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On the geometry of projective immersions

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In two preceeding papers [5] and [6] we have given a new general approach to classical affine differential geometry and established the basic results concerning the geometry of affine immersions. The purpose of the present paper is to begin the study of projective immersions. We shall concentrate our attention to the case of codimension one.

In Section 1 we recall the notion of projective structure on a manifold and state relevant facts. In Section 2 we define the notion of projective and equiprojective immersions and related concepts such as totally geodesic and umbilical immersions. In Section 3 we study equiprojective immersions of a flat projective structure (M, P) into a flat projective structure $(\tilde{M}, \tilde{P})$ of one higher dimension and show that they are umbilical, provided $\dim M \geqq 3$ and the rank of $h \geqq 2$. We derive certain corollaries and determine all connected, compact, umbilical hypersurfaces in $\boldsymbol{R}P^{n+1}$. In Section 4 we prove the projective version of the theorem of Berwald which characterizes quadrics in affine differential geometry. In Section 5 we study the effect of a projective change of the ambiant connection on a nondegenerate hypersurface M, namely, how the affine normal, the Blaschke induced connection, the affine metric, and the cubic form change. We find that the difference tensor between the Blaschke connection and the Levi-Civita connection is a projective invariant. We hope to find some more applications of these formulas in the study of nondegenerate hypersurfaces in $\boldsymbol{R}P^{n+1}$.

1. Projective structure.

We recall from [4] the notion of projective structure P on a differentiable manifold M. It is defined by an atlas of local affine connections $(U_\alpha, \nabla_\alpha)$, where $\{U_\alpha\}$ is an open covering of M and ∇_α is a torsion-free affine connection on U_α such that in any nonempty intersection $U_\alpha \cap U_\beta$ the connections ∇_α and ∇_β are projectively equivalent. Here, in general, two affine connections ∇ and $\bar{\nabla}$ are said to be projectively equivalent if there is a 1-form μ such that

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(1) $\bar{\nabla}_X Y = \nabla_X Y + \mu(X)Y + \mu(Y)X$ for any vector fields X and Y.

As usual, when (M, P) is a projective structure, we consider a maximal atlas of local affine connections and write $(U, \nabla) \in (M, P)$ to mean that an affine connection ∇ on an open subset U of M belongs to the maximal atlas for the projective structure (M, P).

In dealing with projective equivalence and projective structures we normally assume that each affine connection involved is locally equiaffine relative to a certain volume element; this condition is equivalent to the property that the Ricci tensor is symmetric. When two such equiaffine connections are projectively equivalent, it follows that $d\mu=0$ in (1) and that they have the same projective curvature tensor:

(2) $$W(X, Y)Z = R(X, Y)Z - [\gamma(Y, Z)X - \gamma(X, Z)Y],$$

where γ denotes the normalized Ricci tensor $\mathrm{Ric}/(n-1)$, where n is the dimension of the manifold. We also remark that if ∇ is an equiaffine connection and μ is a closed 1-form and thus exact on a neighborhood U, then the projective change of ∇ by μ gives rise to an equiaffine connection on U. It is known (cf. Proposition 4 in [4]) that for a projective structure (M, P) and for any volume element ω on M there is a unique globally defined ∇ compatible with P such that $\nabla\omega=0$.

Let us also recall that if $\dim M \geqq 3$, vanishing of the projective curvature tensor W is a necessary and sufficient condition for ∇ to be projectively flat (i.e. projectively equivalent to a flat affine connection). If $\dim M=2$, then

(3) $$(\nabla_X \gamma)(Y, Z) = (\nabla_Y \gamma)(X, Z)$$

is a necessary and sufficient condition for projective flatness.

We may define the notion of path for a projective structure (M, P). By a *path* we mean a curve x_t in M which, around each of its points, is a pregeodesic relative to some $\nabla \in (M, P)$, that is, $\nabla_t \dot{x}_t = \rho(t)\dot{x}_t$ for some function $\rho(t)$; in this case, x_t is a pregeodesic relative to every $\nabla \in (M, P)$.

2. Projective immersion.

Let (M, P) and $(\tilde{M}, \tilde{P})$ be differentiable manifolds each with a projective structure defined by means of an atlas of local affine connections $(U_\alpha, \nabla_\alpha)$ and $(\tilde{U}_\beta, \tilde{\nabla}_\beta)$, respectively. We set $n=\dim M$ and $n+p=\dim \tilde{M}$.

An immersion $f: M \to \tilde{M}$ is called a *projective immersion* if the following condition is satisfied:

(A) For each point x_0 of M, there exist local affine connections $(U, \nabla) \in$

(M, P) and $(\tilde{U}, \tilde{\nabla})\in(\tilde{M}, \tilde{P})$, where U and $\tilde{U}$ are neighborhoods of x_0 and $f(x_0)$, respectively, such that $f:(U, \nabla)\to(\tilde{U}, \tilde{\nabla})$ is an affine immersion.

This means that there is a field of transversal subspaces $x\to N_x$ on U such that for any vector fields X and Y on U we have

$$\tilde{\nabla}_X(f_*(Y))=f_*(\nabla_X Y)+\alpha(X, Y), \quad \text{where } \alpha(X, Y)\in N. \tag{4}$$

See [**6**]. In the case where codimension $p=1$, there is a transversal vector field ξ on U such that

$$\tilde{\nabla}_X(f_*(Y))=f_*(\nabla_X Y)+h(X, Y)\xi. \tag{5}$$

Let (M, ∇) and $(\tilde{M}, \tilde{\nabla})$ be manifolds with affine connections. An affine immersion $f:(M, \nabla)\to(\tilde{M}, \tilde{\nabla})$ is a projective immersion $(M, P)\to(\tilde{M}, \tilde{P})$, where P and $\tilde{P}$ are the projective structures determined by ∇ and $\tilde{\nabla}$, respectively.

