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Equivariant s-cobordism theorems

Dedicated to Professor Itiro Tamura on his 60th birthday

By Shôrô ARAKI and Katsuo KAWAKUBO

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§ 1. Introduction.

The classical h-cobordism theorem and the s-cobordism theorem have played an important role in studying differential topology [15], [16].

In the present paper, we discuss equivariant versions of these theorems.

Let G be a compact Lie group and X a finite G-CW-complex. In 1974, S. Illman [6] defined the equivariant Whitehead group $Wh_G(X)$ of X and the equivariant Whitehead torsion $\tau_G(f)$ for a G-homotopy equivalence $f: X \to Y$ between finite G-CW-complexes X, Y as an element of $Wh_G(X)$. When $\tau_G(f)=0$, f is called a *simple G-homotopy equivalence*. In this paper, we deal with only smooth G-manifolds.

Let $(W; X, Y)$ be a smooth G-h-cobordism. Namely W is a compact G-manifold with boundary $\partial W = X \amalg Y$ (disjoint union) and the inclusions

$$i_X: X \longrightarrow W \quad \text{and} \quad i_Y: Y \longrightarrow W$$

are G-homotopy equivalences.

When G is a finite group, W admits a unique smooth G-triangulation [7]. Accordingly the equivariant Whitehead torsion $\tau_G(i_X)$ is well-defined. On the other hand, the recent investigation of Matumoto and Shiota [13] enables us to define the equivariant Whitehead torsion $\tau_G(i_X)$ even when G is a compact Lie group. Notice that $\tau_G(i_X)$ is often written as $\tau_G(W, X)$.

As in the non-equivariant case, a G-h-cobordism $(W; X, Y)$ is called a *G-s-cobordism* when $\tau_G(i_X)$ vanishes.

We say that a G-h-cobordism (resp. G-s-cobordism) theorem holds if a G-h-cobordism (resp. G-s-cobordism) $(W; X, Y)$ implies a G-diffeomorphism

$$W \cong X \times I \qquad \text{rel } X$$

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where I is the interval [0, 1] with trivial G-action.

Unfortunately the G-h-cobordism theorem and the G-s-cobordism theorem do not hold in general [12]. Accordingly we need to add some assumptions for a theorem of this sort.

Let H, K be isotropy groups appearing in W and

$$W^H = \coprod_\lambda W_\lambda^H, \qquad W^K = \coprod_\mu W_\mu^K$$

be the decompositions to connected components. We now consider two conditions.

(*1) If $W_\mu^K \supsetneqq W_\lambda^H$, then $\dim W_\mu^K - \dim W_\lambda^H \geqq \dim G + 3$ for any pair of components W_μ^K and W_λ^H.

(*2) If H is a maximal isotropy group, then

$$\dim W_\lambda^H \geqq \dim G + 6$$

for any component W_λ^H.

Then our first theorem is the following

THEOREM 1. *Let G be a compact Lie group and $(W; X, Y)$ a G-s-cobordism. If W satisfies the conditions* (*1) *and* (*2) *above, then we have a G-diffeomorphism*

$$W \cong X \times I \qquad \text{rel } X.$$

In particular, X is G-diffeomorphic to Y.

If we stabilize a G-h-cobordism $(W; X, Y)$ with respect to disks of suitable representations, then the conditions (*1) and (*2) are automatically satisfied. However the restriction homomorphism (to a closed subgroup H of G) $Wh_G(X) \to Wh_H(X)$ is defined only for the case of the index $|G/H|$ being finite, and we need to use such restriction homomorphisms to diagonal actions in stable versions. Thus we assume hereafter that the group G is *finite* and have the following

THEOREM 2 (stable equivariant s-cobordism theorem). *Let G be a finite group and $(W; X, Y)$ a G-s-cobordism. Then there exist an orthogonal G-representation space V and a G-diffeomorphism*

$$W \times V(1) \cong X \times V(1) \times I \qquad \text{rel } X \times V(1).$$

In particular, we have G-diffeomorphisms

$$X \times V(1) \cong Y \times V(1) \quad \text{and} \quad X \times SV(1) \cong Y \times SV(1).$$

Here $V(1)$ (resp. $SV(1)$) denotes the closed unit disk (resp. the unit sphere) of V.

Let M_1, M_2 be closed G-manifolds. A G-homotopy equivalence $f: M_1 \to M_2$ will be called a *tangential G-homotopy equivalence* if there exist a G-representation space V and a G-vector bundle isomorphism:

$$T(M_1)\oplus \underline{V} \cong f^*T(M_2)\oplus \underline{V}$$

where $T(M_i)$ are tangent G-vector bundles of M_i ($i=1, 2$), $\underline{V}$ is the trivial G-vector bundle $M_1\times V\to M_1$ and $f^*T(M_2)$ is the induced G-vector bundle of $T(M_2)$ *via* the map f.

A tangential G-homotopy equivalence $f: M_1\to M_2$ is called a *tangential simple G-homotopy equivalence* if f is a simple G-homotopy equivalence.

Then we have the following equivariant version of [**5**], [**14**].

