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Rings of automorphic forms which are not Cohen-Macaulay, II

By Shigeaki TSUYUMINE

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In [2], [3], Eichler posed the question whether or not a ring of automorphic forms (particularly, Hilbert and Siegel modular forms) is Cohen-Macaulay (C.-M. for short). Freitag [4] first gave the negative answer to the question in the case of a ring of Hilbert modular forms of dimension $\geqq 3$. In our previous papers [18], [19] we have surveyed the question, and as for Siegel modular forms we have got the following results. Let $\Gamma_n := Sp_{2n}(\boldsymbol{Z})$, and let $\Gamma_n(l)$ be its congruence subgroup of level l; $\{M \in \Gamma_n \mid M \equiv 1_{2n} \bmod l\}$. For a congruence subgroup Γ of $Sp_{2n}(\boldsymbol{Z})$ let $A(\Gamma) = \bigoplus_{k \geqq 0} A(\Gamma)_k$ denote the graded ring of Siegel modular forms for Γ, $A(\Gamma)_k$ being the vector space of modular forms of weight k. Let $A(\Gamma)^{(r)}$ denote the ring $\bigoplus_{k \equiv 0 (r)} A(\Gamma)_k$ for an integer r. Then

(i) $A(\Gamma_2(l))^{(r)}$ is not C.-M. for any r if $l \geqq 6$.

(ii) Let Γ be a neat congruence subgroup of $Sp_{2n}(\boldsymbol{R})$ with $n \geqq 3$. Then $A(\Gamma)^{(r)}$ is not C.-M. for any r.

(iii) $A(\Gamma_n)^{(r)}$ is not C.-M. for any r if $n \geqq 4$.

Concerning $A(\Gamma_n)$ $(n \geqq 1)$, it is only a remaining problem if $A(\Gamma_3)^{(r)}$ is C.-M., since $A(\Gamma_1)^{(r)}$, $A(\Gamma_2)^{(r)}$ are known to be C.-M. for any r, or at least it is an easy consequence of the structure theorems of $A(\Gamma_1)$, $A(\Gamma_2)$ (cf. Igusa [12], [13]). In the present paper we show that *$A(\Gamma_3)^{(r)}$ is not C.-M. for any r.*

$A(\Gamma_n)$ $(n \geqq 3)$ has been shown to be U.F.D. by Freitag [5], [6] (cf. Tsuyumine [20]), and so they furnish negative examples of the question whether U.F.D. is C.-M. which is posed by Samuel [16]. In the case of characteristic 0, Freitag and Kiehl [7] first gave the negative example to this question (see also S. Mori [15]).

Our method to prove that $A(\Gamma_3)^{(r)}$ is not C.-M. is as follows. If $A(\Gamma_3)^{(r)}$ is C.-M., then the Satake compactification X_3^* of the quotient space H_3/Γ_3 would be a C.-M. variety, and so the Serre duality would hold on it. Then $\dim H^6(X_3^*, \mathcal{O}_{X_3^*})$ must be equal to one since $H^6(X_3^*, \mathcal{O}_{X_3^*})$ is dual to the group of global sections of the coherent sheaf on X_3^* corresponding to modular forms of weight four, and since there is the unique modular form of weight four up to constant multiples. Thus to prove our assertion it is enough to show the vanishing of $H^6(X_3^*, \mathcal{O}_{X_3^*})$, which is done by making use of Igusa's desingularization [**14**].

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1. Let $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{C}$ denote as usual, the ring of integers, the real number field, the complex number field respectively. We denote by $\boldsymbol{M}_{k,l}(*)$, the set of $k\times l$ matrices with entries in $*$, and by $\boldsymbol{M}_k(*)$, $\boldsymbol{SM}_k(*)$, the set of square matrices of size k, the set of symmetric matrices of size k respectively. 1_k denotes the identity matrix of size k.

Let H_n be the Siegel space of degree n; $\{Z\in\boldsymbol{M}_n(\boldsymbol{C}) \mid {}^tZ=Z, \operatorname{Im}Z>0\}$, and let Γ be a congruence subgroup of $Sp_{2n}(\boldsymbol{R})$. Γ acts on H_n by the usual modular transformation

$$Z \longrightarrow MZ=(AZ+B)(CZ+D)^{-1}, \qquad M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma.$$

Let f be a holomorphic function on H_n. f is called a *(Siegel) modular form* of weight k for Γ if it satisfies

$$f(MZ)=|CZ+D|^k f(Z) \qquad \text{for } M\in\Gamma$$

(when $n=1$, we need an additional condition that f is holomorphic also at cusps). We denote by $A(\Gamma)=\bigoplus_{k\geq 0}A(\Gamma)_k$ (resp. $S(\Gamma)=\bigoplus_{k>0}S(\Gamma)_k$), the graded ring of modular forms (resp. the graded ideal of cusp forms).

Let X be the quotient space H_n/Γ, and X^* be its Satake compactification, which is a normal projective variety isomorphic to $\operatorname{Proj}(A(\Gamma))$. Set-theoretically X^* is the union of X and of the similar pieces H_{n_1}/Γ' $(n_1<n)$ as X. H_{n_1}/Γ' as a cusp of X^*, is called an *n_1-cusp*. Up to conjugacy, we may assume that this cusp is corresponding to the limit of

$$\begin{pmatrix}Z_1 & 0\\ 0 & \lambda 1_{n_2}\end{pmatrix}, \qquad Z_1\in H_{n_1}, \quad n_1+n_2=n,$$

as $\lambda\to\sqrt{-1}\infty$. Let us decompose $Z\in H_n$ as

$$(1) \qquad \begin{pmatrix}Z_1 & \tau\\ {}^t\tau & Z_2\end{pmatrix}\in H_n, \qquad Z_i\in H_{n_i}, \quad \tau\in\boldsymbol{M}_{n_1,n_2}(\boldsymbol{C}).$$

Let us fix a point x of an n_1-cusp corresponding to $Z_1\in H_{n_1}$. Here we assume that Γ is a normal subgroup of Γ_n. Then Γ_n acts on X^*. The stabilizer subgroup P_{n_1} at x of Γ_n is generated by Γ and matrices of the form

$$(2)\qquad \begin{pmatrix} A_1 & 0 & B_1 & B_{12} \\ A_{21} & A_2 & B_{21} & B_2 \\ C_1 & 0 & D_1 & D_{12} \\ 0 & 0 & 0 & D_2 \end{pmatrix} \in \Gamma_n \quad \text{with } M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \Gamma_{n_1}, \quad M_1 Z_1 = Z_1 .$$

