J. Math. Soc. Japan
Vol. 41, No. 3, 1989

# Closed orbits of non-singular Morse-Smale flows on $S^3$

By Masaaki WADA

(Received Sept. 19, 1986)
(Revised March 11, 1988)

The closed orbits of a non-singular Morse-Smale flow ([8], p. 798) on $S^3$ form an indexed link, that is, a link with the index 0, 1 or 2 attached to each component. Although closed orbits are naturally oriented, we do not consider oriented links since the orientation of a closed orbit of a non-singular Morse-Smale flow can be easily reversed by modifying the flow near the closed orbit.

In this paper, we characterize the set of indexed links which arise as the closed orbits of a non-singular Morse-Smale flow on $S^3$ in terms of a generator and six operations. The generator is the Hopf link with indices 0 and 2 attached to the components, and the operations are, roughly speaking, split sum, connected sum, and cabling.

Since the author first obtained the result, several papers dealing with the topic have appeared ([6], [7], [9]). Of these, the works of Sasano [7] and Yano [9] were independently done, and are contained in the results in this paper.

The author is greatly thankful to the referee for many helpful suggestions.

## § 1. Results.

The Hopf link with indices 0 and 2 attached to the components is called the (0, 2)-Hopf link. We prove the following:

THEOREM. *Every indexed link which consists of all the closed orbits of a non-singular Morse-Smale flow on $S^3$ is obtained from (0, 2)-Hopf links by applying the following six operations. Conversely, every indexed link obtained from (0, 2)-Hopf links by applying the operations is the set of all the closed orbits of some non-singular Morse-Smale flow on $S^3$.*

OPERATIONS. For given indexed links $l_1$ and $l_2$, we define six operations as follows. We denote by $l_1 \cdot l_2$ the split sum of $l_1$ and $l_2$, and by $N(k, M)$ the regular neighborhood of $k$ in $M$. For other terminologies of knot theory, refer to [5].

I. To make $l_1 \cdot l_2 \cdot u$, where $u$ is an unknot with index 1.

II. To make $l_1 \cdot (l_2 \setminus k_2) \cdot u$, where $k_2$ is a component of $l_2$ of index 0 or 2.

III. To make $(l_1 \setminus k_1) \cdot (l_2 \setminus k_2) \cdot u$, where $k_1$ is a component of $l_1$ of index 0, and $k_2$ is a component of $l_2$ of index 2.

IV. To make $(l_1 \# l_2) \cup m$. The connected sum $l_1 \# l_2$ is obtained by composing a component $k_1$ of $l_1$ and a component $k_2$ of $l_2$ each of which has index 0 or 2. The index of the composed component $k_1 \# k_2$ is equal to either $\text{ind}(k_1)$ or $\text{ind}(k_2)$. Finally, $m$ is a meridian of $k_1 \# k_2$, and is of index 1.

V. Choose a component $k_1$ of $l_1$ of index 0 or 2, and replace $N(k_1, S^3)$ by $D^2 \times S^1$ with three indexed circles in it; $\{0\} \times S^1$, $k_2$, and $k_3$. Here, $k_2$ and $k_3$ are parallel $(p, q)$-cables on $\partial N(\{0\} \times S^1, D^2 \times S^1)$. The indices of $\{0\} \times S^1$ and $k_2$ are either 0 or 2, and one of them is equal to $\text{ind}(k_1)$. The index of $k_3$ is 1.

VI. Choose a component $k_1$ of $l_1$ of index 0 or 2. Replace $N(k_1, S^3)$ by $D^2 \times S^1$ with two indexed circles in it; $\{0\} \times S^1$, and the $(2, q)$-cable $k_2$ of $\{0\} \times S^1$. The index of $\{0\} \times S^1$ is 1, and $\text{ind}(k_2) = \text{ind}(k_1)$.