When condition (A) is satisfied, we can, in fact, pick (U, ∇) or $(\tilde{U}, \tilde{\nabla})$ and find $(\tilde{U}, \tilde{\nabla})$ or (U, ∇) which satisfies the condition. More precisely, we have

PROPOSITION 1. *If $f:(M, P)\to(\tilde{M}, \tilde{P})$ is a projective immersion, then*
(B) *for any point $x_0\in M$ and for any local affine connection $(\tilde{U}, \tilde{\nabla})\in(\tilde{M}, \tilde{P})$, where $\tilde{U}$ is a neighborhood of $f(x_0)$, there exists a local affine connection (U, ∇), where U is a neighborhood of x_0, such that $f:(U, \nabla)\to(\tilde{U}, \tilde{\nabla})$ is an affine immersion;*
(C) *for any point $x_0\in M$ and for any local affine connection (U, ∇), where U is a sufficiently small neighborhood of x_0, there exists a local affine connection $(\tilde{U}, \tilde{\nabla})$, where $\tilde{U}$ is a neighborhood of $f(x_0)$, such that $f:(U_1, \nabla)\to(\tilde{U}, \tilde{\nabla})$ is an affine immersion, where U_1 is a neighborhood of U such that $U_1\subset U$ and $f(U_1)\subset\tilde{U}$.*

PROOF. Let (U, ∇) and $(\tilde{U}, \tilde{\nabla})$ be as in (A). Let $(\tilde{U}_1, \tilde{\nabla}_1)$ be any affine connection belonging to $(\tilde{M}, \tilde{P})$, where $\tilde{U}_1$ is a neighborhood of $f(x_0)$, where we may assume $\tilde{U}_1\subset\tilde{U}$. Choose a neighborhood $U_1\subset U$ of x_0 such that $f(U_1)\subset\tilde{U}_1$. Then there exists a 1-form $\tilde{\mu}$ on $\tilde{U}_1$ which gives projective equivalence of $\tilde{\nabla}$ and $\tilde{\nabla}_1$. Then

$$\begin{aligned}\tilde{\nabla}_{1X}(f_*Y)&=\tilde{\nabla}_X(f_*Y)+\tilde{\mu}(f_*X)f_*Y+\tilde{\mu}(f_*Y)f_*X\\&=f_*(\nabla_X Y+\mu(X)Y+\mu(Y)X)+\alpha(X, Y),\end{aligned}$$

where $\mu=f^*\tilde{\mu}$ is a 1-form on M and $\alpha(X, Y)$ belongs to the transversal subspace N for $f:(U,\nabla)\to(\tilde{U}_1, \tilde{\nabla}_1)$. Now we may pick the connection $(U_1, \nabla_1)\in(M, P)$, where $\nabla_{1X}Y=\nabla_X Y+\mu(X)Y+\mu(Y)X$. Then the equation above shows that $f:(U_1, \nabla_1)\to(\tilde{U}_1, \tilde{\nabla}_1)$ is an affine immersion.

The proof for (C) is similar. Let (U, ∇) and $(\tilde{U}, \tilde{\nabla})$ be as in (A). For any $(U_1, \nabla_1)\in(M, P)$, where $U_1\subset U$, there is a closed 1-form μ on U_1 such that

$\nabla_{1X}Y=\nabla_XY+\mu(X)Y+\mu(Y)X$ for all vector fields X and Y. Now it is easy to find a closed 1-form $\tilde{\mu}$ on U such that $f^*\tilde{\mu}=\mu$. If we projectively change $\tilde{\nabla}$ to $\tilde{\nabla}_1$ by using the form $\tilde{\mu}$, then $f:(U_1, \nabla_1)\to(\tilde{U}, \tilde{\nabla}_1)$ is an affine immersion. □

REMARK. In stating conditions such as (A), (B) or (C), we shall from now on omit explicit mention of the domains of local affine connections. In many cases, it suffices to say that around a given point there is a local affine connection $\nabla\in(M, P)$ with such and such properties.

A projective immersion $f:(M, P)\to(\tilde{M}, \tilde{P})$ is said to be *totally geodesic* at $x_0\in M$ if for any $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$ around $f(x_0)$, f is *totally geodesic relative to* $\tilde{\nabla}$ at x_0, that is, for any vector fields X and Y around x_0, $[\tilde{\nabla}_X(f_*(Y))]_{x_0}$ is tangent to $f(M)$, that is, there is a vector Z at x_0 such that $f_*(Z)=[\tilde{\nabla}_X(f_*(Y))]_{x_0}$. Now this condition is independent of the choice of $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$, as is easily verified. It is equivalent to the condition that $h=0$ at x_0 in (5). We say that f is totally geodesic if it is so at every point of M.

It is not difficult to see that f is totally geodesic if and only if the image of a path x_t in M is a path for $\tilde{M}$.

From this point on, we shall deal with the case of codimension $p=1$.

Let $f:(M, P)\to(\tilde{M}, \tilde{P})$ be a projective immersion of codimension 1. For any point x_0, there exist $\nabla\in(M, P)$ around x_0 and $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$ around $f(x_0)$ such that f is an affine immersion relative to ∇ and $\tilde{\nabla}$ around x_0. This means that there is a transversal vector field ξ as in the equation (5). We show that the transversal direction $[\xi]$ at x_0 is independent of the pair $(\nabla, \tilde{\nabla})$ if f is not totally geodesic at x_0.

For this purpose, let $(\nabla_1, \tilde{\nabla}_1)$ be another pair we may choose. We can assume that ∇_1 and ∇ are defined in the same domain and projectively related: $\nabla_{1X}Y=\nabla_XY+\mu(X)Y+\mu(Y)X$, where μ is a certain 1-form, and, similarly for $\tilde{\nabla}_1$ and $\tilde{\nabla}$: $\tilde{\nabla}_{1X}Y=\tilde{\nabla}_XY+\tilde{\mu}(X)Y+\tilde{\mu}(Y)X$, where $\tilde{\mu}$ is a certain 1-form. Suppose ξ_1 is a transversal vector field for the affine immersion f relative to $(\nabla_1, \tilde{\nabla}_1)$ and write $\xi_1=f_*(Z)+\varphi\xi$, where Z is a vector field tangent to M and φ is a non-vanishing function. From

$$\tilde{\nabla}_X(f_*(Y))=f_*(\nabla_XY)+h(X, Y)\xi$$

and

$$\tilde{\nabla}_{1X}(f_*(Y))=f_*(\nabla_{1X}Y)+h_1(X, Y)\xi_1$$

we obtain

$$\nabla_{1X}Y+h_1(X, Y)Z=\nabla_XY+(f^*\tilde{\mu})(Y)X+(f^*\tilde{\mu})(X)Y \tag{6}$$

and

$$h(X, Y)=\varphi h_1(X, Y). \tag{7}$$

Now we want to prove that $Z=0$ at x_0. Assume that $Z\neq 0$ at x_0 and take a tangent vector X at x_0 linearly independent from Z. We may take a geodesic x_t for ∇ with initial condition (x_0, X). This curve is a pregeodesic for ∇_1 and so $\nabla_{1X}X$ is a multiple of X by a certain function of t, where X is considered as the tangent vector field of the curve x_t. From (6) we obtain $h_1(X, X)Z=\lambda X$ at x_0, where λ is a scalar. Thus $h_1(X, X)=0$.

