. 197–199), we see that(5)

$$\lim_{T\to+\infty}\int_0^T k(xy)f(y)y^{-1/n}dy=2\lim_{T\to+\infty}\int_0^T k(xy^2)f(y^2)y^{1-2/n}dy$$

exists for almost all $x\in\boldsymbol{R}_+$, since $f(y^2)y^{1/2-1/n}\in L^2(\boldsymbol{R}_+)$. By (2.16) and Corollary 2.2, this coincides with the action of $\pi_1(\tilde{w})$ for $f\in V_n^+$. By Proposition 3.2 and (3.17), we see that this fact holds also for $f\in H_n$ by a standard theorem on the Fourier transformation of $L^2$-functions.

What remains to be shown is the irreducibility of $\pi_1$. Let $V\neq\{0\}$ be a closed invariant subspace of $H_n$ and let $W$ be the orthogonal complement of $V$. Take $f\in V$, $f\neq 0$. Choose $\alpha\in C_c^\infty(\boldsymbol{R}_+)$ and consider the multiplicative convolution $f_0(x)=\int_0^\infty\alpha(t)f(tx)dt$. By (2), we have $f_0\in V\cap C^\infty(\boldsymbol{R}_+)$. We can choose $\alpha$ so that $f_0\neq 0$. Now by the same proof as in Proposition 3.2, we conclude $W=\{0\}$. Hence the irreducibility follows.

## § 4. Metaplectic representations of $\tilde{G}$.

Let $\mathfrak{H}_n$ be the Hilbert space of all measurable functions $F$ on $\boldsymbol{R}_+$ such that

$$\|F\|=\Big(n\int_0^\infty|F(x)|^2x^{n-2}dx\Big)^{1/2}<\infty.$$

By the map $H_n \ni f(x) \to F(x) = f(x^n) \in \mathfrak{H}_n$, $H_n$ and $\mathfrak{H}_n$ are isomorphic as Hilbert spaces. We transport the representation $\pi_1$ of $\tilde{G}$ on $H_n$ to the representation $\tilde{\pi}$ of $\tilde{G}$ on $\mathfrak{H}_n$. Then we obtain the following theorem.

THEOREM 4.1. *There exists a unique irreducible unitary representation* $\tilde{\pi}$ *of* $\tilde{G}$ *on* $\mathfrak{H}_n$ *which satisfies the following conditions for* $F \in \mathfrak{H}_n$.

(1) $$(\tilde{\pi}_1(\widetilde{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}})F)(x) = \psi(bx^n)F(x), \qquad b \in \boldsymbol{R},\ x \in \boldsymbol{R}_+.$$

(2) $$(\tilde{\pi}(\widetilde{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}})F)(x) = \eta_a |a|^{1-1/n} F(|a|^{2/n}x), \qquad a \in \boldsymbol{R}^\times,\ x \in \boldsymbol{R}_+.$$

(3) $$(\tilde{\pi}(\tilde{w})F)(x) = \lim_{T \to +\infty} \int_0^T K(xy)F(y)y^{n-2}dy, \qquad x \in \boldsymbol{R}_+.$$

*Here* $\eta_a$ *is given by* (2.2) *with* $\nu = -\zeta_{2n}$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and*

$$K(z) = nk(z^n) = n\sqrt{2\pi a}\,\zeta_{4n}^{-1}\sqrt{-1}\,z^{1/2}J_{-1/n}(2az^{n/2}), \qquad z > 0.$$

If $n=2$ and $a=\pi$, we have $K(z) = 2\sqrt{2}\zeta_8 \cos(2\pi z)$. Noting $dy$ in (3) is $\sqrt{1/2}$ times the usual Lebesgue measure, we see that $\tilde{\pi}$ coincides with the usual Weil representation realized on even functions in $L^2(\boldsymbol{R})$.

