J. Math. Soc. Japan
Vol. 41, No. 4, 1989

On toroidal groups

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(Received Oct. 7, 1987)
(Revised Aug. 30, 1988)

§0. Introduction.

A toroidal group is a quotient $X=\boldsymbol{C}^n/\Gamma$ of $\boldsymbol{C}^n$ by a lattice Γ such that X has no non-constant holomorphic function ([**11**]). Morimoto considered a connected complex Lie group without non-constant holomorphic functions and called it an (H, C)-group ([**12**]). Since every (H, C)-group is commutative, the (H, C)-groups are exactly the toroidal groups.

It is a well known result that for a complex torus $\boldsymbol{T}$ the following are equivalent:

(1) $\boldsymbol{T}$ is an abelian variety.

(2) $\boldsymbol{T}$ has a positive line bundle.

(3) $\boldsymbol{T}$ is projective algebraic.

In the previous paper [**1**] we obtained a similar result for a toroidal group X under the condition $\dim H^1(X, \mathcal{O}_X)<\infty$.

One of the purpose of this paper is to drop the above condition (see Theorem 4.6). This contains answers to problems of the structure and of the global embedding of weakly 1-complete manifolds in the case of toroidal groups (see [**1**]). Another is to prove the meromorphic reduction theorem for toroidal groups (Theorem 5.1). As a by-product we obtain that for a topologically trivial holomorphic line bundle L over a toroidal group $X=\boldsymbol{C}^n/\Gamma$, $H^0(X, \mathcal{O}(L))\neq\{0\}$ if and only if L is analytically trivial (Corollary 3.3). By different methods Huckleberry and Margulis [7] proved it.

§1. Preliminaries.

A discrete subgroup Γ of $\boldsymbol{C}^n$ is called a lattice in $\boldsymbol{C}^n$. Let $p_1=(p_{11}, p_{21}, \cdots, p_{n1}), \cdots, p_r=(p_{1r}, p_{2r}, \cdots, p_{nr})\in\boldsymbol{C}^n$ be generators of Γ, where $r=\operatorname{rank}\Gamma$. An (n, r)-matrix

$$P=({}^tp_1\ {}^tp_2\ \cdots\ {}^tp_r)$$

is called a period matrix of Γ, or also of $X=\boldsymbol{C}^n/\Gamma$. Two period matrices P and P' are said to be equivalent if there exist $A\in GL(n, \boldsymbol{C})$ and $M\in GL(r, \boldsymbol{Z})$

such that

$$P' = APM.$$

Let Γ and Γ' be lattices in $\boldsymbol{C}^n$ with period matrices P and P' respectively. Then $X=\boldsymbol{C}^n/\Gamma$ and $X'=\boldsymbol{C}^n/\Gamma'$ are isomorphic if and only if P and P' are equivalent. If $X=\boldsymbol{C}^n/\Gamma$ is a toroidal group, then the generators of Γ contain n-vectors linearly independent over $\boldsymbol{C}$ and $r>n$. For the condition on which $X=\boldsymbol{C}^n/\Gamma$ is a toroidal group, we refer the reader to [**10**], [**17**] and [**18**].

Let $X=\boldsymbol{C}^n/\Gamma$ be a toroidal group and let $\operatorname{rank}\Gamma=n+m$, $m\geqq 1$. We can choose a period matrix P of Γ as follows

$$P=\begin{pmatrix} 0 & I_m & U_1+\sqrt{-1}U_2 \\ I_{n-m} & R_1 & R_2 \end{pmatrix},$$