THEOREM 3. *Let G be a finite group. Let M_1 and M_2 be closed G-manifolds and $f: M_1\to M_2$ a G-map. Then f is tangential simple G-homotopy equivalence if and only if there exist an orthogonal G-representation space V and a G-diffeomorphism*

$$\bar{f}: M_1\times V(1) \longrightarrow M_2\times V(1)$$

such that the following diagram

$$\begin{array}{ccc} M_1\times V(1) & \xrightarrow{\bar{f}} & M_2\times V(1) \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is G-homotopy commutative, where π are the projection maps.

REMARK. Browder and Quinn had an isovariant s-cobordism theorem in [**20**].

REMARK. An equivariant s-cobordism theorem is stated in [**17**]. Unfortunately the assumption of the theorem is not stated in terms of the equivariant torsion $\tau_G(W, X)$ in the sense of Illman [**6**]. One of our tasks for the proofs of Theorems 1 and 2 is to show that a filtration inherits the property of G-deformation retractions (see §4). Accordingly we can define equivariant torsions successively. The other task is to show that it follows from the assumption $\tau_G(W, X)=0$ that these successive equivariant torsions also vanish.

REMARK. An equivariant s-cobordism theorem for finite G in the category of PL and Top is studied in [**18**].

§2. Naturality of equivariant Whitehead torsions.

We first review some of the basic facts about equivariant simple homotopy theory for the benefit of the reader. For further details we refer to [**2**].

In [**6**], Illman described the basic properties of the equivariant Whitehead group $Wh_G(X)$ for a finite G-CW-complex X, got a decomposition of $Wh_G(X)$ and described it algebraically for abelian G.

Each element of $Wh_G(X)$ is represented by a finite G-CW-pair (V, X) such that X is a strong G-deformation retract of V. The element represented by such a pair (V, X) is denoted by $\tau_G(V, X)$ and is called the Whitehead G-torsion of (V, X).

By a family $\mathcal{F}$ of closed subgroups of G, we understand a collection of closed subgroups H of G such that $H \in \mathcal{F}$ implies $(H) \subset \mathcal{F}$, where (H) denotes the conjugacy class of H.

For a family $\mathcal{F}$ of closed subgroups of G, Illman introduced the notion of restricted Whitehead group $Wh_G(X, \mathcal{F})$ consisting of those elements $\tau_G(V, X)$ such that all the isotropy groups of $V-X$ belong to $\mathcal{F}$. Then $Wh_G(X, \mathcal{F})$ is a subgroup of $Wh_G(X)$.

In 1978, H. Hauschild [4] gave the natural direct sum decomposition

$$Wh_G(X) \cong \coprod_{(H)} Wh_G(X, (H))$$

where (H) runs over all conjugacy classes of closed subgroups of G. He described $Wh_G(X)$ algebraically based on this decomposition in a way.

Let H be a closed subgroup of G and X a G-space. We denote by X^H the H-fixed point set of X and by WH the quotient group NH/H where NH is the normalizer of H in G.

Then the G-action on X induces a WH action on X^H and there holds the following natural isomorphism

$$Wh_G(X, (H)) \cong Wh_{WH}(X^H, \{e\})$$

which is also due to Hauschild [4].

The WH-action on X^H induces the WH-action on the set of connected components of X^H. Taking WH orbits of the induced action, we get a decomposition

$$X^H = \coprod_{\alpha} WH \cdot X_\alpha^H$$

as a topological sum of WH-subspaces, where the X_α^H's are connected components of X^H. Denote by A_H the index set $\{\alpha\}$ of the above decomposition. We call each summand $WH \cdot X_\alpha^H$ a *WH-component* of X^H and X_α^H a *representative component* of the WH-component $WH \cdot X_\alpha^H$.

Then there holds a direct sum decomposition [2]

$$Wh_{WH}(X^H, \{e\}) \cong \coprod_{\alpha \in A_H} Wh_{WH}(WH \cdot X_\alpha^H, \{e\}).$$

We now put

$$W_\alpha H = \{w \in WH \mid w \cdot X_\alpha^H \subset X_\alpha^H\}$$

which is a closed subgroup of WH. X_α^H is a $W_\alpha H$-space and we can express

$$WH\cdot X_\alpha^H = WH \underset{W_\alpha H}{\times} X_\alpha^H .$$

Then there holds a kind of Shapiro isomorphism [2]

$$Wh_{WH}(WH\cdot X_\alpha^H, \{e\}) \cong Wh_{W_\alpha H}(X_\alpha^H, \{e\}).$$

We are now in a position to pass to universal covering spaces.

Denote by $\tilde{X}_\alpha^H$ the universal covering space of X_α^H. Choose a point x_0 of X_α^H. Then $\pi_1=\pi_1(X_\alpha^H, x_0)$ operates on $\tilde{X}_\alpha^H$ as the covering transformation group.

By [1], [8], we have a Lie group Γ_α satisfying the short exact sequence

$$1 \longrightarrow \pi_1 \longrightarrow \Gamma_\alpha \stackrel{q}{\longrightarrow} W_\alpha H \longrightarrow 1$$

and $\tilde{X}_\alpha^H$ is a Γ_α-space such that the Γ_α-action contains the π_1-action and the covering projection $p:\tilde{X}_\alpha^H \to X_\alpha^H$ is q-equivariant.

Then there holds an isomorphism [2]

$$Wh_{W_\alpha H}(X_\alpha^H, \{e\}) \cong Wh_{\Gamma_\alpha}(\tilde{X}_\alpha^H, \{e\}).$$

We now consider the final step of reductions of $Wh_G(X)$.

