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Analytic functions convex in one direction.

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A necessary and sufficient condition for the convexity of a function $f(z)$, regular for $|z|<r$ such that $f'(0)\neq 0$, is known[1] to be

(1) $$1+\Re\frac{zf''(z)}{f'(z)}>0 \quad \text{for} \quad |z|<r.$$

However, as a sufficient condition for the univalency of such a function $f(z)$, the condition (1) is not sharp enough. This fact has been pointed out by Ozaki[2], who has proved the following theorem:

THEOREM A. *Let $f(z)=z+a_2z^2+\cdots$ be regular for $|z|<r$. If $f(z)$ satisfies in $|z|<r$ one of the following conditions:*

(i) $$1+\Re\frac{zf''(z)}{f'(z)}>-\frac{1}{2},$$ (ii) $$1+\Re\frac{zf''(z)}{f'(z)}<\frac{3}{2},$$

(iii) $$\left|1+\Re\frac{zf''(z)}{f'(z)}\right|<2,$$ (iv) $$\left|\Re\frac{zf''(z)}{f'(z)}\right|<2,$$

then $f(z)$ is univalent for $|z|<r$.

In this paper we shall generalize or make more precise this theorem, by proving that the above conditions (i)–(iv) are also sufficient for $f(z)$ to be convex in one direction[3], i.e. to have the property that it maps $|z|=\rho<r$ for every ρ near r into a contour which may be cut by every straight-line parallel to this direction in not more than two points. Furthermore we shall obtain some more sufficient conditions for the covexity of functions in one direction in generalized forms. It will be proved also that these conditions are equally sufficient for $f(z)$ to be at most k-valent.

§ 1. Univalent functions convex in one direction.

THEOREM 1. *Let*

(2) $$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

be meromorphic for $|z|<1$. *If there exists the relation in* $|z|<1$

(3) $$\alpha > 1+\Re\frac{zf''(z)}{f'(z)} > -\frac{\alpha}{2\alpha-3},$$

where α *is an arbitrary number not less than* 3/2, *then* $f(z)$ *is regular and univalent for* $|z|<1$. *Moreover,* $f(z)$ *maps* $|z|=r$ *for every* $r<1$ *into a curve which is convex in one direction, and*

(4) $$|a_n|\leqq n \quad \text{for all} \quad n.$$

REMARK. If we put $\alpha=\infty$, $\alpha=3/2$, $\alpha=2$, $\alpha=3$ in (3), we have the conditions (i)–(iv) of Theorem A respectively. We see also that when we have these conditions (i)–(iv) for $|z|<1$, the Bieberbach conjecture (4) for the normalized function (2) is true.

To prove the above theorem we need the following lemmas.

LEMMA 1. *Let* $f(z)=z+a_2z^2+a_3z^3+\cdots$ *be regular for* $|z|<1$. *If* $zf'(z)$ *is star-like in one direction*[3], *i.e.* $zf'(z)$ *has the property that it maps* $|z|=r$ *for every* r *near* 1 *onto a contour* C *which is cut by a straight-line passing through the origin in two, and not more than two points, then* (i) $f(z)$ *is convex in one direction,* (ii) $f(z)$ *is univalent for* $|z|<1$, (iii) $|a_n|\leqq n$ *for all* n.

We owe this lemma to M. S. Robertson.[3]

LEMMA 2. *Let* $\varphi(z)$ *be regular for* $|z|\leqq r$ *and* $\varphi(z)\neq 0$ *in* $0<|z|\leqq r$. *Further let* $\varphi(0)=0$. *If there exists the relation*

(5) $$\int_0^{2\pi}\left|\Re\frac{z\,\varphi'(z)}{\varphi(z)}\right|d\theta<4\pi,\quad z=\rho\,e^{i\theta},\quad \text{for every } \rho\leqq r,$$

then $\varphi(z)$ *maps* $|z|=\rho$ *for every* $\rho\leqq r$ *onto a curve which is star-like in one direction.*

This is a special case of Lemma 5 which will be proved in § 2 of this paper.

LEMMA 3. *If* $f(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+\cdots$ (p: *an integer, positive or negative*) *is meromorphic and*

(6) $$\Re\left[e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>K \qquad (\alpha, K\text{: real constants})$$

for $|z|<r$, *then*

$$\frac{f'(z)}{z^{p-1}} \neq \begin{cases} 0 \\ \infty \end{cases} \qquad \textit{for } |z|<r,$$

and hence $f(z)$ *is regular* (*i.e.* $f(z)$ *has no pole*) *for* $0<|z|<r$.

This lemma is due to Ozaki[2].

PROOF OF THEOREM 1.

We see by Lemma 3 that $f(z)$ is regular for $|z|<1$ and that $zf'(z)$ has only one zero at the origin under the hypothesis (3).