where I_k is the (k, k) unit matrix, $(I_m\ U_1+\sqrt{-1}U_2)$ is a period matrix of an m-dimensional complex torus and $(R_1\ R_2)$ is a real matrix (see [**17**] and [**18**]). We denote by $\boldsymbol{C}_\Gamma^m$ the maximal complex linear subspace contained in the real linear subspace $\boldsymbol{R}_\Gamma^{n+m}$ spanned by Γ. In the above normal form of P, $\boldsymbol{C}_\Gamma^m$ is spanned by U_2. There exists a real linear subspace V of $\boldsymbol{C}^n$ such that $\boldsymbol{R}_\Gamma^{n+m}=\boldsymbol{C}_\Gamma^m\oplus V$, $\boldsymbol{C}^n=\boldsymbol{C}_\Gamma^m\oplus V\oplus\sqrt{-1}V$. We take a holomorphic coordinate system (z, w) of $\boldsymbol{C}^n$ so that $z=(z_1, \cdots, z_m)$ is a coordinate system of $\boldsymbol{C}_\Gamma^m$ and that $w=(w_1, \cdots, w_{n-m})$ is a coordinate system of $V\oplus\sqrt{-1}V$. $X=\boldsymbol{C}^n/\Gamma$ is isomorphic onto $\boldsymbol{T}_{\boldsymbol{R}}^{n+m}\times\boldsymbol{R}^{n-m}$ as a real Lie group, where $\boldsymbol{T}_{\boldsymbol{R}}^{n+m}=\boldsymbol{R}_\Gamma^{n+m}/\Gamma$ is an $(n+m)$-dimensional real torus. We have global vector fields $\partial/\partial z_\alpha$, $\partial/\partial\bar{z}_\alpha$, (1, 0)-forms dz_α and (0, 1)-forms $d\bar{z}_\alpha$. We take coordinate charts U_i of X with coordinate systems induced from $\boldsymbol{C}^n$. We have the d-operator $d_z=\partial_z+\bar{\partial}_z$ with respect to the z-variable.

§ 2. d_z-exact (1, 1)-forms.

We write $\boldsymbol{T}=\boldsymbol{T}_{\boldsymbol{R}}^{n+m}$. Let

$$z_\alpha=\xi_\alpha+\sqrt{-1}\eta_\alpha, \qquad w_\alpha=x_\alpha+\sqrt{-1}y_\alpha.$$

Then $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_m, x_1, \cdots, x_{n-m})$ is a real coordinate system of $\boldsymbol{C}_\Gamma^m\oplus V=\boldsymbol{R}_\Gamma^{n+m}$. We denote by $\mathcal{A}^{p,q}=\mathcal{A}_z^{p,q}$ the sheaf of (p, q)-forms on X with respect to $\{dz_\alpha, d\bar{z}_\alpha\}$. An element φ of $H^0(X, \mathcal{A}^{p,q})$ is represented globally as

$$\varphi=\frac{1}{p!q!}\sum\varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}dz_{\alpha_1}\wedge\cdots\wedge dz_{\alpha_p}\wedge d\bar{z}_{\beta_1}\wedge\cdots\wedge d\bar{z}_{\beta_q},$$

where $\varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}$ is a C^∞ function on X. φ is also regarded as a (p, q)-form on $\boldsymbol{T}$ with parameter space $\boldsymbol{R}^{n-m}$. For $\varphi, \psi\in H^0(X, \mathcal{A}^{p,q})$ we define

$$(\varphi, \psi)_y(z, x):=\frac{1}{p!q!}\sum_{\alpha,\beta}\varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}\overline{\psi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}}.$$

Let $\omega=\sqrt{-1}\sum' dz_\alpha\wedge d\bar{z}_\alpha$. We define

$$(\varphi, \psi)_y := \int_T (\varphi, \psi)_y(z, x)\frac{\omega^m}{m!}\wedge dx_1\wedge \cdots \wedge dx_{n-m}$$

for $\varphi, \psi\in H^0(X, \mathcal{A}^{p,q})$. For a fixed $y\in \boldsymbol{R}^{n-m}$, $(\ ,\ )_y$ is an inner product on $H^0(X, \mathcal{A}^{p,q})$ regarded as the space of (p, q)-forms with respect to $\{dz_\alpha, d\bar{z}_\alpha\}$ on $\boldsymbol{T}$. The star operator $*: H^0(X, \mathcal{A}^{p,q})\to H^0(X, \mathcal{A}^{m-q,m-p})$ is defined by the usual way and has properties

$$(\varphi, \psi)_y(z, x)\frac{\omega^m}{m!} = \varphi\wedge *\bar{\psi}\,,$$

$$**\varphi = (-1)^{p+q}\varphi \qquad \text{if } \varphi\in H^0(X, \mathcal{A}^{p,q}).$$

Let ϑ_z, $\bar{\vartheta}_z$ and δ_z be dual operators of $\bar{\partial}_z$, ∂_z and d_z with respect to the inner product $(\ ,\ )_y$, respectively.