Denote by $\Gamma_{\alpha,0}$ the component of Γ_α including the unit element. As is well-known, $\Gamma_{\alpha,0}$ is a closed normal subgroup of Γ_α. Then we have the following isomorphism [2]

$$Wh_{\Gamma_\alpha}(\tilde{X}_\alpha^H, \{e\}) \cong Wh(\Gamma_\alpha/\Gamma_{\alpha,0})$$

where the right hand side is the Whitehead group defined algebraically (see [3]).

Putting all this together, we have the following theorem.

THEOREM 4. *Let X be a finite G-CW-complex. Then we have a direct sum decomposition*

$$Wh_G(X) \cong \coprod_{(H)} \coprod_{\alpha\in A_H} Wh(\Gamma_\alpha/\Gamma_{\alpha,0}).$$

Since one verifies the naturalities of all the processes of the reductions above, one has the following theorem on which our theorems are based.

THEOREM 5 ([2]). *Let $f: X\to Y$ be a G-map between finite G-CW-complexes and H a closed subgroup of G. Suppose that the restriction $f^H: X^H \to Y^H$ gives a bijection of the connected components and induces isomorphisms of fundamental groups for any base points. Then there holds the isomorphism*

$$f_*: Wh_G(X, (H)) \xrightarrow{\cong} Wh_G(Y, (H)).$$

For the detailed proof of Theorems 4 and 5, see [2]. Theorem 4 is proved also by Illman [8] in a different approach.

§ 3. Decomposition of G-manifolds.

In this section, we recall the decomposition theorem of smooth G-manifolds of [11] for the benefit of the reader.

Let G be a compact Lie group. There is a partial order among the set of conjugacy classes of closed subgroups of G, i.e., $(H_1)\leqq(H_2)$ if and only if there exists $g\in G$ such that $gH_1g^{-1}\subset H_2$.

Let W be a compact G-manifold. We shall denote the isotropy group at $x\in W$ by G_x, namely

$$G_x=\{g\in G\,|\,gx=x\}\,.$$

For a closed subgroup H of G, we shall put

$$W(H)=\{x\in W\,|\,(G_x)=(H)\}\,.$$

Since W is compact, there are only finitely many isotropy types, say

$$\{(G_x)\,|\,x\in W\}=\{(H_1),(H_2),\cdots,(H_k)\}\,.$$

It is possible to arrange $\{(H_i)\}$ in such order that $(H_i)\geqq(H_j)$ implies $i\leqq j$.

We get a filtration

$$W=W_1\supset W_2\supset\cdots\supset W_k$$

consisting of compact G-manifolds W_i with corners such that

$$\{(G_x)\,|\,x\in W_i\}=\{(H_i),(H_{i+1}),\cdots,(H_k)\}$$

as follows.

For this, we introduce some notations. Let $\pi:E\rightarrow M$ be a differentiable G-vector bundle over a compact G-manifold M. As is well known, there is a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on E. Concerning the metric $\langle\ ,\ \rangle$, we set

$$\|v\|=\sqrt{\langle v,v\rangle}\qquad\text{for}\quad v\in E\,.$$

Then we put for $r>0$,

$$E(r)=\{v\in E\ \mid\ \|v\|\leqq r\},$$
$$SE(r)=\{v\in E\ \mid\ \|v\|=r\},$$
$$\mathring{E}(r)=E(r)-SE(r)=\{v\in E\ \mid\ \|v\|<r\}\,.$$

Obviously $E(r)$ and $SE(r)$ are compact G-manifolds.

Since (H_1) is a maximal conjugacy class, $W(H_1)$ is a compact G-invariant submanifold of W. We identify the normal bundle ν_1 of $W(H_1)$ in W with an open tubular neighborhood of $W(H_1)$ in W and impose a G-invariant Riemannian metric on ν_1.

Concerning the metric on ν_1, we set

$$W_2 = W - \mathring{\nu}_1(1).$$

Then W_2 is a compact G-manifold with corner and satisfies

$$\{(G_x) \mid x \in W_2\} = \{(H_2), (H_3), \cdots, (H_k)\}.$$

Suppose that we get a filtration

$$W = W_1 \supset W_2 \supset \cdots \supset W_i$$

consisting of compact G-manifolds W_j with corners such that

$$\{(G_x) \mid x \in W_j\} = \{(H_j), (H_{j+1}), \cdots, (H_k)\}$$

for every $j \leqq i$. Since (H_i) is a maximal conjugacy class among the set

$$\{(G_x) \mid x \in W_i\},$$

$W_i(H_i)$ is a compact G-invariant submanifold of W_i. We identify the normal bundle ν_i of $W_i(H_i)$ in W_i with an open tubular neighborhood of $W_i(H_i)$ in W_i and impose a G-invariant Riemannian metric on ν_i. Concerning this metric, we set

$$W_{i+1} = W_i - \mathring{\nu}_i(1).$$

Then W_{i+1} is a compact G-manifold with corner and satisfies

$$\{(G_x) \mid x \in W_{i+1}\} = \{(H_{i+1}), (H_{i+2}), \cdots, (H_k)\}.$$

This completes the inductive construction.

Thus we have shown the following decomposition theorem.