We have shown that for any $X\in T_{x_0}(M)$ linearly independent from $Z\neq 0$, we have $h_1(X, X)=0$. Let $Z=X_1, X_2, \cdots, X_n$ be a basis of $T_{x_0}(M)$. From what we proved, we have for each k, $2\leqq k\leqq n$, $h_1(X_k, X_k)=0$ and $h_1(X_j, X_k)=0$ for all j, $k\geqq 2$. We have also

$$h_1(X_1+X_k,\ X_1+X_k)=h_1(X_1, X_1)+2h_1(X_1, X_k)=0$$

as well as

$$h_1(X_1+2X_k,\ X_1+2X_k)=h_1(X_1, X_1)+4h_1(X_1, X_k)=0\,,$$

which together imply that $h_1(X_1, X_1)=h_1(X_1, X_k)=0$, where $k\geqq 2$. We have shown that $h_1=0$ at x_0. This contradicts the assumption that f is not totally geodesic at x_0.

We may state this result as follows.

PROPOSITION 2. *Let $f:(M, P)\to(\tilde{M}, \tilde{P})$ be a projective immersion for codimension* 1. *There is a uniquely determined transversal direction field $[\xi]$ except in the interior V of the set of points where f is totally geodesic.*

We shall call $[\xi]$ on $M-V$ the *transversal direction field* for the projective immersion f. The symmetric bilinear form h is determined up to a scalar factor on M and is 0 at the points where f is totally geodesic. We call the conformal class $[h]$ the *fundamental form* for the projective immersion f. The rank, uniquely determined at each point, is the rank of f at the point. In particular, if the rank is n, f is said to be *nondegenerate* at the point. We say that f is *nondegenerate* if it is so at every point of M.

Suppose that f is not totally geodesic at x_0 and thus not in a neighborhood of x_0. For any choice of $\nabla\in(M, P)$, $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$ and a transversal field ξ relative to which f is an affine immersion, we write

$$\tilde{\nabla}_X\xi=-f_*(SX)+\tau(X)\xi\,, \tag{8}$$

where S is the shape operator for ξ on M and τ is the transversal connection form for ξ. If we change ξ to $\varphi\xi$, where φ is a function, then S changes to φS and τ into $\tau+d\varphi$. Thus the condition that S is a scalar multiple of the identity: $S=\lambda I$ does not change. Nor does the 2-form $d\tau$.

We may also change $\tilde{\nabla}$ to a projectively equivalent $\tilde{\nabla}_1\in(\tilde{M}, \tilde{P})$: $\tilde{\nabla}_{1X}Y=\tilde{\nabla}_XY+\tilde{\mu}(X)Y+\tilde{\mu}(Y)X$, where $\tilde{\mu}$ is a certain exact 1-form. Then S changes into

$S_1=S-\mu(\xi)I$ and τ into $\tau_1=\tau+f^*\tilde{\mu}$. Thus the condition $S=\lambda I$ does not change. Nor does the form $d\tau$.

In view of this observation, we can make the following definition. A projective immersion f, which is not totally geodesic at a point x_0, is said to be *umbilical at* x_0 if, for some choice of ∇, $\tilde{\nabla}$ and ξ relative to which f is an affine immersion, S is a scalar multiple of the identity at x_0. If f is umbilical at every point, we say that f is *umbilical*.

We now introduce the notion of equiprojective immersion. Let (M, P) and $(\tilde{M}, \tilde{P})$ be two manifolds with projective structures and $\dim\tilde{M}=\dim M+1$. An immersion $f: M\to\tilde{M}$ is said to be *equiprojective* if

(A_1) for each point x_0 of M, there exist local equiaffine connections $\nabla\in(M, P)$ and $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$ such that f is an affine immersion.

Note that one of ∇ and $\tilde{\nabla}$ can be always chosen to be equiaffine. Just like the case of mutually equivalent conditions (A), (B), (C) for projective immersions, we may state the following.

PROPOSITION 3. *If $f: (M, P)\to(\tilde{M}, \tilde{P})$ is an equiprojective immersion, then*
(B) *for any point $x_0\in M$ and for any local equiaffine connection $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$ around $f(x_0)$ there exists a local equiaffine connection $\nabla\in(M, P)$ around x_0 such that f is an affine immersion;*
(C) *for any point $x_0\in M$ and for any local equiaffine connection $\nabla\in(M, P)$ around x_0, there exists a local equiaffine connection $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$ around $f(x_0)$ such that f is an affine immersion.*

When f is an equiprojective immersion, then locally we may choose a transversal vector field ξ to be equiaffine relative to ∇ and $\tilde{\nabla}$ (cf. Proposition 3 in [5]). For such a choice of ξ, we have $\tau=0$. Thus for an equiprojective immersion, we have $d\tau=0$. We now state

PROPOSITION 4. *A projective immersion $f: (M, P)\to(\tilde{M}, \tilde{P})$ is equiprojective if and only if $d\tau=0$.*

PROOF. We prove that $d\tau=0$ implies that f is equiprojective. Let $x_0\in M$ and choose an equiaffine $\tilde{\nabla}\in(\tilde{M}, \tilde{P})$ with a local parallel volume element $\tilde{\omega}$ around $f(x_0)$ and an arbitrary $\nabla\in(M, P)$ around x_0 so that f is an affine immersion (with a transversal vector field ξ). Since $d\tau=0$, there exists a function φ around x_0 such that $\tau=-d\varphi$. Then for $\bar{\xi}=\varphi\xi$ we get the same connection ∇ induced, namely,

$$\tilde{\nabla}_X(f_*Y)=f_*(\nabla_XY)+h(X, Y)\bar{\xi},$$

and, on the other hand, $\bar{\tau}=0$. This means that the local volume element ω defined by

$$\omega(X_1, \cdots, X_n) = \tilde{\omega}(X_1, \cdots, X_n, \xi), \qquad \text{where } X_1, \cdots, X_n \in T_x(M)$$

is parallel relative to ∇ (cf. [3]). Thus ∇ is equiaffine, and f has been shown to be equiprojective. □

REMARK. If $d\tau=0$, then there is a choice of ξ for the affine immersion f so that the equation $h(SX, Y)=h(X, SY)$ holds (see [5]). In the terminology of projective differential geometry, this property is expressed by saying that the normal congruence ξ is conjugate (for instance, see [1], p. 31).