## § 5. Metaplectic representations of $SL(2, \boldsymbol{C})$.

The construction of metaplectic representations of $SL(2, \boldsymbol{C})$ can be carried out in a similar manner as in the case $SL(2, \boldsymbol{R})$. Since $SL(2, \boldsymbol{C})$ is simply connected, its algebraic part is quite simple though analytic part is somewhat more complex.

Let $G = SL(2, \boldsymbol{C})$ and define subgroups $T$, $B$, $N$ of $G$ as in § 1 with $\boldsymbol{C}$ (resp. $\boldsymbol{C}^\times$) in the place of $\boldsymbol{R}$ (resp. $\boldsymbol{R}^\times$). For simplicity, we fix an additive character $\psi$ of $\boldsymbol{C}$ so that $\psi(z) = \exp(\pi\sqrt{-1}(z+\bar{z}))$. Then the usual Lebesgue measure $dxdy$ for $z = x + \sqrt{-1}y$, $x, y \in \boldsymbol{R}$ is the self-dual measure with respect to the self-duality $\langle x, y \rangle = \psi(xy)$ of $\boldsymbol{C}$. We denote this measure simply by $dz$ since no confusion is likely. Set

$$\delta\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = |a|^4, \qquad a \in \boldsymbol{C}^\times,\ b \in \boldsymbol{C},$$

which is the modular function of $B$. For a quasi-character $\chi$ of $\boldsymbol{C}^\times$, let $PS(\chi)$ denote the space of all $C^\infty$-functions $\varphi$ on $G$ which satisfy

(5.1) $$\varphi(tng) = \delta(t)^{1/2}\chi(t)\varphi(g) \qquad \text{for all } t \in T,\ n \in N,\ g \in G.$$

Let $\pi(\chi)$ denote the representation of $G$ realized on $PS(\chi)$ by right translations.

For $\varphi\in PS(\chi)$, let $\Phi=R(\varphi)$ denote the $C^\infty$-function on $\boldsymbol{C}$ defined by

$$\Phi(u)=\varphi(w\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}),\qquad u\in\boldsymbol{C}, \tag{5.2}$$

where $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. We take $\chi$ so that

$$\chi(a)=|a|^s,\qquad a\in\boldsymbol{C}^\times$$

with $s\in\boldsymbol{C}$, $0<\sigma=\Re(s)<3/2$. By

$$\Phi(u)=|u|^{-s-2}\varphi(\begin{pmatrix}1 & 0\\ u^{-1} & 1\end{pmatrix}),\qquad u\in\boldsymbol{C}^\times,$$

we obtain

$$(\partial/\partial u_1)^{m_1}(\partial/\partial u_2)^{m_2}\Phi(u_1+\sqrt{-1}u_2)=O(|u|^{-\sigma-2}),\qquad |u|\to\infty \tag{5.3}$$

for $m_1, m_2\geqq 0$, $u=u_1+\sqrt{-1}u_2$, $u_1, u_2\in\boldsymbol{R}$. Put $f=\mathcal{F}(\Phi)$, that is

$$f(x)=\int_C\varphi(w\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix})\overline{\psi(ux)}du,\qquad x\in\boldsymbol{C}. \tag{5.4}$$

By (5.3), $f$ is a continuous function on $\boldsymbol{C}$ which satisfies

$$f(x)=O(|x|^{-N}),\qquad |x|\to\infty\quad\text{for every } N>0. \tag{5.5}$$

Hence we have $\Phi=\mathcal{F}'(f)$, i.e.,

$$\varphi(w\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix})=\int_C f(x)\psi(ux)dx,\qquad u\in\boldsymbol{C}. \tag{5.6}$$

Let $V_s$ denote the vector space

$$\{f\mid f=\mathcal{F}(R(\varphi))\qquad\text{for some } \varphi\in PS(\chi)\}.$$

