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Knotted homology 3-spheres in S^5

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§1. Introduction.

A 3-*knot* will denote the (oriented) isotopy class of a smooth (oriented) submanifold K of the 5-sphere S^5, where K is a homology 3-sphere. When the diffeomorphism type Σ of K is to be emphasized, we call the 3-knot (S^5, K) a Σ-*knot*. A 3-knot (S^5, K) is *simple* if $\pi_1(S^5-K)\cong \boldsymbol{Z}$. Simple Σ-knots are classified by their Seifert matrices (Theorem 2.2), just as simple S^3-knots are ([**12**]). A 3-knot is *decomposable* if it is the connected sum of two 3-knots, both different from the trivial S^3-knot.

In this paper, we consider the following four problems using the classification of simple Σ-knots.

(A) Fixing Σ, can one define a "trivial" knot among Σ-knots?

(B) When is a simple 3-knot decomposable?

(C) Does there exist a fibered 3-knot which is, though decomposable, not the connected sum of two fibered 3-knots, both different from the trivial S^3-knot?

(D) If a simple 3-knot is algebraically fibered, when is it geometrically fibered?

As for Problem (A), we define a trivial Σ-knot to be a simple Σ-knot with *trivial* Seifert matrix, i.e., a Seifert matrix S-equivalent to the zero matrix, for each Σ with zero Rohlin invariant. This trivial Σ-knot is unique (by Theorem 2.2) and characterized by the property that $\pi_i(S^5-K)\cong\pi_i(S^1)$ for all i. Furthermore, if Σ bounds a compact contractible 4-manifold M, then K bounds M embedded in S^5 (§5).

We can answer Problem (B) in terms of Seifert matrices (§3). From this we can derive the following notable fact: If Σ is not diffeomorphic to S^3 and has zero Rohlin invariant, all simple Σ-knots except the trivial Σ-knot are decomposable. As an application, we shall determine when an algebraic 3-knot is decomposable (Theorem 3.4). As a corollary of this, we shall obtain the existence theorem of decomposable algebraic 3-knots (Corollary 3.8) analogous to a result of Michel and Weber [**13**].

We answer Problem (C) affirmatively using a result of Donaldson [**4**] (Example 4.1). Thus the solution of Problem (B) does not apply directly to the problem of decomposing fibered 3-knots into fibered 3-knots. Note that this is the only odd dimension in which simple fibered knots with the property as in (C) exist ([2], [20]).

Problem (D) is motivated by Kearton's example of a simple S^3-knot which is algebraically fibered but is not geometrically fibered ([9]). Combining the classification theorem with our previous result [18], we can partly answer Problem (D). We show, for example, that an algebraically fibered simple 3-knot is stably fibered (Proposition 4.4).

Throughout the paper, Σ denotes an (oriented) homology 3-sphere. All maps and manifolds are C^∞. The symbol $\approx$ denotes a diffeomorphism between manifolds.

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§ 2. Classification.

Let (S^5, K) be a 3-knot. Then the following holds. See [10, Theorem (2)].

PROPOSITION 2.1. *(S^5, K) is simple if and only if K bounds a 1-connected compact oriented 4-submanifold of S^5.*

Suppose (S^5, K) is a simple 3-knot. Let F^4 be a 1-connected 4-submanifold of S^5 bounded by K as in Proposition 2.1. A *Seifert form* Γ of (S^5, K) (obtained via F) is the bilinear map

$$\Gamma:\ H_2(F)\times H_2(F)\longrightarrow \boldsymbol{Z}$$

defined by $\Gamma(\alpha, \beta)=\mathrm{lk}(\alpha, i_*\beta)$, where lk denotes linking number and $i: F\to S^5-F$ is the map defined by the translation in the positive normal direction. (We always assume that the homology is with integer coefficient unless otherwise indicated.) Since $H_2(F)$ is free abelian, we have a matrix representing the form Γ, called a *Seifert matrix.*

