J. Math. Soc. Japan
Vol. 43, No. 4, 1991

Symmetric solutions of the equation for the scalar curvature under conformal deformation of a Riemannian metric

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(Received Oct. 25, 1990)

1. Introduction.

Let (M, g) be a compact n-dimensional Riemannian manifold without boundary ($n\geqq 3$) and S_g be its scalar curvature. Define a new metric $\hat{g}$ which is pointwise conformal to g by $\hat{g}:=u^{p-2}g$ with a positive smooth function u. Then its scalar curvature is given by

$$S_{\hat{g}} = u^{1-p}(a\Delta_g u+S_g u)$$

where $p=2n/(n-2)$, $a=4(n-1)/(n-2)$ and Δ_g is the Laplacian with respect to g, namely

$$\Delta_g u := -g^{ij}\nabla_i\nabla_j u .$$

Now we are interested in the question of which kind of functions can be realized as scalar curvatures in the conformal class of g. In particular, we assume g is invariant under the action of some isometry group Γ, and consider Γ-invariant scalar curvatures realized with Γ-invariant metrics. From the above formula, f is realized if and only if there exists at least one solution of the following nonlinear elliptic differential equation:

$$(*) \qquad \begin{cases} a\Delta_g u+S_g u = f u^{p-1} \\ u>0 \end{cases} \quad \text{on } M.$$

In the case f is a constant, it is called the Yamabe problem, and if f has the same signature as the Yamabe invariant $\mu(M)$ (see Definitions 1.1), then $(*)$ has at least one solution (cf. [**17**], [**15**], [**1**], [**10**]; see also [8]). On the other hand, several authors discussed also about the case f is not a constant and it is known that in the case $(M, g)=(S^n, g_0)$ (that is the standard sphere), $(*)$ does not always have a solution (cf. [5], [**2**]) even if $f>0$ (namely f has the same signature as $\mu(S^n)>0$). Aubin [**1**] obtained the useful criterion for the existence of a solution of $(*)$, and several authors generalized it to the Γ-invariant case. From this criterion and using Green function, Schoen [**10**] solved the Yamabe problem completely. By the similar method, Escobar and Schoen [4] get a sufficient condition called a flatness condition. In this paper, we consider its generalization.

To describe the above known results and our results, we prepare some definitions. Throughout this paper, we use the same notations as above and the following

DEFINITIONS 1.1.

(1)
$$J(u):=a\|\nabla_g u\|_2{}^2+\int_M S_g u^2 dV_g \qquad (u\in H_1(M)).$$

$$\mu(M):=\inf\left\{\frac{J(u)}{\|u\|_p{}^2}\,\middle|\, u\in H_1(M)\setminus\{0\}\right\}.$$

$\mu(M)$ is called the Yamabe invariant of (M, g) and for convenience, we denote

$$\mu_n:=\mu(S^n) \qquad (=n(n-1)\omega_n{}^{2/n})$$

where $\omega_n=\mathrm{vol}\, S^n$.

(2) For any compact subgroup Γ of the isometry group of (M, g) and any $j\in \boldsymbol{N}\cup\{+\infty\}$, we set

$$F_j:=\{x\in M\,|\,\#\Gamma x=j\},$$

$$D_j:=\{x\in M\,|\,\#\Gamma x\leqq j\}=\bigcup_{i=1}^{j} F_i\,.$$

If Γ is noncompact, replace Γ by its closure.

(3) For any $f\in C^{0,\alpha}(M)$ $(0<\alpha<1)$ which is Γ-invariant and $\max_M f>0$.

$$H_{\Gamma,f}(M):=\left\{u\in H_1(M)\,|\,u:\Gamma\text{-invariant},\ \int_M f|u|^p dV_g>0\right\},$$

$$\|u\|_{p,f}:=\left(\int_M f|u|^p dV_g\right)^{1/p} \qquad (u\in H_{\Gamma,f}(M)),$$

$$\mu_{\Gamma,f}(M):=\inf\left\{\frac{J(u)}{\|u\|_{p,f}{}^2}\,\middle|\, u\in H_{\Gamma,f}(M)\right\},$$

$$\boldsymbol{N}_{\Gamma,f}:=\{j\in\boldsymbol{N}\,|\,F_j\neq\emptyset,\ \max_{D_j} f>0\}.$$

To solve equation (*), we employ the standard variational principle. Namely we try to find $u\in H_{\Gamma,f}(M)$ which realizes $\mu_{\Gamma,f}(M)$. If such u exists, by N. Trudinger's regularity theorem (cf. [**15**]) and the maximal principle, we can show that $|u|$ is a Γ-invariant C^2-solution of (*). About any minimizing sequence of $J(u)/\|u\|_{p,f}{}^2$, we know the following significant criterion obtained at first by T. Aubin (cf. [**1**]) and generalized by P. Lions et al. (cf. [**7**]; see also Vaugon [**16**], Chen [**3**]).

THEOREM A. *Suppose $\mu(M)>0$, and f is a Γ-invariant $C^{0,\alpha}$-function with*

$0<\alpha<1$ *such that* $\max_M f>0$. *If*

$$\text{(A)} \qquad \mu_{\Gamma,f}(M)<j^{2/n}(\max_{D_j} f)^{-2/p}\mu_n$$

for any $j\in N_{\Gamma,f}$, *then any minimizing sequence of* $J(u)/\|u\|_{p,f}^2$ *has a subsequence which converges* H_1*-strongly in* $H_{\Gamma,f}(M)$, *namely equation* $(*)$ *has a* Γ*-invariant* C^2*-solution.*

Actually the Yamabe problem was solved by showing (A) in the case (M, g) is not conformally equivalent to (S^n, g_0), $\Gamma=id$ and f is a constant, that is

$$\mu_{id,1}(M)=\mu(M)<1^{2/n}(\max 1)^{-2/p}\mu_n=\mu(S^n).$$

It has not been shown in general case yet but $n=3$ and $\Gamma=id$. By testing a constant function 1, we obtain the following integral condition:

$$\text{(I)} \qquad \int_M f dV_g>j^{-p/n}\left(\frac{\int_M S_g dV_g}{\mu_n}\right)^{p/2}\max_{D_j} f>0.$$

The method emploied by Schoen [**10**] is more useful. Actually, by using it, J. Escobar and R. Schoen obtained a flatness condition for $n\geqq 3$, and solved this problem completely for $n=3$. They also considered the same condition as above in the case $(M, g)=(S^n, g_0)$ and f is invariant under the action of some nontrivial finite subgroup Γ of the isometry group of (S^n, g_0) (that is $O(n+1)$) without fixed point. By the similar method, Chen [**3**] generalized it to the case with fixed point.