Let W_{n_1} (resp. U_{n_1}) be the group generated by Γ and by matrices of the form

$$\begin{pmatrix} 1_{n_1} & 0 & & {}^t b' \\ v & 1_{n_2} & b & B_2 \\ & & 1_{n_1} & -{}^t v \\ 0 & & & 1_{n_2} \end{pmatrix} \in \Gamma_n \quad \left(\text{resp.} \begin{pmatrix} 1_n & \begin{matrix} 0 & 0 \\ 0 & B_2 \end{matrix} \\ 0 & 1_n \end{pmatrix}\right).$$

Then we have inclusions of normal subgroups;

$$U_{n_1} \subset W_{n_1} \subset P_{n_1} .$$

If $\Gamma=\Gamma_n(l)$, then $U_{n_1}/\Gamma_n(l)$ (resp. W_{n_1}/U_{n_1}, resp. P_{n_1}/W_{n_1}) is canonically isomorphic to $\boldsymbol{SM}_{n_2}(\boldsymbol{Z}/l\boldsymbol{Z}) \cong (\boldsymbol{Z}/l\boldsymbol{Z})^{n_2(n_2+1)/2}$ (resp. $(\boldsymbol{Z}/l\boldsymbol{Z})^{2n_1n_2}$, resp. {the stabilizer subgroup of $\Gamma_{n_1}/\Gamma_{n_1}(l)$ at the image point of Z_1 in $H_{n_1}/\Gamma_{n_1}(l)\} \times GL_{n_2}(\boldsymbol{Z}/l\boldsymbol{Z})$).

2. We discuss the space of differential forms on the fiber space over X^*, which is of use to estimate the dimensions of cohomology groups later. Let n (resp. m) be a positive (resp. nonnegative) integer. Let Γ be a congruence subgroup of $Sp_{2n}(\boldsymbol{R})$. On $H_n \times \boldsymbol{M}_{n,m}(\boldsymbol{C})$, there is a group of automorphisms (M, u), $M \in \Gamma$, $u \in \boldsymbol{M}_{2n,m}(\boldsymbol{Z})$ such that

$$(Z, \zeta) \longrightarrow (M, u)(Z, \zeta) = (MZ, {}^t(CZ+D)^{-1}(\zeta+(Z, 1_n)u)), \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The action is properly discontinuous and the quotient space W' is a normal algebraic variety. W' is a fiber space over $X=H_n/\Gamma$, whose generic fiber is a Kummer variety or an abelian variety according as Γ contains -1_n or not. In the latter case the fiber of a point in X corresponding to a generic $Z \in H_n$, is an abelian variety which is a product of m copies of an abelian variety $\boldsymbol{C}^n/(Z, 1_n)\boldsymbol{Z}^{2n}$.

Let $Z=(z_{ij}) \in H_n$. Let us put $\zeta=(\zeta^1, \cdots, \zeta^m) \in \boldsymbol{M}_{n,m}(\boldsymbol{C})$ and ${}^t\zeta^i=(\zeta_1^i, \cdots, \zeta_n^i)$, and let ω be the differential form

$$\omega = (dz_{11} \wedge dz_{12} \wedge \cdots \wedge dz_{nn}) \wedge \bigwedge_{i=1}^{m} (d\zeta_1^i \wedge \cdots \wedge d\zeta_n^i) .$$

Then the formula $(M, u)\omega = |CZ+D|^{-m-n-1}\omega$ holds for $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let W_0' be the smooth locus of W', and W_{00}', the complement of the set of images of fixed points. Then $W_{00}' \subset W_0'$ ($W_{00}' \subsetneqq W_0'$ occurs possibly only when $n \leqq 2$). If $\tilde{\mathscr{I}}(m+n+1)$

denotes the coherent sheaf on W' which is the inverse image attached to the canonical projection of W' to X, of the coherent sheaf $\mathcal{L}(m+n+1)$ on X corresponding to modular forms of weight $m+n+1$, then $\tilde{\mathcal{L}}(m+n+1)|_{W'_{00}}$ is isomorphic to the canonical invertible sheaf on W'_{00} by $f\to f\omega$, f being a section of $\mathcal{L}(m+n+1)$. Further we have the canonical inclusion

$$K_{W'_0} \hookrightarrow \tilde{\mathcal{L}}(m+n+1)|_{W'_0},$$

$K_{W'_0}$ denoting the canonical invertible sheaf.

LEMMA 1. *Let $\phi: W\to X^*$ be a morphism of projective varieties which is an extension of $W'\to X$. Suppose that W is a* C.-M. *variety and that $W-W_0$ is of codimension at least two, W_0 being the smooth locus of W. Moreover suppose that the fiber of each $(n-1)$-cusp is of codimension one in W. Then*

$$\dim H^{n(n+1)/2+mn}(W, \mathcal{O}_W) \leqq \dim S(\Gamma)_{m+n+1}.$$

PROOF. Since W is a C.-M. variety, we have an isomorphism $H^{n(n+1)/2+mn}(W, \mathcal{O}_W)^{\vee} \cong H^0(W, \omega_W)$ by the Serre duality theorem where ω_W denotes the dualizing sheaf. If K_{W_0} denotes the canonical invertible sheaf on W_0, then the homomorphism $H^0(W, \omega_W)$ to $H^0(W_0, K_{W_0})$ induced by the restriction is an isomorphism by Grauert-Riemenschneider [8], Satz 3.1. So it sufficies to show $\dim H^0(W_0, K_{W_0}) \leqq \dim S(\Gamma)_{m+n+1}$. Let $\eta\in H^0(W_0, K_{W_0})$. On $W_0\cap W'$ we can write as $\eta=f\omega$ with $f\in A(\Gamma)_{m+n+1}$ since $K_{W_0}|_{W_0\cap W'} \cong \tilde{\mathcal{L}}(m+n+1)|_{W_0\cap W'}$. By our assumption, for each $(n-1)$-cusp, there is a nonsingular point w of W which is mapped into the $(n-1)$-cusp by ϕ. We may assume that $\phi(w)$ is a limit point as $Z_2\to\sqrt{-1}\infty$ using the notation (1) with $n_2=1$. f has the Fourier-Jacobi expansion $f(Z)=\sum_{k\geqq 0}\theta_k(Z_1, \tau)\mu^k$ where $\mu=\exp(2\pi\sqrt{-1}az_{nn})$ for a suitable rational number a. Then

$$f\omega = (2\pi\sqrt{-1}a)^{-1}(\mu^{-1}f)(dz_{11}\wedge\cdots\wedge dz_{n\ n-1}\wedge d\mu)\wedge \bigwedge_{i=1}^{m}(d\zeta_1^i\wedge\cdots\wedge d\zeta_n^i).$$

So $f\omega$ is not extendable to a neighborhood at w unless $\theta_0(Z_1, \tau)=0$, in other words, f vanishes at the $(n-1)$-cusp. Thus f is a cusp form. Our assertion follows immediately from this. q. e. d.