Since $R$ is injective, we can transport the representation $\pi(\chi)$ to the representation $\pi_s$ of $G$ on $V_s$. By (5.4), we immediately obtain

$$(\pi_s(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix})f)(x)=\psi(bx)f(x),\qquad b\in\boldsymbol{C},\ x\in\boldsymbol{C}, \tag{5.7}$$

$$(\pi_s(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix})f)(x)=|a|^{2-s}f(a^2x),\qquad a\in\boldsymbol{C}^\times,\ x\in\boldsymbol{C} \tag{5.8}$$

for $f\in V_s$. We can compute the action of $\pi_s(w)$ on $f\in V_s$ as in the real case and obtain

$$(\pi_s(w)f)(x)=\int_C\Big(\int_C f(y)\psi(vy)dy\Big)|v|^{s-2}\psi(v^{-1}x)dv. \tag{5.9}$$

Hence we have

(5.10) $$(\pi_s(w)f)(x) = \lim_{T\to+\infty}\int_C\Big(\int_{|v|\leq T}|v|^{s-2}\psi(vy+v^{-1}x)dv\Big)f(y)dy.$$

To justify the interchange of the limit and the integral, let us first consider the integral

$$I_T = \int_{|v|\leq T}|v|^{s-2}\psi(vy)dv.$$

Using the polar coordinate, put

(5.11) $$v = \rho e^{\sqrt{-1}\phi}, \qquad y = re^{\sqrt{-1}\theta}, \qquad 0\leqq\rho,\ 0<r,\ 0\leqq\phi,\ \theta<2\pi.$$

Then we have

$$I_T = \int_0^T\int_0^{2\pi}\rho^{s-1}\exp(2\pi\sqrt{-1}r\rho\cos(\theta+\phi))d\phi d\rho$$

$$= 2\pi\int_0^T\rho^{s-1}J_0(2\pi r\rho)d\rho = 2\pi r^{-s}\int_0^{rT}x^{s-1}J_0(2\pi x)dx.$$

Here we have used an integration formula

(5.12) $$\int_0^\pi\exp(\sqrt{-1}z\cos\theta)\cos n\theta d\theta = (\sqrt{-1})^n\pi J_n(z), \qquad n\in\boldsymbol{Z},$$

(cf. [1], p. 482). By the asymptotic formula (3.17), we see that $\left|\int_0^{rT}x^{s-1}J_0(2\pi x)dx\right|\leqq C$ with a constant $C$ independent of $r$, $T$. We have

(5.13) $$\int_0^\infty x^\mu J_0(ax)dx = 2^\mu a^{-\mu-1}\frac{\Gamma\Big(\frac{1}{2}+\frac{1}{2}\mu\Big)}{\Gamma\Big(\frac{1}{2}-\frac{1}{2}\mu\Big)}, \qquad a>0,\ -1<\Re(\mu)<1/2.$$

(cf. [1], p. 684.) Hence we obtain

(5.14) $$\lim_{T\to+\infty}\int_{|v|\leq T}|v|^{s-2}\psi(vy)dv = \pi^{1-s}\frac{\Gamma(s/2)}{\Gamma(1-s/2)}|y|^{-s}.$$

Since $|I_T|\leqq 2\pi C|y|^{-\sigma}$ and $|y|^{-\sigma}f(y)\in L^1(\boldsymbol{C})$, it suffices to show the existence of $H\in L^1(\boldsymbol{C})$ such that

(5.15) $$\Big|\Big(\int_{|v|\leq T}|v|^{s-2}\psi(vy)(\psi(v^{-1}x)-1)dv\Big)f(y)\Big| \leqq |H(y)|, \qquad y\neq 0,$$

and also the existence of the limit

(5.16) $$\lim_{T\to+\infty}\int_{|v|\leq T}|v|^{s-2}\psi(vy)(\psi(v^{-1}x)-1)dv, \qquad y\neq 0.$$

We have $|\psi(v^{-1}x)-1|\leqq c_1/|v|$, $|v|\geqq 1$ with some constant $c_1$. If $\sigma<1$, the integral $\int_{|v|\geq 1}|v|^{s-3}dv$ is absolutely convergent. Hence we obtain (5.15) and (5.16).