We prove the theorem in § 3. As an easy consequence of our Lemma 1 in § 2 and Corollary 2.5 in [**3**], we also get the following:

COROLLARY. *Every link which consists of closed orbits of a non-singular Morse-Smale flow on $S^3$ is a graph link* [**4**]. *Conversely, given a graph link, we can always construct a non-singular Morse-Smale flow for which each component of the link is a closed orbit.*

## § 2. Round handle decompositions of $S^3$.

The proof of our theorem is based on round handle decomposition. We have the following by Asimov [**1**]:

PROPOSITION. *If a manifold $M$ admits a round handle decomposition*

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_n = M\,,$$

*there is a non-singular Morse-Smale flow on $M$ such that* (1) *the closed orbits of the flow coincide with the cores of round handles, and* (2) *the flow is pointing outward on $\partial M_j$. Conversely, if $M$ has a non-singular Morse-Smale flow, then $M$ admits a round handle decomposition satisfying* (1) *and* (2).

We will analyze round handle decompositions of $S^3$ together with the cores. For our purpose, it is more convenient to use the fat round handle decomposition described in [**3**], § 3:

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_n = S^3\,,$$
$$M_i = \bigcup_{j=1}^{i} C_j \qquad (i=1, 2, \cdots, n)\,.$$

Each $C_j$ has the form

$$C_j = A\times[0, 1]\bigcup_{\varphi} B_s\oplus B_u\,,$$

where $A$ is a union of components of $\partial M_{j-1}$, $B_s\oplus B_u$ is the Whitney sum of disk bundles $B_s$ and $B_u$ over $S^1$, and the image of $\varphi:(\partial B_s)\oplus B_u\to A\times\{1\}$ intersects every component of $A\times\{1\}$. Let us put $\partial_- C_j = A\times\{0\}$, and consider $C_j$ together with $\partial_- C_j$ and the core $k_j$ which is the 0-section of $B_s\oplus B_u$. The component $C_j$ associated to a round 0- or 2-handle is just a solid torus.

LEMMA 1. *The triple $(C_j, \partial_- C_j, k_j)$ associated to a round* 1-*handle is of one of the following types:*

(a) $C\cong T_1\times[0,1]\# T_2\times[0,1]$ *where $T_1$ and $T_2$ are tori,* $\partial_- C = T_1\times\{0\}\cup T_2\times\{0\}$, *and $k$ is an unknot in $C$.*

(b) $C\cong T^2\times[0, 1]\# D^2\times S^1$, $\partial_- C = T^2\times\{0\}$ *or* $T^2\times\{0\}\cup\partial D^2\times S^1$, *and $k$ is an unknot in $C$.*

(c) $C\cong V_1\# V_2$ *where $V_1$ and $V_2$ are solid tori,* $\partial_- C = \partial V_1$, *and $k$ is an unknot in $C$.*

(d) $C\cong F\times S^1$ *where $F$ is a disk with two holes, $\partial_- C$ is a component or a union of two components of $\partial C$, and $k = *\times S^1$ for some point $*$ in* Int $F$.

(e) $C\cong D^2\times S^1\setminus \mathrm{Ind} W$ *where $W$ is a tubular neighborhood of the* (2, 1)-*cable of* $\{0\}\times S^1$ *in* $D^2\times S^1$, $\partial_- C = \partial W$, *and* $k = \{0\}\times S^1$.

PROOF. We make case-by-case observations of the attaching map

$$\varphi:(\partial B_s)\oplus B_u \longrightarrow A\times\{1\}.$$

The image of $\varphi$ is an annulus or a union of two annuli, depending on whether the round handle is untwisted or twisted. Since each component of $\partial C$ is a torus (Lemma 3.1, [3]), $A$ is a torus or a union of two tori. We call a component of $\varphi(\partial B_s\oplus\{0\})$ an attaching circle.

Let us first suppose that the round 1-handle is untwisted. Then, it is diffeomorphic to $[-1, 1]_s\times[-1, 1]_u\times S^1$. We denote the two attaching circles by $c_1 = \varphi(\{-1\}\times\{0\}\times S^1)$ and $c_2 = \varphi(\{1\}\times\{0\}\times S^1)$.

*Case* 1. Suppose that $c_1$ and $c_2$ are contained in different components. Then, $A$ is a disjoint union of two tori $T_1$ and $T_2$, where $c_j\subset T_j$ ($j=1, 2$).

*Case* 1.1. If both $c_1$ and $c_2$ are essential, we get type (d), where $\partial_- C$ is a union of two components of $\partial C$.