Let L_1 and L_2 be integral square matrices. Then L_1 is *congruent* to L_2 (over $\boldsymbol{Z}$) if $L_1=PL_2{}^tP$ for some integral unimodular matrix P. (${}^tP$ denotes the transposed matrix of P.) Any matrix of the form

$$\left(\begin{array}{c|cc} L_1 & \multicolumn{2}{c}{0} \\ \hline \alpha & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|cc} L_1 & \beta & 0 \\ \hline \multirow{2}{*}{0} & 0 & 1 \\ & 0 & 0 \end{array}\right),$$

where α is a row vector and β is a column vector, is called an *elementary enlargement* of L_1. L_1 is an *elementary reduction* of any of its elementary enlargements. *S-equivalence* is the equivalence relation generated by congruence over $\boldsymbol{Z}$, elementary enlargement, and elementary reduction. If L_1 is S-equivalent to L_2, we write $L_1 \sim L_2$. If L_1 is congruent to L_2 (over $\boldsymbol{Z}$), we write $L_1 \simeq L_2$.

DEFINITION. For a homology 3-sphere Σ, $\mathrm{SK}(\Sigma, S^5)$ denotes the set of isotopy classes of simple Σ-knots in S^5.

DEFINITION. For $\mu \in \boldsymbol{Z}/2\boldsymbol{Z}$ $(=\{0, 1\})$, $\mathrm{SM}(\mu)$ denotes the set of S-equivalence classes of integral square matrices L such that $L+{}^tL$ is unimodular and $\mathrm{sign}(L+{}^tL) \equiv 8\mu \pmod{16}$, where $\mathrm{sign}(L+{}^tL)$ denotes the signature of $L+{}^tL$. Note that $|\det(L+{}^tL)|$ and $\mathrm{sign}(L+{}^tL)$ are invariants of the S-equivalence class of L.

For a homology 3-sphere Σ, $\mu(\Sigma)$ $(\in \boldsymbol{Z}/2\boldsymbol{Z})$ denotes the Rohlin invariant of Σ (see [8]). Then we have the following classification theorem of simple Σ-knots.

THEOREM 2.2. *For any homology* 3-*sphere* Σ, *the map*

$$\Phi_\Sigma : \mathrm{SK}(\Sigma, S^5) \longrightarrow \mathrm{SM}(\mu(\Sigma))$$

which associates with each knot its Seifert matrix is well-defined and bijective.

This theorem is an easy generalization of [**12**, Theorem 1, 2, 3] and can be proved by the same argument as in [**12**]. The most important point lies in the injectivity of Φ_Σ. This property is proved with the help of [**18**, §4]. See also [**7**, p. 601].

§3. Decomposability.

For a simple 3-knot, we can determine when it is decomposable as follows.

PROPOSITION 3.1. *Let* (S^5, K) *be a simple* Σ-*knot with Seifert matrix* L. *Then* (S^5, K) *is decomposable if and only if the following conditions are satisfied.*

(1) $L \sim L_1 \oplus L_2$ *for some integral square matrices* L_1 *and* L_2 *(possibly trivial).*

(2) $\Sigma \approx \Sigma_1 \# \Sigma_2$ *for some homology* 3-*spheres* Σ_1 *and* Σ_2 *(possibly diffeomorphic to* S^3*).*

(3) L_i *is non-trivial or* $\Sigma_i \not\approx S^3$ *for* $i=1, 2$.

(4) $\mathrm{sign}(L_i+{}^tL_i) \equiv 8\mu(\Sigma_i) \pmod{16}$ *for* $i=1, 2$.

PROOF. Suppose $(S^5, K)=(S^5, K_1)\#(S^5, K_2)$. Since $\pi_1(S^5-K) \cong \boldsymbol{Z}$, we see easily that $\pi_1(S^5-K_i) \cong \boldsymbol{Z}$. Thus (S^5, K_i) is simple. Using this fact and Theorem 2.2, we obtain the result easily.

In the following three cases, we can rewrite the above conditions more simply.