THEOREM 6 ([11]). *Let W be a compact G-manifold and $(H_1), \cdots, (H_k)$ the isotropy types appearing in W. Arrange $\{(H_i)\}$ in such order that $(H_i) \geqq (H_j)$ implies $i \leqq j$. Then there exist compact G-manifolds M_i with corners and G-vector bundles $\nu_i \to M_i$ for $1 \leqq i \leqq k$ such that*

$$M_i(H_i) = M_i \underset{G}{\simeq} W(H_i)$$

and that we have a decomposition

$$W \cong \nu_1(1) \cup \nu_2(1) \cup \cdots \cup \nu_k(1).$$

Moreover if we set

$$W_i = \nu_i(1) \cup \nu_{i+1}(1) \cup \cdots \cup \nu_k(1),$$

then we have

$$\{(G_x) \mid x \in W_i\} = \{(H_i), (H_{i+1}), \cdots, (H_k)\}$$

and a G-diffeomorphism

$$M_i \cong W_i(H_i).$$

§ 4. Excision theorem of G-deformation retractions.

Let W be a compact G-manifold with boundary $\partial W = X \amalg Y$ (disjoint union). Let

$$W = W_1 \supset W_2 \supset \cdots \supset W_k$$

be the filtration of Theorem 6.

We now set

$$X_i = X \cap W_i, \qquad Y_i = Y \cap W_i.$$

Then we have the following theorem which is crucial for the inductive proof of our equivariant s-cobordism theorem.

THEOREM 7 (Excision theorem of G-deformation retractions). *Suppose that* $(W; X, Y)$ *is a* G-h-*cobordism. Namely both* X *and* Y *are* G-*deformation retracts of* W. *If* W *satisfies the condition* (*1) *in* § 1, *then both* X_i *and* Y_i *are* G-*deformation retracts of* W_i *for each* i, $1 \leqq i \leqq k$.

PROOF. Although the assumption of Lemma 3.1 of [**11**] is slightly different from the condition (*1), we proved it actually under the condition (*1). Therefore Theorem 7 was already shown in the proof of Lemma 3.1 of [**11**].

REMARK. In [**12**], we have shown that the excision theorem of G-deformation retractions does not hold in general if the condition (*1) is not satisfied. Accordingly the equivariant torsion itself is not defined in general.

The counter example of the equivariant s-cobordism theorem is provided by making use of the failure of the excision theorem of G-deformation retractions.

REMARK. The excision theorem of G-deformation retractions does not follow from [**10**].

The rest of the section will be devoted to showing how to employ Matumoto-Shiota's well-definedness of the equivariant Whitehead torsion in our case. Let $(W; X, Y)$ and $(W_i; X_i, Y_i)$ be the G-h-cobordisms in Theorem 7. Then their method is briefly as follows. Take a G-diffeomorphism $f: (W; X, Y) \to (W'; X', Y')$ such that $(W'; X', Y')$ is an analytic G-h-cobordism embedded in a representation space V of G analytically and equivariantly. Then $(W'; X', Y')$ is endowed with an equivariant analytic stratification by isotropy types. Now the quotient W'/G is embedded in $\boldsymbol{R}^n$ subanalytically for some $n>0$ and has subanalytic stratification by isotropy types. Moreover the quotient map $\pi': W' \to W'/G$ is subanalytic. Next take a subanalytic triangulation of the triple $(W'/G; X'/G, Y'/G)$ compatibly with the stratification. After taking a barycentric subdivision of this triangulation they lift each simplex to a subanalytic simplex embedded in W' satisfying certain conditions (cf. [**13**], Lemma 4.4) and take its G-orbit. The collection of them forms a G-CW-subdivision of $(W'; X', Y')$. Finally take the pull-back of such a G-CW-complex by f, then we get a G-CW-subdivision of $(W; X, Y)$. This type of G-CW-subdivisions of $(W; X, Y)$ is unique up to subdivisions and G-isomorphisms.

In our case we use the above mentioned filtration

$$W = W_1 \supset W_2 \supset \cdots \supset W_k .$$

Construct the same filtration

$$W' = W_1' \supset W_2' \supset \cdots \supset W_k'$$

making use of real analytic induced invariant metric from V, which refines subanalytic stratifications of W' and of W'/G respectively. Take Matumoto-Shiota's construction of a G-CW-subdivision of W' so that it is compatible with these refined stratifications, and pull back to W the filtration and G-CW-subdivision of W' by f. Then we get well-defined equivariant Whitehead torsion at each stage of our inductive argument.

§ 5. Equivariant s-cobordism theorem.

Let A be a G-manifold and B a G-invariant submanifold of A. Denote by I the unit interval $[0, 1]$ with trivial G-action. Then a G-diffeomorphism $f: A \to B\times I$ which is an extension of the canonical G-diffeomorphism $B(\subset A)\to B\times\{0\}$ is called a *G-diffeomorphism relative B* and is denoted by

$$A \cong B\times I \qquad \text{rel } B .$$

For a compact G-manifold M, we denote its boundary by ∂M. Let W, X, Y, Z, be compact G-manifolds with corners satisfying

$$\partial W = (X \amalg Y)\cup Z ,$$

$$\partial X = X\cap Z , \quad \partial Y = Y\cap Z \quad \text{and} \quad \partial X \amalg \partial Y = \partial Z .$$

We prove Theorem 1 in the following form.