REMARK. Let Γ' be a congruence subgroup of $Sp_{2n}(\boldsymbol{R})$ having Γ as a normal subgroup such that Γ'/Γ acts on W. Let V be the quotient of W by Γ'/Γ. Then we have

$$H^{n(n+1)/2+mn}(W, \mathcal{O}_W)^{\Gamma'/\Gamma} \cong H^{n(n+1)/2+mn}(V, \mathcal{O}_V) \hookrightarrow S(\Gamma')_{m+n+1} = S(\Gamma)_{m+n+1}^{\Gamma'/\Gamma}$$

(see Grothendieck [**9**], Cor. to Prop. 5.2.3. and Théorème 5.3.1. for the isomorphism between cohomology groups). In particular, if $S(\Gamma')_{m+n+1}$ is $\{0\}$, then no nontrivial sections of $H^{n(n+1)/2+mn}(W, \mathcal{O}_W)$ are Γ'/Γ-invariant.

3. We put $X_3=H_3/\Gamma_3$, $X_3(l)=H_3/\Gamma_3(l)$ $(l>1)$. X_3^* (resp. $X_3^*(l)$) denotes the Satake compactification of X_3 (resp. $X_3(l)$), and $\bar{X}_3$ (resp. $\bar{X}_3(l)$) denotes Igusa's compactification [14] of X_3 (resp. $X_3(l)$). If $l\geqq 3$, then $\bar{X}_3(l)$ is the monoidal transform of $X_3^*(l)$ along $D^*(l)=X_3^*(l)-X_3(l)$ and it is nonsingular. By construction, obviously Γ_n acts on $\bar{X}_3(l)$ as well as $X_3^*(l)$. $\bar{X}_3$ is given as a quotient space of $\bar{X}_3(l)$ by $\Gamma_3/\Gamma_3(l)$, which is independent of $l\geqq 3$. We put $D^*=X_3^*-X_3$, $D(l)=\bar{X}_3(l)-X_3(l)$. D^* equals $H_2/\Gamma_2\cup H_1/\Gamma_1\cup\{\text{a point }\infty\}$ set-theoretically. $D(l)$ is a divisor with only normal crossings. Let us fix an irreducible component $D_0(l)$ of $D(l)$, and let $\tilde{D}$ be the quotient of the normalization of $\tilde{D}_0(l)$ of $D_0(l)$ by the stabilizer subgroup at $D_0(l)$ of Γ_n where it should be noted that the stabilizer subgroup is regarded as a group of automorphisms of the normalization $\tilde{D}_0(l)$ of $D_0(l)$, and that it is a stabilizer subgroup of Γ_n at the cusp $D_0^*(l)$ of $X_3^*(l)$ associated with $D_0(l)$. Then we have a morphism ψ of $\tilde{D}$ to D which is canonically determined by construction of $\tilde{D}$. ψ is a morphism of normalization, and if π denotes the morphism of $\bar{X}_3$ to X_3^*, then ψ is an isomorphism on the open subset $\pi^{-1}(X_2)$ of D where $X_2=H_2/\Gamma_2$ is considered to be the 2-cusp of X_3^*. Let $\tilde{\pi}$ be the composite of ψ and of $\pi|_D: D\to D^*$.

LEMMA 2. $\dim H^6(X_3^*, \mathcal{O}_{X_3^*})\leqq\dim H^3(D^*, R^2\tilde{\pi}_*\mathcal{O}_{\tilde{D}})$.

PROOF. Let $\mathcal{F}$ be the cokernel of the canonical injective homomorphism $\mathcal{O}_D\to\psi_*\mathcal{O}_{\tilde{D}}$;

$$0\longrightarrow\mathcal{O}_D\longrightarrow\psi_*\mathcal{O}_{\tilde{D}}\longrightarrow\mathcal{F}\longrightarrow 0\quad\text{(exact)}.$$

Then $\mathcal{F}|_{\pi^{-1}(X_2)}=0$. From this we derive a long exact sequence

$$\longrightarrow R^1\pi_*\mathcal{F}\xrightarrow{p}R^2\pi_*\mathcal{O}_D\xrightarrow{q}R^2\tilde{\pi}_*\mathcal{O}_{\tilde{D}}\longrightarrow R^2\pi_*\mathcal{F}\longrightarrow$$

and hence we have

$$0\longrightarrow\mathrm{Ker}(p)\longrightarrow R^2\pi_*\mathcal{O}_D\longrightarrow\mathrm{Ker}(q)\longrightarrow 0,$$
$$0\longrightarrow\mathrm{Ker}(q)\longrightarrow R^2\tilde{\pi}_*\mathcal{O}_{\tilde{D}}\longrightarrow\mathrm{Coker}(q)\longrightarrow 0.$$

From these we derive two long exact sequences

$$\to H^3(D^*,\mathrm{Ker}(p))\to H^3(D^*, R^2\pi_*\mathcal{O}_D)\to H^3(D^*,\mathrm{Ker}(q))\to H^4(D^*,\mathrm{Ker}(p))\to,$$
$$\to H^2(D^*,\mathrm{Coker}(q))\to H^3(D^*,\mathrm{Ker}(q))\to H^3(D^*, R^2\tilde{\pi}_*\mathcal{O}_{\tilde{D}})\to H^3(D^*,\mathrm{Coker}(q))\to.$$

Since $R^1\pi_*\mathcal{F}$, $R^2\pi_*\mathcal{F}$ are supported at the closure of the 1-cusp, $\mathrm{Ker}(p)$, $\mathrm{Coker}(q)$ are also. Then it follows from the above long exact sequence that

$$H^3(D^*, R^2\pi_*\mathcal{O}_D)\cong H^3(D^*,\mathrm{Ker}(q))\cong H^3(D^*, R^2\tilde{\pi}_*\mathcal{O}_{\tilde{D}}).$$

So to show our assertion it is enough to prove $\dim H^6(X_3^*, \mathcal{O}_{X_3^*})\leqq\dim H^3(D^*, R^2\pi_*\mathcal{O}_D)$. $R^\nu\pi_*\mathcal{O}_{\bar{X}_3}$ and $R^\nu\pi_*\mathcal{O}_D$ (extended by zero) are isomorphic for $\nu>0$ at least except on the closure of 1-cusp (Tsuyumine [19], Prop. 2). So by the similar argument as above, $H^3(D^*, R^2\pi_*\mathcal{O}_D)$ is shown to be isomorphic to $H^3(X_3^*, R^2\pi_*\mathcal{O}_{\bar{X}_3})$. So our problem is reduced to show $\dim H^6(X_3^*, \mathcal{O}_{X_3^*}) \leqq \dim H^3(X_3^*, R^2\pi_*\mathcal{O}_{\bar{X}_3})$.