DEFINITION. We define the complex Laplacian with respect to the z-variable

$$\Box_z: H^0(X, \mathcal{A}^{p,q}) \longrightarrow H^0(X, \mathcal{A}^{p,q})$$

by $\Box_z:=\bar{\partial}_z\vartheta_z+\vartheta_z\bar{\partial}_z$.

We can write explicitly

$$(\Box_z\varphi)_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q} = -\sum_{\alpha=1}^{m}\frac{\partial^2}{\partial z_\alpha\partial\bar{z}_\alpha}\varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}$$

for all $\varphi\in H^0(X, \mathcal{A}^{p,q})$. The complex Laplacian $\Box_z$ with respect to the z-variable is self-adjoint, i.e. $(\Box_z\varphi, \psi)_y=(\varphi, \Box_z\psi)_y$, and has the property $\Delta_z=2\Box_z=2\bar{\Box}_z$, where $\bar{\Box}_z:=\partial_z\bar{\vartheta}_z+\bar{\vartheta}_z\partial_z$ and $\Delta_z:=d_z\delta_z+\delta_z d_z$. As usual we have an operator $\Lambda: H^0(X, \mathcal{A}^{p,q})\to H^0(X, \mathcal{A}^{p-1,q-1})$ with $\partial_z\Lambda-\Lambda\partial_z=\sqrt{-1}\vartheta_z$ and $\bar{\partial}_z\Lambda-\Lambda\bar{\partial}_z=-\sqrt{-1}\bar{\vartheta}_z$.

We denote by S the subbundle of the tangent bundle $T(\boldsymbol{T})$ given by vector fields $\{\partial/\partial\xi_1, \cdots, \partial/\partial\xi_m, \partial/\partial\eta_1, \cdots, \partial/\partial\eta_m\}$ on $\boldsymbol{T}$. Clearly S is completely integrable. The complex Laplacian $\Box_z$ with respect to the z-variable is an elliptic, self-adjoint differential operator on $\boldsymbol{T}$ with respect to S ([2]). Using Corollary 5.5 in [2], we obtain the following proposition.

PROPOSITION 2.1. *For any $\varphi\in H^0(X, \mathcal{A}^{p,q})$ there exist $\eta, \psi\in H^0(X, \mathcal{A}^{p,q})$ with $\Box_z\eta=0$ such that*

$$\varphi = \eta+\Box_z\psi\,.$$

PROPOSITION 2.2. *Let $\psi\in H^0(X, \mathcal{A}^{1,1})$. If $\psi=d_z\varphi$, then there exists a C^∞ function f on X such that*

$$\psi = \partial_z\bar{\partial}_z f\,.$$

PROOF. Our proof is along the argument in the proof of Theorem 7.4 in [**13**]. By Proposition 2.1 there exist $\eta, \psi\in H^0(X, \mathcal{A}^{1,1})$ with $\Box_z\eta=0$ such that

$$d_z\varphi = \eta + \Box_z\psi .$$

Since $\Delta_z\eta = 2\Box_z\eta = 0$, we have $(\Delta_z\eta, \eta)_y = 0$. Then $d_z\delta_z d_z\psi = 0$. Hence we have

$$(\delta_z d_z\psi, \delta_z d_z\psi)_y = (d_z\psi, d_z\delta_z d_z\psi)_y = 0 .$$

This means that $\delta_z d_z\psi = 0$. Also we have

$$(\eta, d_z\varphi)_y = (\delta_z\eta, \varphi)_y = 0 .$$

Then $\eta = 0$. Since

$$0 = (\delta_z d_z\psi, \psi)_y = (d_z\psi, d_z\psi)_y ,$$

we have $d_z\psi = 0$. Considering the type of ψ, we obtain that $\partial_z\psi = 0$ and $\bar{\partial}_z\psi = 0$. Therefore we have

$$d_z\varphi = \Box_z\psi = \bar{\partial}_z\partial_z\psi = \sqrt{-1}\partial_z\bar{\partial}_z\Lambda\psi .$$

§ 3. Topologically trivial line bundles.