THEOREM 8. *Let W, X, Y, Z be as above. Suppose that both X and Y are G-deformation retracts of W and*

(i) $Z \cong \partial X\times I$ *rel* ∂X

(ii) $\tau_G(W, X) = 0$

(iii) *the conditions* (*1), (*2) *are satisfied for* W.

Then there exists a G-diffeomorphism

$$W \cong X\times I \qquad \text{rel } X$$

which is an extension of the G-diffeomorphism of (i).

PROOF. We prove Theorem 8 by induction on the number of isotropy types of W.

Suppose that W has only one isotropy type, say (H). In this case, we have the isomorphism

$$Wh_{WH}(X^H, \{e\}) \cong Wh_G(X(H), (H)) \cong Wh_G(X, (H)).$$

Note that both X^H and Y^H are WH-deformation retracts of W^H. It follows from the above isomorphism that

$$\tau_{WH}(W^H, X^H) = \tau_G(W(H), X(H)) = \tau_G(W, X) = 0.$$

Since WH acts freely on W^H, W^H/WH, X^H/WH and Y^H/WH are compact manifolds with corners and satisfy:

$$\begin{aligned} \partial W^H/WH &= (X^H/WH \amalg Y^H/WH) \cup Z^H/WH, \\ \partial X^H/WH &= X^H/WH \cap Z^H/WH, \\ \partial Y^H/WH &= Y^H/WH \cap Z^H/WH, \\ \partial X^H/WH \amalg \partial Y^H/WH &= \partial Z^H/WH. \end{aligned}$$

Obviously we have the induced diffeomorphism

$$Z^H/WH \cong (\partial X^H/WH) \times I \qquad \text{rel } \partial X^H/WH.$$

Moreover one verifies that both X^H/WH and Y^H/WH are deformation retracts of W^H/WH and that

$$\tau(W^H/WH, X^H/WH) = 0$$

by [6]. It follows from the classical s-cobordism theorem that we get a diffeomorphism

$$W^H/WH \cong (X^H/WH) \times I \qquad \text{rel } X^H/WH$$

extending the above diffeomorphism since $\dim(W^H/WH) \geqq 6$. For the relative s-cobordism theorem, see for example [19].

The projection $\pi: W^H \to W^H/WH$ is a principal WH-bundle. Hence by the homotopy property of principal bundles, we have a WH-diffeomorphism

$$W^H \cong X^H \times I \qquad \text{rel } X^H$$

extending the induced WH-diffeomorphism

$$Z^H \cong \partial X^H \times I \qquad \text{rel } \partial X^H.$$

Since W has only one isotropy type (H), there are the canonical G-diffeomorphisms:

$$W \cong G/H \times_{WH} W^H, \qquad X \cong G/H \times_{WH} X^H$$

$$Y \cong G/H \underset{WH}{\times} Y^H, \qquad Z \cong G/H \underset{WH}{\times} Z^H.$$

Thus we get a G-diffeomorphism

$$W \cong X \times I \qquad \text{rel } X$$

extending the given G-diffeomorphism (i).

This completes the first step of the inductive proof.

Next we assume that Theorem 8 holds for the case where the number of the isotropy types is less than k.

Let W, X, Y, Z be as before such that the number of isotropy types of W is k. Let $\{(H_i) | i=1, \cdots, k\}$ be the isotropy types indexed as in §3.

Since (H_1) is maximal among the set of isotropy types of W, $W(H_1)$, $X(H_1)$, $Y(H_1)$ and $Z(H_1)$ are compact G-invariant submanifolds of W. Obviously we have

$$\begin{aligned}
\partial W(H_1) &= (X(H_1) \amalg Y(H_1)) \cup Z(H_1), \\
\partial X(H_1) &= X(H_1) \cap Z(H_1), \\
\partial Y(H_1) &= Y(H_1) \cap Z(H_1), \\
\partial X(H_1) \amalg \partial Y(H_1) &= \partial Z(H_1)
\end{aligned}$$

and we have a G-diffeomorphism

$$Z(H_1) \cong \partial X(H_1) \times I \qquad \text{rel } \partial X(H_1)$$

which is the restriction of the G-diffeomorphism (i). As is well-known, there exist the canonical G-diffeomorphisms

$$W(H_1) \cong G/H_1 \underset{WH_1}{\times} W^{H_1}, \qquad X(H_1) \cong G/H_1 \underset{WH_1}{\times} X^{H_1},$$
$$Y(H_1) \cong G/H_1 \underset{WH_1}{\times} Y^{H_1}, \qquad Z(H_1) \cong G/H_1 \underset{WH_1}{\times} Z^{H_1}.$$

Since both X^{H_1} and Y^{H_1} are WH_1-deformation retracts of W^{H_1}, we may assert that both $X(H_1)$ and $Y(H_1)$ are G-deformation retracts of $W(H_1)$.

It follows from [6] that

$$\tau_G(W, X)=0 \quad \text{implies} \quad \tau_G(W(H_1), X(H_1))=0.$$

We are now in a position to employ the arguments in the case where W has only one isotropy type and we get a G-diffeomorphism

$$W(H_1) \cong X(H_1) \times I \qquad \text{rel } X(H_1)$$

extending the above G-diffeomorphism

$$Z(H_1) \cong \partial X(H_1) \times I \qquad \text{rel } \partial X(H_1).$$

Next we consider the normal bundle ν_1 of $W(H_1)$ in W. By the G-homotopy property of G-vector bundles, we have an isomorphism of G-vector bundles

$$\nu_1 \cong (\nu_1 | X(H_1)) \times I$$

which is an extension of the canonical bundle isomorphism

$$\nu_1 | Z(H_1) \cong (\nu_1 | \partial X(H_1)) \times I$$

induced from the product structure above.