Let $K_{\bar{X}_3}$ be the canonical coherent sheaf (in the sense of Grauert-Riemenschneider [8]), which gives the dualizing sheaf on $\bar{X}_3$ since $\bar{X}_3$ has only quotient singularities. So we have

$$H^0(\tilde{X}_3, K_{\tilde{X}_3}) \cong H^0(\bar{X}_3, K_{\bar{X}_3}) \cong H^6(\bar{X}_3, \mathcal{O}_{\bar{X}_3})^\vee$$

where $\tilde{X}_3$ is a nonsingular model of $\bar{X}_3$. Since $\tilde{X}_3$ is unirational (see for instance, Tsuyumine [22]), the above cohomology groups vanish. So to prove our assertion, we show that $H^6(\bar{X}_3, \mathcal{O}_{\bar{X}_3})$ $(=\{0\})$ is isomorphic to $H^6(X_3^*, \mathcal{O}_{X_3^*})/d(H^3(X_3^*, R^2\pi_*\mathcal{O}_{\bar{X}_3}))$ where d is some homomorphism of $H^3(X_3^*, R^2\pi_*\mathcal{O}_{\bar{X}_3})$ to $H^6(X_3^*, \mathcal{O}_{X_3^*})$. Let $E_2^{p,q}=H^p(X_3^*, R^q\pi_*\mathcal{O}_{\bar{X}_3}) \Rightarrow H^{p+q}(\bar{X}_3, \mathcal{O}_{\bar{X}_3})$ be the Leray spectral sequence. Since π is a proper morphism and since $\dim\pi^{-1}(x)=2$ if x is a point of the 2-cusp, $\dim\pi^{-1}(x)=3$ if x is a point of the 1- or 0-cusp, $H^p(X_3^*, R^q\pi_*\mathcal{O}_{\bar{X}_3})$ vanishes for (p, q) with $p+q=5$ except for $(p, q)=(3, 2)$ or $(0, 5)$. By an elementary consideration on the Leray spectral sequence, we get the desired homomorphism d. q. e. d.

LEMMA 3. $\dim H^6(X_3^*, \mathcal{O}_{X_3^*}) \leqq \dim H^1(D^*, R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}})$.

PROOF. Let us consider the Leray spectral sequence $E_2^{p,q}=H^p(D^*, R^q\tilde{\pi}_*\mathcal{O}_{\tilde{D}}) \Rightarrow H^{p+q}(\tilde{D}, \mathcal{O}_{\tilde{D}})$. Since $\dim\tilde{\pi}^{-1}(x)=3$ if x is in the 0- or 1-cusp of D^*, and $\dim\tilde{\pi}^{-1}(x)=2$ if otherwise, we have an inequality

$$\dim H^3(D^*, R^2\tilde{\pi}_*\mathcal{O}_{\tilde{D}}) \leqq \dim H^5(\tilde{D}, \mathcal{O}_{\tilde{D}})+\dim H^1(D^*, R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}})$$

by the similar argument as in the proof of Lemma 2. $\tilde{D}$ is normal and C.-M. because it is a quotient of a smooth variety $\tilde{D}_0(l)$ by a finite group. Then Lemma 1 is applicable, and we get $\dim H^5(\tilde{D}, \mathcal{O}_{\tilde{D}})=0$ because $\dim S(\Gamma_2)_4=0$. q. e. d.

Let us denote by E^*, the closure of H_1/Γ_1 in D^* which is isomorphic to $X_1^*=(H_1/\Gamma_1)^*$, and by $\tilde{\tilde{\pi}}: E \to E^*$, some fiber space which is isomorphic to $\tilde{\pi}^{-1}(E^*) \to E^*$ except at the 0-cusp. We want to reduce the computation of $H^1(D^*, R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}})$ to that of $H^1(E^*, R^3\tilde{\tilde{\pi}}_*\mathcal{O}_E)$ by the similar method as in [19], Prop. 2. We do it in Sect. 5. Before it we make some preliminaries.

4. Let $D_0^*(l)$ be the cusp of $X_3^*(l)$ associated with $D_0(l) \subset \bar{X}_3(l)$. By taking normalization, we get a morphism $\pi(l): \tilde{D}_0(l) \to \tilde{D}_0^*(l)$ from $D_0(l) \to D_0^*(l)$ where $\tilde{D}_0^*(l)$ denotes the normalization of $D_0^*(l)$. We observe the fibers of $\pi(l)$. In Igusa [14], the fiber at each point, of the morphism of $\bar{X}_3(l)$ to $X_3^*(l)$ was completely described, and the following is a direct consequence of it. In this section we always suppose $l \geqq 3$.

$\tilde{D}_0^*(l)$ is isomorphic to the Satake compactification of $H_2/\Gamma_2(l)$, and its 1-cusps are the unions of copies of $H_1/\Gamma_1(l)$. $\pi(l)^{-1}(H_2/\Gamma_2(l)) \to H_2/\Gamma_2(l)$ is the universal family of 2-dimensional principally polarized abelian varieties with level l-structure, and $\pi^{-1}(l)(x)$ is an abelian variety

$$\boldsymbol{C}^2/(Z_1, 1_2)(l\boldsymbol{Z})^4$$

where Z_1 is a point of H_2 corresponding to $x \in H_2/\Gamma_2(l)$. If $x \in X_1(l) = H_1/\Gamma_1(l) \subset \tilde{D}_0^*(l)$, then $\pi(l)^{-1}(x)$ is a δ-bundle over an abelian variety $A(l)^2$ where δ is an l-gon composed of $\boldsymbol{P}^1$ and $A(l)$ is an elliptic curve

$$A(l) := \boldsymbol{C}/(z_1, 1)(l\boldsymbol{Z})^2,$$

z_1 being a point of H_1 corresponding to x. We put $A := A(1)$. If x is a point of 0-cusps, then $\pi(l)^{-1}(x)$ is a reducible rational variety whose irreducible component is smooth of dimension three.