Put $zf'(z)=Re^{i\Theta}$, $z=re^{i\theta}$, then we have

$$1+\Re\frac{zf''(z)}{f'(z)}=\Re\frac{z(zf'(z))'}{zf'(z)}=\frac{d\Theta}{d\theta} \quad \text{for} \quad |z|=r<1. \tag{7}$$

Hence the assumption (3) can be rewritten as follows:

$$\alpha>\frac{d\Theta}{d\theta}>-\frac{\alpha}{2\alpha-3} \qquad \text{for} \quad |z|=r<1. \tag{3'}$$

Now let us denote by C_2 the part of $|z|=r$ on which

$$\frac{d\Theta}{d\theta}>0 \qquad \text{and put} \quad \int_{C_1} d\arg z=x,$$

and by C_2 the part of $|z|=r$ on which

$$\frac{d\Theta}{d\theta}\leqq 0 \qquad \text{and hence} \quad \int_{C_2} d\arg z=2\pi-x.$$

Furthermore put

$$y_1=\int_{C_1} d\Theta, \qquad -y_2=\int_{C_2} d\Theta, \tag{8}$$

then we have

$$\int_C d\Theta=y_1-y_2=2\pi, \tag{9}$$

$$\int_C |d\Theta|=y_1+y_2=2y_1-2\pi. \tag{10}$$

Since we have the hypothesis (3′), we have

$$y_1=\int_{C_1}\frac{d\Theta}{d\theta}\,d\theta<\alpha x, \tag{11}$$

(12) $$y_2=\int_{C_2}\left(-\frac{d\Theta}{d\theta}\right)d\theta<(2\pi-x)\frac{\alpha}{2\alpha-3}=\frac{2\pi\alpha-\alpha x}{2\alpha-3}.$$

Let us show that, under these circumstances, $y_1<3\pi$.

Suppose that $y_1\geqq 3\pi$, then by (9)

(13) $$y_2\geqq\pi$$

and by (11)

(14) $$\alpha x>3\pi.$$

But on the other hand by (12) and (14)

$$y_2<\frac{2\pi\alpha-3\pi}{2\alpha-3}=\pi.$$

This contradicts (13). Hence

(15) $$y_1<3\pi.$$

By (10) and (15) $$\int_C\left|\frac{d\Theta}{d\theta}\right|d\theta<4\pi.$$

By (7) we obtain

$$\int_C\left|\Re\frac{z(zf'(z))'}{zf'(z)}\right|d\theta<4\pi \quad \text{for} \quad |z|=r<1.$$

Hence by Lemma 2, $zf'(z)$ is star-like in one direction for every $r<1$. Consequently $f(z)$ is convex in one direction for every $r<1$. Hence $f(z)$ is univalent for $|z|<1$ and $|a_n|\leqq n$ for all n, by Lemma 1.

Q. E. D.

By the above argument we can also state the following:

THEOREM 2. *Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be regular for $|z|\leqq 1$. Furthermore let C_1 be the part of $|z|=1$ on which*

$$1+\Re\frac{zf''(z)}{f'(z)}>0 \qquad \textit{and put} \quad x=\int_{C_1}d\arg z$$

and C_2 be the part of $|z|=1$ on which

$$1+\Re\frac{zf''(z)}{f'(z)}\leqq 0 \qquad \textit{and hence}\ 2\pi-x=\int_{C_2}d\arg z.$$

If $f(z)$ satisfies for $|z|=1$ one of the following conditions

$$\text{i)}\quad \int_{C_1}\left(1+\Re\frac{zf''(z)}{f'(z)}\right)d\theta<3\pi, \qquad \text{i}')\quad 1+\Re\frac{zf''(z)}{f'(z)}<\frac{3\pi}{x},$$

$$\text{ii)}\quad \int_{C_2}\left(1+\Re\frac{zf''(z)}{f'(z)}\right)d\theta>-\pi \qquad \text{ii}')\quad 1+\Re\frac{zf''(z)}{f'(z)}>-\frac{\pi}{2\pi-x}$$

and $f'(z)\neq 0$ in $|z|\leqq 1$, and $f'(z)\neq 0$ in $|z|\leqq 1$,

$$\text{iii)}\quad \int_{C_1+C_2}\left|1+\Re\frac{zf''(z)}{f'(z)}\right|d\theta<4\pi, \qquad \text{iii}')\quad \left|1+\Re\frac{zf''(z)}{f'(z)}\right|<2,$$

then $f(z)$ is convex in one direction and univalent for $|z|\leqq 1$ and $|a_n|\leqq n$ for every n.

