J. Math. Soc. Japan
Vol. 39, No. 4, 1987

Singular hyperbolic systems, VI. Asymptotic analysis for Fuchsian hyperbolic equations in Gevrey classes

By Hidetoshi TAHARA

(Received March 22, 1986)

In the previous papers [11, 12, 13], the author has investigated Fuchsian hyperbolic equations in C^{∞} function spaces. But, here, Fuchsian hyperbolic equations are studied in Gevrey function spaces.

The motivation is as follows. Let

$$P=(t\partial_t)^2-t^{2\kappa_1}\partial_{x_1}^2-t^{2\kappa_2}\partial_{x_2}^2+t^{l_1}a_1(t, x)\partial_{x_1}+t^{l_2}a_2(t, x)\partial_{x_2}+b(t, x)(t\partial_t)+c(t, x),$$

where $(t, x)=(t, x_1, x_2)\in[0, T]\times\boldsymbol{R}^2$, $2\kappa_1, 2\kappa_2, l_1, l_2\in\boldsymbol{N}$ $(=\{1, 2, 3, \cdots\})$, $a_1(t, x)$, $a_2(t, x)$, $b(t, x)$, $c(t, x)\in C^{\infty}([0, T]\times\boldsymbol{R}^2)$, $a_1(0, x)\not\equiv 0$ and $a_2(0, x)\not\equiv 0$. Let $\rho_1(x)$, $\rho_2(x)$ be the roots of $\rho^2+b(0, x)\rho+c(0, x)=0$ and assume that $\rho_1(x), \rho_2(x)\notin\boldsymbol{Z}_+$ $(=\{0, 1, 2, \cdots\})$ for any $x\in\boldsymbol{R}^2$. Then, by Tahara [11] and Mandai [7] we can see the following: $Pu=f$ is well-posed in $C^{\infty}([0, T]\times\boldsymbol{R}^2)$, if and only if "$l_1\geqq\kappa_1$ and $l_2\geqq\kappa_2$" holds. Hence, if we want to treat P without "$l_1\geqq\kappa_1$ and $l_2\geqq\kappa_2$", we must restrict ourselves to the study in suitable subclasses of $C^{\infty}([0, T]\times\boldsymbol{R}^2)$. For this purpose, Gevrey classes seem to be very fitting. This is the reason why the author has come to treat the equation in Gevrey classes.

§1. Main Theorem.

First, we state our Main Theorem and its background.

Let $(t, x)\in[0, T]\times\boldsymbol{R}^n$ $(T>0)$, and let us consider

$$P(t, x, t\partial_t, \partial_x)=(t\partial_t)^m+\sum_{\substack{j+|\alpha|\leq m\\ j<m}}t^{l(j,\alpha)}a_{j,\alpha}(t, x)(t\partial_t)^j\partial_x^{\alpha}, \tag{1.1}$$

where $x=(x_1, \cdots, x_n)$, $\partial_t=\partial/\partial t$, $\partial_x=(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $m\in\boldsymbol{N}$ $(=\{1, 2, 3, \cdots\})$, $\alpha=(\alpha_1, \cdots, \alpha_n)\in\boldsymbol{Z}_+^n$ $(=\{0, 1, 2, \cdots\}^n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $\partial_x^{\alpha}=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. Assume the following conditions:

(A_{κ}) $l(j, \alpha)\in\boldsymbol{R}$ $(j+|\alpha|\leqq m$ and $j<m)$ satisfy

This research was partially supported by Grant-in-Aid for Scientific Research (No. 60740098), Ministry of Education, Science and Culture.

$$\begin{cases} l(j,\alpha)=\kappa_1\alpha_1+\cdots+\kappa_n\alpha_n, & \text{when } j+|\alpha|=m \text{ and } j<m, \\ l(j,\alpha)>0, & \text{when } j+|\alpha|<m \text{ and } |\alpha|>0, \\ l(j,\alpha)\geqq 0, & \text{when } j+|\alpha|<m \text{ and } |\alpha|=0 \end{cases}$$

for some $\kappa=(\kappa_1, \cdots, \kappa_n)\in \boldsymbol{R}^n$ such that $\kappa_i>0$ $(1\leqq i\leqq n)$.

(B) All the roots $\lambda_i(t, x, \xi)$ $(1\leqq i\leqq m)$ of

$$\lambda^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t, x)\lambda^j\xi^\alpha=0$$

are *real, simple* and *bounded* on $\{(t, x, \xi)\in[0, T]\times \boldsymbol{R}^n\times \boldsymbol{R}^n; |\xi|=1\}$.

Then, this operator $P(t, x, t\partial_t, \partial_x)$ $(=P)$ is a good generalization of the operator in the introduction. In this case, the characteristic exponents $\rho_1(x), \cdots, \rho_m(x)$ of P are defined by the roots of

$$\rho^m+\sum_{j<m} a_j(x)\rho^j=0,$$

where $a_j(x)=[t^{l(j,(0,\cdots,0))}a_{j,(0,\cdots,0)}(t, x)]|_{t=0}$ $(j<m)$.

In Tahara [**11, 12**], we have discussed Fuchsian hyperbolic equations in C^∞ function spaces, and established the following result.

THEOREM (Tahara [**11, 12**]). *Assume that* $l(j, \alpha)\in \boldsymbol{Z}_+$ $(j+|\alpha|\leqq m$ *and* $j<m)$, *that* $a_{j,\alpha}(t, x)\in C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$ $(j+|\alpha|\leqq m$ *and* $j<m)$, *and that* (A_κ), (B) *and the condition*

(T) $l(j, \alpha)\geqq\kappa_1\alpha_1+\cdots+\kappa_n\alpha_n$, *when* $j+|\alpha|<m$ *and* $|\alpha|>0$

are satisfied. Then, we have the following results.

(I) (Unique solvability, [**11**]). *If* $\rho_i(x)\notin \boldsymbol{Z}_+$ *holds for any* $x\in \boldsymbol{R}^n$ *and* $1\leqq i\leqq m$, *the equation*

$$P(t, x, t\partial_t, \partial_x)u=f \quad in\ C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$$

is uniquely solvable.

(II) (Asymptotic expansions, [**12**]). *If* $\rho_i(x)-\rho_j(x)\notin \boldsymbol{Z}$ *holds for any* $x\in \boldsymbol{R}^n$ *and* $1\leqq i\neq j\leqq m$, *the general solution of*

$$P(t, x, t\partial_t, \partial_x)u=0 \quad in\ C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$$

is characterized as follows. (II-1) *Any solution* $u(t, x)\in C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$ *can be expanded asymptotically into the form*

$$u(t, x)\sim\sum_{i=1}^m\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^\infty\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big)$$

(as $t\to+0$*) for some unique* $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)\in\mathcal{E}(\boldsymbol{R}^n)$. (II-2) *Conversely, for any* $\varphi_1(x), \cdots, \varphi_m(x)\in\mathcal{E}(\boldsymbol{R}^n)$ *there exist a unique solution* $u(t, x)\in C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$ *and unique coefficients* $\varphi_{k,h}^{(i)}(x)\in\mathcal{E}(\boldsymbol{R}^n)$ $(1\leqq i\leqq m, 1\leqq k<\infty$ *and* $0\leqq h\leqq mk)$ *such that the asymptotic relation in* (II-1) *holds.*

Here, $\mathcal{E}(\boldsymbol{R}^n)$ means the space of all C^∞ functions on $\boldsymbol{R}^n$ equipped with the usual topology, and $C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$ (resp. $C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n)))$ means the space of all infinitely differentiable functions on $[0, T]$ (resp. $(0, T)$) with values in $\mathcal{E}(\boldsymbol{R}^n)$.

In the above theorem, we have assumed the condition (T). But, here, we want to consider the case without (T).

When (T) is not satisfied, it seems impossible to have good results in $C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$ or $C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$. In fact, we can see the following: under the conditions $a_{j,\alpha}(0, x)\not\equiv 0$ $(j+|\alpha|<m$ and $|\alpha|>0)$, (T) is the necessary and sufficient condition for the flat Cauchy problem for P to be C^∞ well-posed (by Tahara [**11**] and Mandai [**7, 8**]). Therefore, if we want to discuss the case without (T), we must restrict ourselves to the study in suitable subclasses of $C^\infty([0, T], \mathcal{E}(\boldsymbol{R}^n))$ or $C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$. This is the starting point of this paper. Especially, our interest lies in the following. *What kind of subclasses are suitable? By what quantities are the admissible classes characterized?*

A function $f(x)\in C^\infty(\boldsymbol{R}^n)$ is said to belong to the *Gevrey class* $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$, if $f(x)$ satisfies the following: for any compact subset K of $\boldsymbol{R}^n$, there are $C>0$ and $h>0$ such that

$$\sup_{x\in K}|\partial_x^\alpha f(x)| \leqq Ch^{|\alpha|}(|\alpha|!)^s \quad \text{for any } \alpha\in \boldsymbol{Z}_+^n. \tag{1.2}$$

As a locally convex space, $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ is defined as follows. For $h>0$ and a regular compact subset K of $\boldsymbol{R}^n$, we denote by $\mathcal{E}^{\{s\},h}(K)$ the space of all functions $f(x)\in C^\infty(K)$ satisfying (1.2) for some $C>0$. By the norm $\|f\|=\sup\{|\partial_x^\alpha f(x)|/h^{|\alpha|}(|\alpha|!)^s;\ x\in K \text{ and } \alpha\in\boldsymbol{Z}_+^n\}$, $\mathcal{E}^{\{s\},h}(K)$ becomes a Banach space. Then,

$$\mathcal{E}^{\{s\}}(K)=\varinjlim_{h\to\infty}\mathcal{E}^{\{s\},h}(K),$$

$$\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)=\varprojlim_{K\Subset\boldsymbol{R}^n}\mathcal{E}^{\{s\}}(K)$$

(see Komatsu [**5**]). By $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (resp. $C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)))$ we denote the space of all infinitely differentiable functions on $[0, T]$ (resp. $(0, T)$) with values in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ equipped with the locally convex topology above.