In [3] we defined a certain refined Chern class $\tilde{c}_z(L)$ for C^∞ complex line bundle L holomorphic with respect to the z-variable over a toroidal group X. The following proposition was stated without proof to give an application of the obtained results in [3]. Here we give its proof.

PROPOSITION 3.1. *Let L be a holomorphic line bundle over a toroidal group X. If L is topologically trivial, then $\tilde{c}_z(L) = 0$.*

PROOF. Since L is topologically trivial, there exists a summand of automorphy $a(\gamma; z, w)$ which gives L (Vogt [17], Proposition 5). Moreover, we may assume by Proposition 8 in [17] that $a(\gamma; z, w) = a(\gamma, w)$ for all $\gamma \in \Gamma$, and $a_\gamma(w) := a(\gamma, w)$ is $\boldsymbol{Z}^{n-m}$-periodic for all $\gamma \in \Gamma$.

Let $\{a_i\}$ be a hermitian metric along the fibres of L. Then there exists a corresponding real-valued C^∞ function $A(z, w)$ on $\boldsymbol{C}^n$ such that

$$(*) \qquad A((z, w)+\gamma) - A(z, w) = 2\,\mathrm{Re}\, a(\gamma, w)$$

for all $\gamma \in \Gamma$ and $(z, w) \in \boldsymbol{C}^n$ (see [3], Remark 1). By the above equality (*) we obtain that $\{\bar{\partial}_z \log a_i\}$ gives a $\varphi \in H^0(X, \mathcal{A}^{0,1})$. Therefore we have

$$\partial_z\bar{\partial}_z \log a_i = d_z\varphi .$$

By Proposition 2.2 there exists a C^∞ function f on X such that

$$\partial_z\bar{\partial}_z \log a_i = \partial_z\bar{\partial}_z f ,$$

this implies $\tilde{c}_z(L) = 0$.

Let L be a holomorphic line bundle over X. Then L is isomorphic onto

$L_0 \otimes L_\rho$, where L_0 is a topologically trivial holomorphic line bundle and L_ρ is a theta bundle with theta factor ρ of type $(\mathcal{H}, \psi, Q, \mathcal{L})$ ([18], cf. also [1]). For a hermitian form $\mathcal{H}$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$, we define

$$\mathrm{Ker}(\mathcal{H}) := \{z \in \boldsymbol{C}_\Gamma^m ; \mathcal{H}(z, z')=0 \quad \text{for all } z' \in \boldsymbol{C}_\Gamma^m\}.$$

By Theorem 1 in [3] and Proposition 3.1 we obtain the following theorem.

THEOREM 3.2. *Let L be a holomorphic line bundle over a toroidal group X. Suppose that L is isomorphic onto $L_0 \otimes L_\rho$. Then there exists a C^∞ function h on $\boldsymbol{C}^n$ which is holomorphic with respect to the z-variable such that fe^h is constant on* $\mathrm{Ker}(\mathcal{H})$ *for any holomorphic function f corresponding to a holomorphic section of L.*

By Proposition 3.1 and Corollary in [3] we have the following corollary.

COROLLARY 3.3 (Huckleberry and Margulis [7], Corollary 3). *Let L be a topologically trivial holomorphic line bundle over a toroidal group X. Then $H^0(X, \mathcal{O}(L)) \neq \{0\}$ if and only if L is analytically trivial.*

§ 4. Quasi-abelian varieties.

The following definition is due to Gherardelli and Andreotti [4].