If $1\leqq\sigma<3/2$, we take the first order term of the expansion of $\phi(v^{-1}x)-1$ in $v^{-1}$ and $\bar{v}^{-1}$ into consideration. Using the polar coordinate (5.11), we obtain

$$\int_{T_1\leq|v|\leq T_2}|v|^{s-2}v^{-1}\phi(vy)dv=\int_{T_1}^{T_2}\int_0^{2\pi}\rho^{s-2}e^{-\sqrt{-1}\phi}\exp(2\pi\sqrt{-1}r\rho\cos(\theta+\phi))d\phi d\rho$$
$$=2\pi\sqrt{-1}e^{\sqrt{-1}\theta}r^{-s+1}\int_{rT_1}^{rT_2}x^{s-2}J_1(2\pi x)dx$$

by (5.12). Similarly we obtain

$$\int_{T_1\leq|v|\leq T_2}|v|^{s-2}\bar{v}^{-1}\phi(vy)dv=2\pi\sqrt{-1}e^{-\sqrt{-1}\theta}r^{-s+1}\int_{rT_1}^{rT_2}x^{s-2}J_1(2\pi x)dx.$$

By the asymptotic formula (3.17) for $J_1$, we see that the integral $\int_1^\infty x^{s-2}J_1(2\pi x)dx$ is absolutely convergent. This proves the existence of the limit (5.16). If $\sigma>1$, $x^{s-2}J_1(2\pi x)$ is locally integrable at $x=0$ and this proves (5.15). If $\sigma=1$, we have

$$\left|\int_A^1 x^{s-2}J_1(2\pi x)dx\right|=O(|\log A|),\qquad A\rightarrow+0.$$

Since $f(y)|y|^{-s+1}\log|y|$ is integrable, (5.15) follows. Therefore we obtain

$$(5.17)\qquad \begin{aligned}(\pi_s(w)f)(x)&=\int_C\lim_{T\rightarrow+\infty}\Big(\int_{|v|\leq T}|v|^{s-2}\phi(vy+v^{-1}x)dv\Big)f(y)dy\\&=\int_C\lim_{T\rightarrow+\infty}\Big(\int_{|v|\leq T}|v|^{s-2}\phi(v+v^{-1}xy)dv\Big)f(y)|y|^{-s}dy.\end{aligned}$$

Put

$$(5.18)\qquad k(z)=\lim_{T\rightarrow+\infty}\int_{|v|\leq T}|v|^{s-2}\phi(v+v^{-1}z)dv,\qquad z\in\boldsymbol{C}.$$

LEMMA 5.1. *We have, for* $s\in\boldsymbol{R}$, $0<s<3/2$,

$$k(z)=-\pi^2|z|^{s/2}\mathfrak{J}(e^{\sqrt{-1}s\pi/2}H_{s/2}^{(1)}(2\pi z^{1/2})H_{s/2}^{(1)}(2\pi\bar{z}^{1/2}))$$
$$=\frac{\pi^2}{\sin(s\pi/2)}|z|^{s/2}(|J_{-s/2}(2\pi z^{1/2})|^2-|J_{s/2}(2\pi z^{1/2})|^2),\qquad z\in\boldsymbol{C}.$$

PROOF. Set $z=re^{\sqrt{-1}\theta}$, $v=\rho e^{\sqrt{-1}\phi}$ in the polar coordinate. Here we choose $\theta$ so that $-\pi<\theta\leqq\pi$. Then we have

$$\int_{|v|\leq T}|v|^{s-2}\phi(v+v^{-1}z)dv=\int_0^T\int_0^{2\pi}\rho^{s-1}\exp\Big(2\pi\sqrt{-1}\Big(\rho\cos\phi+\frac{r}{\rho}\cos(\phi-\theta)\Big)\Big)d\phi d\rho$$
$$=2\pi\int_0^T\rho^{s-1}J_0\Big(2\pi\sqrt{\rho^2+\Big(\frac{r}{\rho}\Big)^2+2r\cos\theta}\Big)d\rho$$
$$=2\pi r^{s/2}\int_0^{T/\sqrt{r}}\rho^{s-1}J_0\Big(2\pi\sqrt{r}\sqrt{\rho^2+\Big(\frac{1}{\rho}\Big)^2+2\cos\theta}\Big)d\rho$$