*Case* 1.2. If one of $c_1$ and $c_2$ is essential and the other is inessential, we get type (b), where $\partial_- C = T^2\times\{0\}\cup\partial D^2\times S^1$.

*Case* 1.3. Suppose that both $c_1$ and $c_2$ are inessential. For $j=1, 2$, let $D_j$ be a 2-disk in $T_j\times\{1\}$ which bounds $c_j$. We may assume that $D_1$ contains $\varphi(\{-1\}\times\{-1\}\times S^1)$. Then, $D_2$ must contain $\varphi(\{1\}\times\{1\}\times S^1)$, for otherwise we would have a 2-sphere as a component of $\partial C$. Let $D_j'$ be a properly embedded 2-disk in $T_j\times[0, 1]$ which bounds $c_j$. Then, $C$ splits along the 2-sphere $D_1'\cup([-1, 1]\times\{0\}\times S^1)\cup D_2'$ as a connected sum of $T_1\times[0, 1]$ and $T_2\times[0, 1]$. This leads to (a).

*Case* 2. Suppose that $c_1$ and $c_2$ are contained in the same component.

*Case* 2.1. If both $c_1$ and $c_2$ are essential, they are parallel circles in $T^2\times\{1\}$. Let $E$ be a properly embedded annulus in $T^2\times[0, 1]$ which bounds $c_1\cup c_2$. Since the surface $E\cup([-1, 1]\times\{0\}\times S^1)$ embeds in $S^3$, it can not be a Klein bottle, hence is a torus, and is 2-sided. From this fact, the attaching map $\varphi$ is determined up to diffeomorphisms of $T^2\times\{1\}$. We then get type (d), where $\partial_- C$ is a component of $\partial C$.

*Case* 2.2. Suppose that one of $c_1$ and $c_2$ is essential and the other is inessential. In this case, the attaching map $\varphi$ is unique up to diffeomorphisms of $T^2\times\{1\}$, since there is a diffeomorphism of $T^2\times\{1\}$ to itself which preserves $c_1$ and $c_2$ setwisely, preserves an orientation of the essential one, and reverses an orientation of the other. We then get (c).

*Case* 2.3. Suppose that both $c_1$ and $c_2$ are inessential. For $j=1, 2$, let $D_j$ be a 2-disk in $T^2\times\{1\}$ which bounds $c_j$. We first assume that $D_1$ and $D_2$ are disjoint. Let $E$ be a properly embedded annulus in $T^2\times[0, 1]$ which bounds $c_1\cup c_2$. From that $E\cup([-1, 1]\times\{0\}\times S^1)$ is a 2-sided torus, the attaching map $\varphi$ is determined up to diffeomorphisms of $T^2\times\{1\}$. But in this case, a 2-sphere appears as a component of $\partial C$. Therefore, $D_1$ and $D_2$ must intersect. We may assume that $D_1$ contains $D_2$. Since $(D_1\setminus \mathrm{Int}\, D_2)\cup([-1, 1]\times\{0\}\times S^1)$ is a 2-sided torus, $C$ is uniquely determined, and is of type (b) where $\partial_- C=T^2\times\{0\}$.

If the round 1-handle is twisted, $B_s$ and $B_u$ are non-orientable $D^1$-bundles over $S^1$. The attaching circle is $c=\varphi(\partial B_s\oplus\{0\})$. If $c$ bounded a 2-disk $D$ in $T^2\times\{1\}$, then $D\cup(B_s\oplus\{0\})$ would be a project plane embedded in $S^3$. Hence, $c$ is essential in $T^2\times\{1\}$. This leads to (e).

These cases cover all the possibilities, and this completes the proof.

In each case, it is easily verified that both $C$ and $C\setminus \mathrm{Int}\, N(k, C)$ are graph manifolds. This proves the former part of the corollary.

## § 3. Proof of Theorem.

We denote by $r$ the number of closed orbits of index 1 of the non-singular Morse-Smale flow. We prove the former part of our theorem by induction on $r$. Associated to the non-singular Morse-Smale flow is a decomposition

$$S^3 = \bigcup_{j=1}^{n} C_j .$$

We denote by $l$ the indexed link which consists of the cores of this decomposition.