In this paper, we employ the class $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ or $C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ as a framework of our discussion, following the case of the Cauchy problem for analogous operators in Ivrii [**4**], Igari [**3**], Uryu [**16**]. Then, we can set up our problem as follows: determine the precise bound of the index s of the Gevrey class $\mathcal{E}^{\{s\}}$ for which the results—the unique solvability in $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ and the asymptotic expansions of solutions in $C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$—are valid.

The conclusion of this paper is as follows. Let $P(t, x, t\partial_t, \partial_x)$ be as in (1.1), and assume that (A_κ) and (B) are satisfied. Define the *irregularity index* σ ($\geqq 1$) of P by

$$\sigma = \max\left[1, \max_{\substack{j+|\alpha|\leq m \\ |\alpha|>0}} \left\{\min_{\tau\in\mathfrak{S}_n}\left(\max_{1\leq r\leq n} M_{j,\alpha}(\tau, r)\right)\right\}\right], \tag{1.3}$$

where $\mathfrak{S}_n$ is the permutation group of n-numbers and

$$M_{j,\alpha}(\tau, r) = \frac{\sum_{i=1}^{r}(\kappa_{\tau(i)}-\kappa_{\tau(r)})\alpha_{\tau(i)}+(m-j)\kappa_{\tau(r)}-l(j, \alpha)}{(m-j-|\alpha|)\kappa_{\tau(r)}}. \tag{1.4}$$

Then, the results desired in our problem are valid, if s satisfies the following condition:

(C) $1<s<\sigma/(\sigma-1)$.

MAIN THEOREM. *Assume that* $l(j, \alpha)\in \boldsymbol{Z}_+$ ($j+|\alpha|\leqq m$ *and* $j<m$), *that* $a_{j,\alpha}(t, x)\in C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ ($j+|\alpha|\leqq m$ *and* $j<m$), *and that* (A_κ), (B) *and* (C) *are satisfied. Then, we have the following results.*

(I) (Unique solvability). *If* $\rho_i(x)\notin \boldsymbol{Z}_+$ *holds for any* $x\in\boldsymbol{R}^n$ *and* $1\leqq i\leqq m$, *the equation*

$$P(t, x, t\partial_t, \partial_x)u = f \quad in \ C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$$

is uniquely solvable.

(II) (Asymptotic expansions). *If* $\rho_i(x)-\rho_j(x)\notin\boldsymbol{Z}$ *holds for any* $x\in\boldsymbol{R}^n$ *and* $1\leqq i\neq j\leqq m$, *the general solution of*

$$P(t, x, t\partial_t, \partial_x)u = 0 \quad in \ C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$$

is characterized as follows. (II-1) *Any solution* $u(t, x)\in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ *can be expanded asymptotically into the form*

$$u(t, x) \sim \sum_{i=1}^{m}\left(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\right) \tag{1.5}$$

(*as* $t\to+0$) *for some unique* $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$. (II-2) *Conversely, for any* $\varphi_1(x), \cdots, \varphi_m(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ *there exist a unique solution* $u(t, x)\in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ *and unique coefficients* $\varphi_{k,h}^{(i)}(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ ($1\leqq i\leqq m$, $1\leqq k<\infty$ *and* $0\leqq h\leqq mk$) *such that the asymptotic relation in* (II-1) *holds.*

Here, the meaning of the asymptotic relation (1.5) is as follows: for any $a>0$ and any compact subset K of $\boldsymbol{R}^n$, there is an $N_0\in\boldsymbol{N}$ such that

$$t^{-a}(t\partial_t)^l R_N(t, x)|_K \to 0 \quad \text{in } \mathcal{E}^{\{s\}}(K)$$

(as $t\to+0$) for any $N\geqq N_0$ and $l\in\boldsymbol{Z}_+$, where

$$R_N(t, x) = u(t, x) - \sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)} + \sum_{k=1}^{N}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big).$$

REMARK. (1) $\sigma=1$ is equivalent to (T) (see Lemma 1). In this case, (C) is read as $1<s<\infty$.

(2) In the case $\kappa_1=\cdots=\kappa_n$ $(=k)$, σ is given by

$$\sigma = \max\Big[1, \max_{\substack{j+|\alpha|\leq m \\ |\alpha|>0}}\Big(\frac{m-j-l(j, \alpha)/k}{m-j-|\alpha|}\Big)\Big].$$

(3) In the case $\kappa_1=\cdots=\kappa_n$, Uryu [**16**] has defined an index σ_u $(\geqq 1)$ and obtained the unique solvability of $P(t, x, t\partial_t, \partial_x)u=f$ in $C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ for $1<s<(\sigma_u/(\sigma_u-1))$. But, even in this case, our condition (C) is better than his. In fact, we can see the following: (i) $1\leqq\sigma\leqq\sigma_u$ holds in general, and (ii) $1<\sigma<\sigma_u$ holds in the case $P=(t\partial_t)((t\partial_t)^2-t^{2k}\partial_x^2)+t^p\partial_x^2+t^q\partial_x+c$ with $(k, p, q)=$ (5, 9, 3), (6, 10, 3), (6, 11, 4), $\cdots$.

(4) Also in the case $l(j, \alpha)\in\boldsymbol{Q}$ $(j+|\alpha|\leqq m$ and $j<m)$, we can obtain the same result as in (II). To see this, we have only to apply the change of variables $t^{1/N}\to t$ and $x\to x$. See §7 of Tahara [**12**].

(5) See also Ivrii [**4**], Igari [**3**], Wasow [**17**], Tahara [**10**], Bove-Lewis-Parenti [**2**], and the remarks and references in [**12**].

EXAMPLE. (1) Let P_1 be of the form

$$P_1 = (t\partial_t)^2 - t^{2\kappa}\partial_x^2 + t^l a(t, x)\partial_x + b(t, x)(t\partial_t) + c(t, x),$$

where $(t, x)\in[0, T]\times\boldsymbol{R}$ and $2\kappa, l\in\boldsymbol{N}$. Then, σ is given by

$$\sigma = \max\Big\{1, \frac{2\kappa-l}{\kappa}\Big\}.$$

(2) Let P_2 be of the form

$$P_2 = (t\partial_t)^2 - t^{2\kappa_1}\partial_{x_1}^2 - t^{2\kappa_2}\partial_{x_2}^2 + t^{l_1}a_1(t, x)\partial_{x_1} + t^{l_2}a_2(t, x)\partial_{x_2} + b(t, x)(t\partial_t) + c(t, x),$$

where $(t, x)\in[0, T]\times\boldsymbol{R}^2$ and $2\kappa_1, 2\kappa_2, l_1, l_2\in\boldsymbol{N}$. Then, σ is given by

$$\sigma = \max\Big\{1, \frac{2\kappa_1-l_1}{\kappa_1}, \frac{2\kappa_2-l_2}{\kappa_2}\Big\}.$$

(3) Let P_3 be of the form

$$P_3 = (t\partial_t)((t\partial_t)^2 - t^{2\kappa_1}\partial_{x_1}^2 - t^{2\kappa_2}\partial_{x_2}^2) + t^l a(t, x)\partial_{x_1}\partial_{x_2},$$

where $(t, x)\in[0, T]\times\boldsymbol{R}^2$ and $2\kappa_1, 2\kappa_2, l\in\boldsymbol{N}$. Then, σ is given by

$$\sigma=\begin{cases}\max\left\{1,\ \dfrac{3\kappa_1-l}{\kappa_1},\ \dfrac{\kappa_1+2\kappa_2-l}{\kappa_2}\right\}, & \text{when } 0<\kappa_1\leqq\kappa_2,\\ \max\left\{1,\ \dfrac{2\kappa_1+\kappa_2-l}{\kappa_1},\ \dfrac{3\kappa_2-l}{\kappa_2}\right\}, & \text{when } 0<\kappa_2\leqq\kappa_1.\end{cases}$$

APPLICATION. Let $A(t, x, \partial_t, \partial_x)$ be a linear partial differential operator of order m with $\{t=0\}$ as a non-characteristic hypersurface, and assume that $t^m A(t, x, \partial_t, \partial_x)$ satisfies our conditions in Main Theorem. Then, by (I) in Main Theorem we can obtain the following result (see Tahara [**14**]): the Cauchy problem for A is well-posed in $C^\infty([0, T], \mathcal{E}^{(s)}(\boldsymbol{R}^n))$, if s satisfies the condition (C).

As to the necessity of (C), by Ivrii [**4**] we can see the following: if the coefficients of A are analytic and if $\kappa_1=\cdots=\kappa_n$, then (C) is necessary for the Cauchy problem for A to be $\mathcal{E}^{(s)}$ well-posed.

The author believes that (C) is necessary also in the general case. But, as far as the author knows, it is still open. The following example seems to be very instructive. Let A be of the form

$$A=\partial_t^2-t^{2\nu_1}\partial_{x_1}^2-t^{2\nu_2}\partial_{x_2}+t^{p_1}a_1(t, x)\partial_{x_1}+t^{p_2}a_2(t, x)\partial_{x_2}+b(t, x)\partial_t+c(t, x).$$

Then, our condition (C) for t^2A is $1<s<\sigma/(\sigma-1)$ with

$$\sigma=\max\left\{1,\ \frac{2\nu_1-p_1}{\nu_1+1},\ \frac{2\nu_2-p_2}{\nu_2+1}\right\}.$$

In this case, by [**4**] we can see that (C) is also the necessary condition (under the assumptions that $a_1(t, x)$, $a_2(t, x)$, $b(t, x)$, $c(t, x)$ are analytic and that $a_1(0, x)\not\equiv 0$, $a_2(0, x)\not\equiv 0$).