Let G be the stabilizer subgroup at $D_0^*(l) \subset X_3^*(l)$ of $\Gamma_3/\Gamma_3(l)$. Then G acts on $\tilde{D}_0^*(l)$, also on $D_0(l)$ and on $\tilde{D}_0(l)$. G contains a subgroup $\Gamma_2/\Gamma_2(l)$ whose element corresponds to $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ if we use the notation (2). For $x \in \tilde{D}_0^*(l)$, G_x denotes the stabilizer subgroup at x which acts on the fiber $\pi(l)^{-1}(x)$. We can see how G_x acts on $\pi(l)^{-1}(x)$ by virtue of [**14**]. Let x be a point of 1-cusp which, we may assume, is given by $Z_2 \to \sqrt{-1}\infty 1_2$ in (1). Then G_x is a subgroup of P_1 (see Sect. 1 for definition) of index $(1/2)l^2 \prod_{p|l}(1-p^2)$ since the same number of analytic components meets at each of 1-cusps (Cartan seminar [1], 13). G_x contains $U_1/\Gamma_3(l)$, $W_1/\Gamma_3(l)$. The group $U_1/\Gamma_3(l)$, which is isomorphic to $\boldsymbol{SM}_2(\boldsymbol{Z}/l\boldsymbol{Z}) \cong (\boldsymbol{Z}/l\boldsymbol{Z})^3$, acts only on δ as follows. Let t be a coordinate on $\boldsymbol{P}^1$ which takes $0, \infty$ at points of intersections with other components. Then there is a character of the additive group $U_1/\Gamma_3(l)$ such that $M^*t = \chi(M)t$ for $M \in U_1$. So the quotient of $\boldsymbol{P}^1$ by $U_1/\Gamma_3(l)$ is again $\boldsymbol{P}^1$ and the quotient of δ is also isomorphic to δ itself. The group W_1/U_1, which is isomorphic to $(\boldsymbol{Z}/l\boldsymbol{Z})^4$, acts only on $A(l)^2$ as

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \tau_1 + v_1 z_1 + b_1 \\ \tau_2 + v_2 z_1 + b_2 \end{pmatrix}, \qquad b_i, v_i \in \boldsymbol{Z} \pmod{l}.$$

Thus the quotient of $A(l)^2$ by W_1/U_1 is A^2 with $A = A(1)$, which is isomorphic to $A(l)^2$. Hence the quotient of $\pi(l)^{-1}(x)$ by W_1 is isomorphic to itself. The group G_x/W_1 is isomorphic to the direct product of the stabilizer subgroup at $z_1 \in H_1/\Gamma_1(l)$ of $\Gamma_1/\Gamma_1(l)$, z_1 being a point corresponding to x, and of the dihedral group Δ_l of an l-gon. G_x/W_1 acts simultaneously on $A(l)^2$ and on δ. Δ_l, whose element corresponds to the part 'A_2' or 'D_2' in (2), acts on $A(l)^2$ as

$$\binom{\tau_1}{\tau_2} \longrightarrow U\binom{\tau_1}{\tau_2}, \quad U \in GL_2(\boldsymbol{Z}) \bmod l,$$

and acts on an l-gon δ in the usual manner. We denote by Δ_l' the cyclic subgroup of Δ_l of order l which acts on δ as rotations.

Let ∞ be a 0-cusp of $\tilde{D}_0^*(l)$ $(\cong (H_2/\Gamma_2(l))^*)$. Then $\pi(l)^{-1}(\infty)$ is covered by an open affine varieties which are product of open affine subvarieties of δ, and of those of a limit variety of $A(l)^2$ where an elliptic curve $A(l)$ degenerates also to an l-gon of $\boldsymbol{P}^1$ in the natural way as in Shioda [**16**]. This can be easily seen, indeed we can take coordinates at $\pi(l)^{-1}(\infty)$ explicitly by the observation of Igusa [**14**].

5. Let $E^*(l)$ denote a closure of the set of 1-cusps of $\tilde{D}_0^*(l)$, and let $E(l)=\pi(l)^{-1}(E^*(l))$. Further let $E_0(l)$ be an irreducible component of $E(l)$, and $E_0^*(l)$, the corresponding irreducible component of $E^*(l)$. The normalization $\tilde{E}_0^*(l)$ of $E_0^*(l)$ is isomorphic to compactified modular curve $X_1^*(l)=(H_1/\Gamma_1(l))^*$.

$$\begin{array}{ccccccc} & & E_0(l) & \hookrightarrow & E(l) & \hookrightarrow & \tilde{D}_0(l) \\ & & \downarrow & & \downarrow & & \downarrow \\ H_1/\Gamma_1(l) & \hookrightarrow & E_0^*(l) & \hookrightarrow & E^*(l) & \hookrightarrow & \tilde{D}_0^*(l) \cong (H_2/\Gamma_2(l))^* . \end{array}$$

As we saw in the preceding section, $E_0(l) \to E_0^*(l)$ is a fiber space of relative dimension three whose generic fiber is a δ-bundle over an abelian variety $A(l)^2$. $E(l)$ is a Cartier divisor of $\tilde{D}_0(l)$. ($D_0(l)$ and other irreducible components of $D(l)$, or $D_0(l)$ and itself are crossing normally at the image of $E_0(l)$ by $\tilde{D}_0(l) \to D_0(l)$. Hence the image of $E_0(l)$ is a Cartier divisor of $D_0(l)$. For detail we refer the reader to Igusa [**14**]. Our statement follows from this.) Let $\mathcal{G}$ be the sheaf of ideals in $\mathcal{O}_{\tilde{D}_0(l)}$ defining $E(l)$, which is invertible. Let $\mathcal{M} := \mathcal{G}/\mathcal{G}^2$, which is supported on $E(l)$. From an exact sequence on $\tilde{D}_0(l)$

$$0 \longrightarrow \mathcal{G}^{j+1} \longrightarrow \mathcal{G}^j \longrightarrow \mathcal{M}^j \longrightarrow 0,$$

we get an exact sequence

$$(3) \qquad \longrightarrow R^3\pi(l)_*\mathcal{G}^{j+1} \longrightarrow R^3\pi(l)_*\mathcal{G}^j \longrightarrow R^3\pi(l)_*\mathcal{M}^j \longrightarrow 0.$$

$R^3\pi(l)$ (coherent sheaf) is supported at $E(l)$ since the dimension of the fiber $\pi(l)^{-1}(x)$ is two for $x \in \tilde{D}_0^*(l)-E^*(l)$, and so it is an $\mathcal{O}_{\tilde{D}_0^*(l)}/\mathcal{K}^M$-module for a sufficiently large M where $\mathcal{K} := \pi(l)_*\mathcal{G}$ is a sheaf of ideals concentrated at $E^*(l)$. By Grothendieck [**10**], (3.3.1), (3.3.2), we have

$$R^3\pi(l)_*\mathcal{G}^{i+i_0} = \mathcal{K}^i R^3\pi(l)_*\mathcal{G}^{i_0} \subset R^3\pi(l)_*\mathcal{G}^{i_0}$$

for some $i_0 \geqq 0$. Hence $R^3\pi(l)_*\mathcal{G}^j$ vanishes if j is large enough, and so $R^3\pi(l)_*\mathcal{M}^j$

does. We note that $E^*(l)$ is a reducible curve with singularities only at 0-cusps, and that $E(l)\to E^*(l)$ is flat outside of 0-cusps. Taking a point x from there, we have an isomorphism

$$(R^3\pi(l)_*\mathcal{M}^j)_x \cong H^3(\pi(l)(x),\ \mathcal{M}^j|_{\pi(l)^{-1}(x)}).$$