First, if there is no closed orbit of index 1, it is easily seen that $n=2$, $C_1$ is a round 2-handle, and $C_2$ is a round 0-handle. Hence, $l$ is a (0, 2)-Hopf link.

Let us assume $r \geqq 1$ and that every indexed link which consists of all the closed orbits of a non-singular Morse-Smale flow and which has less than $r$ components of index 1 is obtained from (0, 2)-Hopf links by applying the operations I–VI. Since $r \geqq 1$, there is a component $C=C_j$ associated to a round 1-handle. We divide the proof into five cases according to the type of $C$ in Lemma 1.

*Case* (a). Suppose that $C \cong T_1 \times [0, 1] \# T_2 \times [0, 1]$ where $T_1$ and $T_2$ are tori. For $j=1, 2$, let $N_{j-}$ and $N_{j+}$ be the components of the complement of $C$ in $S^3$ which bound $T_j \times \{0\}$ and $T_j \times \{1\}$ respectively. Since the 2-sphere splitting $C$ as a connected sum of $T_1 \times [0, 1]$ and $T_2 \times [0, 1]$ bounds 3-balls on both sides by Schönflies' theorem, we see that $C \cup N_{2-} \cup N_{2+} \cong T^2 \times [0, 1]$. Therefore, $N_{1-} \cup N_{1+} \cong S^3$. This gives a round handle decomposition of $S^3$. Let $l_1$ denote the indexed link which consists of the cores of this round handle decomposition. Similarly, we get $N_{2-} \cup N_{2+} \cong S^3$. Let $l_2$ be the indexed link which consists of the cores of this round handle decomposition. Both $l_1$ and $l_2$ have fewer components of index 1 than $l$. By the assumption of induction, $l_1$ and $l_2$ are obtained from (0, 2)-Hopf links by applying operations I–VI. We have $l=l_1 \cdot l_2 \cdot u$, where $u$ is an unknot with index 1. Namely, $l$ is obtained from $l_1$ and $l_2$ by the operation I.

*Case* (b). Suppose that $C \cong T^2 \times [0, 1] \# D^2 \times S^1$. Let $N_-$, $N_+$, and $N_0$ be the components of the complement of $C$ in $S^3$ whose boundaries are $T^2 \times \{0\}$, $T^2 \times \{1\}$, and $\partial D^2 \times S^1$ respectively. In the same manner as in Case (a), we get $N_- \cup N_+ \cong S^3$. Let $l_1$ denote the indexed link which consists of the cores of this round handle decomposition. We also obtain $C \cup N_- \cup N_+ \cong D^2 \times S^1$. Hence $N_0$ together with a round $i$-handle $(D^2 \times S^1, k_2)$ form a round handle decomposition of $S^3$, where $i=0$ or 2 according as $\partial D^2 \times S^1 \subset \partial_- C$ or not. Let $l_2$ denote the indexed link which consists of the cores of this decomposition. Then, $l=l_1 \cdot (l_2 \setminus k_2) \cdot u$.

*Case* (c). Suppose that $C \cong V_1 \# V_2$, where $V_1$ and $V_2$ are solid tori. Let $N_1$ and $N_2$ be the components of the complement of $C$ in $S^3$ which bound $\partial V_1$ and $\partial V_2$ respectively. Since $C \cup N_2 \cong D^2 \times S^1$, we can construct a 3-sphere by attaching a round 0-handle $(D^2 \times S^1, k_1)$ to $N_1$. Let $l_1$ denote the indexed link which consists of the cores of this round handle decomposition. Similarly, let $l_2$ denote the indexed link which consists of the cores of the round handle decomposition made by attaching $N_1$ and a round 2-handle $(D^2 \times S^1, k_2)$ to each other. Then,

$l=(l_1 \searrow k_1)\cdot(l_2 \searrow k_2)\cdot u$.

So far we have proved the induction step for the cases where there is a component $C$ of type (a), (b), or (c). Now, let us assume that there is no component of type (a), (b), or (c). To proceed further, we have to choose $C$ more carefully.