The paper is organized as follows. In §2 we state two theorems (Theorems 1 and 2) without proofs. In §3 we show that Main Theorem above is obtained from Theorems 1 and 2. So, from §4 to §8 we confine ourselves to proving Theorems 1 and 2. In §4 we discuss the condition (C), in §5 we prepare formal norms in Leray-Ohya [**6**], and in §6 we establish two kinds of a priori estimates. After these preparations, we prove Theorem 1 in §7 and Theorem 2 in §8. Thus, at the end of §8, the proof of Main Theorem is completed in the true sense.

Throughout this paper, we use the following notations: $\boldsymbol{N}=\{1, 2, 3, \cdots\}$, $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$ and $\boldsymbol{Z}_+^n=\{0, 1, 2, \cdots\}^n$.

§2. Basic results in asymptotic analysis.

Secondly, we state two theorems (Theorems 1 and 2) from which Main Theorem is obtained. Our asymptotic analysis in this paper consists mainly of these two theorems. As to the C^∞-versions, see Tahara [**13**].

Let $P(t, x, t\partial_t, \partial_x)$ $(=P)$ be the operator in (1.1). In Theorems 1 and 2 given below, we treat

(S) $$P(t, x, t\partial_t, \partial_x)u = f \quad \text{in } C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$$

under (A_κ), (B), (C) and the following condition:

(D) $a_{j,\alpha}(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ $(j+|\alpha| \leqq m$ and $j<m)$ and they satisfy $(t\partial_t)^l a_{j,\alpha}(t, x) \in C^0([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ for any $l \in \boldsymbol{Z}_+$.

To state Theorems 1 and 2, we prepare some terminologies. For $v(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$, $\rho(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ and $K \Subset \boldsymbol{R}^n$, we define

$$v(t, x) = o(t^{\rho(x)} ;\ \nabla^\infty, \mathcal{E}^{\{s\}}(K)) \quad (\text{as } t \to +0)$$

by the following: $(t\partial_t)^l (t^{-\rho(x)} v(t, x))|_K \to 0$ in $\mathcal{E}^{\{s\}}(K)$ (as $t \to +0$) for any $l \in \boldsymbol{Z}_+$. Here and hereafter, $K \Subset \boldsymbol{R}^n$ means that K is a compact subset of $\boldsymbol{R}^n$. Also, we define

$$v(t, x) = o(t^{\rho(x)} ;\ \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)) \quad (\text{as } t \to +0)$$

by the following: $v(t, x) = o(t^{\rho(x)}; \nabla^\infty, \mathcal{E}^{\{s\}}(K))$ (as $t \to +0$) for any $K \Subset \boldsymbol{R}^n$. We say that $w(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ is *tempered* in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ (as $t \to +0$), if $w(t, x) = o(t^{\lambda(x)}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for some $\lambda(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$, or equivalently, if $w(t, x)$ satisfies the following: for any $K \Subset \boldsymbol{R}^n$, there is an $a \in \boldsymbol{R}$ such that $w(t, x) = o(t^a; \nabla^\infty, \mathcal{E}^{\{s\}}(K))$ (as $t \to +0$). Then, we can state Theorems 1 and 2 as follows.

THEOREM 1 (Unique solvability with bounds). *Assume that P and s satisfy* (A_κ), (B), (C) *and* (D). *Let $\lambda(x)$, $\mu(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ such that*

$$\max_{1 \leqq i \leqq m} \mathrm{Re}(\rho_i(x)) < \mathrm{Re}(\mu(x)) < \mathrm{Re}(\lambda(x)) \quad \text{on } \boldsymbol{R}^n.$$

Then, if $f(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ satisfies $f(t, x) = o(t^{\lambda(x)+A}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for some $A \geqq 0$, (S) has a unique solution $u(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ such that $u(t, x) = o(t^{\mu(x)+A}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$).

THEOREM 2 (Tempered growth condition). *Assume that P and s satisfy* (A_κ), (B), (C) *and* (D). *Then, P has the tempered growth condition in the following sense. If $u(t, x)$, $f(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ satisfy* (S) *and if $f(t, x)$ is tempered in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ (as $t \to +0$), then $u(t, x)$ is also tempered in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ (as $t \to +0$).*

§3. From Theorems 1 and 2 to Main Theorem.

Thirdly, we show that Main Theorem is obtained from Theorems 1 and 2.

Assume that Theorems 1 and 2 are true, let $\lambda(x)$, $\mu(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ be as in Theorem 1, and let $\phi_j(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n) \cap C_0^\infty(\boldsymbol{R}^n)$ $(j=1, 2, \cdots)$ such that $\sum_{j=1}^\infty \phi_j(x)$ is a locally finite sum and that $\sum_{j=1}^{\infty}\phi_j(x)=1$ on $\boldsymbol{R}^n$. Then, we can prove Main Theorem as follows.

PROOF OF (I) IN MAIN THEOREM. Let $f(t, x)\in C^{\infty}([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$. Then, we can uniquely determine the coefficients $u_k(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ $(0\leqq k<\infty)$ so that for any $N\in\boldsymbol{N}$

$$f(t, x)-P(t, x, t\partial_t, \partial_x)\Big(\sum_{k=0}^{N} u_k(x)t^k\Big)=o(t^N; \nabla^{\infty}, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)) \quad (\text{as } t\to+0).$$

Therefore, by choosing $N_j\in\boldsymbol{N}$ $(j=1, 2, \cdots)$ so that $N_j>\mathrm{Re}(\lambda(x))$ on $\mathrm{supp}(\phi_j)$, we have the following:

$$U_l(t, x)=\sum_{j=1}^{\infty}\phi_j(x)\Big(\sum_{k=0}^{N_j+l}u_k(x)t^k\Big)$$

$(l=1, 2, \cdots)$ satisfy $(f-PU_l)(t, x)=o(t^{\lambda(x)+l}; \nabla^{\infty}, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t\to+0$). Hence, by Theorem 1 we have a solution $V_l(t, x)\in C^{\infty}((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ of $P(t, x, t\partial_t, \partial_x)V_l=(f-PU_l)$ such that $V_l(t, x)=o(t^{\mu(x)+l}; \nabla^{\infty}, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t\to+0$). In addition, by Theorem 1 we can see that $(U_l+V_l)=(U_k+V_k)$ for any $l, k\in\boldsymbol{N}$, because $W=(U_l+V_l)-(U_k+V_k)$ satisfies $P(t, x, t\partial_t, \partial_x)W=0$ and $W(t, x)=o(t^{\mu(x)+l}; \nabla^{\infty}, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t\to+0$) for $k>l$. Thus, by putting $u(t, x)=(U_l+V_l)(t, x)$ $(l=1, 2, \cdots)$ we obtain a solution $u(t, x)\in C^{\infty}((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ of $P(t, x, t\partial_t, \partial_x)u=f$ such that

$$u(t, x)=U_l(t, x)+o(t^{\mu(x)+l}; \nabla^{\infty}, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$$

(as $t\to+0$) for any $l\in\boldsymbol{N}$, that is,

$$u(t, x)\sim\sum_{k=0}^{\infty}u_k(x)t^k \quad \text{in } \mathcal{E}^{\{s\}}(\boldsymbol{R}^n) \quad (\text{as } t\to+0). \tag{3.1}$$

This leads us to the existence of a solution in $C^{\infty}([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$, because (3.1) is equivalent to $u(t, x)\in C^{\infty}([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$.

The uniqueness of solutions is proved as follows. Let $u_1(t, x)$, $u_2(t, x)\in C^{\infty}([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ be two solutions of $P(t, x, t\partial_t, \partial_x)u=f$. Then, we have

$$\begin{cases} P(t, x, t\partial_t, \partial_x)(u_1-u_2)=0, \\ (u_1-u_2)(t, x)\sim 0 \quad \text{in } \mathcal{E}^{\{s\}}(\boldsymbol{R}^n) \quad (\text{as } t\to+0), \end{cases} \tag{3.2}$$

because the Taylor coefficients (in t) of the solution of $P(t, x, t\partial_t, \partial_x)u=f$ are uniquely determined by $f(t, x)$. Therefore, by applying Theorem 1 to (3.2) we have $(u_1-u_2)(t, x)=0$, that is, $u_1(t, x)=u_2(t, x)$ on $[0, T]\times\boldsymbol{R}^n$. Q.E.D.

PROOF OF (II-1) IN MAIN THEOREM. Note that under the assumptions in (II-1) we have the following facts.

(1) (Theorem 2 in §2). Any solution $u(t, x)\in C^{\infty}((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^{n}))$ of $P(t, x, t\partial_{t}, \partial_{x})u=0$ is tempered in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^{n})$ (as $t\rightarrow+0$).

(2) (Tahara [12, Proposition 3]). If $\phi_{k,h}^{(i)}(x)\in \mathcal{E}^{\{s\}}(U)$ $(1\leqq i\leqq m,\ 0\leqq k<\infty$ and $0\leqq h\leqq mk)$ satisfy

$$0\sim\sum_{i=1}^{m}\sum_{k=0}^{\infty}\sum_{h=0}^{mk}\phi_{k,h}^{(i)}(x)t^{\rho_{i}(x)+k}(\log t)^{mk-h}\quad \text{in } \mathcal{E}^{\{s\}}(U)\quad (\text{as } t\rightarrow+0)$$

(where U is an open subset of $\boldsymbol{R}^{n}$), then we have $\phi_{k,h}^{(i)}(x)=0$ on U for any i, k and h.

(3) (by a calculation). For $\phi(x),\ \rho(x)\in \mathcal{E}^{\{s\}}(U)$ and $l\in \boldsymbol{Z}_{+}$, we have $\phi(x)t^{\rho(x)}(\log t)^{l}\in C^{\infty}((0, T), \mathcal{E}^{\{s\}}(U))$ and $\phi(x)t^{\rho(x)}(\log t)^{l}=o(t^{\rho(x)-\varepsilon};\nabla^{\infty}, \mathcal{E}^{\{s\}}(U))$ (as $t\rightarrow+0$) for any $\varepsilon>0$.