3. Equiprojective immersions between flat projective structures.

Now we recall (see [4]) that a projective structure (M, P) is said to be *flat* if each local affine connection $\nabla \in (M, P)$ is projectively flat; in other words, if the atlas (M, P) contains a flat affine connection around each point. We now prove

THEOREM 5. *Let $f:(M, P) \to (\tilde{M}, \tilde{P})$ be an equiprojective immersion, where* $\dim M = n \geqq 3$, $\dim \tilde{M} = n+1$. *Assume that $(\tilde{M}, \tilde{P})$ is flat. Then (M, P) is flat if and only if at each point $x_0 \in M$ we have either*

1) $S = \rho I$, *or*
2) $\operatorname{rank} h = 1$ *and* $S = \rho I$ *on* $\operatorname{Ker} h$, *or*
3) $h = 0$.

PROOF. Assume that (M, P) is flat. For $x_0 \in M$, choose equiaffine connections $\nabla \in (M, P)$ and $\tilde{\nabla} \in (\tilde{M}, \tilde{P})$ such that f is an affine immersion with an equiaffine transversal vector field ξ.

Since $\tilde{\nabla}$ is projectively flat, we have

$$\tilde{R}(X, Y)Z = \tilde{\gamma}(Y, Z)X - \tilde{\gamma}(X, Z)Y, \tag{9}$$

where $\tilde{\gamma}$ is the normalized Ricci tensor. Thus the Gauss equation (see [5]) says

$$R(X, Y)Z = \tilde{\gamma}(Y, Z)X - \tilde{\gamma}(X, Z)Y + h(Y, Z)SX - h(X, Z)SY. \tag{10}$$

From this we find that the normalized Ricci tensor γ of ∇ is given by

$$\gamma(Y, Z) = \tilde{\gamma}(Y, Z) + [h(Y, Z)\operatorname{tr} S - h(SY, Z)]/(n-1). \tag{11}$$

Since we assume that ∇ is also projectively flat, we have an equation similar to (9):

$$R(X, Y)Z = \gamma(Y, Z)X - \gamma(X, Z)Y. \tag{12}$$

Using (11) in (12) and comparing it with (10) we find

$$(n-1)[h(Y,Z)SX-h(X,Z)SY] \tag{13}$$
$$= [(\mathrm{tr}\, S)h(Y,Z)-h(SY,Z)]X-[(\mathrm{tr}\, S)h(X,Z)-h(SX,Z)]Y\,.$$

It is easy to see that, conversely, this equation implies that ∇ is projectively flat.

Now assume that $\mathrm{rank}\, h_x \geqq 2$, and we'll show that $S=\rho I$. Let $\{X_1, \cdots, X_r, X_{r+1}, \cdots, X_n\}$ be a basis such that the last $n-r$ vectors form a basis of $\mathrm{Ker}\, h_x$ and the first r vectors are orthonormal: $h(X_i, X_j)=\pm\delta_{ij}$, for $1\leqq i,\ j\leqq r$.

Let $Y\neq Z$ be from $\{X_1, \cdots, X_r\}$. If $r=n$ $(\geqq 3)$, choose $X\neq Y, Z$ from $\{X_1, \cdots, X_r\}$. If $r<n$, choose X from $\{X_{r+1}, \cdots, X_n\}$. From (13) we get $h(SY, Z)X=h(SX, Z)Y$. Since X and Y are linearly independent, we get $h(SY, Z)=h(SX, Z)=0$. This means that there exist constants $\rho_1, \cdots, \rho_r$, such that $SX_j=\rho_j X_j \bmod \mathrm{Ker}\, h_x$ for $1\leqq j\leqq r$. By a similar argument to [**6**, Lemma 2] we see that all ρ_j's are equal, say, to ρ.

Now take $X\neq Y$ from $\{X_1, \cdots, X_r\}$, and set $Z=X$. (13) implies

$$-(n-1)h(X,X)SY = -h(SY,X)X-(\mathrm{tr}\, S)h(X,X)Y+h(SX,X)Y\,. \tag{14}$$

Write

$$SY = \rho Y+W, \qquad SX=\rho X+V, \qquad \text{where } W,\ V\in \mathrm{Ker}\, h_x\,. \tag{15}$$

Since $h(SY, X)=0$, $h(SX, X)=\rho h(X, X)$, we get

$$(n-1)(\rho Y+W) = (\mathrm{tr}\, S)Y-\rho Y\,. \tag{16}$$

This implies that $W=0$, as well as, $\mathrm{tr}\, S=n\rho$, and $SY=\rho Y$. Since Y is arbitrary from $\{X_1, \cdots, X_r\}$, we have $SX_j=\rho X_j$, $1\leqq j\leqq r$.

Now take X from $\{X_{r+1}, \cdots, X_n\}$ and $y\neq Z$ from $\{X_1, \cdots, X_r\}$. (13) implies $h(SY, Z)X=h(SX, Z)Y$. But $h(SY, Z)=\rho h(Y, Z)=0$ and thus $h(SX, Z)=0$. Since Z is arbitrary in $\{X_1, \cdots, X_r\}$, we see that $SX\in \mathrm{Ker}\, h_x$. Since X is arbitrary in $\mathrm{Ker}\, h_x$, we have $S(\mathrm{Ker}\, h_x)\subset \mathrm{Ker}\, h_x$.

Finally, take X from $\{X_{r+1}, \cdots, X_n\}$ and $Y=Z$ from $\{X_1, \cdots, X_r\}$. (13) implies

$$(n-1)h(Y,Y)SX = (\mathrm{tr}\, S)h(Y,Y)X-h(SY,Y)X\,.$$

From $h(SY, Y)=\rho h(Y, Y)$, $h(Y, Y)\neq 0$, we see $SX=\rho X$ for $X\in\{X_{r+1}, \cdots, X_n\}$. We have thus proved that $S=\rho I$, under the assumption that $\mathrm{rank}\, h_x\geqq 2$.

We now consider the case where $\mathrm{rank}\, h_x=1$. Let $\{X_1, X_2, \cdots, X_n\}$ be a basis of $T_x(M)$ such that $h(X_1, X_1)=\pm 1$ and $\{X_2, \cdots, X_n\}$ is a basis for $\mathrm{Ker}\, h_x$. Taking $X\neq Y$ from $\{X_2, \cdots, X_n\}$ and $Z=X_1$, we get from (13) $h(SY, X_1)X= h(SX, X_1)Y$. Since X and Y are linearly independent, we have $h(SX, X_1)=0$, which implies that $SX\in \mathrm{Ker}\, h_x$. Thus $S(\mathrm{Ker}\, h_x)\subset \mathrm{Ker}\, h_x$.