by (5.12). Hence it suffices to show

(5.19)
$$\int_0^\infty v^{s-1}H_0^{(1)}\left(a\sqrt{v^2+\frac{1}{v^2}+2\cos\theta}\right)dv$$
$$=\frac{\pi\sqrt{-1}}{2}e^{\sqrt{-1}s\pi/2}H_{s/2}^{(1)}(ae^{\sqrt{-1}\theta/2})H_{s/2}^{(1)}(ae^{-\sqrt{-1}\theta/2}),\qquad a>0,\ -\pi<\theta\leqq\pi.$$

We employ an integral representation for Hankel functions (cf. [1], p. 956).

(5.20)
$$H_\nu^{(1)}(z)=-\frac{\sqrt{-1}}{\pi}e^{-\sqrt{-1}\nu\pi/2}\int_0^\infty\exp\left[\frac{1}{2}\sqrt{-1}z\left(t+\frac{1}{t}\right)\right]t^{-\nu-1}dt,\quad \nu\in\boldsymbol{C},\ \Im z>0.$$

Take $\alpha,\beta\in\boldsymbol{C}$, $\Im\alpha>0$, $\Im\beta>0$. By (5.20), we have

$$H_\nu^{(1)}(\alpha)H_\nu^{(1)}(\beta)$$
$$=-\frac{1}{\pi^2}e^{-\sqrt{-1}\nu\pi}\int_0^\infty\int_0^\infty\exp\left[\frac{1}{2}\sqrt{-1}\alpha\left(t+\frac{1}{t}\right)+\frac{1}{2}\sqrt{-1}\beta\left(u+\frac{1}{u}\right)\right]t^{-\nu-1}u^{-\nu-1}dtdu$$

with the absolutely convergent double integral. Changing variables so that $v=\sqrt{tu}$, $w=\sqrt{t/u}$, we obtain

(5.21)
$$H_\nu^{(1)}(\alpha)H_\nu^{(1)}(\beta)=-\frac{2}{\pi^2}e^{-\sqrt{-1}\nu\pi}$$
$$\int_0^\infty\int_0^\infty\exp\left[\frac{\sqrt{-1}}{2}\left\{\left(\alpha v+\frac{\beta}{v}\right)w+\left(\frac{\alpha}{v}+\beta v\right)w^{-1}\right\}\right]w^{-1}v^{-2\nu-1}dwdv.$$

Let $I$ denote the inner integral of (5.21). Assume $|\alpha|=|\beta|$ and set

$$\alpha=\rho e^{\sqrt{-1}\phi_1},\quad \beta=\rho e^{\sqrt{-1}\phi_2}\qquad\text{with } \rho>0,\ 0<\phi_1,\ \phi_2<\pi.$$

Then we find

(5.22)
$$I=\int_0^\infty\exp\left[\frac{\sqrt{-1}}{2}\rho\rho_1e^{\sqrt{-1}(\phi_1+\phi_2)/2}\{e^{\sqrt{-1}\eta}w+e^{-\sqrt{-1}\eta}w^{-1}\}\right]w^{-1}dw,$$

where $\rho_1$ and $\eta$, $-\pi<\eta\leqq\pi$, are determined by

$$\rho_1=\sqrt{v^2+\frac{1}{v^2}+2\cos(\phi_1-\phi_2)},$$

$$\rho_1\cos\eta=\cos\frac{\phi_1-\phi_2}{2}\left(v+\frac{1}{v}\right),\qquad \rho_1\sin\eta=\sin\frac{\phi_1-\phi_2}{2}\left(v-\frac{1}{v}\right).$$