Here we need to prove that the δ-bundle $\pi(l)^{-1}(x)$ over $A(l)^2$ has the structure sheaf as its dualizing sheaf. Let us put $Z:=\pi(l)^{-1}(x)$, and denote by $\gamma: Z\to A(l)^2$, the projection. Since Z is locally of complete intersection, the dualizing sheaf ω_Z is invertible and is given by $\omega_Z=\omega_{Z/A(l)^2}\otimes\gamma^*\omega_{A(l)^2}\cong\omega_{Z/A(l)^2}$ (Hartshorne [11], Chap. III, Sect. 1) because $\omega_{A(l)^2}\cong\mathcal{O}_{A(l)^2}$, where $\omega_{Z/A(l)^2}$ denotes the relative dualizing sheaf. The dualizing sheaf of δ is trivial, and hence the restriction of $\omega_{Z/A(l)^2}$ to each fiber is trivial. So there is the invertible sheaf $\mathcal{N}$ on $A(l)^2$ such that $\gamma^*\mathcal{N}\cong\omega_{Z/A(l)^2}\cong\omega_Z$. Noticing that Z is locally a direct product of δ and an affine open subset of $A(l)^2$, we have $R^1\gamma_*\mathcal{O}_Z=H^1(\delta, \mathcal{O}_\delta)\otimes_C\mathcal{O}_Z=\mathcal{O}_Z$. Then for $\mathcal{F}$ any locally free sheaf on $A(l)^2$, there are isomorphisms $H^0(A(l)^2, \mathcal{F}^{-1})$ $\cong H^0(Z, \gamma^*\mathcal{F}^{-1})\cong H^3(Z, \gamma^*\mathcal{F}\otimes\gamma^*\mathcal{N})^\vee\cong H^2(A(l)^2, R^1\gamma_*(\gamma^*\mathcal{F}\otimes\gamma^*\mathcal{N}))^\vee\cong H^2(A(l)^2, \mathcal{F}\otimes\mathcal{N})^\vee$, where we have applied to the argument, the Leray spectral sequence $E_2^{p,q}=H^p(A(l)^2, R^q\gamma_*(\gamma^*\mathcal{F}\otimes\gamma^*\mathcal{N}))\Rightarrow H^{p+q}(Z, \gamma^*\mathcal{F}\otimes\gamma^*\mathcal{N})$. In particular, we have $H^0(A(l)^2, \mathcal{N})$ $\cong H^2(A(l)^2, \mathcal{O}_{A(l)^2})^\vee\cong\boldsymbol{C}\cong H^2(A(l)^2, \mathcal{N}\otimes\mathcal{N}^{-1})^\vee\cong H^0(A^2(l), \mathcal{N}^{-1})$. So $\mathcal{N}\cong\mathcal{O}_{A(l)^2}$ and $\omega_Z\cong\mathcal{O}_Z$.

Now we return to our argument. $(R^3\pi(l)_*\mathcal{M}^j)_x$ and the dual of $H^0(\pi(l)^{-1}(x), \mathcal{M}^{-j}|_{\pi(l)^{-1}(x)})$ are isomorphic, and they vanish for $j\gg 0$ by the above argument. It is easily deduced from this that $H^0(\pi(l)^{-1}(x), \mathcal{M}^{-j}|_{\pi(l)^{-1}(x)})$ vanishes for any $j>0$. Then it follows that the homomorphism $R^3\pi(l)_*\mathcal{G}^{j+1}$ $\to R^3\pi(l)_*\mathcal{G}^j$ is surjective for any $j>0$ outside of 0-cusps, and hence $R^3\pi(l)_*\mathcal{G}^j$ vanishes there by descending induction. Using (3) for $j=0$, we have

$$R^3\pi(l)_*\mathcal{O}_{\tilde{D}_0(l)} \cong R^3\pi(l)_*\mathcal{O}_{E(l)} \tag{4}$$

at least except at 0-cusps, since $\mathcal{O}_{E(l)}\cong\mathcal{O}_{\tilde{D}_0(l)}/\mathcal{G}$.

We have got (4) for $l\geqq 3$. But it can be proved also for '$l=1$'. Let E^* be the closure of the 1-cusp $X_1=H_1/\Gamma_1$ in $D^*=(H_2/\Gamma_2)^*$, and let $E'=\tilde{\pi}^{-1}(E^*)$. Let us take any point $x\in X_1(l)=E^*(l)-\{\text{0-cusps}\}$, and the stabilizer subgroup G_x at x of G (cf. Sect. 4), and a sufficiently small neighborhood U at x stable under G_x. Then if V is a sufficiently small neighborhood at the image point of x by the map $X_1(l)\to X_1\subset D^*$, then $(R^3\pi(l)_*\mathcal{O}_{\tilde{D}(l)}|_U)^{G_x}|_V\cong(R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}})|_V$ and $(R^3\pi(l)_*\mathcal{O}_{E(l)}|_U)^{G_x}|_V\cong(R^3\tilde{\pi}_*\mathcal{O}_{E'})|_V$ (Grothendieck [9], Cor. to Prop. 5.2.3, Théorème 5.3.1 and its Cor.). Hence by (4),

$$R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}} \cong R^3\tilde{\pi}_*\mathcal{O}_{E'}$$

at least except at the 0-cusps. Since we have the canonical surjection

$\alpha: R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}} \to R^3\tilde{\pi}_*\mathcal{O}_{E'}$ (because the fibers of $\tilde{\pi}$ are of dimension $\leqq 3$), there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}} \longrightarrow R^3\tilde{\pi}_*\mathcal{O}_{E'} \longrightarrow 0,$$

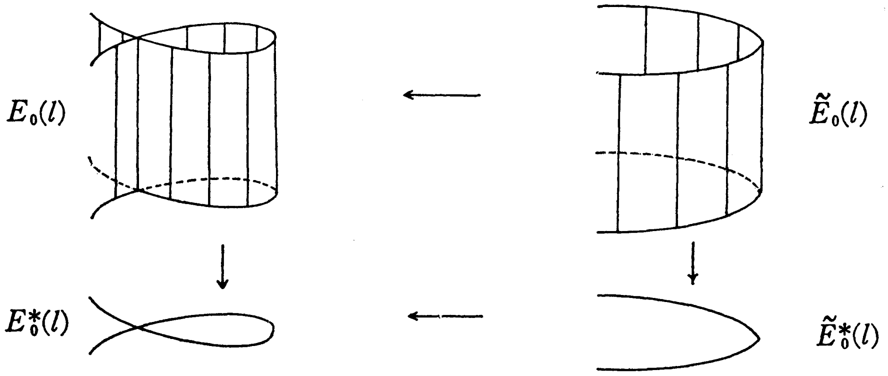
where $\mathrm{Ker}(\alpha)$ is supported possibly at the 0-cusp. So

$$H^1(D^*, R^3\tilde{\pi}_*\mathcal{O}_{\tilde{D}}) \cong H^1(D^*, R^3\tilde{\pi}_*\mathcal{O}_{E'}) = H^1(E^*, R^3\tilde{\pi}_*\mathcal{O}_{E'}).$$