Now take $X=Z=X_1$ and $Y\in \mathrm{Ker}\, h_x$ (so that $SY\in \mathrm{Ker}\, h_x$). (13) implies

$SY=\rho Y$, where $\rho=[1/(n-1)][(\operatorname{tr} S)-h(SX_1, X_1)/h(X_1, X_1)]$. Hence we have seen that $S=\rho I$ on $\operatorname{Ker} h_x$.

The converse part of Theorem 5 is easy, because either $S=\rho I$, or $S=\rho I$ on $\operatorname{Ker} h_x$, or $h=0$ implies (13). $\square$

In order to take care of the case where the rank of h is $\leqq 1$, we make use of a result by Ferus [**2**, Theorem 1]. Let $f: M^n \to S^{n+1}$ be an isometric immersion of a complete Riemannian manifold M^n, $n\geqq 2$, into the unit sphere S^{n+1}. Let t_0 be the maximum type number and assume $t_0 \leqq n-1$. Then t_0 is an even number and $t_0>0$ implies that $t_0 \geqq n/2$.

Rephrasing this result, we get the following. *Denote by $r(n)$ the smallest even integer $\geqq n/2$. Then if the rank of the second fundamental form h is $<r(n)$, then f is totally geodesic.*

We now observe that we have the projective version of this result. To state it, let M^n be a connected compact differentiable manifold, $n\geqq 2$, and let $f: M^n \to \boldsymbol{R}P^{n+1}$ be an immersion. The notion of the rank of f at each point is well defined as follows. For $x_0 \in M$, let ∇ be any local affine connection belonging to the canonical projective structure of $\boldsymbol{R}P^{n+1}$ around $f(x_0)$, and let ξ be a transversal vector field around x_0. Write the transversal component of $\tilde{\nabla}_X Y$ as $h(X, Y)\xi$. The form h is defined up to a scalar multiple and its rank is independent of the choice of ξ. We call it the *rank* of the immersion f. When the rank is 0, f is totally geodesic.

REMARK. The definitions of the rank of h and the rank of the immersion f at each point are valid when $\boldsymbol{R}P^{n+1}$ is replaced by any manifold with a projective structure $(\tilde{M}, \tilde{P})$.

We have now

PROPOSITION 6. *Let $f: M^n \to \boldsymbol{R}P^{n+1}$ be an immersion of a connected, compact differentiable manifold, where $n\geqq 2$. If the rank of f is $<r(n)$ at every point, then f is totally geodesic.*

PROOF. Let g_0 be the Riemannian metric of $\boldsymbol{R}P^{n+1}$ as well as that of the unit sphere S^{n+1}. Denote by $g=f^*g_0$ the Riemannian metric induced on M^n, and let $\tilde{M}^n$ be the universal covering manifold of M^n with the natural complete metric $\tilde{g}$. We can then find an isometric immersion $\tilde{f}: \tilde{M}^n \to S^{n+1}$ such that $\pi \circ \tilde{f} = f \circ \pi_1$, where $\pi: S^{n+1} \to \boldsymbol{R}P^{n+1}$ and $\pi_1: \tilde{M} \to M$ are the natural projections. Since the rank of $\tilde{f}$ at $\tilde{x} \in \tilde{M}^{n+1}$ coincides with the rank of f at $\pi(\tilde{x}) \in M$, we may now apply the result of Ferus to conclude that $\tilde{f}$ is totally geodesic. It follows that f is totally geodesic. $\square$

Combining Theorem 5 and Proposition 6 we obtain

THEOREM 7. *Let* $f:(M, P)\to \boldsymbol{R}P^{n+1}$ *be a real analytic and equiprojective immersion with codimension* 1 *of a flat projective structure* (M, P), *where* $\dim M \geqq 3$. *If* M *is connected and compact, then* f *is totally geodesic or umbilical.*

PROOF. If the rank of h is $\leqq 1$ everywhere, then Proposition 6 applies. If the rank of h is $\geqq 2$ somewhere (and so on some open subset W), then $S=\rho I$ on W by Theorem 5. By analyticity this holds on the whole M. □

PROPOSITION 8. *Let* $f:(M, P)\to \boldsymbol{R}P^{n+1}$ *be an equiprojective immersion.* f *is umbilical if and only if the projective lines in the transversal directions* $[\xi]$ *go through a point in* $\boldsymbol{R}P^{n+1}$.

PROOF. Assume $S=\lambda I$, where λ is a function. For each point x of M, there is an open neighborhood U of x such that $f(U)$ lies in $A^{n+1}=\boldsymbol{R}P^{n+1}-H$, where H is a certain projective hyperplane. The flat affine connection $\tilde{\nabla}_0$ in the affine space A^{n+1} belongs to the atlas of local affine connections for $\boldsymbol{R}P^{n+1}$. Relative to $\tilde{\nabla}_0$ and $\nabla\in(M, P)$ and for a choice of an equiaffine transversal vector field ξ, f is an affine immersion which is umbilical $S=\lambda I$, with constant λ. If $\lambda\neq 0$, then for the mapping $x\in M\to y=x+\xi/\lambda$, we have $D_X y=0$, showing that the lines in the direction of ξ meet at one single point. If $\lambda=0$, then the lines in the direction of ξ are parallel. In either case, these lines, when considered in $\boldsymbol{R}P^{n+1}$, meet at a single point. We have shown that all lines in the transversal direction through the points of U (neighborhood of x) meet at a single point.

We now show that all projective lines $[\xi]$ in the transversal direction through the points of M meet at a single point. For each point p in $\boldsymbol{R}P^{n+1}$, let W_p be the set of points $x\in M$ such that in a neighborhood of x, all $[\xi]$ go through p. Each W_p is an open subset (possibly empty). If $p\neq q$, then W_p and W_q are disjoint. Suppose $x\in W_p\cap W_q$. If they are on one line, then we can take a point $y\in M$ near x such that the lines $y\cup p$ and $y\cup q$ are distinct, and this contradicts the fact that $[\xi]_y$ must coincide with the line $y\cup p$ as well as with the line $y\cup q$, since $y\in W_p\cap W_q$. Since M is the union of all W_p, it follows that $M=W_p$ for one point p. Thus all $[\xi]$ go through p. □

We now give an analytic description of a connected, compact hypersurface in $\boldsymbol{R}P^{n+1}$ with transversal directions $[\xi]$ that go through a single point.