Since

$$-\pi/2<\eta<\pi/2,\qquad 0<\frac{\phi_1-\phi_2}{2}\pm\eta<\pi$$

for $v>0$, we see easily that the path of integration in (5.22) can be altered to $\int_0^{e^{-\sqrt{-1}\eta}(+\infty)}$. Then, by (5.20), we get

$$I=\sqrt{-1}\pi H_0^{(1)}(\rho\rho_1 e^{\sqrt{-1}(\psi_1+\psi_2)/2}).$$

Hence we have

$$H_\nu^{(1)}(\alpha)H_\nu^{(1)}(\beta)=-\frac{2\sqrt{-1}}{\pi}e^{-\sqrt{-1}\nu\pi}$$

$$\int_0^\infty v^{-2\nu-1}H_0^{(1)}\Big(|\alpha|e^{\sqrt{-1}(\psi_1+\psi_2)/2}\sqrt{v^2+\frac{1}{v^2}+2\cos(\psi_1-\psi_2)}\Big)dv.$$

Changing the variable $v$ to $v^{-1}$ and putting $\nu=s/2$, we obtain the formula[6]

$$\int_0^\infty v^{s-1}H_0^{(1)}\Big(\sqrt{\alpha^2+\beta^2+\alpha\beta\Big(v^2+\frac{1}{v^2}\Big)}\Big)dv=\frac{\pi\sqrt{-1}}{2}e^{\sqrt{-1}s\pi/2}H_{s/2}^{(1)}(\alpha)H_{s/2}^{(1)}(\beta). \tag{5.23}$$

This formula holds whenever $\Im(\alpha)>0$, $\Im(\beta)>0$, $|\alpha|=|\beta|$ for arbitrary $s\in\boldsymbol{C}$ when $\sqrt{\ }$ is taken to have positive imaginary part. Recall the asymptotic formula

$$H_\nu^{(1)}(z)=\Big(\frac{2}{\pi z}\Big)^{1/2}\exp\Big(\sqrt{-1}\Big(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\Big)\Big)[1+O(|z|^{-1})],\quad \Re(\nu)>-\frac{1}{2} \tag{5.24}$$

when $|z|\to\infty$ satisfying $-\pi+\delta\leqq\operatorname{Arg}(z)\leqq\pi$ for fixed $0<\delta<\pi/2$. We see that the integral (5.23) is absolutely convergent and defines an analytic function of two variables $\alpha$ and $\beta$ in the domain $0<\operatorname{Arg}(\alpha+\beta)<\pi/2$, $\Im(\alpha\beta)>0$ when $\sqrt{\ }$ is taken to have positive imaginary part. Hence (5.23) holds in this domain by analytic continuation. If $0<\Re(s)<3/2$, we see, by continuity, that (5.23) holds also for the domain $0\leqq\operatorname{Arg}(\alpha+\beta)\leqq\pi/2$, $\Im(\alpha\beta)\geqq 0$, $\alpha\beta\neq 0$ if $\sqrt{\ }$ is taken to have non-negative real and imaginary parts. Putting $\alpha=ae^{\sqrt{-1}\theta/2}$, $\beta=ae^{-\sqrt{-1}\theta/2}$, we obtain (5.19). This completes the proof.

For $\varphi\in PS(\chi)$, set

$$(T_w(\varphi))(g)=\int_C\varphi\Big(w^{-1}\begin{pmatrix}1&u\\0&1\end{pmatrix}g\Big)du,\qquad g\in G. \tag{5.25}$$

This integral is absolutely convergent. As in the real case, we see that $T_w(\varphi)\in PS(\chi^{-1})$ and that $T_w$ defines an intertwining operator from $PS(\chi)$ to $PS(\chi^{-1})$. Assume $s\in\boldsymbol{R}$, i.e., $0<s<3/2$. Set

$$\langle\varphi_1,\varphi_2\rangle=\int_{B\backslash G}(T_w(\varphi_1))(g)\overline{\varphi_2(g)}dg,\qquad \varphi_1,\varphi_2\in PS(\chi). \tag{5.26}$$