COROLLARY 3.2. *Let (S^5, K) be a simple Σ-knot with Seifert matrix L.*

(1) *If $\mu(\Sigma)\neq 0$ and Σ is an irreducible 3-manifold, then (S^5, K) is decomposable if and only if $L\sim L_1\oplus L_2$ for some non-trivial integral square matrices L_1 and L_2.*

(2) *If $\mu(\Sigma)\neq 0$ and Σ is a reducible 3-manifold, then (S^5, K) is decomposable.*

(3) *If $\mu(\Sigma)=0$, $\Sigma\not\approx S^3$, and L is non-trivial, then (S^5, K) is decomposable.*

PROOF. (1) Using Proposition 3.1, one can prove this without difficulty.

(2) Let $\Sigma\approx\Sigma_1\#\Sigma_2$, where $\Sigma_i\not\approx S^3$. We may assume that $\mu(\Sigma_1)=0$ and $\mu(\Sigma_2)\neq 0$. Let L_1 be zero and $L_2=L$. Then L_i and Σ_i satisfy the conditions (1) through (4) of Proposition 3.1.

(3) Let $\Sigma_1=S^3$ and $\Sigma_2=\Sigma$. Further let $L_1=L$ and L_2 be zero. Then these satisfy the conditions (1) through (4) of Proposition 3.1. This completes the proof.

Thus, for any Σ with $\mu(\Sigma)=0$ and $\Sigma\not\approx S^3$, all simple Σ-knots except the trivial Σ-knot are decomposable. This is because no Σ-knot is the trivial S^3-knot if $\Sigma\not\approx S^3$. This motivated the definition of the trivial Σ-knot. We shall characterize the trivial Σ-knot in §5.

DEFINITION. A simple Σ-knot (S^5, K) is *strictly decomposable* if $(S^5, K)=(S^5, K_1)\#(S^5, K_2)$ for some non-trivial Σ_i-knots (S^5, K_i) $(i=1, 2)$.

Under this definition, the following proposition is direct from Proposition 3.1.

PROPOSITION 3.3. *Let (S^5, K) be a simple Σ-knot with Seifert matrix L. Then (S^5, K) is strictly decomposable if and only if the following conditions are satisfied.*

(1) *$L\sim L_1\oplus L_2$ for some non-trivial integral square matrices L_1 and L_2.*

(2) *$\Sigma\approx\Sigma_1\#\Sigma_2$ for some homology 3-spheres Σ_1 and Σ_2 (possibly diffeomorphic to S^3).*

(3) $\mathrm{sign}(L_i+{}^tL_i)\equiv 8\mu(\Sigma_i) \pmod{16}$ *for $i=1, 2$.*

Next we consider algebraic 3-knots. An algebraic knot is a knot that arises around an isolated singular point of a complex hypersurface. More precisely, let f be an analytic function on some neighborhood of the origin 0 in $\boldsymbol{C}^{n+1}$ with $f(0)=0$. We suppose that f has an isolated critical point at the origin. Then the *algebraic knot* associated with f is defined to be the isotopy class of $K_f^{2n-1}=S_\varepsilon^{2n+1}\cap f^{-1}(0)\subset S_\varepsilon^{2n+1}$ for $\varepsilon>0$ sufficiently small, where S_ε^{2n+1} is the $(2n+1)$-sphere of radius ε about the origin. (As a general reference for this see [**14**].)

Algebraic knots are always simple fibered knots. A *fibered knot* is a knot (S^{2n+1}, K^{2n-1}) whose complement $S^{2n+1}-K$ is a smooth fiber bundle over S^1 such that the closure $\overline{M}$ of the fiber M is a compact manifold with boundary $\partial\overline{M}=K$

($\overline{M}=M\cup K$). A fibered knot (S^{2n+1}, K^{2n-1}) is *simple* if $\overline{M}$ is $(n-1)$-connected and K is $(n-2)$-connected (see [5], [18]).

Let (S^{2n+1}, K^{2n-1}) be a simple fibered knot, and let F^{2n} be its fiber, i.e., $F=\overline{M}=M\cup K$. A Seifert matrix of (S^{2n+1}, K^{2n-1}) obtained via F is called *special.* A special Seifert matrix of a simple fibered knot is always unimodular by virtue of the Alexander duality.