It should be noticed that, in i′) and ii′)

$$\frac{3\pi}{x}\geqq\frac{3}{2}\quad\text{and}\quad -\frac{\pi}{2\pi-x}\leqq-\frac{1}{2}.$$

By the way, we can calculate the radius of convexity in one direction for the class of normalized univalent functions.

THEOREM 3. *Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be regular and univalent for $|z|<1$. Then $f(z)$ is convex in one direction for $|z|<4-\sqrt{13}$.*

PROOF. From the Bieberbach's inequality

$$\left|\frac{zf''(z)}{f'(z)}-\frac{2|z|^2}{1-|z|^2}\right|\leqq\frac{4|z|}{1-|z|^2},$$

we obtain

$$1+\Re\frac{zf''(z)}{f'(z)}\geqq 1+\frac{2|z|^2-4|z|}{1-|z|^2}=\frac{1-4|z|+|z|^2}{1-|z|^2}. \tag{16}$$

If $|z|<4-\sqrt{13}$, then the right hand side of the above inequality (16) is surely larger than $-1/2$. Hence by Theorem 1 (in the case where $\alpha=\infty$) $f(z)$ is convex in one direction.

§2. Functions convex of order p in one direction.

To generalize the above results to the case of p-valency, we extend the concepts of star-likeness and convexity in one direction as follows:

DEFINITION. Let $F(z)$ be meromorphic for $|z|\leqq 1$ and continuous on $|z|=1$. Furthermore let C be the image curve of $|z|=1$.

i) If C is cut by a straight-line passing through the origin in $2p$, and not more than $2p$ points, then $F(z)$ is said to be star-like of order p in one direction. Let the class of such functions be denoted by $\mathfrak{S}_1(p)$.

ii) If every straight-line parallel to a direction cuts C in not more than $2p$ points and there exists at least one such straight-line which cuts C in $2p$ points, then $F(z)$ is said to be convex of order p in one direction. Let the class of such functions be denoted by $\mathfrak{K}_1(p)$.

LEMMA 4. *If* $zF'(z) \in \mathfrak{S}_1(p)$, *then* (i) $F(z)$ *is convex of order at most* p *in one direction,* (ii) $F(z)$ *is at most* $\big(p+n(\infty)\big)$*-valent and at least* $\mathrm{Max}\,[n(\infty)-p, 1]$*-valent for* $|z| \leqq 1$, *where* $n(\infty)$ *denotes the number of poles of* $F(z)$ *in* $|z|<1$.

PROOF. (i) It will be sufficient to prove the lemma in the case where the direction of star-likeness of order p is that of the real axis, since otherwise we may consider $e^{i\alpha}F(z)$ instead of $F(z)$, with a suitable choice for the real parameter α.

By the hypothesis, $\mathfrak{I}\{zF'(z)\}$ changes its sign $2p$ times as z moves along the circumference of the unit circle. In view of the identity

$$\mathfrak{I}\{zF'(z)\} = -\frac{\partial\, \mathfrak{R}\, F(z)}{\partial\theta} \qquad \text{for} \qquad z=e^{i\theta},$$

$\mathfrak{R}\, F(z)$ then increases and decreases alternatively p times, respectively, whence it follows that $F(z)$ is convex of order at most p in the direction of the imaginary axis.

(ii) Let $n(w)$ be the number of roots of the equation $F(z)=w$ in $|z| \leqq 1$, and let C be the image of $|z|=1$ by the function $F(z)$. Then we make use of the following geometrical principle[4]: $n(w)$ *can change its value only when* w *arrives at a value assumed by* $F(z)$ *on* $|z|=1$, *and hence, if* w *moves along a continuous curve without crossing the line* C, *then* $n(w)$ *is invariant. When* w *crosses the line* C, *then the saltus of* $n(w)$ *is an integer.*

Since $F(z) \in \mathfrak{K}_1(p')$ with $p' \leqq p$, every straight-line parallel to the direction of convexity of order p' cuts C in not more than $2p$ points. Let us consider the change of $n(w)$ when w moves along any line of this kind. It starts with $n(\infty)$ and ends at the same value $n(\infty)$ after changing its value at most $2p$ times by integer, the multiplicity being taken into account. Hence the maximum value of $n(w)$ is at most

$p+n(\infty)$ and the minimum value of $n(w)$ is at least $\operatorname{Max}[n(\infty)-p, 0]$. Consequently $F(z)$ is at most $(p+n(\infty))$-valent and at least $\operatorname{Max}[n(\infty)-p, 1]$-valent for $|z|\leqq 1$.

LEMMA 5. *Let $\varphi(z)$ be meromorphic for $|z|\leqq r$ and continuous and $\neq 0$ on $|z|=r$. If there holds the relation*

$$\int_0^{2\pi} \left| \Re \frac{z\varphi'(z)}{\varphi(z)} \right| d\theta < 2\pi(p+1), \quad z=re^{i\theta}, \quad p: \text{positive integer}, \tag{17}$$

then $\varphi(z)$ is star-like of order at most p in one direction.