Here we have normalized the invariant measure $dg$ on $B\backslash G$ so that

$$\langle\varphi_1,\varphi_2\rangle=\int_C(T_w(\varphi_1))\Big(w\begin{pmatrix}1&u\\0&1\end{pmatrix}\Big)\overline{\varphi_2\Big(w\begin{pmatrix}1&u\\0&1\end{pmatrix}\Big)}du. \tag{5.27}$$

Then we have

$$(T_w(\varphi))(w\begin{pmatrix}1 & u\\0 & 1\end{pmatrix}) = \int_C |v|^{s-2}\varphi(w\begin{pmatrix}1 & u+v\\0 & 1\end{pmatrix})dv, \qquad u\in\boldsymbol{C}.$$

Define a locally integrable function $T$ on $\boldsymbol{C}$ by

(5.28) $$T(x) = |x|^{s-2}, \qquad x\in\boldsymbol{C}.$$

Then we get

$$R(T_w(\varphi)) = \check{T}*\Phi, \qquad \Phi = R(\varphi) \qquad \text{for } \varphi\in PS(\chi).$$

Now it follows immediately that

(5.29) $$\langle\varphi_1, \varphi_2\rangle = T(\Phi_1*\tilde{\Phi}_2) \qquad \text{for } \varphi_i\in PS(\chi),\ \Phi_i=R(\varphi_i),\ i=1, 2.$$

Let

$$(\mathcal{F}'T)(x) = \lim_{A\to+\infty}\int_{|y|\leq A} T(y)\psi(xy)dy$$

be the inverse Fourier transformation of $T$. We have

(5.30) $$(\mathcal{F}'T)(x) = \pi^{1-s}\frac{\Gamma(s/2)}{\Gamma(1-s/2)}|x|^{-s}$$

by (5.14). When transferred to the Fourier transformation, (5.29) yields

(5.31) $$\langle\varphi_1, \varphi_2\rangle = c_s\int_C f_1(x)\overline{f_2(x)}|x|^{-s}dx, \qquad \varphi_1, \varphi_2\in PS(\chi),$$

where $c_s=\pi^{1-s}\Gamma(s/2)/\Gamma(1-s/2)$, $f_i=\mathcal{F}(\Phi_i)$, $\Phi_i=R(\varphi_i)$, $i=1, 2$. Dropping the constant $c_s$, put

(5.32) $$(\varphi_1, \varphi_2) = \int_C f_1(x)\overline{f_2(x)}|x|^{-s}dx.$$

Then (5.32) defines an invariant positive definite hermitian form on $PS(\chi)$. Let $H_s$ denote the Hilbert space with the norm $\|\ \|_s$ of all measurable functions $f(x)$ on $\boldsymbol{C}$ such that

$$\|f\|_s^2 = \int_C |f(x)|^2|x|^{-s}dx < \infty.$$

As in Proposition 3.2, we see that $V_s$ is a dense subspace of $H_s$ (the fact $V_s \neq \{0\}$ is trivial in this case). Hence $\pi_s$ extends to a unitary representation of $G$ on $H_s$ which we denote by the same symbol $\pi_s$. We see that $\pi_s$ is continuous and irreducible as in the real case. Summing up, we have obtained:

PROPOSITION 5.2. *For every real number $s$, $0<s<3/2$, there exists a unique irreducible unitary representation $\pi_s$ of $G$ on $H_s$ which satisfies the following conditions for every $f\in H_s$.*

(1) $$(\pi_s(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix})f)(x)=\psi(bx)f(x), \qquad b\in \boldsymbol{C},\ x\in \boldsymbol{C}.$$

(2) $$(\pi_s(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix})f)(x)=|a|^{2-s}f(a^2x), \qquad a\in \boldsymbol{C}^{\times},\ x\in \boldsymbol{C}.$$

(3) $$(\pi_s(w)f)(x)=\lim_{T\to+\infty}\int_{|y|\leq T}k(xy)f(y)|y|^{-s}dy, \qquad x\in \boldsymbol{C}.$$

*where*

$$k(z)=\frac{\pi^2}{\sin(s\pi/2)}|z|^{s/2}(|J_{-s/2}(2\pi z^{1/2})|^2-|J_{s/2}(2\pi z^{1/2})|^2), \qquad z\in \boldsymbol{C}.$$

We remark that (3) can be shown in a similar manner as in the real case using (5.24) and the first expression of $k(z)$ given in Lemma 5.1.