Let $\mathfrak{N}$ be the collection of all the submanifolds of $S^3$ each of which is a union of $C_j$'s including at least one $C_j$ of index 1, and whose boundary is a torus.

ASSERTION 1. *The set $\mathfrak{N}$ contains a solid torus.*

PROOF. Let $C=C_j$ be a component in our decomposition of $S^3$ which is associated to a round 1-handle. First, assume that $C$ is of type (d) in Lemma 1. Namely, $C\cong F\times S^1$ where $F$ is a disk with two holes. Let $\partial_0$, $\partial_1$ and $\partial_2$ be the boundary components of $C$. For $j=0$, 1 and 2, let $N_j$ be the component of the complement of $C$ in $S^3$ whose boundary is $\partial_j$. If one of $N_0$, $N_1$ and $N_2$ is not a solid torus, by the solid torus theorem ([**5**], p. 107), its complement in $S^3$ is a solid torus which belongs to $\mathfrak{N}$. Hence, we may assume that $N_0$, $N_1$ and $N_2$ are solid tori.

We fix our notation for $\pi_1(C)$ as follows: $\partial_1\cap(F\times\{1\})$ represents $d_1$, $\partial_2\cap(F\times\{1\})$ represents $d_2$, $\partial_0\cap(F\times\{1\})$ represents $d_1d_2$, and $*\times S^1$ represents $t$. Then, we have

$$\pi_1(C)\cong\langle d_1,\ d_2,\ t\mid[d_1,\ t]=[d_2,\ t]=1\rangle,$$

where $[d,\ t]$ denotes the commutator of $d$ and $t$.

Suppose that the meridians of $N_0$, $N_1$ and $N_2$ represent $(d_1d_2)^{p_0}t^{q_0}$, $d_1^{p_1}t^{q_1}$, and $d_2^{p_2}t^{q_2}$ respectively. Then, $\pi_1(C\cup N_0\cup N_1\cup N_2)$ is isomorphic to

$$G=\langle d_1,\ d_2,\ t\mid[d_1,\ t]=[d_2,\ t]=(d_1d_2)^{p_0}t^{q_0}=d_1^{p_1}t^{q_1}=d_2^{p_2}t^{q_2}=1\rangle.$$

Since $C\cup N_0\cup N_1\cup N_2\cong S^3$, $G$ is trivial. By the Coxeter's theorem ([**2**], p. 67), the triviality of the group

$$\begin{aligned}G/\langle t\rangle &\cong\langle d_1,\ d_2\mid d_1^{p_1}=d_2^{p_2}=(d_1d_2)^{p_0}=1\rangle\\ &\cong\langle d_1,\ d_2,\ d_3\mid d_1^{p_1}=d_2^{p_2}=d_3^{p_0}=d_1d_2d_3=1\rangle\end{aligned}$$

implies that at least one of $p_1$, $p_2$ and $p_0$ is equal to 1. We may assume $p_1=1$. Since there is a diffeomorphism $f: C\to C$ which preserves $\partial_0$, $\partial_1$ and $\partial_2$ setwisely such that $f_*:\pi_1(C)\to\pi_1(C)$ satisfies $f_*(d_1)=d_1t^{q_1}$, $f_*(d_2)=d_2$, and $f_*(t)=t$, we may also assume that the meridian of $N_1$ represents $d_1$. Then, $C\cup N_1\cong T^2\times[0,1]$. Therefore, $C\cup N_1\cup N_0$ and $C\cup N_1\cup N_2$ are solid tori which belong to $\mathfrak{N}$.

The proof for the case where $C$ is of type (e) is similar to the above.

We take a solid torus $N$ in $\mathfrak{N}$ which consists of the least number of $C_j$'s. Let $C$ be the component $C_j$ contained in $N$ which contains $\partial N$. The component $C$ is associated to a round 1-handle.

*Case* (d). Suppose that $C \cong F \times S^1$ where $F$ is a disk with two holes. Let us follow the same notation as in the proof of Assertion 1, and assume that $\partial N = \partial_0$. Then, $N = C \cup N_1 \cup N_2$.