Combining this with Lemma 3, we get

$$\text{(5)} \qquad \dim H^6(X_3^*, \mathcal{O}_{X_3^*}) \leqq \dim H^1(E^*, R^3\tilde{\pi}_*\mathcal{O}_{E'}).$$

Let G_0 be the stabilizer subgroup at $E_0^*(l)$ of G where G_0 is considered to be also a group of automorphisms of $E_0^*(l)$, or of its normalization $\tilde{E}_0^*(l)$, or of $E_0(l)$. G_0 contains a subgroup $\Gamma_1/\Gamma_1(l)$ whose elements correspond to matrices $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ if we use the notation (2) with $n_1=1$. Let $\tilde{E}_0(l) := E_0(l) \times_{E_0^*(l)} \tilde{E}_0^*(l)$.



$\tilde{E}_0(l)$ possesses naturally an action of G_0. Let E be the quotient of $\tilde{E}_0(l)$ by G_0. Then E is a fiber space over E^*, and we denote it by $\tilde{\tilde{\pi}}: E \to E^*$. By construction there is a finite morphism of E to E' as fiber spaces over E^*, which is an isomorphism except at the fiber of the 0-cusp. Then the similar argument as above shows that $H^1(E^*, R^3\tilde{\tilde{\pi}}_*\mathcal{O}_E) \cong H^1(E^*, R^3\tilde{\pi}_*\mathcal{O}_{E'})$. By (5) we get

$$\dim H^6(X_3^*, \mathcal{O}_{X_3^*}) \leqq \dim H^1(E^*, R^3\tilde{\tilde{\pi}}_*\mathcal{O}_E).$$

Let $E_2^{p,q} = H^p(E^*, R^q\tilde{\tilde{\pi}}_*\mathcal{O}_E) \Rightarrow H^{p+q}(E, \mathcal{O}_E)$ be the Leray spectral sequence. It is easy to see that $H^1(E^*, R^3\tilde{\tilde{\pi}}_*\mathcal{O}_E)$ is isomorphic to $H^4(E, \mathcal{O}_E)$ since E is a fiber space over E^* of relative dimension three. We have proved the following:

LEMMA 4. $\dim H^6(X_3^*, \mathcal{O}_{X_3^*}) \leqq \dim H^4(E, \mathcal{O}_E)$.

6. $\tilde{E}_0(l)$ is a fiber space over $\tilde{E}_0^*(l) \cong X_1^*(l)$. $E_0(l)$ is a C.-M. variety since it is a complete intersection of two nonsingular varieties. So $\tilde{E}_0(l)$ is also C.-M.

(see the figure above). We see the action of G_0 on $\tilde{E}_0(l)$. Let $x \in E_0^*(l) - \{0\text{-cusps}\}$ and let U_1, W_1 be the subgroups of the stabilizer group as in Sect. 1, which acts on the fibers, and trivially on the base space $\tilde{E}_0^*(l)$. The quotient of $\tilde{E}_0(l)$ by W_1 is isomorphic to itself. We saw it for each fiber over $\tilde{E}_0^*(l) - \{0\text{-cusps}\}$, and hence it is true for the fibers at 0-cusps because they are limit varieties. G_0/W_1 has a cyclic subgroup Δ_l' of order l as its normal subgroup which acts effectively on the fibers, especially on δ as rotations, trivially on the base space $\tilde{E}_0^*(l)$. Let H be the composite of Δ_l' and W_1, and let $B = \tilde{E}_0(l)/H$. B is a fiber space

$$\rho : B \longrightarrow \tilde{E}_0^*(l) = X_1^*(l).$$

The fiber $\rho^{-1}(x)$, $x \in X_1(l)$, is an extension of an abelian variety A^2 (cf. Sect. 4) by γ where γ denotes a projective line with one node. For a point ∞ of 0-cusps, $\rho^{-1}(\infty)$ is its limit variety where A degenerates to an l-gon δ of $\boldsymbol{P}^1$ at ∞. Let us consider the closed subfiber space B' of B given by {a point of a node}$\times A^2$ or its limit variety. B' is a C.-M. variety because it is locally defined by a single element in the C.-M. variety B. ρ is a projective morphism, i.e., ρ is factored as $B \longrightarrow \boldsymbol{P}^M_{\tilde{E}_0^*(l)} \longrightarrow \tilde{E}_0^*(l)$, where $\boldsymbol{P}^M_{\tilde{E}_0^*(l)}$ is a projective M-space over $\tilde{E}_0^*(l)$. Let us take a blowing up along B', of $\boldsymbol{P}^M_{\tilde{E}_0^*(l)}$, and moreover take the strict transform $\tilde{B}$ of B. If $\sigma : \tilde{B} \to B$ denotes the projection, then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \sigma_*\mathcal{O}_{\tilde{B}} \longrightarrow \mathcal{O}_{B'} \longrightarrow 0,$$

and hence a long exact sequence derived from this;

$$\longrightarrow R^2\rho_*\mathcal{O}_{B'} \longrightarrow R^3\rho_*\mathcal{O}_B \longrightarrow R^3\rho_*\sigma_*\mathcal{O}_{\tilde{B}} \longrightarrow 0.$$

Each of $(\rho\cdot\sigma)^{-1}(x)$, $x \in \tilde{E}_0^*(l)$, is a $\boldsymbol{P}^1$-bundle over two dimensional variety. Then by using the Leray spectral sequence, it is shown that $(R^3\rho_*\sigma_*\mathcal{O}_{\tilde{B}})_x$ vanishes at each point $x \in \tilde{E}_0^*(l)$, hence $R^3\rho_*\sigma_*\mathcal{O}_{\tilde{B}}$ vanishes. Then

$$H^1(\tilde{E}_0^*(l),\, R^2\rho_*\mathcal{O}_{B'}) \longrightarrow H^1(\tilde{E}_0^*(l),\, R^3\rho_*\mathcal{O}_B) \longrightarrow 0$$

is exact. Here $H^1(\tilde{E}_0^*(l), R^2\rho_*\mathcal{O}_{B'})$ is isomorphic to $H^3(B', \mathcal{O}_{B'})$ by using the Leray spectral sequence. Since $B' \to \tilde{E}_0^*(l) = X_1^*(l)$ satisfies the condition in Lemma 1, $\dim H^3(B', \mathcal{O}_{B'})$ is at most equal to $\dim S(\Gamma_1(l)_4)$ and further

$$\dim H^3(B', \mathcal{O}_{B'})^{\Gamma_1/\Gamma_1(l)} \leqq \dim S(\Gamma_1)_4 = 0.$$