Consider $\boldsymbol{R}P^{n+1}$ *as the quotient of the unit sphere* S^{n+1} *by identification of antipodal points. Let* $e_{n+2}=(0, \cdots, 0, 1)$ *be the north pole of* S^{n+1}. *Let* $S^n=\{x=(x_1, \cdots, x_{n+1}, 0); \sum x_i^2=1\}$ *be the unit sphere in the tangent space* $T_{e_{n+2}}(S^{n+1})$. *Let* $r=r(x)$ *be a positive differentiable function on* S^n. *Define*

$$f(x)=(\cos r(x))e_{n+2}+(\sin r(x))x \in S^{n+1}, \qquad x\in S^n \tag{17}$$

$$g(x)=\pi\circ f(x)\in \boldsymbol{R}P^{n+1}. \tag{18}$$

Then the hypersurface $g: S^n\to \boldsymbol{R}P^{n+1}$ has the desired property.

To see this, let $X\in T_x(S^n)$. We have

$$f_*(X)=(-\sin r(x))(Xr)e_{n+1}+(\cos r(x))(Xr)x+(\sin(r(x))X. \tag{19}$$

If $f_*(X)=0$, it follows that $X=0$. Thus f is an immersion, and so is g. The curve $t\to x_t=(\cos t)e_{n+2}+(\sin t)x$ is a great circle and $\pi(x_t)$ is a path in $\boldsymbol{R}P^{n+1}$. The tangent vector $\vec{x}_t$ at $t=r(x)$ is

$$-(\sin r(x))e_{n+2}+(\cos r(x))x, \tag{20}$$

which is transversal to $f_*(T_xS^n)$, since it is linearly dependent from (19). Going to $\boldsymbol{R}P^{n+1}$ we see that the path from $\pi(e_{n+2})$ to $g(x)$ is transversal to $g(S^n)$. Relative to this choice of transversal direction field, M acquires a projective structure. Namely, for any local equiaffine connection compatible with $\boldsymbol{R}P^{n+1}$ we may induce a local equiaffine connection on M, which is determined up to a projective change. Clearly, g is an equiprojective immersion which is umbilical. (We might say that a hypersurface M in $\boldsymbol{R}P^{n+1}$ is umbilical if it is umbilical in the manner above relative to a choice of transversal direction field.)

REMARK. In the model discussed above, suppose r is an odd function on S^n, that is, $r(-x)=-r(x)$. Then $f(-x)=f(x)$. This means that f induces an umbilical hypersurface $g: \boldsymbol{R}P^n\to \boldsymbol{R}P^{n+1}$.

PROPOSITION 9. *A connected compact umbilical hypersurface M in $\boldsymbol{R}P^{n+1}$ may be obtained in the form $g(S^n)$ described above, up to a projective transformation of $\boldsymbol{R}P^{n+1}$.*

PROOF. We may assume that the paths in the transversal directions meet at $\pi(e_{n+2})$. By taking $x\in M$ into the unit tangent vector at e_{n+2} of the geodesic in S^{n+1} projecting on the path in the transversal direction, we get a mapping of M into S^n (the unit sphere in $T_{e_{n+2}}(S^{n+1})$), which is a local diffeomorphism. Since M is connected and compact, it follows that the M is of the form $g(S^n)$ described above. □

As another application of Theorem 5 we obtain a result related to the possibilities of isometric immersion between riemannian or pseudo-riemannian manifolds each of constant sectional curvature. For example, it is known that there is no isometric immersion of a Euclidean space E^n into S^{n+1} of constant curvature 1, while E^n can be isometrically imbedded (as a horosphere) into the hyperbolic space H^{n+1} of constant curvature -1. We now consider a manifold M with flat affine connection ∇ and show that it cannot be immersed as a non-degenerate Blaschke hypersurface (in the classical sense, see [5], Example 6) in

S^{n+1}. More precisely, we have

THEOREM 10. *Let* $\tilde{M}$ *be an* $(n+1)$*-dimensional pseudo-riemannian manifold with metric* $\tilde{g}$ *of constant sectional curvature* $c\neq 0$ *and its Levi-Civita connection* $\tilde{\nabla}$, *where* $n\geqq 3$. *If there exists a nondegenerate hypersurface* M^n *whose induced Blaschke connection* ∇ *is flat, then* $c<0$. *In particular,* S^{n+1}, $n\geqq 3$, *does not admit any nondegenerate flat affine hypersurface.*

PROOF. Suppose M^n is a nondegenerate Blaschke hypersurface with flat induced connection. We show that its affine normal is perpendicular to M^n relative to the metric $\tilde{g}$ and that M^n is umbilical in the metric sense.

The Gauss equation for the affine hypersurface M^n is

$$R(X, Y)Z = \tilde{\gamma}(Y, Z)X-\tilde{\gamma}(X, Z)Y+h(Y, Z)SX-h(X, Z)SY\,,$$

where $\tilde{\gamma}$ is the normalized Ricci tensor of $\tilde{M}$, which is $c\tilde{g}$ by assumption, and h and S are the affine fundamental form and the affine shape operator. Now since $R=0$, ∇ is projectively flat in particular. From Theorem 5 applied to the affine immersion of (M^n, ∇) into $(\tilde{M}, \tilde{\nabla})$, which is also projectively flat, we know that $S=\rho I$, where ρ is a constant. The Gauss equation above now reduces to

$$[c\tilde{g}(Y, Z)+\rho h(Y, Z)]X+[c\tilde{g}(X, Z)+\rho h(X, Z)]Y = 0\,.$$

For arbitrary X and Z tangent to M, choose Y to be linearly independent of X. We get $c\tilde{g}(X, Z)=-\rho h(X, Z)$. Thus $\rho\neq 0$ and $h=-(c/\rho)g_0$, where g_0 is the restriction of $\tilde{g}$ to M^n. Since h is nondegenerate, so is g_0.

If we denote by ξ_0 the unit normal vector field for M^n (relative to $\tilde{g}$) and by h_0 the second fundamental form in the metric sense, we know that $h_0=\lambda h$, where λ is a certain scalar function. But then $h_0=-(\lambda c/\rho)g_0$. As is well known, it now follows that $k=-\lambda c/\rho$ and thus λ are constants. Hence M^n is umbilical in $\tilde{M}$ in the metric sense.

Recall now how the affine normal ξ is determined for a nondegenerate hypersurface (cf. [**3**], proof of Theorem 1). We now have $h_0=kg_0$. Let $\{X_1, \cdots, X_n\}$ be an orthonormal basis relative to g_0. When we take the absolute value of $\det[h_0(X_i, X_j)]$, we get the constant $|k|^n$. This means that the affine normal ξ is in the same direction as the unit normal vector ξ_0 and hence the induced connection ∇ on M^n coincides with the Levi-Civita connection ∇_0 of g_0. The metric shape operator S_0 is equal to kI, because $h_0=kg_0$. By the Gauss equation in the metric sense we now see that $c+k^2=0$, which implies that $c<0$. □

REMARK. For $n=2$, there are many Blaschke immersions of a flat torus (T^2, ∇) into S^3, for example, all Clifford tori. In view of Theorem 5 for $n\geqq 3$, it will be an interesting problem to study projectively flat surfaces in $\boldsymbol{R}P^3$.