ASSERTION 2. *Either $N_1$ or $N_2$ is a solid torus.*

PROOF. If both $N_1$ and $N_2$ are not solid torus, they are non-trivial knot complements. Hence, $\partial_1$ and $\partial_2$ are incompressible in $N_1$ and $N_2$ respectively. Since $\partial_1$ and $\partial_2$ are also incompressible in $C$, every incompressible surface in $C$ is incompressible in $N = C \cup N_1 \cup N_2$. Especially, $\partial_0$ is incompressible in $N$. This contradicts the fact that $N$ is a solid torus.

Then, there are two possibilities.

*Case* (d.1). Suppose that one of $N_1$ and $N_2$ is a solid torus, and the other is not.

We may assume that $N_2$ is a solid torus. Since $N$ consists of the least number of $C_j$'s, $N_2$ is not in $\mathfrak{N}$. Hence, $N_2$ is either a round 0-handle or a round 2-handle depending on whether $\partial_2 \subset \partial_- C$ or not.

Let $\alpha$ be an essential arc properly embedded in $F \times \{1\}$ whose end points lie in $\partial_2$. The properly embedded annulus $E = \alpha \times S^1$ cuts $C$ into two components $P_0$ and $P_1$ which contain $\partial_0$ and $\partial_1$ respectively. The two components of $\partial E$, $a_1$ and $a_2$, are meridians of $N_2$, since otherwise $C \cup N_2$ would be a Seifert fibered space over an annulus, and each component of the boundary would be incompressible. Then, $(C \cup N_2) \cup N_1$ would not be a solid torus. Let $D_1$ and $D_2$ be disjoint meridian disks of $N_2$ bounding $a_1$ and $a_2$ respectively. We also assume for $j=1, 2$ that $D_j \cap k'$ is a point, where $k'$ is the core of $N_2$. Two disks $D_1$ and $D_2$ together split $N_2$ into two 3-balls $B_0$ and $B_1$. Since $\partial(N_0 \cup P_0 \cup B_0) = E \cup D_1 \cup D_2$ is a 2-sphere, by the Schönflies' theorem, $N_0 \cup P_0 \cup B_0$ is a 3-ball.

We can make a 3-sphere by attaching the standard ball pair $(D^3, D^1)$ to $(N_0 \cup P_0 \cup B_0, k' \cap B_0)$ by a diffeomorphism $\varphi : (\partial D^3, \partial D^1) \to (E \cup D_1 \cup D_2, k' \cap (D_1 \cup D_2))$. Then, $P_0 \cup B_0 \cup D^3$ is a solid torus, and $(k' \cap B_0) \cup D^1$ is its core. Regard this solid torus as a round 0-handle or a round 2-handle according as $\partial_0 \subset \partial_- C$ or not. Then, we obtain a round handle decomposition for $S^3$. Let $l_1$ denote the indexed link which consists of the cores of this round handle decomposition.

Since $N_1 \cup P_1 \cup B_1$ is also a 3-ball, we can obtain an indexed link $l_2$ in the same way as the above. We see that $l$ is obtained from $l_1$ and $l_2$ by the operation IV.

*Case* (d.2). Suppose that both $N_1$ and $N_2$ are solid tori. Since $N_j$ $(j=1, 2)$

contains fewer $C_j$'s than $N$, $N_j$ does not contain a piece of index 1. Hence, $N_j$ is a round 0-handle or a round 2-handle according as $\partial_j \subset \partial_- C$ or not. Let us assume that the meridians of $N_1$ and $N_2$ represent $d_1^{p_1}t^{q_1}$ and $d_2^{p_2}t^{q_2}$ respectively. Then, $\pi_1(N)$ is isomorphic to

$$G = \langle d_1, d_2, t \mid [d_1, t]=[d_2, t]=d_1^{p_1}t^{q_1}=d_2^{p_2}t^{q_2}=1\rangle .$$