Thus

$$\dim H^1(\tilde{E}_0^*(l),\, R^3\rho_*\mathcal{O}_B)^{\Gamma_1/\Gamma_1(l)} = 0.$$

Since $H^1(\tilde{E}_0^*(l), R^3\rho_*\mathcal{O}_B)$, $H^4(B, \mathcal{O}_B)$ and $H^4(\tilde{E}_0(l), \mathcal{O}_{\tilde{E}_0(l)})^H$ are all isomorphic as G_0/H-modules, $H^4(\tilde{E}_0(l), \mathcal{O}_{\tilde{E}_0(l)})$ has no nontrivial invariant sections under G_0 since G_0 contains $\Gamma_1/\Gamma_1(l)$. So $H^4(E, \mathcal{O}_E)$ vanishes. By Lemma 4 we have proved the following:

PROPOSITION 1. $\dim H^6(X_3^*, \mathcal{O}_{X_3^*}) = 0$.

7. We show that the graded ring $A(\Gamma_3)^{(r)}$ is not C.-M. for any integer r.

Let $\mathcal{L}(4)$ be the coherent sheaf on X_3^* corresponding to modular forms of weight four. Let X_3° be the smooth locus of X_3^*. Then $\operatorname{codim}(X_3^*-X_3^\circ)=2$. $\mathcal{L}(4)|_{X_3^\circ}$ is isomorphic to the canonical invertible sheaf $K_{X_3^\circ}$ on X_3°. By Grauert-Riemenschneider [8], the dualizing sheaf $\omega_{X_3^*}$ is given by $i_*K_{X_3^\circ}=i_*\mathcal{L}(4)|_{X_3^\circ}$, i being the inclusion of X_3° into X_3^*. By the extendability of holomorphic functions across a subvariety of codimension two, and by Koecher's principle $i_*\mathcal{L}(4)|_{X_3^\circ}$ is equal to $\mathcal{L}(4)$, hence

$$\omega_{X_3^*} \cong \mathcal{L}(4).$$

THEOREM. *X_3^* is not a Cohen-Macaulay variety, and the ring $A(\Gamma_3)^{(r)}$ is not Cohen-Macaulay for any integer r.*

PROOF. If $A(\Gamma_3)^{(r)}$ is C.-M., then $X_3^*=\operatorname{Proj}(A(\Gamma_3)^{(r)})$ is a C.-M. variety. So it is enough to prove the first assertion. Suppose that X_3^* is a C.-M. variety. Then $H^6(X_3^*, \mathcal{O}_{X_3^*})$ is just dual to $H^0(X_3^*, \mathcal{L}(4))$, however the former is of dimension 0 by Proposition 1, and the latter is of dimension one since there is the unique modular form of weight four up to constant multiples, a contradiction. So X_3^* is not a C.-M. variety. q.e.d.

By the dimension formula for the space of modular forms of degree three (Tsuyumine [21]), the arithmetic genus of X_3^* is known to be two. So some cohomology group $H^k(X_3^*, \mathcal{O}_{X_3^*})$ with even $k>0$ does not vanish. Also by the result of [21], it is shown that the depth of $A(\Gamma_3)$ is at least five. From these we get the following (cf. Watanabe [23], Cor. (2.3));

$$\dim H^4(X_3^*, \mathcal{O}_{X_3^*}) \neq 0, \qquad \operatorname{depth} A(\Gamma_3) = 5,$$

and

$$\dim H^k(X_3^*, \mathcal{O}_{X_3^*}) = 0 \quad (1 \leqq k \leqq 3),$$
$$\dim H^4(X_3^*, \mathcal{O}_{X_3^*}) - \dim H^5(X_3^*, \mathcal{O}_{X_3^*}) = 1.$$

References

[1] H. Cartan, Fonctions automorphes, Ecole Normale Supérier Seminaire, 1957/58.

[2] M. Eichler, Projective varieties and modular forms, Lecture Notes in Math., **210**, Springer, 1970.

[3] ———, On the graded ring of modular forms, Acta Arith., **18** (1971), 87-92.

[4] E. Freitag, Lokale und globale Invarianten der Hilbertschen Modulgruppen, Invent. Math., **17** (1972), 106-134.

[5] ———, Stabile Modulformen, Math. Ann., **230** (1977), 197-211.

[6] ———, Die Irreducibilität der Schottky relation (Bemerkungen zu einem Satz von

J. Igusa), Arch. Math., **40** (1983), 255-259.
[7] E. Freitag und R. Kiehl, Algebraiche Eigenschaften der lokal Ringe in den Spitzen der Hilbertschen Modulgruppen, Invent. Math., **24** (1974), 121-146.
[8] H. Grauert und O. Riemenschneider, Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen, Invent. Math., **11** (1970), 263-292.
[9] A. Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Math. J., **9** (1957), 119-221.
[10] ———, Éléments de géométrie algébrique, III, Publ. Math. IHES., **11** (1961).
[11] R. Hartshorne, Residue and duality, Lecture Notes in Math., **20**, Springer, 1966.
[12] J. Igusa, On Siegel modular forms of genus two, Amer. J. Math., **84** (1962), 175-200; (II), ibid., **86** (1964), 392-412.
[13] ———, Modular forms and projective invariants, Amer. J. Math., **89** (1967), 817-855.
[14] ———, A desingularization problem in the theory of Siegel modular functions, Math. Ann., **168** (1967), 228-260.
[15] S. Mori, On affine cones associated with polarized varieties, Japan. J. Math., **1** (1975), 301-309.
[16] P. Samuel, On unique factorization domain, Illinois J. Math., **5** (1961), 1-17.
[17] T. Shioda, On elliptic modular surfaces, J. Math. Soc. Japan, **24** (1972), 20-59.
[18] S. Tsuyumine, Rings of modular forms (On Eichler's problem), Nagoya Math. J., **99** (1985), 31-44.
[19] ———, Rings of automorphic forms which are not Cohen-Macaulay, I, J. Math. Soc. Japan, **38** (1986), 147-162.
[20] ———, Factorial property of a ring of automorphic forms, Trans. Amer. Math. Soc., **296** (1986), 111-123.
[21] ———, On Siegel modular forms of degree three, Amer. J. Math., **108** (1986), 755-862; Addendum, ibid., 1001-1003.
[22] ———, On the Siegel modular function field of degree three, Compositio Math., **63** (1987), 83-98.
[23] K. Watanabe, Some remarks concerning Demazure's construction of normal graded rings, Nagoya Math. J., **83** (1981), 203-211.

Shigeaki TSUYUMINE
Department of Mathematics
Mie University
Tsu 514
Japan