In particular, we have G-diffeomorphisms

$$\nu_1(1) \cong (\nu_1(1) | X(H_1)) \times I \qquad \text{rel } \nu_1(1) | X(H_1),$$
$$S\nu_1(1) \cong (S\nu_1(1) | X(H_1)) \times I \quad \text{rel } S\nu_1(1) | X(H_1).$$

Therefore we have

$$\tau_G(\nu_1(1),\ \nu_1(1) | X(H_1)) = 0$$

and

$$\tau_G(S\nu_1(1),\ S\nu_1(1) | X(H_1)) = 0.$$

We now set

$$W_2 = W - \mathring{\nu}_1(1), \quad X_2 = X \cap W_2, \quad Y_2 = Y \cap W_2$$

and

$$Z_2 = (Z - \mathring{\nu}_1(1) | Z(H_1)) \cup S\nu_1(1).$$

Then W_2, X_2, Y_2, Z_2 are compact G-manifolds with corners and Z_2 has the induced product structure

$$Z_2 \cong \partial X_2 \times I \qquad \text{rel } \partial X_2.$$

Moreover it is easy to see that

$$\partial W_2 = (X_2 \amalg Y_2) \cup Z_2,$$
$$\partial X_2 = X_2 \cap Z_2, \qquad \partial Y_2 = Y_2 \cap Z_2,$$
$$\partial X_2 \amalg \partial Y_2 = \partial Z_2.$$

It follows from Theorem 7 that both X_2 and Y_2 are G-deformation retracts of W_2.

Next we will show that

$$\tau_G(W_2,\ X_2) = 0.$$

For this, we make use of the following geometric sum theorem due to Illman [6].

THEOREM 9 (Illman). *Let (A, B) be a finite G-CW-pair and A_1, A_2 G-subcomplexes of A such that $A = A_1 \cup A_2$. Set*

$$A_0 = A_1 \cap A_2 \quad \textit{and} \quad B_k = B \cap A_k \qquad (k=0, 1, 2).$$

Denote by $i_k : B_k \to B$ the inclusion maps $(k=0, 1, 2)$. Suppose that the inclusion maps $j_k : B_k \to A_k$ are all G-homotopy equivalences.

Then the inclusion $B \to A$ is also a G-homotopy equivalence and we have the equality

$$\tau_G(A, B) = i_{1*}\tau_G(A_1, B_1) + i_{2*}\tau_G(A_2, B_2) - i_{0*}\tau_G(A_0, B_0).$$

Apply Theorem 9 to the following case:

$$A = W, \quad B = X, \quad A_1 = \nu_1(1), \quad A_2 = W_2.$$

Then we have

$$A_0 = A_1 \cap A_2 = S\nu_1(1), \qquad B_0 = X \cap S\nu_1(1) = S\nu_1(1) | X(H_1),$$

$$B_1 = X \cap \nu_1(1) = \nu_1(1) | X(H_1) \quad \text{and} \quad B_2 = X \cap W_2 = X_2.$$

The maps corresponding to the maps in Theorem 9 are the following inclusion maps:

$$i_0 : S\nu_1(1) | X(H_1) \longrightarrow X,$$

$$i_1 : \nu_1(1) | X(H_1) \longrightarrow X,$$

$$i_2 : X_2 \longrightarrow X,$$

$$j_0 : S\nu_1(1) | X(H_1) \longrightarrow S\nu_1(1),$$

$$j_1 : \nu_1(1) | X(H_1) \longrightarrow \nu_1(1),$$

$$j_2 : X_2 \longrightarrow W_2.$$

Note that j_k are all G-homotopy equivalences ($k=0, 1, 2$). It follows from Theorem 9 that

$$\tau_G(W, X) = i_{1*}\tau_G(\nu_1(1), \nu_1(1) | X(H_1)) + i_{2*}\tau_G(W_2, X_2) - i_{0*}\tau_G(S\nu_1(1), S\nu_1(1) | X(H_1)).$$

Thus we have

$$i_{2*}\tau_G(W_2, X_2) = 0.$$

Consider the Hauschild decomposition:

$$Wh_G(X_2) \cong \coprod_{(H)} Wh_G(X_2, (H)).$$

Since the set of the isotropy types of W_2 is $\{(H_2), (H_3), \cdots, (H_k)\}$, the element $\tau_G(W_2, X_2)$ can be written as

$$\tau_G(W_2, X_2) = \coprod_{i=2}^{k} \tau_G(W_2, X_2)(H_i)$$

corresponding to the Hauschild decomposition. By the assumption (*1), the inclusion map

$$i_2^{H_i} : X_2^{H_i} \longrightarrow X^{H_i} \qquad i \geqq 2$$

gives a bijection of the connected components and induces isomorphisms of fundamental groups for any base points. It follows from Theorem 5 that the homomorphism

$$i_{2*}:\ Wh_G(X_2, (H_i)) \longrightarrow Wh_G(X, (H_i)), \qquad i \geqq 2,$$

is an isomorphism. Since $i_{2*}\tau_G(W_2, X_2)(H_i)=0$ for any $i \geqq 2$, we have

$$\tau_G(W_2, X_2)(H_i) = 0 \qquad \text{for} \quad 2 \leqq i \leqq k.$$

It turns out that

$$\tau_G(W_2, X_2) = 0.$$

Clearly W_2 satisfies the conditions (*1), (*2).