Hence, we can obtain (II-1) in the same way as [12, Theorem 1]. In other words, the proof of [12, Theorem 1] becomes a proof of (II-1), if we replace $C^{\infty}(U)$, $C^{\infty}((0, T)\times U)$, $u\sim w$ on U (as $t\rightarrow+0$), $u=o(t^{\rho(x)};\nabla^{\infty})$ on U (as $t\rightarrow+0$), $\cdots$ by $\mathcal{E}^{\{s\}}(U)$, $C^{\infty}((0, T), \mathcal{E}^{\{s\}}(U))$, $u\sim w$ in $\mathcal{E}^{\{s\}}(U)$ (as $t\rightarrow+0$), $u=o(t^{\rho(x)};\nabla^{\infty}, \mathcal{E}^{\{s\}}(U))$ (as $t\rightarrow+0$), $\cdots$, respectively. So, we may omit the details. Q. E. D.

PROOF OF (II-2) IN MAIN THEOREM. Let $\varphi_{1}(x), \cdots, \varphi_{m}(x)\in \mathcal{E}^{\{s\}}(\boldsymbol{R}^{n})$. Then, we can uniquely determine the coefficients $\varphi_{k,h}^{(i)}(x)\in \mathcal{E}^{\{s\}}(\boldsymbol{R}^{n})$ $(1\leqq i\leqq m,\ 1\leqq k<\infty$ and $0\leqq h\leqq mk)$ in (1.5) so that the following condition is satisfied: for any $a>0$ and any $K\Subset \boldsymbol{R}^{n}$, there is an $N\in \boldsymbol{N}$ such that for any $l\in \boldsymbol{N}$

$$\begin{aligned}&P(t, x, t\partial_{t}, \partial_{x})\Big[\sum_{i=1}^{m}\Big(\varphi_{i}(x)t^{\rho_{i}(x)}+\sum_{k=1}^{N+l}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_{i}(x)+k}(\log t)^{mk-h}\Big)\Big]\\ &=o(t^{a+l}\ ;\ \nabla^{\infty}, \mathcal{E}^{\{s\}}(K))\quad (\text{as } t\rightarrow+0)\end{aligned}$$

(see [12, Proposition 7]). Therefore, we can choose $N_{j}\in \boldsymbol{N}$ $(j=1, 2, \cdots)$ so that

$$U_{l}(t, x)=\sum_{j=1}^{\infty}\phi_{j}(x)\Big\{\sum_{i=1}^{m}\Big(\varphi_{i}(x)t^{\rho_{i}(x)}+\sum_{k=1}^{N_{j}+l}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_{i}(x)+k}(\log t)^{mk-h}\Big)\Big\}$$

$(l=1, 2, \cdots)$ satisfy $(PU_{l})(t, x)=o(t^{\lambda(x)+l};\nabla^{\infty}, \mathcal{E}^{\{s\}}(\boldsymbol{R}^{n}))$ (as $t\rightarrow+0$). Hence, by the same argument as in the proof of (I) we obtain a solution $u(t, x)\in C^{\infty}((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^{n}))$ of $P(t, x, t\partial_{t}, \partial_{x})u=0$ such that

$$u(t, x)=U_{l}(t, x)+o(t^{\mu(x)+l}\ ;\ \nabla^{\infty}, \mathcal{E}^{\{s\}}(\boldsymbol{R}^{n}))$$

(as $t\rightarrow+0$) for any $l\in \boldsymbol{N}$. This immediately leads us to the existence part of (II-2). The uniqueness part of (II-2) may be proved in the same way as (I). Q. E. D.

Hence, from now on we confine ourselves to proving Theorems 1 and 2. In §§4$\sim$8, we use the following notations for $a=(a_{1}, \cdots, a_{n})$, $b=(b_{1}, \cdots, b_{n})\in \boldsymbol{R}^{n}$:

$a \leqq b$ means that $a_i \leqq b_i$ for any i, $a<b$ means that $a_i<b_i$ for any i, $|a| = |a_1| + \cdots + |a_n|$ and $\langle a, b\rangle = a_1b_1 + \cdots + a_nb_n$.

§ 4. Interpretation of the condition (C).

Fourthly, we give an interpretation of the following condition:

(C′) $1 \leqq s < \sigma/(\sigma-1)$.

PROPOSITION 1. *Let* σ *be as in* (1.3) *with* $l(j, \alpha)>0$ ($j+|\alpha|<m$ *and* $|\alpha|>0$) *and* $\kappa_i>0$ ($1 \leqq i \leqq n$). *Assume that* s *satisfies* (C′). *Then, there are* $z(j, \alpha) \in \boldsymbol{R}^n$ ($j+|\alpha|<m$ *and* $|\alpha|>0$) *such that* $(0, \cdots, 0) \leqq z(j, \alpha) \leqq \alpha$, $|z(j, \alpha)|<|\alpha|$, $\langle \kappa, z(j, \alpha)\rangle \leqq l(j, \alpha)$ (*where* $\kappa=(\kappa_1, \cdots, \kappa_n)$), *and*

$$s < \frac{m-j-|z(j, \alpha)|}{|\alpha|-|z(j, \alpha)|}. \tag{4.1}$$

Before the proof, we present some discussions. Let $M_{j,\alpha}(\tau, r)$ be as in (1.4) and $\mathfrak{S}_n$ be the permutation group of n-numbers. For simplicity, we use the following notations for $\kappa=(\kappa_1, \cdots, \kappa_n)$, $\alpha=(\alpha_1, \cdots, \alpha_n)$, $1 \leqq r \leqq n$ and $\tau \in \mathfrak{S}_n$: $\langle \kappa, \alpha\rangle_0=0$, $\langle \kappa, \alpha\rangle_r=\kappa_1\alpha_1+\cdots+\kappa_r\alpha_r$, $\kappa^\tau=(\kappa_{\tau(1)}, \cdots, \kappa_{\tau(n)})$, $\alpha^\tau=(\alpha_{\tau(1)}, \cdots, \alpha_{\tau(n)})$, $\langle \kappa^\tau, \alpha^\tau\rangle_0=0$ and $\langle \kappa^\tau, \alpha^\tau\rangle_r=\kappa_{\tau(1)}\alpha_{\tau(1)}+\cdots+\kappa_{\tau(r)}\alpha_{\tau(r)}$. Note that $\langle \kappa, \alpha\rangle=\langle \kappa, \alpha\rangle_n =\langle \kappa^\tau, \alpha^\tau\rangle_n$.

LEMMA 1. *Assume that* $0<\kappa_1 \leqq \kappa_2 \leqq \cdots \leqq \kappa_n$ *and that* $l(j, \alpha)>0$. *Denote by* id *the identity element of* $\mathfrak{S}_n$. *Then, we have the following results.*

(1) *If* $l(j, \alpha) \geqq \langle \kappa, \alpha\rangle$, *then*

$$\min_{\tau\in\mathfrak{S}_n}\Big(\max_{1\leq r\leq n} M_{j,\alpha}(\tau, r)\Big) \leqq M_{j,\alpha}(\mathrm{id}, n) \leqq 1. \tag{4.2}$$

(2) *If* $l(j, \alpha)<\langle \kappa, \alpha\rangle$ *and if* $\langle \kappa, \alpha\rangle_{p-1}<l(j, \alpha) \leqq \langle \kappa, \alpha\rangle_p$ *for some* p ($1 \leqq p \leqq n$), *then*

$$\min_{\tau\in\mathfrak{S}_n}\Big(\max_{1\leq r\leq n} M_{j,\alpha}(\tau, r)\Big) = M_{j,\alpha}(\mathrm{id}, p) > 1. \tag{4.3}$$

PROOF. The proof of (1) is as follows. Since

$$M_{j,\alpha}(\mathrm{id}, r+1)-M_{j,\alpha}(\mathrm{id}, r) = \frac{(\kappa_{r+1}-\kappa_r)(l(j, \alpha)-\langle \kappa, \alpha\rangle_r)}{(m-j-|\alpha|)\kappa_r\kappa_{r+1}}, \tag{4.4}$$

we have $M_{j,\alpha}(\mathrm{id}, r+1) \geqq M_{j,\alpha}(\mathrm{id}, r)$ for $1 \leqq r<n$, that is,

$$\max_{1\leq r\leq n} M_{j,\alpha}(\mathrm{id}, r) = M_{j,\alpha}(\mathrm{id}, n).$$

This immediately leads us to (4.2).

The proof of (2) is as follows. Since $\langle\kappa, \alpha\rangle_{p-1}<l(j, \alpha)\leqq\langle\kappa, \alpha\rangle_{p}$, by (4.4) we have

$$\max_{1\leqq r\leqq n} M_{j,\alpha}(\mathrm{id}, r)=M_{j,\alpha}(\mathrm{id}, p).$$

Therefore, to obtain (4.3) it is sufficient to show that

$$\max_{1\leqq r\leqq n} M_{j,\alpha}(\tau, r)\geqq M_{j,\alpha}(\mathrm{id}, p) \tag{4.5}$$

holds for any $\tau\in\mathfrak{S}_n$. Take any $\tau\in\mathfrak{S}_n$ and fix it. Define $\tau_k\in\mathfrak{S}_n$ $(k=1, 2, \cdots, n)$ by the following: $\tau_k(i)=\tau(i)$ for $1\leqq i<k$, and $\tau_k(k)<\tau_k(k+1)<\cdots<\tau_k(n)$. Then, we have $\tau_1=\mathrm{id}$, $\tau_n=\tau$ and $\tau_k(i)=\tau_{k+1}(i)$ for $1\leqq i<k<n$. Since $l(j, \alpha)<\langle\kappa, \alpha\rangle$ $(=\langle\kappa, \alpha\rangle_n)$, we can choose $p_k\in\boldsymbol{N}$ $(k=1, 2, \cdots, n)$ such that $1\leqq p_k\leqq n$, $p_1=p$ and

$$\langle\kappa^{\tau_k}, \alpha^{\tau_k}\rangle_{p_k-1}<l(j, \alpha)\leqq\langle\kappa^{\tau_k}, \alpha^{\tau_k}\rangle_{p_k}.$$

Hence, by Lemma 2 given below we obtain $M_{j,\alpha}(\tau_k, p_k)\leqq M_{j,\alpha}(\tau_{k+1}, p_{k+1})$ for $1\leqq k<n$, that is, $M_{j,\alpha}(\mathrm{id}, p)\leqq M_{j,\alpha}(\tau, p_n)$. This immediately leads us to (4.5). Q. E. D.