In this paper, we shall consider the case (M, g) is not necessarily conformally equivalent to (S^n, g_0) and Γ is not necessarily finite, and obtain the following flatness condition:

(F) there exists $x_j\in D_j$ such that

$$f(x_j)=\max_{D_j} f>0,$$
$$f(x)\geqq f(x_j)-C(d_g(x, x_j))^{n-1} \text{ near } x_j$$

for some $C>0$.

Parts of our main results are stated as follows.

THEOREM 1.2. *Let* (M, g) *be a locally conformally flat manifold which is not conformally equivalent to* (S^n, g_0) *and* $\mu(M)>0$, *and let* $f\in C^{0,\alpha}(M)$ $(0<\alpha<1)$ *be* Γ*-invariant and* $\max_M f>0$. *Then there exists a* C^2*-solution of equation* $(*)$, *provided that conditions* (F) *and* (S) *hold for each* $j\in N_{\Gamma,f}$ *where*
(S) *there exists a* Γ_0*-invariant smooth function* $\lambda_0>0$ *which is defined in some neighborhood* $B_\varepsilon(x_j)$ *such that* $\lambda_0^{p-2}g|_{B_\varepsilon(x_j)}$ *is flat where* $\Gamma_0:=\{h\in\Gamma \mid hx_j=x_j\}$.

THEOREM 1.3. *Let* (M, g) *be a compact* 3*-dimensional manifold which is not* *conformally equivalent to* (S^3, g_0) *and* $\mu(M)>0$, *and let* $f\in C^{2,\alpha}(M)$ $(0<\alpha<1)$ *be* Γ*-invariant and* $\max_M f>0$. *Then there exists a* C^4*-solution of equation* $(*)$.

Sections 2 and 3 are devoted to the full statements in Theorems 2.1 and 3.1 and the proofs for these. The above assertions are involved in these theorems. After observations on the condition (S) are made in Section 4, some relations between the symmetry and the nonuniqueness of solutions of equation $(*)$ are given in Section 5.

The author would like to express his sincere gratitude to Prof. S. Murakami and Prof. A. Kasue for their constant encouragement and valuable suggestion, and also to Prof. O. Kobayashi and Dr. S. Nayatani for their useful comments.

2. The locally conformally flat case.

In this section, we treat the case (M, g) is locally conformally flat. Before stating the main existence theorem, we prove the following crucial

LEMMA 2.1. *Suppose* (M, g) *is locally conformally flat and* $j\in \boldsymbol{N}_{\Gamma,f}$. *If conditions* (F) *and* (S) *hold, then* (A) *is true except the case* $(M, g)=(S^n, g_0)$ *and* $j=1$ *simultaneously, where* (S) *is as in Theorem* 1.2.

PROOF. We shall show that for each $j\in \boldsymbol{N}_{\Gamma,f}$, there exists $u\in H_{\Gamma,f}(M)$ such that

$$\text{(A}'\text{)}\qquad \frac{J(u)}{\|u\|_{p,f}^{\,2}} < j^{2/n}(\max_{D_j} f)^{-2/p}\mu_n\,.$$

STEP 1. Since $x_j\in D_j$, there exists $j'\leqq j$ such that $x_j\in F_{j'}$. Then Γ_0 is compact and $|\Gamma:\Gamma_0|=j'$, so there exist $h_1(=e)$, $h_2, \cdots, h_{j'}\in\Gamma$ such that

$$\Gamma=\sum_{i=1}^{j'} h_i\Gamma_0\,.$$

We put $y_i:=h_ix_j$ $(1\leqq\forall i\leqq j')$, then

$$\Gamma x_j=\{x_j, h_2x_j, \cdots, h_{j'}x_j\}=\{y_1, y_2, \cdots, y_{j'}\}.$$

We may take ε to satisfy $B_\varepsilon(y_i)\cap B_\varepsilon(y_{i'})=\emptyset$ for any $i\neq i'$, and define a positive smooth function λ by

$$\lambda(x):=\sum_{i=1}^{j'}(\eta(\lambda_0-1))\circ h_i^{-1}(x)+1\qquad(\forall x\in M)$$

where η is a cut-off function which is defined by

$$\eta(x)=\begin{cases}1 & \text{for } d_g(x, x_j)\leqq \varepsilon/2\\ 0 & \text{for } d_g(x, x_j)\geqq \varepsilon\,.\end{cases}$$

and depends only on $d_g(x, x_j)$. Since $d_g(x, x_j)$ is Γ_0-invariant, so is η. Now for any $h\in\Gamma$,

$$\sum_{i=1}^{j'} h_i\Gamma_0=\Gamma=h^{-1}\Gamma=\sum_{i=1}^{j'} h^{-1}h_i\Gamma_0\,.$$

This implies there exist $k_1, \cdots, k_{j'}\in\Gamma_0$ such that

$$\{h^{-1}h_1, \cdots, h^{-1}h_{j'}\}=\{h_1k_1, \cdots, h_{j'}k_{j'}\}.$$

Hence for any $x\in M$,

$$\lambda(hx)=\sum_{i=1}^{j'}(\eta(\lambda_0^{-1}))\circ h_i^{-1}h(x)+1=\sum_{i'=1}^{j'}(\eta(\lambda_0^{-1}))\circ k_{i'}^{-1}h_{i'}^{-1}(x)+1$$

$$=\sum_{i'=1}^{j'}(\eta(\lambda_0^{-1}))\circ h_{i'}^{-1}(x)+1=\lambda(x)\,,$$

namely λ is Γ-invariant. So we can define a new Γ-invariant metric $\hat{g}:=\lambda^{p-2}g$ which is pointwise conformal to g and flat near x_j.

STEP 2. Now since $\mu(M)>0$, there exists a unique Green function for $a\Delta_{\hat{g}}+S_{\hat{g}}$ at x_j, and we denote it by G_0. For its uniqueness, it must be Γ_0-invariant. Denote the distance $d_{\hat{g}}(x, y)$ between x and y with respect to the metric $\hat{g}$ by $|x-y|$ for simplicity, and take $\rho_0>0$ such that $\hat{g}|_{B_{2\rho_0}}(y_i)$ is flat for any i. If we replace G_0 by some multiple of G_0, then we have

$$G_0(x)=|x-x_j|^{2-n}+A_0+\alpha(x)\qquad(\forall x\in B_{2\rho_0}(x_j))$$

with some constant $A_0\geqq 0$ and some α such that $\Delta_{\hat{g}}\alpha=0$, $\alpha(x_j)=0$ (cf. [**10**]). For any $k\in\Gamma_0$ and $x\in B_{2\rho_0}(x_j)\setminus\{x_j\}$.