Let $\mathfrak{H}_s$ be the Hilbert space with the norm $\|\ \|_{(s)}$ of all measurable functions $F$ on $\boldsymbol{C}$ such that

$$F(\zeta_n x)=F(x), \quad x\in \boldsymbol{C}, \qquad \|F\|_{(s)}=(n\int_{\boldsymbol{C}}|F(x)|^2|x|^{2n-2-ns}dx)^{1/2}<\infty.$$

By the map $H_s\ni f(x)\to F(x)=f(x^n)\in\mathfrak{H}_s$, $H_s$ and $\mathfrak{H}_s$ are isomorphic as Hibert spaces. We transport the representation $\pi_s$ of $G$ on $H_s$ to the representation $\tilde{\pi}_s$ of $G$ on $\mathfrak{H}_s$. Then we obtain the following theorem.

THEOREM 5.3. *For every real number $s$, $0<s<3/2$ and a natural number $n$, there exists a unique irreducible unitary representation $\tilde{\pi}_s$ of $G$ on $\mathfrak{H}_s$ which satisfies the following conditions for every $F\in\mathfrak{H}_s$.*

(1) $$(\tilde{\pi}_s(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix})F)(x)=\psi(bx^n)F(x), \qquad b\in \boldsymbol{C},\ x\in \boldsymbol{C}.$$

(2) $$(\tilde{\pi}_s(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix})F)(x)=|a|^{2-s}F(a^{2/n}x), \qquad a\in \boldsymbol{C}^{\times},\ x\in \boldsymbol{C}.$$

(3) $$(\tilde{\pi}_s(w)F)(x)=\lim_{T\to+\infty}\int_{|y|\leq T}K(xy)F(y)|y|^{2n-2-ns}dy, \qquad x\in \boldsymbol{C},$$

*where* $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$,

$$K(z)=nk(z^n)=\frac{n\pi^2}{\sin(s\pi/2)}|z|^{ns/2}(|J_{-s/2}(2\pi z^{n/2})|^2-|J_{s/2}(2\pi z^{n/2})|^2), \quad z\in \boldsymbol{C}.$$

If $s=2/n$, the representation $\tilde{\pi}_s$ coincides with the Weil type representation constructed in Kubota [3].

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### Notes

(1) This simple observation must have been noticed by specialists. The author himself had this idea long time ago.

(2) It is probable that we can compute the cocycle also for non-archimedean local fields by Shimura's method using an analogue of the complex upper half plane; the same idea may apply to some higher dimensional cases.

(3) These formulas can be summalized by a single formula

$$\int_0^\infty v^{s-1}\exp\left(a\sqrt{-1}\left(v+\frac{b^2}{v}\right)\right)dv=\sqrt{\frac{a}{2\pi}}\pi\sqrt{-1}e^{s\pi\sqrt{-1}/2}b^s H_s^{(1)}(2ab),$$

$$a>0,\ 0\leqq \arg b\leqq \pi/2,\ 0<s<1$$

involving the Hankel function $H_s^{(1)}$, which can be derived from (5.20).

(4) A similar inner product formula for $\mathfrak{p}$-adic groups of higher rank is considered by the author [11].

(5) If $\nu=-1/2$, (3.17) holds as the identity without the $O$-term. We also note that $k(|z|)$ is continuous on $\boldsymbol{R}$.

(6) This formula may be derived from [1], p. 722, 6.648. But we think it better to give a proof here.

Hiroyuki YOSHIDA
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto 606
Japan