THEOREM 3.4. *Let (S^5, K) be an algebraic 3-knot (different from the trivial S^3-knot) with K a homology 3-sphere. Let L be its special Seifert matrix. Then the following holds.*

(1) *If $\mu(K)=0$, then (S^5, K) is decomposable.*

(2) *If $\mu(K)=0$, then (S^5, K) is strictly decomposable if and only if $L\simeq L_1\oplus L_2$ for some non-trivial unimodular matrices L_1 and L_2 with* $\operatorname{sign}(L_1+{}^tL_1)\equiv 0 \pmod{16}$.

(3) *If $\mu(K)\neq 0$, then the following conditions are equivalent.*

(a) *(S^5, K) is decomposable.*

(b) *(S^5, K) is strictly decomposable.*

(c) *$L\simeq L_1\oplus L_2$ for some non-trivial unimodular matrices L_1 and L_2.*

We need the following two lemmas for the proof of Theorem 3.4.

LEMMA 3.5. *Any integral square matrix L with $L+{}^tL$ unimodular is S-equivalent to a non-singular matrix (i.e. with non-zero determinant) or zero.*

LEMMA 3.6. *Suppose that L_1 and L_2 are S-equivalent non-singular matrices. Then* $\det L_1=\det L_2$. *Furthermore, if L_1 and L_2 are unimodular, L_1 is congruent to L_2 over $\boldsymbol{Z}$.*

These lemmas are due to Trotter ([12]).

PROOF OF THEOREM 3.4. (1) Since (S^5, K) is non-trivial, $\operatorname{rank} L\geqq 1$ (see [14, §7]). Hence L is non-trivial by Lemma 3.6. Furthermore, K is not diffeomorphic to S^3 by [15]. It follows from Corollary 3.2 (3) that (S^5, K) is decomposable.

(2), (3) By Neumann [16], K is irreducible as a 3-manifold. Using this fact, Lemmas 3.5, 3.6, and Proposition 3.3, one can prove these results without difficulty.

EXAMPLE 3.7. Let $f(x, y, z)=x^2+y^3+z^{13}$ and let (S^5, K_f) be the algebraic knot associated with f. It is well-known that K_f is the Brieskorn manifold $\Sigma(2, 3, 13)$ which is a homology 3-sphere with zero Rohlin invariant ([1]). Thus (S^5, K_f) is decomposable. On the other hand, the Alexander polynomial $\Delta(t)$ of (S^5, K_f) is the cyclotomic polynomial $\phi_{78}(t)$, which is irreducible in $\boldsymbol{Z}[t]$ ([14]). Since $\Delta(t)=\det(tL+{}^tL)$ (L is a special Seifert matrix of (S^5, K_f)), L cannot be the direct sum of two non-trivial matrices. Thus (S^5, K_f) is not strictly decomposable.

As a corollary of Theorem 3.4, we obtain the following existence theorem of decomposable algebraic 3-knots. This is an extension to dimension three of the theorem of Michel and Weber [**13**].

COROLLARY 3.8. *Let* $g:(\boldsymbol{C}^2, 0)\to(\boldsymbol{C}, 0)$ *be an analytic function which has an isolated critical point and is locally irreducible at* 0. *Suppose that the Puiseux expansion of* g *contains at least two pairs. Define* $f:(\boldsymbol{C}^3, 0)\to(\boldsymbol{C}, 0)$ *by* $f(x, y, z)=g(x, y)+z^r$, *where* $r\geqq 2$ *is an integer. Then if the algebraic* 3*-knot* (S^5, K_f) *associated with* f *is a homology* 3*-sphere knot* (*i.e. if* K_f *is a homology* 3*-sphere*), (S^5, K_f) *is decomposable.*

PROOF. Let L be a special Seifert matrix of (S^5, K_f). It is shown in [**13**] that $L\simeq L_1\oplus L_2$ for some non-trivial matrices L_1 and L_2. Then Theorem 3.4 shows that (S^5, K_f) is decomposable.

EXAMPLE 3.9. Let $g(x, y)=y^4-2x^3y^2-4x^5y+x^6-x^7$ and let $f_r(x, y, z)=g(x, y)+z^r$ for an integer $r\geqq 2$ (see [**13**, §4]). If g.c.d.$(2, r)=$g.c.d.$(3, r)=$g.c.d.$(13, r)=1$, then the algebraic knot (S^5, K_r) associated with f_r is a homology 3-sphere knot ([**18**]). Since g satisfies the condition of Corollary 3.8, (S^5, K_r) is decomposable for any r prime to 2, 3 and 13.