$$\alpha(kx)=G_0(kx)-(|kx-x_j|^{2-n}+A_0)=G_0(x)-(|x-x_j|^{2-n}+A_0)=\alpha(x).$$

which shows α is also Γ_0-invariant. Set

$$G(x):=\sum_{i=1}^{j'} G_0\circ h_i^{-1}(x)\,.$$

Then it is clearly well defined and Γ-invariant. In addition, for any i_1 and $x\in B_{2\rho_0}(y_{i_1})$, we get

$$G(x)=|x-y_{i_1}|^{2-n}+A+\beta_{i_1}(x)$$

where

$$A:=A_0+\sum_{i\neq i_1} G_0\circ h_i^{-1}(y_{i_1})\,.$$

$$\beta_{i_1}(x):=\alpha\circ h_{i_1}^{-1}(x)+\sum_{i\neq i_1}(G_0\circ h_i^{-1}(x)-G_0\circ h_i^{-1}(y_{i_1}))\,.$$

Since there exist $k_1, \cdots, k_{j'} \in \Gamma_0$ such that

$$\{h_{i_1}^{-1}h_1, \cdots, h_{i_1}^{-1}h_{j'}\} = \{h_1k_1, \cdots, h_{j'}k_{j'}\}$$

as before, and in paticular $h_{i_1}^{-1}h_{i_1}=e=h_1k_1$,

$$\begin{aligned} A &= A_0+\sum_{i\neq i_1} G_0\circ h_i^{-1}(h_{i_1}x_j) = A_0+\sum_{i'\neq 1}(G_0\circ k_{i'}^{-1})\circ h_{i'}^{-1}(x_j) \\ &= A_0+\sum_{i'\neq 1} G_0\circ h_{i'}^{-1}(x_j) \geqq 0 . \end{aligned}$$

This shows A is a nonnegative constant independent of i_1. On the other hand, for any $h\in\Gamma$, there exists i_2 such that $hy_{i_1}=y_{i_2}=h_{i_2}x_j$, namely $h_{i_2}^{-1}hh_{i_1}\in\Gamma_0$. Now there exist $k_1, \cdots, k_{j'}\in\Gamma_0$ such that

$$\{h^{-1}h_1, \cdots, h^{-1}h_{j'}\} = \{h_1k_1, \cdots, h_{j'}k_{j'}\}$$

as before, and in paticular $h^{-1}h_{i_2}=h_{i_1}k_{i_1}$. Hence for any $x\in B_{2\rho_0}(y_{i_1})$.

$$hx \in B_{2\rho_0}(y_{i_2})$$

and

$$\begin{aligned} \beta_{i_2}(hx) &= \alpha\circ h_{i_2}^{-1}(hx)+\sum_{i\neq i_2}(G_0\circ h_i^{-1}(hx)-G_0\circ h_i^{-1}(y_{i_2})) \\ &= \alpha\circ(h_{i_2}^{-1}hh_{i_1})\circ h_{i_1}^{-1}(x)+\sum_{i\neq i_2}(G_0\circ h_i^{-1}(hx)-G_0\circ h_i^{-1}(hy_{i_1})) \\ &= \alpha\circ h_{i_1}^{-1}(x)+\sum_{i\neq i_1}((G_0\circ k_{i'}^{-1})\circ h_{i'}^{-1}(x)-(G_0\circ k_{i'}^{-1})\circ h_{i'}^{-1}(y_{i_1})) \\ &= \alpha\circ h_{i_1}^{-1}(x)+\sum_{i\neq i_1}(G_0\circ h_{i'}^{-1}(x)-G_0\circ h_{i'}^{-1}(y_{i_1})) \\ &= \beta_{i_1}(x) . \end{aligned}$$

It is clear that $\Delta_{\hat{g}}\beta_{i_1}=0$, $\beta_{i_1}(y_{i_1})=0$.

STEP 3. For any $\varepsilon>0$, let

$$u_\varepsilon(\rho) := \left(\frac{\varepsilon}{\varepsilon^2+\rho^2}\right)^{(n-2)/2} \qquad (\forall\rho\in\boldsymbol{R}).$$

Take a smooth function ψ satisfying

$$\psi(t) = \begin{cases} 1 & \text{for } t\leqq 1 \\ 0 & \text{for } t\geqq 2 \\ 0\leqq\psi\leqq 1 & \text{for } 1\leqq t\leqq 2 . \end{cases}$$

and let

$$\psi_0(\rho) := \psi\left(\frac{\rho}{\rho_0}\right) \qquad (\forall\rho\in\boldsymbol{R})$$

where ρ_0 is as in Step 1. Then ψ_0 is a smooth function such that

$$\psi_0(\rho)=\begin{cases}1 & \text{for } \rho\leqq\rho_0\\ 0 & \text{for } \rho\geqq 2\rho_0\\ 0\leqq\psi_0\leqq 1 & \text{for } \rho_0\leqq\rho\leqq 2\rho_0\\ \left|\dfrac{d\psi_0}{d\rho}(\rho)\right|\leqq C\rho_0^{-1} & \text{for some } C>0.\end{cases}$$

Now we define ϕ by

$$\phi(x):=\begin{cases}u_\varepsilon(|x-y_i|) & \text{on } B_{\rho_0}(y_i)\\ \varepsilon_0(G(x)-\psi_0(|x-y_i|)\beta_i(x)) & \text{on } B_{2\rho_0}(y_i)\setminus B_{\rho_0}(y_i)\\ \varepsilon_0 G(x) & \text{elsewhere}\end{cases}$$

where ε_0 is chosen so as to satisfy

$$u_\varepsilon(|x-y_i|)=\varepsilon_0(G(x)-\psi_0(|x-y_i|)\beta_i(x)) \qquad \text{on } \partial B_{\rho_0}(y_i)$$

that is

$$\left(\frac{\varepsilon}{\varepsilon^2+\rho^2}\right)^{(n-2)/2}=\varepsilon_0(\rho_0^{2-n}+A).$$

Clearly ϕ is a positive Γ-invariant C^0-function which is smooth in $M\setminus\bigcup_{i=1}^{j'}\partial B_{\rho_0}(y_i)$.

By direct computation and the definition of Green function, we see

$$a\Delta_{\hat{g}}\phi+S_{\hat{g}}\phi=\begin{cases}4n(n-1)\phi^{p-1} & \text{on } \bigcup_{i=1}^{j'}B_{\rho_0}(y_i)\\ 0 & \text{on } M\setminus\bigcup_{i=1}^{j'}B_{2\rho_0}(y_i).\end{cases}$$

Now let $u:=\lambda\phi$. Then u is also Γ-invariant and $u\in H_{\Gamma,f}(M)$ for any sufficiently small $\varepsilon>0$, since we assume $f(x_j)>0$, that is, $f(y_i)>0$ for any i. Moreover u satisfies

$$a\Delta_g u+S_g u=\begin{cases}4n(n-1)u^{p-1} & \text{on } \bigcup_{i=1}^{j'}B_{\rho_0}(y_i)\\ 0 & \text{on } M\setminus\bigcup_{i=1}^{j'}B_{2\rho_0}(y_i).\end{cases}$$