LEMMA 2. *Assume that* $0<\kappa_1\leqq\kappa_2\leqq\cdots\leqq\kappa_n$. *Let* $\tau, \nu\in\mathfrak{S}_n$ *and* $p, q, k\in\{1, 2, \cdots, n\}$ *such that* $\tau(i)=\nu(i)$ *for* $1\leqq i<k$, $\tau(k)<\tau(k+1)<\cdots<\tau(n)$, $\nu(k+1)<\nu(k+2)<\cdots<\nu(n)$, $\langle\kappa^\tau, \alpha^\tau\rangle_{p-1}<l(j, \alpha)\leqq\langle\kappa^\tau, \alpha^\tau\rangle_p$ *and* $\langle\kappa^\nu, \alpha^\nu\rangle_{q-1}<l(j, \alpha)\leqq\langle\kappa^\nu, \alpha^\nu\rangle_q$. *Then, we have* $M_{j,\alpha}(\tau, p)\leqq M_{j,\alpha}(\nu, q)$.

PROOF. Since $\nu(k)\in\{\tau(i); k\leqq i\leqq n\}$, we have $\nu(k)=\tau(h)$ for some h $(k\leqq h\leqq n)$. Then,

$$\nu(i)=\begin{cases}\tau(i), & \text{for } 1\leqq i<k \text{ or } h<i\leqq n,\\ \tau(h), & \text{for } i=k,\\ \tau(i-1), & \text{for } k<i\leqq h.\end{cases} \tag{4.6}$$

When $h=k$, we have $\tau=\nu$, $p=q$ and hence $M_{j,\alpha}(\tau, p)=M_{j,\alpha}(\nu, q)$. Therefore, we may assume $h>k$ from now on.

When $1\leqq p<k$ or $h<p\leqq n$, we have $\langle\kappa^\nu, \alpha^\nu\rangle_{p-1}(=\langle\kappa^\tau, \alpha^\tau\rangle_{p-1})<l(j, \alpha)\leqq\langle\kappa^\nu, \alpha^\nu\rangle_p(=\langle\kappa^\tau, \alpha^\tau\rangle_p)$ and hence $p=q$; therefore, by (4.6) we have $M_{j,\alpha}(\tau, p)=M_{j,\alpha}(\nu, q)$. When $1\leqq q<k$ or $h<q\leqq n$, we can obtain $M_{j,\alpha}(\tau, p)=M_{j,\alpha}(\nu, q)$ in the same way. When $k\leqq p\leqq h$ and $q=k$, $M_{j,\alpha}(\tau, p)\leqq M_{j,\alpha}(\nu, q)$ is verified by the following facts: $\kappa_{\nu(q)}=\kappa_{\nu(k)}=\kappa_{\tau(h)}\geqq\kappa_{\tau(p)}$, $\kappa_{\tau(i)}\leqq\kappa_{\tau(p)}$ for $k\leqq i\leqq p$ and

$$\begin{aligned}&M_{j,\alpha}(\nu, q)-M_{j,\alpha}(\tau, p)\\&\quad=\frac{(\kappa_{\nu(q)}-\kappa_{\tau(p)})(l(j, \alpha)-\langle\kappa^\nu, \alpha^\nu\rangle_{q-1})}{(m-j-|\alpha|)\kappa_{\tau(p)}\kappa_{\nu(q)}}+\frac{\sum_{i=k}^{p}(\kappa_{\tau(p)}-\kappa_{\tau(i)})\alpha_{\tau(i)}}{(m-j-|\alpha|)\kappa_{\tau(p)}}.\end{aligned}$$

When $k\leqq p\leqq h$, $k<q\leqq h$ and $p=q-1$, $M_{j,\alpha}(\tau, p)\leqq M_{j,\alpha}(\nu, q)$ is verified by the following facts: $\kappa_{\tau(h)}\geqq\kappa_{\tau(p)}$ and

$$M_{j,\alpha}(\nu, q)-M_{j,\alpha}(\tau, p)=\frac{(\kappa_{\tau(h)}-\kappa_{\tau(p)})\alpha_{\tau(h)}}{(m-j-|\alpha|)\kappa_{\tau(p)}}.$$

When $k\leqq p\leqq h$, $k<q\leqq h$ and $p\geqq q$, $M_{j,\alpha}(\tau, p)\leqq M_{j,\alpha}(\nu, q)$ is verified by the following facts: $\kappa_{\nu(q)}=\kappa_{\tau(q-1)}\leqq\kappa_{\tau(p)}\leqq\kappa_{\tau(h)}$, $\kappa_{\tau(i)}\leqq\kappa_{\tau(p)}$ for $q\leqq i\leqq p$ and

$$\begin{aligned}M_{j,\alpha}(\nu, q)-M_{j,\alpha}(\tau, p)&=\frac{(\kappa_{\tau(p)}-\kappa_{\nu(q)})(\langle\kappa^{\nu}, \alpha^{\nu}\rangle_{q}-l(j, \alpha))}{(m-j-|\alpha|)\kappa_{\tau(p)}\kappa_{\nu(q)}}\\&\quad+\frac{(\kappa_{\tau(h)}-\kappa_{\tau(p)})\alpha_{\tau(h)}+\sum_{i=q}^{p}(\kappa_{\tau(p)}-\kappa_{\tau(i)})\alpha_{\tau(i)}}{(m-j-|\alpha|)\kappa_{\tau(p)}}.\end{aligned}$$

Here, we note the following: if $k\leqq q-2<h$, we have

$$\begin{aligned}\langle\kappa^{\tau}, \alpha^{\tau}\rangle_{q-2}&\leqq\langle\kappa^{\tau}, \alpha^{\tau}\rangle_{q-2}+\kappa_{\tau(h)}\alpha_{\tau(h)}\\&=\langle\kappa^{\nu}, \alpha^{\nu}\rangle_{q-1}<l(j, \alpha)\leqq\langle\kappa^{\tau}, \alpha^{\tau}\rangle_{p}\end{aligned}$$

and hence $q-2<p$. Therefore, we need not consider the case: $k\leqq p\leqq h$, $k<q\leqq h$ and $p\leqq q-2$. Thus, all the cases are covered. Q.E.D.

PROOF OF PROPOSITION 1. Without loss of generality, we may assume that $0<\kappa_1\leqq\kappa_2\leqq\cdots\leqq\kappa_n$. When $l(j, \alpha)\geqq\langle\kappa, \alpha\rangle$, we choose $z(j, \alpha)\in\boldsymbol{R}^n$ so that $(0, \cdots, 0)\leqq z(j, \alpha)\leqq\alpha$, $|z(j, \alpha)|<|\alpha|$ and that $z(j, \alpha)$ is sufficiently close to α. Then, $\langle\kappa, z(j, \alpha)\rangle\leqq\langle\kappa, \alpha\rangle\leqq l(j, \alpha)$ is clear, and (4.1) is verified by the fact that the right hand side of (4.1) tends to $+\infty$ as $|z(j, \alpha)|\rightarrow|\alpha|$.

When $l(j, \alpha)<\langle\kappa, \alpha\rangle$, we take p $(1\leqq p\leqq n)$ such that $\langle\kappa, \alpha\rangle_{p-1}<l(j, \alpha)\leqq\langle\kappa, \alpha\rangle_{p}$, and then define $z(j, \alpha)=(z_1, \cdots, z_n)\in\boldsymbol{R}^n$ by

$$z_i=\begin{cases}\alpha_i, & \text{for } 1\leqq i<p,\\ \dfrac{l(j, \alpha)-\langle\kappa, \alpha\rangle_{p-1}}{\kappa_p}, & \text{for } i=p,\\ 0, & \text{for } p<i\leqq n.\end{cases}$$

Then, $(0, \cdots, 0)\leqq z(j, \alpha)\leqq\alpha$, $|z(j, \alpha)|<|\alpha|$ and $\langle\kappa, z(j, \alpha)\rangle=l(j, \alpha)$ are clear, and (4.1) is verified by the following: by Lemma 1 we have $\sigma\geqq M_{j,\alpha}(\mathrm{id}, p)$ and this is equivalent to

$$\sigma/(\sigma-1)\leqq\frac{m-j-|z(j, \alpha)|}{|\alpha|-|z(j, \alpha)|}. \qquad \text{Q.E.D.}$$

As a corollary, let us give a variation. Put $l_{\varepsilon}(j, \alpha)=l(j, \alpha)-\varepsilon$ and let σ_{ε} be the one defined by (1.3) with being $l(j, \alpha)$ replaced by $l_{\varepsilon}(j, \alpha)$. Then, in the situation of Proposition 1, we can choose $\varepsilon>0$ and $s_0<s_1$ such that

$$\begin{aligned}&l_{\varepsilon}(j, \alpha)>0 \quad (j+|\alpha|<m \text{ and } |\alpha|>0),\\&1\leqq s<s_0<s_1<\sigma_{\varepsilon}/(\sigma_{\varepsilon}-1).\end{aligned}\tag{4.7}$$