Thus we have shown that W_2, X_2, Y_2, Z_2 instead of W, X, Y, Z in Theorem 8 satisfy all the conditions of Theorem 8. Since the number of the isotropy types of W_2 is equal to $k-1$, we get a G-diffeomorphism

$$W_2 \cong X_2 \times I \qquad \text{rel } X_2$$

which is an extension of the product structure on Z_2, by the inductive hypothesis.

Note that

$$Z \subset (\nu_1(1) \mid Z(H_1)) \cup Z_2$$

and that the product structure on the right hand side agrees with that of Z.

Thus we obtain a product structure on W which is an extension of the product structure on Z.

This makes the proof of Theorem 8 complete.

§ 6. Equivariant stable s-cobordism theorem.

In this section, we assume that G is a finite group.

First we state the following lemma which follows directly from the definition of an elementary G-collapse and an elementary G-expansion.

LEMMA 10. *Let (W, X) be a finite G-CW-pair such that X is a G-deformation retract of W. If $\tau_G(W, X)=0$, then*

$$\tau_G(W \times Y, X \times Y) = 0$$

for any finite G-CW-complex Y.

In view of [11], any compact G-manifold has a finite G-CW-structure. Hence we have the following corollary.

COROLLARY 11. *Let W be a compact G-manifold and X a compact G-submanifold of W such that X is a G-deformation retract of W. If $\tau_G(W, X)=0$, then we have*

$$\tau_G(W \times Y, X \times Y) = 0$$

for any compact G-manifold Y.

PROOF OF THEOREM 2. In [11], it is shown that there exists an orthogonal G-representation space V such that $W\times V(1)$ and $W\times SV(1)$ satisfy the conditions (*1), (*2) in § 1.

First we will apply Theorem 8 to the triad

$$(W\times SV(1);X\times SV(1),\ Y\times SV(1)).$$

It follows from Corollary 11 that $\tau_G(W, X)=0$ implies

$$\tau_G(W\times SV(1),\ X\times SV(1))=0.$$

Hence by Theorem 8 we get a G-diffeomorphism

$$W\times SV(1)\cong X\times SV(1)\times I \qquad \text{rel } X\times SV(1).$$

Next we will apply Theorem 8 to the triad

$$(W\times V(1);X\times V(1),\ Y\times V(1)).$$

As above, we get

$$\tau_G(W\times V(1),\ X\times V(1))=0.$$

Appealing to Theorem 8 again, we have a G-diffeomorphism

$$W\times V(1)\cong X\times V(1)\times I \qquad \text{rel } X\times V(1)$$

which is an extension of the above product structure on $W\times SV(1)$.

This makes the proof of Theorem 2 complete.

§ 7. Stable equivalence of G-manifolds.

In this section, we assume that G is a finite group.

Let M_1, M_2 be closed G-manifolds and $f: M_1\rightarrow M_2$ a tangential simple G-homotopy equivalence. It is well-known that there exist an orthogonal G-representation space V_1 and a G-embedding $e: M_1\rightarrow V_1$. We assume that V_1 includes $\boldsymbol{R}$ with trivial G-action as a direct summand. For any positive integer m, we denote by V_1^m the direct sum of m-copies of V_1 and by $j: V_1\rightarrow V_1^m$ the inclusion to the first factor. Set $V=V_1^m$.

Then the composition

$$M_1 \xrightarrow{f\times e} M_2\times V_1 \xrightarrow{\text{id}\times j} M_2\times V$$

is a G-embedding. One verifies that the normal bundle of the G-embedding is isomorphic to the product bundle

$$M_1\times V \longrightarrow M_1,$$

if V_1 is sufficiently large. Thus we get a G-embedding

$$i : M_1 \times V(1) \longrightarrow M_2 \times V .$$

By this embedding, we identify $M_1 \times V(1)$ with the image $i(M_1 \times V(1))$. If we choose a sufficiently large number r, there holds the following inclusion

$$M_1 \times V(1) \subset M_2 \times \mathring{V}(r).$$

We now set

$$W = M_2 \times V(r) - M_1 \times \mathring{V}(1)$$

and get a triad

$$(W ; M_1 \times SV(1),\ M_2 \times SV(r)).$$

The proof that the triad above is a G-h-cobordism for $m \geqq 3$ is shown in the proof of Lemma 3.2 in [**11**].

Next we will show that the G-h-cobordism is in fact a G-s-cobordism.

For this, we first show that the G-embedding

$$i : M_1 \times V(1) \longrightarrow M_2 \times V(r)$$

is a simple G-homotopy equivalence. Consider the following G-homotopy commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\ f\ } & M_2 \\ \uparrow \pi & & \downarrow j \\ M_1 \times V(1) & \xrightarrow{\ i\ } & M_2 \times V(r) \end{array}$$

where π is the projection map and j is the natural inclusion map. In view of [**3**], [**6**], we have

$$\begin{aligned} \tau_G(i) &= \tau_G(j \cdot f \cdot \pi) = \tau_G(j \cdot f) + (j \cdot f)_* \tau_G(\pi) \\ &= \tau_G(j) + j_* \tau_G(f) + (j \cdot f)_* \tau_G(\pi) . \end{aligned}$$

Since π, f and j are all simple G-homotopy equivalences, we may conclude that i is also a G-simple homotopy equivalence.