Actually this is derived easily from the following formula:

$$(a\Delta_g+S_g)(\lambda v)=\lambda^{p-1}(a\Delta_{\hat{g}}+S_{\hat{g}})v.$$

STEP 4. First by the above formula, we see

$$\frac{J(u)}{\|u\|_{p,f}^2}=\frac{a\|\nabla_g u\|_2^2+\int_M S_g u^2 dV_g}{\left(\int_M f|u|^p dV_g\right)^{2/p}}=\frac{a\|\nabla_{\hat{g}}\phi\|_2^2+\int_M S_{\hat{g}}\phi^2 dV_{\hat{g}}}{\left(\int_M f|\phi|^p dV_{\hat{g}}\right)^{2/p}}.$$

Now applying the same argument as in Escobar and Schoen [4] and Chen [3], we can obtain the estimate (A′). Actually

$$
\begin{aligned}
&a\|\nabla_{\hat g}u_\varepsilon\|_{2,B_{\rho_0}(y_i)}^2+\int_{B_{\rho_0}(y_i)}S_{\hat g}u_\varepsilon^2dV_{\hat g}\\
&=\int_{B_{\rho_0}(y_i)}(a\Delta_{\hat g}u_\varepsilon+S_{\hat g}u_\varepsilon)u_\varepsilon dV_{\hat g}+a\int_{\partial B_{\rho_0}(y_i)}u_\varepsilon\frac{\partial u_\varepsilon}{\partial n}ds\\
&=4n(n-1)\int_{B_{\rho_0}(x_j)}u_\varepsilon^p dV_{\hat g}+a\int_{\partial B_{\rho_0}(x_j)}u_\varepsilon\frac{\partial u_\varepsilon}{\partial n}ds\\
&\leqq \mu_n\Big(\int_{B_{\rho_0}(x_j)}u_\varepsilon^p dV_{\hat g}\Big)^{2/p}+a\int_{\partial B_{\rho_0}(x_j)}u_\varepsilon\frac{\partial u_\varepsilon}{\partial n}ds
\end{aligned}
$$

because

$$
\int_{B_{\rho_0}(x_j)}u_\varepsilon^p dV_{\hat g}\leqq\int_{R^n}u_\varepsilon^p dx=4^{-n/2}\omega_n=\Big(\frac{\mu_n}{4n(n-1)}\Big)^{1-2/p}.
$$

Similarly

$$
\begin{aligned}
&a\|\nabla_{\hat g}(\varepsilon_0^2G)\|_{2,M\setminus\bigcup_i B_{\rho_0}(y_i)}+\int_{M\setminus\bigcup_i B_{\rho_0}(y_i)}S_{\hat g}(\varepsilon_0 G)^2dV_{\hat g}\\
&=\int_{M\setminus\bigcup_i B_{\rho_0}(y_i)}(a\Delta_{\hat g}(\varepsilon_0G)+S_{\hat g}(\varepsilon_0G))\varepsilon_0G dV_{\hat g}-a\int_{\cup\partial B_{\rho_0}(y_i)}\varepsilon_0G\frac{\partial}{\partial n}(\varepsilon_0G)ds\\
&=-j'a\varepsilon_0^2\int_{\partial B_{\rho_0}(x_j)}G\frac{\partial G}{\partial n}ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&a\|\nabla_{\hat g}\phi\|_2^2+\int_M S_{\hat g}\phi^2dV_{\hat g}\\
&\leqq j'\mu\Big(\int_{B_{\rho_0}(x_j)}u_\varepsilon^p dV_{\hat g}\Big)^{2/p}+j'a\int_{\partial B_{\rho_0}(x_j)}\Big\{u_\varepsilon\frac{\partial u_\varepsilon}{\partial n}-\varepsilon_0^2G\frac{\partial G}{\partial n}\Big\}ds\\
&\quad+j'\varepsilon_0^2\int_{B_{2\rho_0}(x_j)\setminus B_{\rho_0}(x_j)}[a\{|\nabla_{\hat g}(G-\psi_0\beta_1)|^2-|\nabla_{\hat g}G|^2\}\\
&\qquad\qquad+S_{\hat g}\{(G-\psi_0\beta_1)^2-G^2\}]dV_{\hat g}\\
&=:\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
$$

Now as Schoen [**10**] estimated, since

$$
\left.\begin{aligned}
&|(G-\psi_0\beta_1)^2-G^2|\leqq C\rho_0^{3-n}\\
&||\nabla_{\hat g}(G-\psi_0\beta_1)|^2-|\nabla_{\hat g}G|^2|\leqq C\rho_0^{1-n}
\end{aligned}\right.\quad \text{on } B_{2\rho_0}(x_j)\setminus B_{\rho_0}(x_j),
$$

we see

$$
|\mathrm{III}|\leqq C\varepsilon_0^2\rho_0.
$$

for some $C>0$. On the other hand, if we take $\varepsilon<\rho_0$, then

$$u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} - \varepsilon_0^2 G \frac{\partial G}{\partial n} \leqq -C_0 A \varepsilon_0^2 \rho_0^{1-n} + C\varepsilon_0^p \rho_0^{1-2n} \qquad \text{on } \partial B_{\rho_0}(x_j)$$

since $\varepsilon \leqq C\varepsilon_0^{2/(n-2)}$. Hence we also have

$$\mathrm{II} \leqq -C_0 A\varepsilon_0^2 + C\varepsilon_0^p \rho_0^{-n}$$

for some $C>0$ and $C_0>0$. Now from our hypothesis, we can assume

$$f(x_j) \leqq f(x) + C\rho^{n-1} \qquad \text{on } B_{\rho_0}(x_j)$$

so that

$$f(x_j)\int_{B_{\rho_0}(x_j)} u_\varepsilon^p dV_{\hat g} \leqq \int_{B_{\rho_0}(x_j)} f u_\varepsilon^p dV_{\hat g} + C\int_{B_{\rho_0}(x_j)} \rho^{n-1} u_\varepsilon^p dV_{\hat g}\,.$$

Since

$$\int_{B_{\rho_0}(x_j)} \rho^{n-1} u_\varepsilon^p dV_{\hat g} = \int_{B_{\rho_0}(x_j)} \rho^{n-1}\Big(\frac{\varepsilon}{\varepsilon^2+\rho^2}\Big)^n dV_{\hat g}$$
$$= \omega_{n-1}\int_0^{\rho_0} \rho^{n-1}\Big(\frac{\varepsilon}{\varepsilon^2+\rho^2}\Big)^n \rho^{n-1} d\rho \leqq C\varepsilon^{n-1}\,,$$

if we take sufficiently small ρ_0, then

$$\int_{M\setminus \cup_i B_{\rho_0}(y_i)} f\phi^p dV_{\hat g} \leqq \max_M |f|\,\{\varepsilon_0(\rho_0^{2-n}+A)\}^p(\mathrm{vol}\,(M, \hat g)) \leqq C\varepsilon_0^p \rho_0^{-2n}.$$

