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On the zeros of integral functions of integral order.

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1. Let $f(z)$ be an integral function of integral order $\rho>0$, and $M(r)$ be its maximum modulus on the circumference $|z|=r$. Further, let $n(r, \alpha)$ denote the number of zeros of $f(z)-\alpha$ for any complex α. In this note we shall prove the following two theorems:

THEOREM 1. *If* $\log_2 M(r)/\log r$ *has the limit* ρ *for* $r\rightarrow\infty$, *then* $\log n(r, \alpha)/\log r$ *has the same limit for* $r\rightarrow\infty$, *except possibly for some values of* α *belonging to a set of inner logarithmic capacity zero.*

THEOREM 2. *If* $\log M(r)/r^{\rho}$ *is bounded from zero and infinity, so is* $n(r, \alpha)/r^{\rho}$, *except possibly for some values of* α *belonging to a set of inner capacity zero.*

It is known that these theorems hold with an exceptional set whose projection on any straight-line is of zero content[1)].

Our proof is based on the following well-known fact:

LEMMA 1. *For an integral function of non-integral order, above two theorems hold for any* α *without exception*[1)].

2. Let $f(z)$ be meromorphic in $|z|<+\infty$ and s be a positive integer. We put

$$F_{\alpha}(W)=\prod_{k=0}^{s-1}\left[f(zt^{k})-\alpha\right], \quad W=z^{s} \text{ and } R=r^{s},$$

where t is a primitive s-th root of 1, so that $F_{\alpha}(W)$ is meromorphic in $|W|<+\infty$. Then,

LEMMA 2. *There holds*

$$T(R, F_{\alpha})\sim sT(r, f)$$

except possibly for some values of α *belonging to a set of inner capacity zero.*

$T(R, F_\alpha)$ and $T(r,f)$ denote the Nevanlinna's characteristic functions of F_α and f, and $p\sim q$ means $\lim_{r\to\infty} p/q=1$.

PROOF. Let z_n $(n=1, 2, \cdots)$ be the poles of $f(z)$. Then, if $\alpha \neq f(z_n t^k)$ $(n=1, 2, \cdots;\ k=0, \cdots, s-1)$, the zeros of $F_\alpha(W)$ in the W-plane correspond exactly to those of $f(z)-\alpha$ in the z-plane, so that we have

$$n(r, \alpha, f)=n(R, 0, F_\alpha)\,. \tag{1}$$

Hence, we have

$$N(R, 0, F_\alpha)=\int_0^R n(R, 0, F_\alpha)\,\frac{dR}{R}=s\int_0^r n(r, \alpha, f)\,\frac{dr}{r}=sN(r, \alpha, f)\,.$$

Since there holds

$$N(r, \alpha, f)\sim T(r,f) \tag{2}$$

except for some α belonging to a set of inner capacity zero[2], we have

$$N(R, 0, F_\alpha)\sim sT(r,f) \tag{3}$$

with similar exceptions.

On the other hand, we have

$$\begin{aligned} m(R, 0, F_\alpha)&=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\frac{1}{|F_\alpha(Re^{i\Theta})|}\,d\Theta \\ &\leqq \frac{1}{2\pi}\sum_{k=0}^{s-1}\int_0^{2\pi}\overset{+}{\log}\frac{1}{|f(re^{i\theta}t^k)-\alpha|}\,d\theta+O(1) \\ &=\frac{s}{2\pi}\int_0^{2\pi}\overset{+}{\log}\frac{1}{|f(re^{i\Theta})-\alpha|}\,d\theta+O(1)=sm(r, \alpha, f)+O(1)\,, \end{aligned}$$

so that, by (2),

$$m(R, 0, F_\alpha)=o\Big[T(r,f)\Big] \tag{4}$$

with exception of a set of α of inner capacity zero.

By (2), (3) and (4), we have

$$\begin{aligned} sT(r,f)&\sim N(R, 0, F_\alpha)\leqq T(R, F_\alpha) \\ &=N(R, 0, F_\alpha)+m(R, 0, F_\alpha)=sT(r,f)+o\Big[T(r,f)\Big], \end{aligned}$$

so that $T(R, F_\alpha)\sim sT(r,f)$ with exception of a set of α of inner capacity zero, q. e. d.

3. If $f(z)$ is an integral function, so is $F_\alpha(W)$. Putting

$$M(r,f)=\max_{|z|=r}|f(z)| \quad \text{and} \quad M(R,F_\alpha)=\max_{|W|=R}|F_\alpha(W)|,$$

we have

$$T(r,f)<\log M(r,f)<3T(2r,f)$$

and
$$T(R,F_\alpha)<\log M(R,F_\alpha)<3T(2R,F_\alpha).$$

Hence, by Lemma 2, we can state:

LEMMA 3. *Let $f(z)$ be an integral function, and $F_\alpha(W)$ be the function defined in $n°2$. Then, for any α with exception of a set of inner capacity zero, there exist two positive functions $h_\alpha(r)$ and $H_\alpha(r)$ bounded from zero and infinity, such that*

$$\log M(R,F_\alpha)=H_\alpha \log M(h_\alpha r,f).$$

R. C. Young proved the same with an exceptional set of α whose projection on any straight-line is of zero content[1].

4. Proof of Theorems 1 and 2.

We take an integer s greater than ρ, so that ρ/s is not an integer. Suppose that $\lim_{r\to\infty}\log_2 M(r,f)/\log r=\rho$ exists, then, by Lemma 3, we see that, for any non-exceptional α, $\lim_{R\to\infty}\log_2 M(R,F_\alpha)/\log R$ exists and $=\rho/s$. Hence, by Lemma 1, $\lim_{R\to\infty}\log n(R,0,F_\alpha)/\log R$ exists and $=\rho/s$, so that, by (1) in $n°2$, $\lim_{r\to\infty}\log n(r,\alpha,f)/\log r=\rho$.

Theorem 2 can be proved similarly.

References.

1) Borel: Leçons sur les fonctions entières, Paris, 1921. Valiron: Lectures on the general theory of integral functions, Toulouse, 1923.

2) Nevanlinna: Eindeutige analytische Funktionen, Berlin, 1936.