Furthermore if $r\equiv 5 \pmod{78}$, (S^5, K_r) is the connected sum of two non-trivial simple fibered 3-knots ([**18**]). Note that a decomposable simple fibered 3-knot is not always the connected sum of two fibered 3-knots, both different from the trivial S^3-knot. See Example 4.1.

§4. Fibered knots.

In this section we consider simple fibered 3-knots. First we construct an example of a (strictly) decomposable simple fibered S^3-knot which is not the connected sum of two non-trivial fibered S^3-knots.

EXAMPLE 4.1. Set

$$A=\begin{pmatrix} 1&1&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&1&1&0&1\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix} 0&1&-1&-1&1&0&0&1\\ 0&0&0&-1&-1&0&1&0\\ 1&0&0&1&0&-1&-1&1\\ 1&1&0&0&1&1&1&0\\ -1&1&0&-1&0&0&0&-1\\ 0&0&1&-1&1&0&0&-1\\ 0&-1&1&-1&0&0&0&1\\ -1&0&-1&0&1&1&0&0 \end{pmatrix}.$$

Then both A and B are unimodular, and we have

$$\det(A+{}^tA)=1, \qquad \operatorname{sign}(A+{}^tA)=8,$$
$$\det(B+{}^tB)=1 \text{ and } \operatorname{sign}(B+{}^tB)=0.$$

Set $L=A\oplus A\oplus B$. By [**18**, §3] there exists a simple fibered S^3-knot (S^5, K) whose special Seifert matrix is L. By Proposition 3.3, (S^5, K) is strictly decomposable.

ASSERTION. *(S^5, K) cannot be the connected sum of two non-trivial fibered S^3-knots.*

PROOF. Suppose $(S^5, K)=(S^5, K_1)\#(S^5, K_2)$, where (S^5, K_i) is a non-trivial fibered S^3-knot $(i=1, 2)$. As in the proof of Proposition 3.1, (S^5, K_i) is simple. Let L_i be a special Seifert matrix of (S^5, K_i). Then $L\simeq L_1\oplus L_2$. Since

$$\operatorname{sign}(L_i+{}^tL_i)\equiv 0 \pmod{16}$$

and

$$\operatorname{sign}(L_1+{}^tL_1)+\operatorname{sign}(L_2+{}^tL_2)=\operatorname{sign}(L+{}^tL)=16,$$

we may assume that $\operatorname{sign}(L_1+{}^tL_1)=16$ and $\operatorname{sign}(L_2+{}^tL_2)=0$.

On the other hand, $\det(tL+{}^tL)=\Delta_A(t)^2\Delta_B(t)$, where

$$\begin{aligned}\Delta_A(t)&=\det(tA+{}^tA)\\&=t^8+t^7-t^5-t^4-t^3+t+1\\&=\phi_{30}(t) \qquad \text{(cyclotomic polynomial)}\end{aligned}$$

and

$$\begin{aligned}\Delta_B(t)&=\det(tB+{}^tB)\\&=t^8-19t^7+95t^6-221t^5+289t^4-221t^3+95t^2-19t+1.\end{aligned}$$

Thus $\Delta_A(t)$ is irreducible. It is an easy exercise to show that $\Delta_B(t)$ is also irreducible (see [**21**, §22]). Since

$$\det(tL+{}^tL)=\det(tL_1+{}^tL_1)\cdot\det(tL_2+{}^tL_2)$$

and both L_1 and L_2 are non-trivial, $\operatorname{rank} L_1=16$ and $\operatorname{rank} L_2=8$.

Thus $L_1+{}^tL_1$ is a positive definite unimodular symmetric matrix of even type. It is a well-known fact that $L_1+{}^tL_1$ is an intersection matrix of the fiber F of the fibered knot (S^5, K_1). Thus $V=F\cup D^4$ (identified along $\partial F=K\approx S^3=\partial D^4$) is a smooth closed 1-connected 4-manifold with a positive definite intersection form which is not the standard form. This contradicts the result of Donaldson [**4**]. This completes the proof.