This implies

$$I \leqq j'\mu_n\Big\{f(x_j)^{-1}\Big(\int_{B_{\rho_0}(x_j)} f u_\varepsilon^p dV_{\hat g} + C\int_{B_{\rho_0}(x_j)} \rho^{n-1} u_\varepsilon^p dV_{\hat g}\Big)\Big\}^{2/p}$$
$$\leqq j' f(x_j)^{-2/p}\mu_n\Big(j'^{-1}\int_{\cup_i B_{\rho_0}(y_i)} f\phi^p dV_{\hat g}\Big)^{2/p} + C\varepsilon^{n-1}$$
$$= j'^{1-2/p} f(x_j)^{-2/p}\mu_n\Big(\int_M f\phi^p dV_{\hat g} - \int_{M\setminus\cup_i B_{\rho_0}(y_i)} f\phi^p dV_{\hat g}\Big)^{2/p} + C\varepsilon^{n-1}\varepsilon$$
$$\leqq j'^{2/n} f(x_j)^{-2/n}\mu_n\Big(\int_M f\phi^p dV_{\hat g}\Big)^{2/p} + C\varepsilon_0^p\rho_0^{-2n} + C\varepsilon_0^2\rho_0$$

for some $C>0$. It follows that

$$a\|\nabla_{\hat g}\phi\|_2^2 + \int_M S_{\hat g}\phi^2 dV_{\hat g} \leqq \mathrm{I} + \mathrm{II} + |\mathrm{III}|$$
$$\leqq j'^{2/n} f(x_j)^{-2/p}\mu_n\Big(\int_M f\phi^p dV_{\hat g}\Big)^{2/p}$$
$$+ C\varepsilon_0^p\rho_0^{-2n} + C\varepsilon_0^2\rho_0 - C_0 A\varepsilon_0^2 + C\varepsilon_0^p\rho_0^{-n} + C\varepsilon_0^2\rho_0$$
$$\leqq j'^{2/n} f(x_j)^{-2/p}\mu_n\Big(\int_M f\phi^p dV_{\hat g}\Big)^{2/p} + C(\varepsilon_0^p\rho_0^{-2n} + \varepsilon_0^2\rho_0) - C_0 A\varepsilon_0^2\,.$$

If (M, g) is not conformally equivalent to (S^n, g_0), then $A_0>0$ because of the positive mass theorem (cf. [13]). On the other hand if $j\neq 1$, then obviously

$$\sum_{i\neq 1} G_0\circ h_i^{-1}(x_j) > 0 .$$

Hence

$$A = A_0+\sum_{i\neq 1} G_0\circ h_i^{-1}(x_j) > 0 .$$

Thus if we take sufficiently small $\rho_0>0$ and smaller $\varepsilon_0>0$, we obtain the following estimate

$$a\|\nabla_{\hat{g}}\phi\|_2^2+\int_M S_{\hat{g}}\phi^2 dV_{\hat{g}} < j'^{2/n} f(x_j)^{-2/p}\mu_n\Big(\int_M f\phi^p dV_{\hat{g}}\Big)^{2/p}$$

which shows

$$\begin{aligned}\frac{J(u)}{\|u\|_{p,f}^2} &= \frac{a\|\nabla_{\hat{g}}\phi\|_2^2+\int_M S_{\hat{g}}\phi^2 dV_{\hat{g}}}{\Big(\int_M f|\phi|^p dV_{\hat{g}}\Big)^{2/p}} < j'^{2/n} f(x_j)^{-2/p}\mu_n \\ &= j'^{2/n}(\max_{D_j} f)^{-2/p}\mu_n \leqq j^{2/n}(\max_{D_j} f)^{-2/p}\mu_n . \qquad \text{Q.E.D.}\end{aligned}$$

Now we obtain the following

THEOREM 2.2. *Let (M, g) be a compact locally conformally flat manifold with $\mu(M)>0$, and let $f\in C^{0,\alpha}(M)$ $(0<\alpha<1)$ be Γ-invariant and* $\max_M f>0$. *Then there exists a Γ-invariant C^2-solution of equation $(*)$, provided that for each $j\in N_{\Gamma,f}$, f satisfies one of the following:*

(C-1) *conditions* (F) *and* (S) *hold unless (M, g) is conformally equivalent to (S^n, g_0) and $j=1$, and condition* (F′) *holds in the above exceptional case where*

(F′) *there exists $x_1\in D_1=F_1$ such that*

$$f(x_1) = \max_{D_1} f>0 .$$

$$f(x) \geqq f(x_1)+C(d_g(x, x_1))^2 \text{ near } x_1$$

for some $C>0$,

(C-2) *condition (I) holds,*

(C-3) *there exists $j_0>j$ such that*

$$\max_{D_{j_0}} f \geqq \Big(\frac{j_0}{j}\Big)^{p/n} \max_{D_j} f$$

and (C-1) *or* (C-2) *holds for j_0.*

PROOF. In the exceptional case in (C-1), since $A=0$, the condition

$$f(x) \geqq f(x_1)-C(d_g(x, x_1))^{n-1}$$

is not available but

$$f(x) \geqq f(x_1)+C(d_g(x, x_1))^2$$

as W. Chen showed by estimating solutions of the Yamabe problem (cf. [**3**]). From this result, Lemma 2.1 and the results in Section 1, it follows that the estimate (A′) holds under the condition (C-1) or (C-2).

On the other hand, under the condition (C-3), since f satisfies (C-1) or (C-2) for $j_0>j$, we get

$$\frac{J(u)}{\|u\|_{p,f}^2} < j_0^{2/n}(\max_{D_{j_0}} f)^{-2/p}\mu_n$$

$$\leqq j_0^{2/n}\Big\{\Big(\frac{j_0}{j}\Big)^{p/n}\max_{D_j} f\Big\}^{-2/p}\mu_n = j^{2/n}(\max_{D_j} f)^{-2/p}\mu_n. \qquad \text{Q.E.D.}$$

3. The Case $n=3$.

In this section, we consider the case $n=3$, and show a stronger result than that of the previous section.