Therefore, by applying Proposition 1 to (4.7) we have $z_\varepsilon(j, \alpha)\in \boldsymbol{R}^n$ $(j+|\alpha|<m$ and $|\alpha|>0)$ such that $(0, \cdots, 0)\leqq z_\varepsilon(j, \alpha)\leqq\alpha$, $|z_\varepsilon(j, \alpha)|<|\alpha|$, $\langle\kappa, z_\varepsilon(j, \alpha)\rangle\leqq l_\varepsilon(j, \alpha)$ and

$$s_1<\frac{m-j-|z_\varepsilon(j, \alpha)|}{|\alpha|-|z_\varepsilon(j, \alpha)|}.$$

COROLLARY TO PROPOSITION 1. *Let* $\varepsilon>0$ *and* $s_0<s_1$ *be as above, and let* $d\in \boldsymbol{N}$ *be sufficiently large. Then, for any sequence* $(j_i, \alpha_{(i)})\in \boldsymbol{Z}_+\times \boldsymbol{Z}_+^n$ $(i=1, 2, \cdots)$ *such that* $j_i+|\alpha_{(i)}|<m$ *and* $|\alpha_{(i)}|>0$, *we can choose a sequence* $\beta_{(i)}\in \boldsymbol{Z}_+^n$ $(i=1, 2, \cdots)$ *so that the following* (i)$\sim$(iv) *are valid for any* i:

(i) $(0, \cdots, 0)\leqq\beta_{(i)}\leqq\alpha_{(i)}$,

(ii) $\langle\kappa, \beta_{(1)}+\cdots+\beta_{(i)}\rangle\leqq l(j_1, \alpha_{(1)})+\cdots+l(j_i, \alpha_{(i)})-\varepsilon i$,

(iii) $(|\alpha_{(i)}|-|\beta_{(i)}|)+\cdots+(|\alpha_{(i+d-1)}|-|\beta_{(i+d-1)}|)\geqq 1$,

(iv) $\dfrac{(m-j_i-|\beta_{(i)}|)+\cdots+(m-j_{i+d-1}-|\beta_{(i+d-1)}|)}{(|\alpha_{(i)}|-|\beta_{(i)}|)+\cdots+(|\alpha_{(i+d-1)}|-|\beta_{(i+d-1)}|)}>s_0$.

PROOF. Put $z_{(i)}=z_\varepsilon(j_i, \alpha_{(i)})$ $(i=1, 2, \cdots)$ and define $\beta_{(i)}\in \boldsymbol{Z}_+^n$ $(i=1, 2, \cdots)$ inductively by the following formulae:

$$\beta_{(i)}=[z_{(1)}+\cdots+z_{(i)}-\beta_{(1)}-\cdots-\beta_{(i-1)}]$$

$(i=1, 2, \cdots)$, where $\beta_0=(0, \cdots, 0)$, $[(x_1, \cdots, x_n)]=([x_1], \cdots, [x_n])$ and $[x_i]=\max\{k\in \boldsymbol{Z}; k\leqq x_i\}$. Then, we can see that the sequence $\beta_{(i)}$ $(i=1, 2, \cdots)$ satisfies (i)$\sim$(iv) for any i in the following way.

(i) is verified by

$$\begin{aligned}(0, \cdots, 0)&\leqq z_{(1)}+\cdots+z_{(i)}-\beta_{(1)}-\cdots-\beta_{(i-1)}\\&<z_{(i)}+(1, \cdots, 1)\leqq\alpha_{(i)}+(1, \cdots, 1).\end{aligned}$$

(ii) is verified by $\beta_{(1)}+\cdots+\beta_{(i)}\leqq z_{(1)}+\cdots+z_{(i)}$ and $\langle\kappa, z_{(k)}\rangle\leqq l_\varepsilon(j_k, \alpha_{(k)})=l(j_k, \alpha_{(k)})-\varepsilon$ $(1\leqq k\leqq i)$. Since $0\leqq(|z_{(1)}|+\cdots+|z_{(k)}|-|\beta_{(1)}|-\cdots-|\beta_{(k)}|)<n$ for any k, we have $-n<(|z_{(i)}|+\cdots+|z_{(i+d-1)}|-|\beta_{(i)}|-\cdots-|\beta_{(i+d-1)}|)<n$. Therefore, (iii) is verified by the following:

$$\begin{aligned}&(|\alpha_{(i)}|-|\beta_{(i)}|)+\cdots+(|\alpha_{(i+d-1)}|-|\beta_{(i+d-1)}|)\\&\geqq(|\alpha_{(i)}|-|z_{(i)}|)+\cdots+(|\alpha_{(i+d-1)}|-|z_{(i+d-1)}|)-n\\&\geqq cd-n,\end{aligned}$$

where $c=\min\{|\alpha|-|z_\varepsilon(j, \alpha)|; j+|\alpha|<m$ and $|\alpha|>0\}$ (>0). Since $(m-j_k-|z_{(k)}|)>s_1(|\alpha_{(k)}|-|z_{(k)}|)$ for any k, (iv) is verified by the following:

$$(m-j_i-|\beta_{(i)}|)+\cdots+(m-j_{i+d-1}-|\beta_{(i+d-1)}|)$$
$$\geqq(m-j_i-|z_{(i)}|)+\cdots+(m-j_{i+d-1}-|z_{(i+d-1)}|)-n$$
$$>s_1(|\alpha_{(i)}|-|z_{(i)}|)+\cdots+s_1(|\alpha_{(i+d-1)}|-|z_{(i+d-1)}|)-n$$
$$\geqq s_0\{(|\alpha_{(i)}|-|z_{(i)}|)+\cdots+(|\alpha_{(i+d-1)}|-|z_{(i+d-1)}|)\}+(s_1-s_0)cd-n$$
$$\geqq s_0\{(|\alpha_{(i)}|-|\beta_{(i)}|)+\cdots+(|\alpha_{(i+d-1)}|-|\beta_{(i+d-1)}|)\}+(s_1-s_0)cd-(s_0+1)n\,.$$

Q. E. D.

§ 5. Formal norms.

Fifthly, we prepare formal norms to estimate functions in Gevrey classes. Our formulation is a variation of Leray-Ohya [6].

Let p, $l\in \boldsymbol{Z}_+$, $r\in\boldsymbol{R}$, and $\kappa=(\kappa_1, \cdots, \kappa_n)\in\boldsymbol{R}^n$ such that $\kappa_i>0$ $(1\leqq i\leqq n)$. For $f(t, x)\in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ we define $\|\nabla_r^{p,\infty}f(t)\|$ by

$$\|\nabla_r^{p,\infty}f(t)\|=\sum_{q=0}^{\infty}\sum_{j+|\alpha|\leq p}\frac{1}{j!\,\alpha!}\sum_{|\beta|=q}\|(t\partial_t+r)^j\partial_x^{\alpha+\beta}f(t)\|_{L^2(\boldsymbol{R}^n)}\frac{\rho^q}{q!}$$

(which is a formal power series in ρ whose coefficients are functions in t), and $\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{l}f(t)\|$ by

$$\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{l}f(t)\|=\sum_{j+|\alpha|\leq l}\|\nabla_r^{p,\infty}t^{\langle\kappa,\alpha\rangle}(t\partial_t+r)^j\partial_x^\alpha f(t)\|\,.$$

Similarly, for $a(t, x)\in C^\infty((0, T)\times\boldsymbol{R}^n)$ satisfying $(t\partial_t)^l\partial_x^\beta a(t, x)\in B^0([0, T]\times\boldsymbol{R}^n)$ (the space of all bounded continuous functions on $\Omega=[0, T]\times\boldsymbol{R}^n$) for any $(l, \beta)\in\boldsymbol{Z}_+\times\boldsymbol{Z}_+^n$, we define $\|\nabla^{p,\infty}a\|_\infty$ by

$$\|\nabla^{p,\infty}a\|_\infty=\sum_{q=0}^{\infty}\sum_{j+|\alpha|\leq p}\frac{1}{j!\,\alpha!}\sum_{|\beta|=q}\|(t\partial_t)^j\partial_x^{\alpha+\beta}a\|_{L^\infty(\Omega)}\frac{\rho^q}{q!}$$

(which is a formal power series in ρ). Also, for a differential operator $R(t, x, t\partial_t, \partial_x)$ $(=R)$ of the form

$$R(t, x, t\partial_t, \partial_x)=\sum_{j+|\alpha|\leq m}t^{\langle\kappa,\alpha\rangle}a_{j,\alpha}(t, x)(t\partial_t)^j\partial_x^\alpha\,, \tag{5.1}$$

we define $\|\nabla^{p,\infty}R\|_\infty$ by

$$\|\nabla^{p,\infty}R\|_\infty=\sum_{j+|\alpha|\leq m}\|\nabla^{p,\infty}a_{j,\alpha}\|_\infty\,.$$

The convenience of introducing these formal norms lies in Lemmas 3, 4 and 5 given below.

LEMMA 3. *The following formulae are valid.*

(1) $\|\nabla_r^{p,\infty}(af)(t)\|\ll\|\nabla^{p,\infty}a\|_\infty\times\|\nabla_r^{p,\infty}f(t)\|\,.$

(2) $\partial_\rho\|\nabla_r^{p,\infty}f(t)\|\ll\sum_{j=1}^{n}\|\nabla_r^{p,\infty}\partial_{x_j}f(t)\|.$

(3) $\|\nabla_r^{p,\infty}\partial_{x_j}f(t)\| \ll \partial_\rho\|\nabla_r^{p,\infty}f(t)\|$ $(1\leqq j\leqq n)$.

(4) *If* $p\geqq 1$ *and if* R *(in (5.1)) satisfies* $a_{m\,(0,\cdots,0)}(t, x)=1$ *on* $[0, T]\times \boldsymbol{R}^n$, *then*

$$\|[\nabla_r^{p,\infty}, R(t, x, t\partial_t+r, \partial_x)]f(t)\|$$
$$\ll c\|\nabla^{p,\infty}R\|_\infty\times(1+t^\mu\rho\partial_\rho)\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m,-1}f(t)\|,$$

where $\mu=\min\{\kappa_1, \cdots, \kappa_n\}$ (>0), *and* $c>0$ *is a constant depending only on* n, m, p, κ_i $(1\leqq i\leqq n)$ *and* T.