DEFINITION. A toroidal group $X=\boldsymbol{C}^n/\Gamma$ is said to be a quasi-abelian variety, if there exists a hermitian form $\mathcal{H}$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ satisfying the following conditions:

(a) $\mathcal{A} := \mathrm{Im}\,\mathcal{H}$ is $\boldsymbol{Z}$-valued on $\Gamma \times \Gamma$.

(b) $\mathcal{H}$ is positive definite on $\boldsymbol{C}_\Gamma^m \times \boldsymbol{C}_\Gamma^m$.

A quasi-abelian variety $X=\boldsymbol{C}^n/\Gamma$ is said to be of kind p, if there exists a hermitian form $\mathcal{H}$ on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ with the conditions (a) and (b) such that $\mathcal{A}|_{\boldsymbol{R}_\Gamma^{n+m}}$ has rank $2m+2p$, $0 \leqq 2p \leqq n-m$.

PROPOSITION 4.1 (Gherardelli and Andreotti [4], Theorem 1). *Every quasi-abelian variety is a covering space on an abelian variety.*

PROPOSITION 4.2 (Gherardelli and Andreotti [4], Theorem 2). *Let $X=\boldsymbol{C}^n/\Gamma$ be a quasi-abelian variety of kind p. Then X is a fibre bundle over an $(m+p)$-dimensional abelian variety with fibres $\boldsymbol{C}^p \times (\boldsymbol{C}^*)^{n-m-2p}$.*

By the embedding theorem of Kodaira and the above proposition, every quasi-abelian variety is quasi-projective. For a quasi-abelian variety of kind 0 Nakano and Rhai proved the embedding theorem directly in [15].

PROPOSITION 4.3. *Let $X=\boldsymbol{C}^n/\Gamma$ be a toroidal group. If X has a positive line bundle, then it is meromorphically separable.*

Proof. It is well known that X has a C^∞ plurisubharmonic exhaustion function (cf. [9]). Since the canonical bundle of X is trivial, we get the conclusion by the standard argument about weakly 1-complete manifolds (cf. [14]).

Proposition 4.4. *Let*

$$\alpha(\gamma\,;z,w)=\exp(a(\gamma,w))\rho(\gamma\,;z,w)$$

be a factor of automorphy for Γ on $\boldsymbol{C}^n$, where a is a summand of automorphy and ρ is a theta factor of type $(\mathcal{H},\psi,Q,\mathcal{L})$. Let f be a holomorphic function on $\boldsymbol{C}^n$ with

$$f((z,w)+\gamma)=\alpha(\gamma\,;z,w)f(z,w)$$

for all $\gamma\in\Gamma$ and $(z,w)\in\boldsymbol{C}^n$. If $f\not\equiv 0$, then $\mathcal{H}$ is positive semi-definite on $\boldsymbol{C}_\Gamma^m\times\boldsymbol{C}_\Gamma^m$.

Proof. Without loss of generality, we may assume that ρ is of type $(\mathcal{H},\psi,0,0)$. Since $\exp(a(\gamma,w))$ gives a topologically trivial holomorphic line bundle, there exists a real-valued C^∞ function φ on $\boldsymbol{C}^n$ which is pluriharmonic with respect to the z-variable by Remark 2 in [3] and Proposition 3.1, so that

$$\varphi((z,w)+\gamma)-\varphi(z,w)=\operatorname{Re}a(\gamma,w)$$

for all $\gamma\in\Gamma$ and $(z,w)\in\boldsymbol{C}^n$. There exists a C^∞ function h on $\boldsymbol{C}^n$ which is holomorphic with respect to the z-variable such that $\operatorname{Re}h=\varphi$. We set

$$F(z,w):=|f(z,w)\exp(-h(z,w))|^2\boldsymbol{e}\left[\frac{\sqrt{-1}}{2}\mathcal{H}((z,w),(z,w))\right],$$

where $\boldsymbol{e}(*)=\exp(2\pi\sqrt{-1}*)$. We see by a straight calculation that F is Γ-periodic. Hence there exists a constant C such that

$$|f(z,w)\exp(-h(z,w))|^2\leqq C\exp\{\pi\mathcal{H}((z,w),(z,w))\}\qquad\text{on } \boldsymbol{R}_\Gamma^{n+m}.$$