LEMMA 3.1. *Suppose $n=3$ and $j\in \boldsymbol{N}_{\Gamma,f}$. If* (F) *holds, then* (A) *is true, except the case $(M, g)=(S^3, g_0)$ and $j=1$ simultaneously.*

PROOF. Since $\mu(M)>0$, there exists a unique Green function for $a\Delta_g+S_g$ at x_j, and we denote it by G_0. For its uniqueness, it must be Γ_0-invariant. Denote the distance $d_g(x, y)$ between x and y with respect to the metric g by $|x-y|$ for simplicity. If we replace G_0 by some multiple of G_0, then for any $\rho_0>0$.

$$G_0(x) = |x-x_j|^{-1}+A_0+\alpha(x) \qquad (\forall x\in B_{2\rho_0}(x_j))$$

with some constant $A_0\geqq 0$ and some α such that $\Delta_{\hat{g}}\alpha=0$, $\alpha(x_j)=0$ (cf. [**10**]). In addition, if (M, g) is not conformally equivalent to (S^3, g_0), then $A_0>0$ because of the positive mass theorem (cf. [**12**]). Construct G and u as before and observe that

$$a\|\nabla_g u_\varepsilon\|_{2,B_{\rho_0}(y_i)}^2+\int_{B_{\rho_0}(y_i)} S_g u_\varepsilon^2 dV_g \leqq a\|\nabla_0 u_\varepsilon\|_{2,B_{\rho_0}(y_i)}^2+C\rho_0\varepsilon_0^2,$$

$$\int_{B_{\rho_0}(x_j)} u_\varepsilon^p dx \leqq \int_{B_{\rho_0}(x_j)} u_\varepsilon^p dV_g+C\varepsilon_0^4$$

for some $C>0$, where ∇_0 is the gradient with respect to the flat metric. Thus we obtain the same estimate as (A′). Q.E.D.

THEOREM 3.2. *Let (M, g) be a compact 3-dimensional manifold with $\mu(M)>0$, and let $f\in C^{2,\alpha}(M)$ $(0<\alpha<1)$ be Γ-invariant and $\max_M f>0$. Then there exists a Γ-invariant C^4-solution of equation* (*) *if either of the following two* *conditions holds:*

(1) *(M, g) is not conformally equivalent to (S^3, g_0).*

(2) *(M, g) is conformally equivalent to (S^3, g_0) and additionally in the case $1 \in \boldsymbol{N}_{\Gamma, f}$, f satisfies one of the conditions* (C-1), (C-2) *and* (C-3) *for $j=1$.*

PROOF. From Lemma 3.1, it is enough to show that (C-1) is satisfied whenever $j \in \boldsymbol{N}_{\Gamma, f}$, unless (M, g) is conformally equivalent to (S^3, g_0) and $j=1$ simultaneously.

If $F_j \neq \emptyset$, then obviously there exists $x_j \in D_j$ such that $f(x_j) = \max_{D_j} f$, and there exists $j' \leqq j$ such that $x_j \in F_{j'}$. Set $\Gamma_0 := \{h \in \Gamma \mid h x_j = x_j\}$ as before. Then it is compact, $|\Gamma : \Gamma_0| = j'$ and there exist $h_1 (=e)$, $h_2, \cdots, h_{j'} \in \Gamma$ such that

$$\Gamma = \sum_{i=1}^{j'} h_i \Gamma_0 .$$

Set

$$M_{\Gamma_0} := \{x \in M \mid hx = x \text{ for any } h \in \Gamma_0\}.$$

Then for any $x \in M_{\Gamma_0}$,

$$\Gamma x = \{h_1 x (=x), h_2 x, \cdots, h_{j'} x\}.$$

Hence $\# \Gamma x \leqq j'$, that is, $x \in D_{j'}$. This implies $M_{\Gamma_0} \subset D_{j'} \subset D_j$ and $f(x_j) = \max_{D_j} f = \max_{M_{\Gamma_0}} f$, hence $\nabla_{g \mid M_{\Gamma_0}} f(x_j) = 0$. Then because of the principle of symmetric criticality (cf. [**9**]), $\nabla_g f(x_j) = 0$, from which it turns out that

$$f(x) \leqq f(x_j) - C(d_g(x, x_j))^2 \text{ near } x_j$$

for some $C > 0$. Q.E.D.

Under the same hypothesis as in Theorem 3.1 (1), the existence of a solution has already been shown by [**4**], but we assert the existence of a Γ-invariant solution. Solutions do not always have Γ-invariance (see Section 5). On the other hand, about (S^3, g_0), we see the assertion in [**3**] is valid with any infinite Γ too.

As above, (F) is not needed at all in the case $n=3$. In the case $n=4$ or 5, though we also see the following similar expansion

$$G_0(x) = |x - x_j|^{2-n} + A_0 + \alpha(x),$$

we cannot remove (F) by the same method. But it is not necessary as we saw in (I). Moreover when we regard $\mu_{\Gamma, f}(M)$ as a functional on

$$\{f \in C^{0,\alpha}(M) \mid \Gamma\text{-invariant}, \max_M f > 0\}.$$

it is trivially upper semicontinuous with respect to C^0-norm. Hence for any locally conformally flat manifold M and any Γ which satisfies (S) for any $x \in M$, we see the set

$$\{f \in C^{0,\alpha}(M) \mid \Gamma\text{-invariant, } (*) \text{ has a } \Gamma\text{-invariant } C^2\text{-solution}\}$$

is at least open dense in

$$\{f \in C^{0,\alpha}(M) \mid \Gamma\text{-invariant, } \max_M f > 0\}.$$

4. Remarks on the condition (S).

The condition (S) was required to construct a Γ-invariant metric which is flat near x_j. In this section, we will see two examples about it.

PROPOSITION 4.1. *When* $(M, g)=(S^n, g_0)$, (S) *is automatically satisfied.*

PROOF. We may regard (M, g) as an imbedded submanifold of $\boldsymbol{R}^{n+1}$:

$$M = S^n \subset \boldsymbol{R}^{n+1},$$

$$g_0 = \sum_{i=1}^{n+1} (dx^i)^2 |_{S^n}.$$

Moreover we can take a coordinate system so that $x_j=(0, \cdots, 0, 1)$ without loss of generality, and then

$$\Gamma_0 \subset \left\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in O(n+1) \mid A \in O(n)\right\}.$$

Define

$$\sigma : S^n \setminus \{(0, \cdots, 0, -1)\} \longrightarrow \boldsymbol{R}^n$$

by

$$\sigma(x^1, \cdots, x^n, x^{n+1}) := \left(\frac{x^1}{1+x^{n+1}}, \cdots, \frac{x^n}{1+x^{n+1}}\right) = (y^1, \cdots, y^n).$$

Then

$$\sigma^*\left(\sum_{i=1}^{n} (dy^i)^2\right) = \left(\frac{1}{1+x^{n+1}}\right)^2 g_0.$$

so σ is a conformal diffeomorphism. Hence if we set

$$\lambda_0 := \left(\frac{1}{1+x^{n+1}}\right)^{2/(p-2)},$$

it is obviously Γ_0-invariant. Q.E.D.

But (S) is not always satisfied as follows.

PROPOSITION 4.2. *When* $(M, g)=(S^1 \times S^{n-1}, d\theta^2+g_0)$, (S) *is equivalent to the following condition:*

(S′) $\Gamma_0 \subset SO(2)\times O(n)$.