Since $N$ is a solid torus, $G$ is isomorphic to $\boldsymbol{Z}$, and

$$G/\langle t\rangle \cong \langle d_1, d_2 \mid d_1^{p_1}=d_2^{p_2}=1\rangle \cong \boldsymbol{Z}/p_1 * \boldsymbol{Z}/p_2$$

is a factor group of $\boldsymbol{Z}$. Hence, either $p_1$ or $p_2$ is equal to 1. We may assume $p_2=1$. We may also assume that the meridian of $N_2$ represents $d_2$, and hence $C\cup N_2 \cong T^2\times[0, 1]$. Therefore, $N$ is as an indexed link equivalent to $D^2\times S^1$ with three indexed circles $\{0\}\times S^1$, $k_2$ and $k_3$, where $k_2$ and $k_3$ are parallel $(p, q)$-cables on $\partial N(\{0\}\times S^1)$. It is now easily seen that $l$ is obtained from an indexed link which has fewer components of index 1 than $l$ by applying the operation V.

*Case* (e). In this case, we put $P=C\setminus \operatorname{Int} N(k, C)$. Then, $P\cong F\times S^1$, where $F$ is a disk with two holes. Let $N_2$ be the complement of $C$ in $N$. We denote the components of $\partial P$ by $\partial_0$, $\partial_1$ and $\partial_2$, so that $\partial_0=\partial N$, $\partial_1=\partial N(k, C)$, and $\partial_2=\partial N_2$. We may assume that the meridian of $N(k, C)$ represents $d_1^2 t$. If $N_2$ is not a solid torus, the same argument as in Case (d.1) shows that $*\times S^1$ represents $d_1$. But, this contradicts the fact that the meridian of $N(k, C)$ represents $d_1^2 t$. Hence, $N_2$ is a solid torus. Let $d_2^p t^q$ be represented by the meridian of $N_2$. We can show as in Case (d.2) that $p=1$. We obtain $P\cup N_2\cong T^2\times[0, 1]$. Therefore, $C\cup N_2$ is equivalent to $D^2\times S^1$ with two indexed circles $\{0\}\times S^1$ of index 1, and $k_2$, the $(2, q)$-cable around $\{0\}\times S^1$ of index $i$, where $i=0$ or 2 according as $\partial_0\subset\partial_- C$ or not. We can construct a round handle decomposition of $S^3$ by replacing $C\cup N_2$ by a round $i$-handle $(D^2\times S^1, k_1)$. Let $l_1$ denote the indexed link which consists of the cores of this round handle decomposition. Then, $l$ is obtained from $l_1$ by the operation VI.

It only remains to prove the latter part of Theorem. Obviously, (0, 2)-Hopf link is the set of cores of a round handle decomposition of $S^3$.

Suppose that we have two round handle decomposition of $S^3$,

$$S^3 = \bigcup_{j=1}^{s} C_j$$

whose cores form $l_1$, and

$$S^3 = \bigcup_{j=1}^{t} C'_j$$

whose cores form $l_2$. First, suppose that $l$ is obtained from $l_1$ and $l_2$ by the operation I. Replace $C_1$ by $C_1\cup(\partial C_1\times[0, 1])$, and $C'_1$ by $C'_1\cup(\partial C'_1\times[0, 1])$. Then, perform the connected sum operation using $\partial C_1 \times [0, 1]$ and $\partial C_1' \times [0, 1]$. We obtain a decomposition

$$S^3 = C_1 \cup C_1' \cup (\partial C_1 \times [0, 1] \# \partial C_1' \times [0, 1]) \cup \left(\bigcup_{j=2}^{s} C_j\right) \cup \left(\bigcup_{j=2}^{t} C_j'\right).$$

If we regard $(\partial C_1 \times [0, 1] \# \partial C_1' \times [0, 1])$ in this decomposition as $C$ of type (a) in Lemma 1, we have a round handle decomposition of $S^3$. The set of cores of this round handle decomposition is $l = l_1 \cdot l_2 \cdot u$. A similar argument applies to $l$ obtained by the operation II, III, or IV by using $C$ of type (b), (c) or (d) respectively. For $l$ obtained from $l_1$ by applying V or VI, it is easy to construct a round handle decomposition of $S^3$ which has $l$ as the set of cores, since there is a round handle decomposition of $D^2 \times S^1$ whose cores are $\{0\} \times S^1$, $k_2$ and $k_3$ defined in V, or $\{0\} \times S^1$ and $k_2$ defined in VI.

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Masaaki WADA

Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104
U. S. A