Since $f\not\equiv 0$, there exists $a=(a_1,a_2)\in\boldsymbol{C}^n$ such that $f(a)\neq 0$. Let

$$f_a(z,w):=f((z,w)+a).$$

Then we have

$$f_a((z,w)+\gamma)=\exp(a(\gamma,w+a_2))\rho_1(\gamma\,;z,w)f_a(z,w),$$

where ρ_1 is a theta factor of type $(\mathcal{H},\psi_1,0,\mathcal{L}_1)$ with

$$\psi_1(\gamma)=\psi(\gamma)\boldsymbol{e}[-\operatorname{Im}\mathcal{H}(a,\gamma)],\qquad \mathcal{L}_1(z,w)=\frac{1}{2\sqrt{-1}}\mathcal{H}(a,(z,w)).$$

Take a trivial theta function $\boldsymbol{\theta}_0$ of type $(0,1,0,-\mathcal{L}_1)$. Let $\tilde{f}_a:=f_a\boldsymbol{\theta}_0$ and let $\varphi_a(z,w):=\varphi(z,w+a_2)$. Then we have

$$\varphi_a((z,w)+\gamma)-\varphi_a(z,w)=\operatorname{Re}a(\gamma,w+a_2),$$

$$\tilde{f}_a((z, w)+\gamma) = \exp(a(\gamma, w+a_2))\tilde{\rho}_1(\gamma\,; z, w)\tilde{f}_a(z, w),$$

where $\tilde{\rho}_1$ is a theta factor of type $(\mathcal{H}, \psi_1, 0, 0)$. Let $h_a(z, w) := h(z, w+a_2)$. Then, by the same reason as above, there exists a constant C' such that

$$|\tilde{f}_a(z, w)\exp(-h_a(z, w))|^2 \leqq C'\exp\{\pi\mathcal{H}((z, w), (z, w))\} \qquad \text{on } \boldsymbol{R}_\Gamma^{n+m}.$$

Assume that there exists $z_0 \in \boldsymbol{C}_\Gamma^m$ with $\mathcal{H}(z_0, z_0)<0$. For all $\lambda \in \boldsymbol{C}$ it holds that

$$|\tilde{f}_a(\lambda z_0)\exp(-h_a(\lambda z_0))|^2 \leqq C'\exp\{\pi|\lambda|^2\mathcal{H}(z_0, z_0)\}.$$

Since $\tilde{f}_a\exp(-h_a)$ is holomorphic on $\boldsymbol{C}_\Gamma^m$, we have $\tilde{f}_a(0)=0$. This contradicts the assumption $f(a)\neq 0$.

We stated the following proposition without proof in the last section of [**3**]. Here we give its detailed proof.

PROPOSITION 4.5. *Let $X=\boldsymbol{C}^n/\Gamma$ be a toroidal group. If X is meromorphically separable, then it is a quasi-abelian variety.*

PROOF. Let Θ be the set of all theta factors ρ of type $(\mathcal{H}_\rho, \psi_\rho, Q_\rho, \mathcal{L}_\rho)$ with $\mathcal{H}_\rho \geqq 0$ on $\boldsymbol{C}_\Gamma^m \times \boldsymbol{C}_\Gamma^m$. We define

$$K := \bigcap_{\rho\in\Theta} \operatorname{Ker}(\mathcal{H}_\rho).$$

It sufficies to show that $K=\{0\}$. Let $\pi : \boldsymbol{C}^n \to X$ be the projection. Assume that $\dim_{\boldsymbol{C}} K>0$. Then there exist $z^1, z^2 \in K$ such that $\pi(z^1)\neq\pi(z^2)$. Since X is meromorphically separable, there exists a meromorphic function f on X such that $f(\pi(z^1))\neq f(\pi(z^2))$. The zero-divisor of f and the pole-divisor of f give the same line bundle L. Let $\alpha(\gamma\,; z, w)$ be a factor of automorphy which gives L. Then there exist holomorphic functions g_1 and g_2 on $\boldsymbol{C}^n$ with