REMARK 4.2. Let (S^5, K_A) and (S^5, K_B) be the non-trivial simple S^3-knots having Seifert matrices $A\oplus A$ and B respectively. Then $(S^5, K)=(S^5, K_A)\#(S^5, K_B)$. By [**18**] (S^5, K_B) is smoothly fibered. Thus (S^5, K_A) is not smoothly fibered. However, (S^5, K_A) does fiber topologically. See [**7**], [**9**].

REMARK 4.3. In the other odd dimensions, anomalous phenomena like Example 4.1 cannot occur (see Stallings [**20**], Browder-Levine [**2**]).

As we have already mentioned, special Seifert matrices of fibered knots are unimodular. Now we consider the converse.

DEFINITION. A simple 3-knot is *algebraically fibered* if its Seifert matrix is S-equivalent to a unimodular matrix or zero.

Note that an algebraically fibered simple 3-knot is not always smoothly fibered. In fact, Kearton [**9**] has constructed a counter-example using the result of Donaldson [**4**].

Now we consider Problem (D) in § 1. Partial answers are given as follows.

Let (S^5, K_S) be a simple fibered S^3-knot whose fiber is diffeomorphic to $(S^2\times S^2\# S^2\times S^2)^\circ(=S^2\times S^2\# S^2\times S^2-\operatorname{Int} D^4)$. Fibered knot with this property exists by [**18**, § 6]. We call (S^5, K_S) a *stabilizer* by virtue of the next proposition.

PROPOSITION 4.4. *Let (S^5, K) be an algebraically fibered simple Σ-knot. Then $(S^5, K)\# k(S^5, K_S)$ is a simple fibered* 3*-knot for some non-negative integer k.*

PROOF. Let L_k be a unimodular Seifert matrix of $(S^5, K)\# k(S^5, K_S)$. If k is large enough, there exists a simple fibered Σ-knot (S^5, K_k) whose special Seifert matrix is L_k by [**18**, § 3]. By Theorem 2.2, $(S^5, K)\# k(S^5, K_S)$ is isotopic to (S^5, K_k). Thus $(S^5, K)\# k(S^5, K_S)$ is fibered.

$$\text{Set}\quad E_8=\begin{pmatrix} 2&1&0&0&0&0&0&0\\ 1&2&1&0&0&0&0&0\\ 0&1&2&1&0&0&0&0\\ 0&0&1&2&1&0&0&0\\ 0&0&0&1&2&1&0&1\\ 0&0&0&0&1&2&1&0\\ 0&0&0&0&0&1&2&0\\ 0&0&0&0&1&0&0&2 \end{pmatrix}\quad\text{and}\quad U=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

PROPOSITION 4.5. *Let (S^5, K) be an algebraically fibered simple Σ-knot with unimodular Seifert matrix L. Suppose that Σ bounds a compact contractible* 4*-manifold and that $L+{}^tL\simeq\alpha E_8\oplus\beta U$, where α is even and $\beta\geqq(3/2)|\alpha|+1$. Then (S^5, K) is a simple fibered* 3*-knot.*

PROOF. Let M be a compact contractible 4-manifold whose boundary is Σ. Set $\alpha'=\alpha/2$, $\beta'=\beta-3|\alpha'|$ and $F=M\#(-\alpha')V_4\#\beta'(S^2\times S^2)$, where V_4 is the non-singular hypersurface of degree 4 in $\boldsymbol{CP}_3$. Then by [**18**, § 3], there exists a simple fibered Σ-knot (S^5, K') with special Seifert matrix L and with fiber diffeomorphic to F. By Theorem 2.2, (S^5, K) is isotopic to (S^5, K'). Thus (S^5, K)

is fibered.