Next we will show that

$$\tau_G(W,\ M_1 \times SV(1)) = 0 .$$

For this, we apply Theorem 9 to the following case:

$$\begin{aligned} A &= M_2 \times V(r), & B &= M_1 \times V(1), \\ A_1 &= M_1 \times V(1) = B, & A_2 &= W . \end{aligned}$$

Then we have

$$\begin{aligned} A_0 &= A_1 \cap A_2 = M_1 \times SV(1), \\ B_0 &= B \cap A_0 = M_1 \times SV(1) = A_0 . \end{aligned}$$

$$B_1 = B \cap A_1 = A_1 = B = M_1 \times V(1),$$
$$B_2 = B \cap A_2 = M_1 \times SV(1) = A_0.$$

The maps corresponding to those maps in Theorem 9 are the following inclusion maps:

$$i_0 = i_2 : M_1 \times SV(1) \longrightarrow M_1 \times V(1),$$
$$i_1 = \mathrm{id} : M_1 \times V(1) \longrightarrow M_1 \times V(1)$$

and

$$j_0 = \mathrm{id} : M_1 \times SV(1) \longrightarrow M_1 \times SV(1),$$
$$j_1 = \mathrm{id} : M_1 \times V(1) \longrightarrow M_1 \times V(1),$$
$$j_2 : M_1 \times SV(1) \longrightarrow W.$$

Note that j_k are all G-homotopy equivalences ($k=0, 1, 2$). It follows from Theorem 9 that there holds

$$\tau_G(M_2 \times V(r), M_1 \times V(1)) = i_{1*}\tau_G(M_1 \times V(1), M_1 \times V(1)) + i_{2*}\tau_G(W, M_1 \times SV(1)) - i_{0*}\tau_G(M_1 \times SV(1), M_1 \times SV(1)).$$

By definition, we have

$$\tau_G(M_1 \times V(1), M_1 \times V(1)) = 0,$$
$$\tau_G(M_1 \times SV(1), M_1 \times SV(1)) = 0.$$

Thus we have

$$i_{2*}\tau_G(W, M_1 \times SV(1)) = \tau_G(M_2 \times V(r), M_1 \times V(1)) = \tau_G(i) = 0.$$

Finally we will show that i_{2*} is an isomorphism. If the m above is greater than two, we have

$$\dim V^G = m \dim V_1^G \geqq m \geqq 3.$$

Hence for any subgroup H of G, $SV(1)^H$ is connected and simply connected. It turns out that the inclusion map

$$i_2^H : (M_1 \times SV(1))^H = M_1^H \times SV(1)^H \longrightarrow (M_1 \times V(1))^H = M_1^H \times V(1)^H$$

gives a bijection of the connected components and induces isomorphisms of fundamental groups for any base points. Accordingly there holds an isomorphism

$$i_{2*}: Wh_G(M_1 \times SV(1), (H)) \cong Wh_G(M_1 \times V(1), (H))$$

by Theorem 5. Since H is an arbitrary subgroup of G, it follows from the Hauschild decomposition that

$$i_{2*}: Wh_G(M_1 \times SV(1)) \cong Wh_G(M_1 \times V(1))$$

is an isomorphism.

Since $i_{2*}\tau_G(W, M_1\times SV(1))=0$, we conclude that

$$\tau_G(W, M_1\times SV(1))=0.$$

Namely the triad $(W; M_1\times SV(1), M_2\times SV(r))$ is a G-s-cobordism.

If we take m as $m\geqq 6$, then the conditions (*1), (*2) of Theorem 1 are satisfied and we have a G-diffeomorphism

$$W\cong M_1\times SV(1)\times I \qquad \text{rel } M_1\times SV(1).$$

Therefore we obtain the following G-diffeomorphisms

$$\begin{aligned} M_2\times V(r) &= M_1\times V(1)\cup W\cong M_1\times V(1)\cup(M_1\times SV(1)\times I) \\ &\cong M_1\times V(1). \end{aligned}$$

Obviously $M_2\times V(r)$ and $M_2\times V(1)$ are G-diffeomorphic and we have the required G-diffeomorphism

$$\bar{f}: M_1\times V(1)\longrightarrow M_2\times V(1).$$

The G-homotopy commutativity of the following diagram:

$$\begin{array}{ccc} M_1\times V(1) & \xrightarrow{\bar{f}} & M_2\times V(1) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

is obvious.

Conversely suppose that there exists a G-diffeomorphism

$$\bar{f}: M_1\times V(1)\longrightarrow M_2\times V(1)$$

so that the diagram above is G-homotopy commutative. Since two projection maps $\pi: M_1\times V(1)\to M_1$, $\pi: M_2\times V(1)\to M_2$, and $\bar{f}$ are all simple G-homotopy equivalences, one can show that f is also a simple G-homotopy equivalence as before.

This makes the proof of Theorem 3 complete.

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Shôrô ARAKI

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558
Japan

Katsuo KAWAKUBO

Department of Mathematics
Osaka University
Toyonaka
Osaka 560
Japan