$$g_i((z, w)+\gamma) = \alpha(\gamma\,; z, w)g_i(z, w), \qquad i=1, 2,$$

for all $\gamma \in \Gamma$ and $(z, w) \in \boldsymbol{C}^n$ such that

$$f\circ\pi = \frac{g_2}{g_1}.$$

We may assume that $\alpha = e^a\rho$, where a is a summand of automorphy and ρ is a theta factor of type $(\mathcal{H}_\rho, \psi_\rho, Q_\rho, \mathcal{L}_\rho)$. By Theorem 3.2 there exists a C^∞ function h on $\boldsymbol{C}^n$ which is holomorphic with respect to the z-variable such that g_ie^h is constant on $\operatorname{Ker}(\mathcal{H}_\rho)$ for $i=1, 2$. Therefore

$$f\circ\pi = \frac{g_2}{g_1} = \frac{g_2e^h}{g_1e^h}$$

is constant on $\operatorname{Ker}(\mathcal{H}_\rho)$. Since $g_1\not\equiv 0$ and $g_2\not\equiv 0$, $\mathcal{H}_\rho$ is positive semi-definite on $\boldsymbol{C}_\Gamma^m \times \boldsymbol{C}_\Gamma^m$ by Proposition 4.4. Then $f\circ\pi$ is constant on K. This is a contradiction.

Then we get the following theorem which was obtained in [1] under the condition $\dim H^1(X, \mathcal{O}_X)<\infty$.

THEOREM 4.6. *Let $X=\boldsymbol{C}^n/\Gamma$ be a toroidal group with* $\operatorname{rank}\Gamma=n+m$. *Then the following statements are equivalent:*

(1) *X has a positive line bundle.*

(2) *X is meromorphically separable.*

(3) *X is a quasi-abelian variety.*

(4) *X is a covering space on an abelian variety.*

(5) *There exists a fibration of X over an $(m+p)$-dimensional abelian variety with fibres $\boldsymbol{C}^p\times(\boldsymbol{C}^*)^{n-m-2p}$, where p is a natural number depending on X $(0\leqq 2p \leqq n-m)$.*

(6) *X is quasi-projective.*

§ 5. Meromorphic reduction fibrations.

In [5], Grauert and Remmert proved the meromorphic reduction theorem for compact homogeneous complex manifolds. It was extended to complex-homogeneous manifolds by Huckleberry and Snow [8]. The following theorem is the meromorphic reduction theorem for toroidal groups.

THEOREM 5.1. *Let $X=\boldsymbol{C}^n/\Gamma$ be a toroidal group. Then there exists a holomorphic fibration $\rho: X\to X_1$ over a quasi-abelian variety X_1 with fibres connected commutative complex Lie groups, which has the following properties:*

(1) *ρ is a homomorphism between toroidal groups.*

(2) *$\mathcal{M}(X)\cong\mathcal{M}(X_1)$, where $\mathcal{M}(X)$ and $\mathcal{M}(X_1)$ are meromorphic function fields on X and X_1 respectively.*

(3) *If $\tau: X\to Y$ is a homomorphism into a quasi-abelian variety Y, then there exists a unique homomorphism $\sigma: X_1\to Y$ such that $\tau=\sigma\circ\rho$. This means that such a quasi-abelian variety X_1 exists uniquely.*

PROOF. We may assume that X is not a quasi-abelian variety. Let K be a complex linear subspace of $\boldsymbol{C}_\Gamma^m$ as in the proof of Proposition 4.5. Then $K\neq\{0\}$ and

$$\overline{K+\Gamma}\subset \boldsymbol{R}_\Gamma^{n+m}=\boldsymbol{C}_\Gamma^m\oplus V.$$