PROOF. We may regard (M, g) as an imbedded submanifold of $\boldsymbol{R}^2\times\boldsymbol{R}^n$:

$$M = S^1\times S^{n-1}\subset \boldsymbol{R}^2\times\boldsymbol{R}^n\,,$$

$$d\theta^2+g_0 = \sum_{i=1}^{2}(dx'^i)^2|_{S^2}+\sum_{i=1}^{n}(dx^i)^2|_{S^{n-1}}\,.$$

Moreover we can take a coordinate system so that $x_j=(1, 0; 0, \cdots, 0, 1)$ without loss of generality, and then

$$\Gamma_0\subset\left\{\left(\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix}\right)\in O(2)\times O(n)\mid A\in O(n-1)\right\}.$$

For any open subsets $U\subset S^1$ and $V\subset S^{n-1}$, choose an argument of the polar coordinate of $U\subset\boldsymbol{R}^2$ and define

$$\tau: U\times V \longrightarrow \tau(U\times V)\subset \boldsymbol{R}^n$$

by

$$\tau(\theta : x^1, \cdots, x^n) := (e^{\theta}x^1, \cdots, e^{\theta}x^n) = (y^1, \cdots, y^n)\,.$$

Then

$$\tau^*\Big(\sum_{i=1}^{n}(dy^i)^2\Big) = e^{2\theta}(d\theta^2+g_0),$$

so τ is a conformal diffeomorphism. Hence if we set

$$\lambda_0 := e^{2\theta/(p-2)}\,,$$

it is obviously invariant by the action of

$$\left\{\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix}\right)\in O(2)\times O(n)\mid A\in O(n-1)\right\}.$$

On the other hand, by Liouville's theorem (cf. [**14**]), we see any metric on $\tau(U\times V)$ which is pointwise conformal to

$$\sum_{i=1}^{n}(dy^i)^2$$

can be written as

$$\Big(\sum_{i=1}^{n}(y^i-y^i{}_0)^2\Big)^{-2}\sum_{i=1}^{n}(dy^i)^2$$

for some $y_0\in\boldsymbol{R}^n\setminus\tau(U\times V)$ up to a constant. Hence by setting $y_0=:\tau(\theta_0; x_0)$, we see any metric on $U\times V$ which is pointwise conformal to $d\theta^2+g_0$ can be written as

$$\Big(e^{2\theta}+e^{2\theta_0}-2e^{\theta+\theta_0}\sum_{i=1}^{n}x^ix^i{}_0\Big)^{-2}e^{2\theta}(d\theta^2+g_0)$$

for some $(\theta_0; x_0)\in(\boldsymbol{R}\times S^{n-1})\setminus(U\times V)$, or as

$e^{-2\theta}(d\theta^2+g_0)$ (this case can be regarded as for $\theta_0=-\infty$)

up to a constant. Now the assertion is clear. Q.E.D.

5. The symmetry and the nonuniqueness of solutions.

In the previous sections, we considered the existence of a Γ-invariant solution of equation $(*)$, but we do not know whether solutions given under different sufficient conditions coincide or not in general. At least in the case f is a constant, if $\mu(M)<0$, then $(*)$ has a unique solution, if $\mu(M)=0$, then a unique solution up to a constant, but if $\mu(M)>0$, there are examples that $(*)$ has nonunique solutions. Hence we may expect nonunique solutions also in the case f is not a constant. In this section, we consider some relation between the symmetry and the nonuniqueness of solutions.

For example, suppose (M, g) is Γ-invariant and there exist $\Gamma'\subsetneqq\Gamma$ and $l\in N_{\Gamma,f}\cap N_{\Gamma',f}$ such that $f=S_g$ does not satisfy (C-1) but (C-2) for $j=l$, and satisfies one of (C-1), (C-2) and (C-3) for $j\in(N_{\Gamma,f}\cup N_{\Gamma',f})\setminus\{l\}$. Then trivially $u\equiv 1$ is a solution of $(*)$ for $f=S_g$:

$$a\Delta_g u+S_g u = S_g u^{p-1}.$$

On the other hand,

$$\int_M S_g dV_g \leqq l^{-p/n}\left(\frac{\int_M S_g dV_g}{\mu_n}\right)^{p/2}\max_{D_l} S_g,$$

namely

$$\int_M S_g dV_g \geqq l(\max_{D_l} S_g)^{-n/p}\mu_n{}^{n/2}.$$

Hence

$$\frac{J(1)}{\|1\|_{p,S_g}{}^2}=\frac{a\|\nabla_g 1\|_2{}^2+\int_M S_g\cdot 1^2 dV_g}{\left(\int_M S_g|1|^p dV_g\right)^{2/p}}=\left(\int_M S_g dV_g\right)^{1-2/p}$$
$$\geqq l^{2/n}(\max_{D_l} S_g)^{-2/p}\mu_n>\mu_{\Gamma',f}(M).$$

Now clearly $u\equiv 1$ does not realize $\mu_{\Gamma',f}(M)$, so there exists some other solution which realizes $\mu_{\Gamma',f}(M)$. Namely in this case, $(*)$ has at least two solutions.

EXAMPLE 5.1. Suppose $(M, g)=(S^1\times S^2, (\sin m\theta+C)^4(d\theta^2+g_0))$ $(m\in \boldsymbol{N})$. Then by direct computation, we see

$$S_g=\frac{2\{(1+4m^2)\sin m\theta+C\}}{(\sin m\theta+C)^5},$$

$$\int_M S_g dV_g=8\pi^2(1+4m^2+2C^2),$$

and for $C\geqq 4+5/(m^2-1)$ (<6),

$$\max_M S_g=\frac{2(1+4m^2+C)}{(1+C)^5}.$$

Hence for $C\geqq 6$, sufficiently large m and $l|m$ such that m/l is also sufficient large,

$$l(\max_M S_g)^{-1/2}\mu_n{}^{3/2}\leqq\int_M S_g dV_g,$$

namely for such l, (C-2) is not satisfied with $\Gamma=\boldsymbol{Z}_l$. On the other hand, since $n=3$, (C-1) is automatically satisfied for any j. Thus equation

$$8\Delta_g u+S_g u=S_g u^5$$

has a trivial solution $u\equiv 1$ and at least another solution which has a period $2\pi/l$.

If $\mu_{\Gamma',f}(M)<\mu_{\Gamma,f}(M)$ for $\Gamma'\subsetneqq\Gamma$, any solution which realizes $\mu_{\Gamma',f}(M)$ is different from any solution which realizes $\mu_{\Gamma,f}(M)$, but it is difficult to show $\mu_{\Gamma',f}(M)<\mu_{\Gamma,f}(M)$ in general. In the case f is a constant, Kobayashi [**6**] and Schoen [**11**] independently obtained the following result.