REMARK 4.6. In Proposition 4.5, if L is non-trivial and $\Sigma \not\approx S^3$, (S^5, K) is the connected sum of two simple fibered 3-knots, both different from the trivial S^3-knot. In fact, there exist a simple fibered Σ-knot (S^5, K_1) whose fiber is diffeomorphic to M (see §5) and a simple fibered S^3-knot (S^5, K_2) with Seifert matrix L. By Theorem 2.2, $(S^5, K)=(S^5, K_1)\#(S^5, K_2)$. Note that (S^5, K_1) is the trivial Σ-knot.

§5. Characterization of the trivial Σ-knot.

THEOREM 5.1. *Let (S^5, K) be a simple Σ-knot. Then the following conditions are equivalent.*

(1) *S^5-K is homotopy equivalent to S^1.*

(2) *(S^5, K) is the trivial Σ-knot.*

In the case that Σ bounds a compact contractible 4-manifold, (1) *and* (2) *are also equivalent to the following condition.*

(3) *K bounds a compact contractible 4-submanifold in S^5.*

PROOF. (1)⇒(2) This follows from [**12**, §23].

(2)⇒(1) By Proposition 4.4, $(S^5, K')=(S^5, K)\#k(S^5, K_S)$ is a simple fibered 3-knot for some k. Set $X'=S^5-K'$, $X=S^5-K$ and $X_S=S^5-K_S$, and let $\tilde{X}'$, $\tilde{X}$ and $\tilde{X}_S$ be their infinite cyclic coverings respectively. Then we have $\tilde{H}_q(\tilde{X}')\cong \tilde{H}_q(\tilde{X})\oplus(\oplus_k\tilde{H}_q(\tilde{X}_S))$. Since (S^5, K') is a simple fibered 3-knot, $\tilde{H}_q(\tilde{X})=0$ $(q\neq 2)$ and $\tilde{H}_2(\tilde{X})$ is free abelian. If we let L be a Seifert matrix of (S^5, K), $tL+{}^tL$ is a presentation matrix of $\tilde{H}_2(\tilde{X}; \boldsymbol{Q})$ over $\boldsymbol{Q}\langle t\rangle=\boldsymbol{Q}[t, t^{-1}]$, where t is a generator of the covering transformation group of $\tilde{X}$ (see [**11**]). Since L is S-equivalent to the zero matrix, $\det(tL+{}^tL)=\pm t^{\alpha}$ for some α. Hence we have $\tilde{H}_2(\tilde{X}; \boldsymbol{Q})= \tilde{H}_2(\tilde{X})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=0$. Thus $\tilde{H}_*(\tilde{X})=0$. Since $\pi_1(\tilde{X})=1$, $\tilde{X}$ is contractible. Thus X is homotopy equivalent to S^1.

Next we consider the case that Σ bounds a compact contractible 4-manifold M.

(3)⇒(2) This is clear.

(2)⇒(3) Let $N=M\times S^1/\sim$, where $(x, 1)\sim(x, \theta)$ for every $x\in\partial M$ and $\theta\in S^1$. It is easily seen that N is a homotopy 5-sphere. Thus N is diffeomorphic to S^5. Hence $(N, \partial M)=(S^5, K')$ is a simple (fibered) Σ-knot with a trivial Seifert matrix. By Theorem 2.2, (S^5, K) is isotopic to (S^5, K'). Thus K bounds M in S^5. This completes the proof.

REMARK 5.2. For Σ with $\mu(\Sigma)\neq 0$, there is no trivial Σ-knot by the very definition. However, if we work in the topological category, a "trivial Σ-knot" can be defined. In fact, there exists a topological (locally flat) Σ-knot (S^5, K)

with S^5-K homotopy equivalent to S^1. For example this is obtained as $(S^2\Sigma, K)$, where $S^2\Sigma$ is the double suspension of Σ and K is the image of the canonical embedding of Σ into $S^2\Sigma$. By a result of Edwards and Cannon, $S^2\Sigma$ is homeomorphic to S^5 ([**3**], [**6**]). For other constructions of (S^5, K) see [**7**] and [**17**].

Furthermore, if (S^5, K_i) are topological Σ-knots with S^5-K_i homotopy equivalent to S^1 $(i=1, 2)$, (S^5, K_1) is isotopic to (S^5, K_2). For details see [**19**, § 2].

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