There exists a real linear subspace E and a lattice Γ_0 in $\boldsymbol{R}_\Gamma^{n+m}$ such that $\overline{K+\Gamma}=E\oplus\Gamma_0$. Let $E_0:=E\cap\boldsymbol{C}_\Gamma^m$ and let $E_1:=E\cap V$. Then we have that $E=E_0\oplus E_1$ and $\sqrt{-1}E_1\cap\boldsymbol{R}_\Gamma^{n+m}=\{0\}$. We define a complex linear subspace $W:=E\oplus\sqrt{-1}E_1$. Then

$$W+\Gamma_0=W\oplus\Gamma_0\supset\overline{K+\Gamma}.$$

Let $\mu: \boldsymbol{C}^n\to\boldsymbol{C}^n/W$ be the canonical projection. Let $\Gamma^*:=\mu(\Gamma_0)$. Then Γ^* is a lattice in $\boldsymbol{C}^n/W$ and $\Gamma^*\cong\Gamma_0$. We see easily that $\Gamma^*=\mu(\Gamma)$. Let $\dim_{\boldsymbol{C}}W=n-k$. We may assume that $\boldsymbol{C}^n=\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$, $\boldsymbol{C}^k\cong\boldsymbol{C}^n/W$, $\boldsymbol{C}^{n-k}\cong W$. We regard Γ^* as a lattice in $\boldsymbol{C}^{n-k}$. Let P be a period matrix of Γ. Then P is equivalent to the following period matrix

$$\begin{pmatrix} P^* & 0 \\ A & B \end{pmatrix},$$

where P^* is a period matrix of Γ^* and B is an $(n-k, l)$-matrix, $l=\operatorname{rank}\Gamma-\operatorname{rank}\Gamma^*$. Since column vectors of B are linearly independent over $\boldsymbol{R}$, $l\leqq 2(n-k)$. Hence B gives a lattice Γ_B in $\boldsymbol{C}^{n-k}$. By the projection $\mu: \boldsymbol{C}^n\to\boldsymbol{C}^k$, we get a fibre bundle $\tilde{\mu}: X=\boldsymbol{C}^n/\Gamma\to X_1=\boldsymbol{C}^k/\Gamma^*$ with fibres $\boldsymbol{C}^{n-k}/\Gamma_B$. By the proof of Proposition 4.5 every meromorphic function on $\boldsymbol{C}^n$ with period Γ is constant on $W\oplus\Gamma_0$. Then $\tilde{\mu}^*: \mathcal{M}(X_1)\to\mathcal{M}(X)$ is an isomorphism. Since X is a toroidal group, X_1 is also a toroidal group.

Repeating this procedure when X_1 is not quasi-abelian, we obtain a fibration of X over a quasi-abelian variety with properties (1) and (2). The property (3) is proved by the standard argument (see [**5**]) because a quasi-abelian variety is meromorphically separable (Theorem 4.6).

Combining with a result of Pothering [**16**] and Hefez [**6**], we obtain the following corollary.

COROLLARY 5.2. *Let $X=\boldsymbol{C}^n/\Gamma$ be a non-compact toroidal group. If* $\dim\mathcal{M}(X)\geqq n$, *then* $\dim\mathcal{M}(X)=\infty$, *where* $\dim\mathcal{M}(X)$ *is the transcendental degree of $\mathcal{M}(X)$ over $\boldsymbol{C}$.*

REMARK. Let $X=\boldsymbol{C}^n/\Gamma$ be a toroidal group with $\operatorname{rank}\Gamma=2n-1$. Then X is a $\boldsymbol{C}^*$-principal bundle over an $(n-1)$-dimensional complex torus $\boldsymbol{T}^{n-1}$. By the proof of Theorem 5.1 we see that X_1 is the abelian image of $\boldsymbol{T}^{n-1}$ or a $\boldsymbol{C}^*$-principal bundle over it, especially X_1 is 0-dimensional when $\boldsymbol{T}^{n-1}$ has the trivial abelian image. Therefore, we can construct an example of non-compact weakly 1-complete manifold which has no non-constant meromorphic function.

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