PROOF. Let D denote the image domain of $|z|<r$ mapped by $\varphi(z)$ and let C denote the contour of the domain D. From our hypothesis it is evident that the contour C is a regular closed curve which has neither cusp nor angular point.

Now let $n(\psi)$ be the number of the points of intersection of the straight-line $\arg w=\psi$ (started from the origin) with C. Then it is evident that the total variation of $\arg w$ on C is given by $\int_0^{\pi} n(\psi)\, d\psi$. On the other hand it is given by

$$\int_{|z|=r} |d \arg \varphi(z)| = \int_{|z|=r} \left| \Re \frac{z\varphi'(z)}{\varphi(z)} \right| d\theta \quad \text{where} \quad \theta=\arg z\,.$$

Hence

$$\int_0^{\pi} n(\psi)\, d\psi = \int_{|z|=r} \left| \Re \frac{z\varphi'(z)}{\varphi(z)} \right| d\theta\,. \tag{18}$$

By (18) and our hypothesis (17) we have

$$\int_0^{\pi} n(\psi)\, d\psi < 2\pi(p+1)\,.$$

Hence, we see $n(\psi)<2(p+1)$ for at least one ψ. For this ψ, $n(\psi)$ being an even integer, C is cut by $\arg w=\psi$ in at most $2p$ points, i. e. $\varphi(z)$ is star-like of order at most p in this direction. Q. E. D.

By using Lemmas 4 and 5 we easily obtain the following:

THEOREM 4. *Let $F(z)$ be meromorphic for $|z|\leqq 1$ and continuous and $F'(z)\neq 0$ on $|z|=1$. If there holds the relation*

$$\int_0^{2\pi} \left| 1+\Re \frac{zF''(z)}{F'(z)} \right| d\theta < 2\pi\left(k-n(\infty)+1\right), \quad z=e^{i\theta} \tag{19}$$

then $F(z)$ *is convex of order at most* $k-n(\infty)$ *in one direction in* $|z|\leqq 1$. *Further* $F(z)$ *is at most k-valent and at least* $\mathrm{Max}[2n(\infty)-k, 1]$-*valent for* $|z|\leqq 1$.

We can also easily verify the following theorems by using Lemmas 3, 4, and 5 similarly as in the proof of Theorem 1.

THEOREM 5. *Let* $F(z)$ *be meromorphic for* $|z|\leqq 1$ *and let* $n^*(0)$ *and* $n^*(\infty)$ *denote the number of zeros and poles of* $zF'(z)$ *respectively in* $|z|<1$. *If there holds the relation for* $|z|=1$:

$$\alpha_1 > 1+\Re\frac{zF''(z)}{F'(z)} > -\frac{\alpha_1(k+1-n(\infty)-n^*(0)+n^*(\infty))}{2\alpha_1-k-1+n(\infty)-n^*(0)+n^*(\infty)}, \tag{20}$$

where α_1 *is an arbitrary number not less than* $(k+1-n(\infty)+n^*(0)-n^*(\infty))/2$, *then* $F(z)$ *is convex of order at most* $k-n(\infty)$ *in one direction in* $|z|\leqq 1$. *Further* $F(z)$ *is at most k-valent and at least* $\mathrm{Max}[2n(\infty)-k, 1]$-*valent for* $|z|\leqq 1$.

THEOREM 6. *Let* $F(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ (p: *positive integer*) *be meromorphic for* $|z|<1$. *If there holds the relation for* $|z|<1$

$$\alpha_2 > 1+\Re\frac{zF''(z)}{F'(z)} > -\frac{\alpha_2(k+1-p)}{2\alpha_2-k-1-p}, \tag{21}$$

where α_2 *is an arbitrary number not less than* $(k+p+1)/2$, *then* $F(z)$ *is regular and convex of order at most* p *in one direction in* $|z|<1$, *and at most k-valent for* $|z|<1$.

THEOREM 7. *Let* $G(z)=z^{-p}+\sum_{n=-p+1}^{\infty} a_n z^n$ (p: *positive integer*) *be meromorphic for* $|z|<1$. *If there holds the relation for* $|z|<1$

$$\alpha_3 > 1+\Re\frac{zG''(z)}{G'(z)} > -\frac{\alpha_3(k+1)}{2\alpha_3-k-1+2p} \tag{22}$$

where α_3 *is an arbitrary number not less than* $(k+1-2p)/2$, *then* $G(z)$ *is convex of order at most* $k-p$ *in one direction in* $|z|<1$. *Further* $G(z)$ *is at most k-valent and at least* $\mathrm{Max}[2p-k, 1]$-*valent for* $|z|<1$.

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