4. Extension of the theorem of Berwald.

Let $(\tilde{M}, \tilde{P})$ be an $(n+1)$-dimensional manifold with a projective structure. Let $f: M \to \tilde{M}$ be an immersion of an n-dimensional manifold M into $\tilde{M}$. Without assuming that M is provided a priori with a projective structure, we shall define a certain property extending the classical condition of vanishing cubic form. For the case of an affine immersion this was already discussed in [**6**].

For each point $x \in M$, choose a local affine connection $\tilde{\nabla} \in (\tilde{M}, \tilde{P})$ around $f(x)$. Also choose any transversal field ξ around x. From $\tilde{\nabla}$ and ξ we may obtain a local affine connection ∇ around x so that

$$\tilde{\nabla}_X(f_*(Y)) = f_*(\nabla_X Y) + h(X, Y)\xi \tag{21}$$

and

$$\tilde{\nabla}_X \xi = -f_*(SX) + \tau(X)\xi\,, \tag{22}$$

where h is the fundamental form and τ the transversal connection form for the affine immersion f of a neighborhood U into $\tilde{M}$. In [**6**] we defined the cubic form $C(X, Y, Z) = (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z)$ and defined the notion that C is divisible by h (denoted by $h|C$), meaning that there is a 1-form ρ such that

$$C(X, Y, Z) = \rho(X)h(Y, Z) + \rho(Y)h(Z, X) + \rho(Z)h(X, Y) \tag{23}$$

for all tangent vectors X, Y and Z to M.

PROPOSITION 11. *The property that $h|C$ does not depend on the choice of $\tilde{\nabla} \in (\tilde{M}, \tilde{P})$ nor of a transversal field ξ. Thus it is a property we can speak of for any immersion $f: M \to (\tilde{M}, \tilde{P})$ of a differentiable manifold M into $(\tilde{M}, \tilde{P})$.*

PROOF. Suppose we have chosen $\tilde{\nabla} \in (\tilde{M}, \tilde{P})$. Then the property $h|C$ is independent of the choice of ξ, as is known in Proposition 5 of [**6**]. Now we change $\tilde{\nabla}$ to $\tilde{\nabla}' \in (\tilde{M}, \tilde{P})$ so that

$$\tilde{\nabla}'_X Y = \tilde{\nabla}_X Y + \mu(X)Y + \mu(Y)X\,, \qquad \text{where } \mu \text{ is a certain 1-form.} \tag{24}$$

From (15) and the corresponding equation for $\tilde{\nabla}'$ we obtain

$$\nabla'_X Y = \nabla_X Y + \mu(X)Y + \mu(Y)X \tag{25}$$

and

$$h'(X, Y) = h(X, Y)\,. \tag{26}$$

From (16) and the corresponding equation for $\tilde{\nabla}'$ we obtain

$$\tau'(X) = \tau(X) + \mu(X)\,. \tag{27}$$

Thus the cubic form C' resulting from $\tilde{\nabla}'$ is given by

$$\begin{aligned}C'(X, Y, Z) &= (\nabla'_X h')(Y, Z)+\tau'(X)h'(Y, Z)\\ &= Xh(Y, Z)-h(\nabla'_X Y, Z)-h(Y, \nabla'_X Z)+(\tau(x)+\mu(X))h(Y, Z)\\ &= Xh(Y, Z)-h(\nabla_X Y, Z)-\mu(X)h(Y, Z)-\mu(Y)h(X, Z)-h(Y, \nabla_X Z)\\ &\quad -\mu(X)h(Y, Z)-\mu(Z)h(Y, X)+\tau(X)h(Y, Z)+\mu(X)h(Y, Z)\\ &= (\nabla_X h)(Y, Z)+\tau(X)h(Y, Z)-\mu(X)h(Y, Z)\\ &\quad -\mu(Y)h(X, Z)-\mu(Z)h(Y, X)\,,\end{aligned}$$

that is,

$$C'(X, Y, Z) = C(X, Y, Z)-\mu(X)h(Y, Z)-\mu(Y)h(X, Z)-\mu(Z)h(Y, X)\,. \tag{28}$$

Thus if (23) holds for C, then a similar equation holds for C' with ρ replaced by $\rho-\mu$. Thus the property $h|C$ implies $h'|C'$. □

Recall the notion of the rank for an immersion $M\to(\tilde{M}, \tilde{P})$ in the remark before Proposition 6.

We shall now prove

THEOREM 12. *Let f be an immersion of an n-dimensional connected differentiable manifold into $\boldsymbol{RP}^{n+1}$ such that the rank of h is $\geqq 2$. Then $f(M)$ lies in a quadric Q^n in $\boldsymbol{RP}^{n+1}$ if and only if f has the property that $h|C$.*

PROOF. Assume the property $h|C$. For each point $x\in M$, there is a neighborhood U of x such that $f(U)$ is contained in an affine space $A^{n+1}=\boldsymbol{RP}^{n+1}-H$, where H is a projective hyperplane, say, $x_0=0$. The flat affine connection $\tilde{\nabla}_0$ belongs to the atlas of local connections for $\boldsymbol{RP}^{n+1}$. Now for $f: U\to(A^{n+1}, \tilde{\nabla}_0)$, the conditions $h|C$ and rank $h\geqq 2$ are satisfied by the assumptions of the theorem. By Theorem 10 in [**6**], we see that $f(U)$ lies in a quadric in A^{n+1} and hence in a quadric in $\boldsymbol{RP}^{n+1}$. Since f is locally an immersion into a quadric in $\boldsymbol{RP}^{n+1}$, it follows that it is so globally.

The converse part of the theorem also follows from Theorem 10 of [**6**] and the proof is omitted. □

5. Effect of a projective change on a nondegenerate hypersurface.

Let $(\tilde{M}, \tilde{P})$ be a manifold of dimension $n+1$ with a projective structure and let $f: M\to\tilde{M}$ be an immersion of a manifold of dimension n. We assume that the rank of f is n at every point, that is, f is nondegenerate. Unlike the case of a nondegenerate immersion into a manifold with an equiaffine structure which determines a unique equiaffine structure on the hypersurface, we cannot determine a projective structure on the hypersurface. We have already shown that a certain property such as $h|C$ is an invariant notion for f.