Here, $\sum_{q=0}^\infty a_q\rho^q \ll \sum_{q=0}^\infty b_q\rho^q$ means that $|a_q|\leqq b_q$ for any q, $[A, B]=AB-BA$ and

$$\|[\nabla_r^{p,\infty}, A]f(t)\| = \sum_{q=0}^\infty \sum_{j+|\alpha|\leq p}\frac{1}{j!\,\alpha!}\sum_{|\beta|=q}\|[(t\partial_t+r)^j\partial_x^{\alpha+\beta}, A]f(t)\|_{L^2(\boldsymbol{R}^n)}\frac{\rho^q}{q!}.$$

Note that (1), (2), (3) and (4) in Lemma 3 correspond respectively to the formulae (10.1), (10.2), (10.3) and (10.4) in Leray-Ohya [**6**]. Therefore, we can obtain Lemma 3 by the argument quite parallel to that in [**6**]. So, we omit the details.

Let $1\leqq s<\infty$ and put

$$\theta_s(\rho) = \sum_{q=0}^\infty (q!)^s\frac{\rho^q}{q!} \tag{5.2}$$

For a formal power series $\varphi(t, \rho)$ in ρ, we write

$$\varphi(t, \rho)\in\mathcal{E}^{\{s\}} \text{ uniformly on } [0, T], \tag{5.3}$$

if $\varphi(t, \rho)\ll B\theta_s(k\rho)$ on $[0, T]$ for some $B>0$ and $k>0$. Then, by using Sobolev's lemma (for example, Mizohata [**9**, Theorem 2.8]) we can see

LEMMA 4. *Assume that* $f(t, x)\in C^0([0, T], H^\infty(\boldsymbol{R}^n))$ *and that* $\mathrm{supp}(f)\subset[0, T]\times K$ *for some* $K\Subset\boldsymbol{R}^n$. *Then, the following* (i) *and* (ii) *are equivalent:*

(i) $f(t, x) \in C^0([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$,

(ii) $\|\nabla_r^{0,\infty}f(t)\| \in \mathcal{E}^{\{s\}}$ *uniformly on* $[0, T]$.

We say that $\varphi(t, \rho)=\sum_{q=0}^\infty\varphi_q(t)\rho^q$ satisfies (M_s), if $\varphi(t, \rho)$ satisfies the following condition:

(M_s) $\dfrac{(p!)^s}{p!}\varphi_q(t) \leqq \varphi_{p+q}(t)$ on $(0, T)$ for any $p, q\in\boldsymbol{Z}_+$.

LEMMA 5. *Let* $1\leqq s<\infty$, *and assume that* $\varphi(t, \rho)$ $(\gg 0)$ *satisfies* (M_s). *Then, for any* $0<2k\leqq h$ *we have*

$$\theta_s(k\rho)\varphi(t, h\rho) \ll 2\varphi(t, h\rho).$$

PROOF.

$$\theta_s(k\rho)\varphi(t, h\rho) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(p!)^s}{p!}\varphi_q(t)(k\rho)^p(h\rho)^q$$

$$\ll \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{k}{h}\right)^p\varphi_{p+q}(t)(h\rho)^{p+q}$$

$$\ll (1-k/h)^{-1}\varphi(t, h\rho) \ll 2\varphi(t, h\rho). \qquad \text{Q.E.D.}$$

For simplicity, in §6 we write $\varphi(t, \rho)=\sum_{q=0}^{\infty}\varphi_q(t)\rho^q\in\mathfrak{P}(\rho ; E)$ (where $E=C^0([0, T])$, $C^1((0, T])$ etc.), if $\varphi(t, \rho)$ is a formal power series in ρ whose coefficients $\varphi_q(t)$ $(q=0, 1, 2, \cdots)$ belong to E.

§6. A priori estimates.

Sixthly, we give two kinds of a priori estimates by combining formal norms in §5 with energy inequalities in Tahara [**11, 12**].

The operator $L(t, x, t\partial_t, \partial_x)$ $(=L)$ treated here is as follows:

$$L(t, x, t\partial_t, \partial_x) = (t\partial_t)^m + \sum_{\substack{j+|\alpha|\leq m \\ j<m}} t^{\langle\kappa, \alpha\rangle}a_{j,\alpha}(t, x)(t\partial_t)^j\partial_x^\alpha, \tag{6.1}$$

where $\kappa=(\kappa_1, \cdots, \kappa_n)$ and $\langle\kappa, \alpha\rangle=\kappa_1\alpha_1+\cdots+\kappa_n\alpha_n$. Throughout this section, we assume that $\kappa_i>0$ $(1\leqq i\leqq n)$, that $a_{j,\alpha}(t, x)$ $(j+|\alpha|\leqq m$ and $j<m)$ satisfy $(t\partial_t)^l\partial_x^\beta a_{j,\alpha}(t, x)\in B^0([0, T]\times\boldsymbol{R}^n)$ for any $(l, \beta)\in\boldsymbol{Z}_+\times\boldsymbol{Z}_+^n$, and that the following condition is satisfied:

(B′) (B) is satisfied. In addition, there is a $c>0$ such that

$$|\lambda_i(t, x, \xi)-\lambda_j(t, x, \xi)| \geqq c$$

on $\{(t, x, \xi)\in[0, T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n ; |\xi|=1\}$ for any $1\leqq i\neq j\leqq m$.

First, let us show our first a priori estimate, which is the one for

$$L(t, x, t\partial_t, \partial_x)(t^r v) = t^r g \tag*{(6.2)$_r$}$$

on $[0, T]\times\boldsymbol{R}^n$, where r is a real parameter. Note that $(6.2)_r$ is equivalent to $L(t, x, t\partial_t, \partial_x)u=f$ under $u=t^r v$ and $f=t^r g$. For $a>0$, $b\geqq 0$, $\mu>0$, $q\in\boldsymbol{Z}_+$ and $\theta(t)\in C^0([0, T])$, we define $R_{a,b,\mu}^{(q)}[\theta](t)\in C^0([0, T])\cap C^1((0, T))$ by

$$R_{a,b,\mu}^{(q)}[\theta](t) = t^{-a}e^{bqt^\mu/\mu}\int_0^t \tau^{a-1}e^{-bq\tau^\mu/\mu}\theta(\tau)d\tau.$$

For $\varphi(t, \rho)=\sum_{q=0}^{\infty}\varphi_q(t)\rho^q\in\mathfrak{P}(\rho ; C^0([0, T]))$, we define $R_{a,b,\mu}[\varphi](t, \rho)\in\mathfrak{P}(\rho ; C^0([0, T])\cap C^1((0, T)))$ by

$$R_{a,b,\mu}[\varphi](t, \rho) = \sum_{q=0}^{\infty} R_{a,b,\mu}^{(q)}[\varphi_q](t)\rho^q.$$

Then, we have

PROPOSITION 2. *Let* $1\leqq s<\infty$ *and* $p\in\boldsymbol{N}$. *Assume that* $\|\nabla^{p,\infty}L\|_{\infty}\ll B\theta_{s}(k\rho)$ *for some* $B>0$ *and* $k>0$. *Then, there is an* $a_{1}>0$ *which satisfies the following condition. If* $r>a_{1}$, *if* $v(t, x)$, $g(t, x)\in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^{n}))$ *satisfy* $(6.2)_{r}$ *and* $(t\partial_{t})^{l}v(t, x)$, $(t\partial_{t})^{l}g(t, x)\in C^{0}([0, T], H^{\infty}(\boldsymbol{R}^{n}))$ *for any* $l\in\boldsymbol{Z}_{+}$, *and if* $\|\nabla_{r}^{p,\infty}g(t)\|\ll\varphi(t, h\rho)$ *for some* $h\geqq 2k$ *and some* $\varphi(t, \rho)\in\mathfrak{P}(\rho; C^{0}([0, T]))$ *satisfying* (M_{s}), *then we have*

$$\|\nabla_{r}^{p,\infty}\nabla_{\kappa,r}^{m-j}v(t)\|\ll c^{j}(R_{a,b,\mu})^{j}[\varphi](t, h\rho) \tag{6.3$_j$}$$

on $[0, T]$ *for* $j=1, 2, \cdots, m$, *where* $\mu=\min\{\kappa_{1}, \cdots, \kappa_{n}\}$ (>0), *and* $a>0$, $b>0$, $c>0$ *are constants independent of* r, $v(t, x)$, $g(t, x)$, $\varphi(t, \rho)$ *and* h.

Before the proof, we prepare some results.

LEMMA 6. *There is an* $a_{0}>0$ *which satisfies the following condition. If* $r>a_{0}$, *and if* $v(t, x)$, $g(t, x)\in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^{n}))$ *satisfy* $(6.2)_{r}$ *and* $(t\partial_{t})^{l}v(t, x)$, $(t\partial_{t})^{l}g(t, x)\in C^{0}([0, T], H^{\infty}(\boldsymbol{R}^{n}))$ *for any* $l\in\boldsymbol{Z}_{+}$, *then we have*

$$\|\nabla_{\kappa,r}^{m-1}v(t)\|\leqq c_{0}t^{-r+a_{0}}\int_{0}^{t}\tau^{r-a_{0}-1}\|g(\tau)\|d\tau \tag{6.4}$$

on $[0, T]$, *where*

$$\begin{aligned}\|\nabla_{\kappa,r}^{m-1}v(t)\| &= \sum_{j+|\alpha|\leqq m-1}\|t^{\langle\kappa,\alpha\rangle}(t\partial_{t}+r)^{j}\partial_{x}^{\alpha}v(t)\|_{L^{2}(\boldsymbol{R}^{n})},\\ \|g(t)\| &= \|g(t)\|_{L^{2}(\boldsymbol{R}^{n})},\end{aligned} \tag{6.5}$$

and $c_{0}>0$ *is a constant independent of* r, $v(t, x)$ *and* $g(t, x)$.