THEOREM B. *Let $(M, g)=(S^1(r)\times S^{n-1}, r^2d\theta^2+g_0)$. For any positive constant f and any $k\in N$.*

(1) *if $0<r\leqq 1/\sqrt{n-2}$, then $(*)$ has a unique solution $u\equiv$constant,*

(2) *if $k/\sqrt{n-2}<r\leqq(k+1)/\sqrt{n-2}$, then $(*)$ has a trivial solution $u\equiv$constant and an S^1-parameter family of solutions which has a primitive period $2\pi r/i$ for any $i\leqq k$. In particular, solutions which has a primitive period $2\pi r$ (namely with no period) is minimizing, and*

$$\lim_{r\to\infty}\mu(S^1(r)\times S^{n-1})=\mu_n. \tag{3}$$

By (3), we can show some generalization of (1) and (2) to the case f is not a constant as follows.

THEOREM 5.2. *Let $(M, g)=(M_r, g_r)=(S^1(r)\times S^{n-1}, r^2d\theta^2+g_0)$, and suppose $\boldsymbol{Z}_j$ acts on $S^1(r)$ for any $j\in\boldsymbol{N}$. For any $m\in\boldsymbol{N}$, there exists $r_m(n)>0$ such that for any $r>r_m(n)$ and any $f\in C^{0,\alpha}(M)$ $(0<\alpha<1)$ which is $\boldsymbol{Z}_m$-invariant and* $\max_M f>0$, *there exists a $\boldsymbol{Z}_l$-invariant C^2-solution of $(*)$ for any $l|m$ which is different from each other if*

(1) $n=3$

or

(2) $n\geqq 4$ *and* f *satisfies each of the following conditions:*

(C-1) *there exists* $x_0\in M$ *such that*

$$f(x_0)=\max_M f\,.$$

$$f(x)\geqq f(x_0)-C(d_g(x,\,x_0))^{n-1}\ \text{near}\ x_0$$

for some $C>0$.

$$\text{(C-2)}\qquad \int_M f\,dV_g>\left(2\pi r\frac{(n-2)\omega_{n-1}}{n\omega_n{}^{2/n}}\right)^{p/2}\max_M f\,.$$

PROOF. For any $j\in\boldsymbol{N}$, define a covering map

$$\pi_j: M_r \longrightarrow M_{r/j}$$

by $\pi_j(\theta,\,x):=(j\theta,\,x)$. If f is $\boldsymbol{Z}_j$-invariant, then there exists $f'\in C^{0,\alpha}(M_{r/j})$ such that $f=f'\circ\pi_j$ and for any $u\in H_{\boldsymbol{Z}_j,f}(M_r)$, there exists $u'\in H_{id,f}(M_{r/j})$ such that $u=u'\circ\pi_j$. Hence

$$\begin{aligned}\mu_{\boldsymbol{Z}_j,f}(M_r)&=\inf\left\{\frac{J_{M_r}(u)}{\|u\|_{p,f,M_r}{}^2}\,\middle|\,u\in H_{\boldsymbol{Z}_j,f}(M_r)\right\}\\&=\inf\left\{\frac{jJ_{M_{r/j}}(u')}{j^{2/p}\|u'\|_{p,f',M_{r/j}}{}^2}\,\middle|\,u'\in H_{id,f'}(M_{r/j})\right\}\\&=j^{1-2/p}\inf\left\{\frac{J_{M_{r/j}}(u')}{\|u'\|_{p,f',M_{r/j}}{}^2}\,\middle|\,u'\in H_{id,f'}(M_{r/j})\right\}\\&=j^{2/n}\mu_{id,f'}(M_{r/j})\geqq j^{2/n}(\max_M f)^{-2/p}\mu(M_{r/j})\,.\end{aligned}$$

Now denote the set of divisors of m by $\{l_1,\cdots,l_k\}$ so that $l_1=1<l_2<\cdots<l_{k-1}<l_k=m$. Then by Theorem B. (3), it is clear that there exists $r_m(n)>0$ such that any $r>r_m(n)$ satisfies

$$\mu(M_{r/l_{i+1}})\geqq(l_i/l_{i+1})^{2/n}\mu_n$$

for any $i\leqq k-1$. On the other hand, from the hypothesis, we can apply Theorem 2.2 or 3.2 and see there exists a $\boldsymbol{Z}_{l_i}$-invariant C^2-solution of $(*)$. Denote it by u_{l_i}. Then it satisfies

$$\begin{aligned}\frac{J_{M_r}(u_{l_i})}{\|u_{l_i}\|_{p,f,M_r}{}^2}&=\mu_{\boldsymbol{Z}_{l_i},f}(M_r)<l_i{}^{2/n}(\max_M f)^{-2/p}\mu_n\\&\leqq l_{i+1}{}^{2/n}(\max_M f)^{-2/p}\mu(M_{r/l_{i+1}})\leqq\mu_{\boldsymbol{Z}_{l_{i+1}},f}(M_r)=\frac{J_{M_r}(u_{l_{i+1}})}{\|u_{l_{i+1}}\|_{p,f,M_r}{}^2}\end{aligned}$$

for any $i\leqq k-1$, namely solutions $u_{l_1},\cdots,u_{l_k}$ are different from each other.

Q.E.D.

References

[1] T. Aubin, The scalar curvature: Differential Geometry and Relativity, ed. M. Cahen and M. Flato, Reidel, 1976, pp. 5-18.

[2] J.P. Bourguignon and J.P. Ezin, Scalar curvature functions in a conformal class of metric and conformal transformations, Trans. Am. Math. Soc., **301** (1987), 723-736.

[3] W. Chen, Scalar curvatures on S^n, Math. Ann., **283** (1989), 353-365.

[4] J.F. Escobar and R.M. Schoen, Conformal metrics with prescribed scalar curvature, Invent. Math., **86** (1986), 243-254.

[5] J.L. Kazdan and F.W. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Differential Geom., **10** (1975), 113-134.

[6] O. Kobayashi, On large scalar curvature, Research Report, Keio Univ., 1985 (unpublished).

[7] P.L. Lions, The concentration-compactness principle in the calculus of variations, the limit case, part. 2, Rev. Mat. Ibero., **1** (1985), 45-121.

[8] J.M. Lee and T.H. Parker, The Yamabe problem, Bull. Amer. Math. Soc, **17** (1987), 37-91.

[9] R.S. Palais, The principle of symmetric criticality, Comm. Math. Phys., **69** (1979), 19-30.

[10] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom., **20** (1984), 479-495.

[11] R.M. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Topics in Calculus of Variations, ed. M. Giaquinta, Lecture Notes in Math., **1365**, Springer, 1989, pp. 120-154.

[12] R. Schoen and S.T. Yau, On the proof of the positive mass conjecture in general relativity, Comm. Math. Phys., **65** (1979), 45-76.

[13] R. Schoen and S.T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math., **92** (1988), 47-71.

[14] M. Spivak, A comprehensive introduction to differential geometry, **4**, Publish or Perish, Inc., Berkeley, 1979.

[15] N.S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa, **22** (1968), 265-274.

[16] M. Vaugon, Transformation conforme de la courbure scalaire, Analyse non lineaire, Ann. Inst. Henri Poincare, **3** (1986), 55-65.

[17] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J., **12** (1960), 21-37.

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