Let $x_0 \in M$. As soon as we pick a local equiaffine connection $\tilde{\nabla} \in (\tilde{M}, \tilde{P})$ with a parallel volume element $\tilde{\omega}$ in a neighborhood $\tilde{U}$ of $f(x_0)$, we can consider a neighborhood U of x_0 as a nondegenerate hypersurface in $(\tilde{U}, \tilde{\nabla})$. By the classical procedure due to Blaschke, we can get an equiaffine structure (∇, ω) in U. Together with this structure we get the fundamental form h, the cubic form $C=\nabla h$, the Levi-Civita connection $\hat{h}$, the difference tensor K between ∇ and $\hat{\nabla}$, and so on. The question is how these quantities change as we pick another $\tilde{\nabla}' \in (\tilde{M}, \tilde{P})$ with a parallel volume element $\tilde{\omega}'$.

In this case, we have

$$\tilde{\omega}' = \varphi\tilde{\omega}, \qquad \text{where } \varphi>0 \tag{29}$$

$$\tilde{\nabla}'_X Y = \tilde{\nabla}_X Y + \mu(X)Y + \mu(Y), \qquad \text{where } \mu = d(\ln\varphi)/(n+2). \tag{30}$$

Relative to the affine connection $\tilde{\nabla}$, let ξ, h, ∇ and ω be the affine normal, the affine metric the induced connection, and the induced volume element (equal to the volume element for h) for M. In order to obtain the corresponding objects for M relative to the affine connection $\tilde{\nabla}'$, let us take ξ as a tentative choice as transversal vector field and follow the standard procedure described in [3]. We write

$$\tilde{\nabla}'_X Y = \nabla^{\#}_X Y + h^{\#}(X, Y)\xi. \tag{31}$$

Because of (30) we see that

$$\nabla^{\#}_X Y = \nabla_X Y + \mu(X)Y + \mu(Y)X \tag{32}$$

$$h^{\#}(X, Y) = h(X, Y). \tag{33}$$

Also from

$$\begin{aligned}\tilde{\nabla}'_X \xi &= \tilde{\nabla}_X \xi + \mu(X)\xi + \mu(\xi)X = -SX + \mu(X)\xi + \mu(\xi)X \\ &= -S^{\#}X + \tau^{\#}(X)\end{aligned}$$

we obtain

$$\tau^{\#}(X) = \mu(X) \tag{34}$$

$$S^{\#} = S - \mu(\xi)I \qquad (I\text{: identity}). \tag{35}$$

The volume element $\theta^{\#}$ given by

$$\theta^{\#}(X_1, \cdots, X_n) = \tilde{\omega}'(X_1, \cdots, X_n, \xi) \tag{36}$$

is equal to $\varphi\omega$. Let $\{X_1, \cdots, X_n\}$ be a basis in $T_x(M)$ with $\theta^{\#}(X_1, \cdots, X_n)=1$. Then

$$\omega(\varphi^{1/n}X_1, \cdots, \varphi^{1/n}X_n) = 1, \quad h_{ij} = h(\varphi^{1/n}X_i, \varphi^{1/n}X_j) = \varphi^{2/n}h(X_i, X_j)$$

so that

$$h^{\#}_{ij} = h^{\#}(X_i, X_j) = h(X_i, X_j) = \varphi^{-n/2} h_{ij}.$$

Setting $H^{\#}=\det[h^{\#}_{ij}]$ we get

$$H^{\#} = (\varphi^{-2/n})^n \det[h_{ij}] = \varphi^{-2},$$

since $\det[h_{ij}]=1$. It follows that the affine metric h' of M relative to $\tilde{\nabla}'$ is given by

$$h' = \varphi^{2/(n+2)} h. \tag{37}$$

In order to find the affine normal vector ξ' of M relative to $\tilde{\nabla}'$ we set

$$\xi' = Z + \varphi^{-2/(n+2)} \xi \tag{38}$$

and choose the tangent vector Z in such a way that $\tilde{\nabla}'_X \xi'$ is tangent to M. Such Z is determined by the following equation to be satisfied for all X:

$$X(\varphi^{-2/(n+2)}) + h(X, Z) + \varphi^{-2/(n+2)} \tau'(X) = 0. \tag{39}$$

The first term equals $-2X\varphi/(n+2)\varphi$. Also using $\mu=d(\ln\varphi)/(n+2)$ and (34), our equation becomes

$$h(X, Z) = \varphi^{-2/(n+2)} \mu(X). \tag{40}$$

If we introduce a vector field U by

(41) $\quad h(X, U) = \mu(X)$ for all X (i.e. U corresponds to μ relative to h),

then we get

$$Z = \varphi^{-2/(n+2)} U \tag{42}$$

and the affine normal ξ' is given by

$$\xi' = \varphi^{-2/(n+2)} (U + \xi). \tag{43}$$

Finally, the affine connection ∇' induced on M by $(\tilde{\nabla}', \tilde{\omega}')$ is given by

$$\nabla'_X Y = \nabla_X Y + \mu(X) Y + \mu(Y) X - h(X, Y) U. \tag{44}$$

This can be easily verified by using (30) and (43).

REMARK. (44) is exactly the same as a general formula for the change of the Levi-Civita connection when a metric h is changed conformally to $\varphi^2 h$, the 1-form μ being $d \ln\varphi$ (see, for example, [7]).

If h is changed to $h'=(\varphi^{1/(n+2)})^2$ as in (37), then $\mu=(d\ln\varphi)/(n+2)$, exactly as in (30). From this fact we get the following theorem.

THEOREM 13. *When an equiaffine structure* $(\tilde{\nabla}, \tilde{\omega})$ *in the ambiant manifold* $\tilde{M}$ *is changed projectively to an equiaffine structure* $(\tilde{\nabla}', \tilde{\omega}')$, *the difference tensor* K *of the induced connection* ∇ *and the Levi-Civita connection* $\hat{\nabla}$ *for the affine metric* h *of a nondegenerate hypersurface* M *in* $\tilde{M}$ *does not change. The cubic form* C *changes conformally with the same factor as the conformal change of the affine metric.*

PROOF. For the cubic form C, recall that $C(X, Y, Z)=-2h(K_X Y, Z)$. $\square$

REMARK. The same conformal change of the affine metric and the cubic form for a nondegenerate hypersurface in $\boldsymbol{R}P^{n+1}$ comes up in the projective theory described by using moving frames (see [8]).

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