PROOF. Since (6.4) is equivalent to

$$\|\nabla_{\kappa,r}^{m-1}v(t)\|\leqq c_{0}\int_{0}^{\infty}e^{-(r-a_{0})y}\|g(te^{-y})\|dy$$

under $\tau=te^{-y}$, Lemma 6 is clear from [**11**, Proposition 2.1] with $a=\sigma=1$, $\mu=\min\{\kappa_{1}, \cdots, \kappa_{n}\}$ and $Q(t, \xi)=t^{2(\kappa_{1}-\mu)}\xi_{1}^{2}+\cdots+t^{2(\kappa_{n}-\mu)}\xi_{n}^{2}$. Q. E. D.

LEMMA 7. *Let* $a>0$, $b\geqq 0$, $\mu>0$ *and* $\varphi(t, \rho)\in\mathfrak{P}(\rho; C^{0}([0, T]))$. *Then, we have the following results.* (1) $R_{a,b,\mu}[t^{A}\varphi](t, \rho)=t^{A}R_{a+A,b,\mu}[\varphi](t, \rho)$ *for any* $A\geqq 0$. (2) *If* $\varphi(t, \rho)$ *satisfies* $\varphi(t, \rho)\gg 0$ *and* (M_{s}), $R_{a,b,\mu}[\varphi](t, \rho)$ *also satisfies* (M_{s}). (3) *The equation*

$$(t\partial_{t}+a-bt^{\mu}\rho\partial_{\rho})\Phi(t, \rho)=\varphi(t, \rho) \tag{6.6}$$

has a unique solution $\Phi(t, \rho)$ *in* $\mathfrak{P}(\rho; C^{0}([0, T])\cap C^{1}((0, T)))$ *and it is given by* $\Phi(t, \rho)=R_{a,b,\mu}[\varphi](t, \rho)$. (4) *If* $\varphi(t, \rho)\gg 0$, *then* $R_{a,b,\mu}[\partial_{\rho}\varphi](t, \rho)\ll\partial_{\rho}R_{a,b,\mu}[\varphi](t, \rho)$. (5) *If* $B\geqq 0$, $h\geqq 2k>0$, $r\geqq a+2B$ *and* $b\geqq 2B$, *if* $\varphi(t, \rho)$ *satisfies* $\varphi(t, \rho)\gg 0$ *and* (M_{s}), *and if* $\Psi(t, \rho)\in\mathfrak{P}(\rho; C^{0}([0, T])\cap C^{1}((0, T)))$ *satisfies* $\Psi(t, \rho)\gg 0$ *and*

$$(t\partial_t+r)\Psi(t, \rho) \ll B\theta_s(k\rho)(1+t^\mu\rho\partial_\rho)\Psi(t, \rho)+\varphi(t, h\rho), \tag{6.7}$$

then $\Psi(t, \rho)\ll R_{a,b,\mu}[\varphi](t, h\rho)$. (6) *If* $r\geqq a$ *and if* $\varphi(t, \rho)\gg 0$, *then*

$$t^{-r}\int_0^t \tau^{r-1}\varphi(\tau, \rho)d\tau \ll R_{a,b,\mu}[\varphi](t, \rho).$$

PROOF. (1) and (2) are clear from the definition. Since (6.6) is equivalent to

$$(t\partial_t+a-bqt^\mu)\Phi_q(t)=\varphi_q(t), \quad q=0, 1, 2, \cdots \tag{6.8}$$

under $\Phi(t, \rho)=\sum_{q=0}^\infty \Phi_q(t)\rho^q$ and $\varphi(t, \rho)=\sum_{q=0}^\infty \varphi_q(t)\rho^q$, by solving (6.8) we have $\Phi_q(t)=R_{a,b,\mu}^{(q)}[\varphi_q](t)$ $(q=0, 1, 2, \cdots)$ and hence (3). If $\varphi(t, \rho)\gg 0$, by operating ∂_ρ on (6.6) we have

$$\begin{aligned} 0 \ll \partial_\rho\varphi(t, \rho) &= (t\partial_t+a-bt^\mu\rho\partial_\rho)\partial_\rho\Phi(t, \rho)-bt^\mu\partial_\rho\Phi(t, \rho) \\ &\ll (t\partial_t+a-bt^\mu\rho\partial_\rho)\partial_\rho\Phi(t, \rho). \end{aligned} \tag{6.9}$$

Therefore, by applying $R_{a,b,\mu}[\ ]$ to (6.9) we have $R_{a,b,\mu}[\partial_\rho\varphi](t, \rho)\ll\partial_\rho\Phi(t, \rho)$. This implies (4). Under the conditions in (5), by (2), (3), Lemma 5 and the fact that $\rho\partial_\rho R_{a,b,\mu}[\varphi](t, \rho)$ also satisfies (M_s) we have

$$\begin{aligned} &\varphi(t, h\rho)+B\theta_s(k\rho)(1+t^\mu\rho\partial_\rho)R_{a,b,\mu}[\varphi](t, h\rho) \\ &\ll (t\partial_t+a-bt^\mu\rho\partial_\rho)R_{a,b,\mu}[\varphi](t, h\rho)+2B(1+t^\mu\rho\partial_\rho)R_{a,b,\mu}[\varphi](t, h\rho) \\ &\ll (t\partial_t+r)R_{a,b,\mu}[\varphi](t, h\rho). \end{aligned} \tag{6.10}$$

Therefore, by combining (6.7) and (6.10) we have

$$\begin{aligned} (t\partial_t+r)\Psi(t, \rho) \ll\ & B\theta_s(k\rho)(1+t^\mu\rho\partial_\rho)\Psi(t, \rho) \\ &+(t\partial_t+r-B\theta_s(k\rho)(1+t^\mu\rho\partial_\rho))R_{a,b,\mu}[\varphi](t, h\rho). \end{aligned}$$

Hence, (5) is verified by Lemma 8 given below. Since

$$(t\partial_t+r)\Big(t^{-r}\int_0^t \tau^{r-1}\varphi(\tau, \rho)d\tau\Big)=\varphi(t, \rho),$$

(6) is verified in the same way as (5) with $B=0$. Q.E.D.

LEMMA 8. *Let* $B(\rho)=\sum_{q=0}^\infty B_q\rho^q\gg 0$, $r>B_0$, $\mu>0$ *and* $\Psi(t, \rho)$, $\Phi(t, \rho)\in \mathfrak{P}(\rho\,; C^0([0, T])\cap C^1((0, T)))$. *If they satisfy* $\Psi(t, \rho)\gg 0$ *and*

$$\begin{aligned} (t\partial_t+r)\Psi(t, \rho) \ll\ & B(\rho)(1+t^\mu\rho\partial_\rho)\Psi(t, \rho) \\ &+(t\partial_t+r-B(\rho)(1+t^\mu\rho\partial_\rho))\Phi(t, \rho), \end{aligned} \tag{6.11}$$

then $\Psi(t, \rho)\ll\Phi(t, \rho)$.

PROOF. Put $a=r-B_0>0$, $b=B_0\geqq 0$, $\Psi(t, \rho)=\sum_{q=0}^\infty \Psi_q(t)\rho^q$ and $\Phi(t, \rho)=\sum_{q=0}^\infty \Phi_q(t)\rho^q$. Then, by (6.11) we have

$$
\begin{aligned}
&(t\partial_t+a-bqt^{\mu})\Psi_q(t)-\sum_{j=0}^{q-1}B_{q-j}(1+jt^{\mu})\Psi_j(t)\\
&\leqq (t\partial_t+a-bqt^{\mu})\Phi_q(t)-\sum_{j=0}^{q-1}B_{q-j}(1+jt^{\mu})\Phi_j(t)
\end{aligned}
\tag{6.12}
$$

($q=0, 1, 2, \cdots$). Therefore, by applying $R_{a,b,\mu}^{(q)}[\ ]$ to (6.12) we have

$$
\begin{aligned}
&\Psi_q(t)-\sum_{j=0}^{q-1}B_{q-j}R_{a,b,\mu}^{(q)}[(1+jt^{\mu})\Psi_j](t)\\
&\leqq \Phi_q(t)-\sum_{j=0}^{q-1}B_{q-j}R_{a,b,\mu}^{(q)}[(1+jt^{\mu})\Phi_j](t)
\end{aligned}
$$

($q=0, 1, 2, \cdots$). Hence, by induction on q we can obtain $\Psi_q(t)\leqq\Phi_q(t)$ for any q. Q. E. D.

PROOF OF PROPOSITION 2. Since $(t\partial_t)(t^r w)=t^r(t\partial_t+r)w$, by operating $(t\partial_t)^j\partial_x^{\alpha+\beta}$ on both sides of $(6.2)_r$ we have

$$
L(t^r((t\partial_t+r)^j\partial_x^{\alpha+\beta}v)) = t^r((t\partial_t+r)^j\partial_x^{\alpha+\beta}g-[(t\partial_t+r)^j\partial_x^{\alpha+\beta}, L_r]v), \tag{6.13}
$$

where $L_r=L(t, x, t\partial_t+r, \partial_x)$. Therefore, if $r>a_0$, by applying Lemma 6 to (6.13) we have

$$
\begin{aligned}
&\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v(t)\|\\
&\ll c_1t^{-r+a_0}\int_0^t\tau^{r-a_0-1}\{\|\nabla_r^{p,\infty}g(\tau)\|+\|[\nabla_r^{p,\infty}, L_r]v(\tau)\|\}d\tau
\end{aligned}
$$

for some $c_1>0$. Hence, by using $\|\nabla_r^{p,\infty}g(t)\|\ll\varphi(t, h\rho)$, $\|\nabla^{p,\infty}L\|_{\infty}\ll B\theta_s(k\rho)$ and the formula (4) in Lemma 3 we obtain

$$
\begin{aligned}
&\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v(t)\|\\
&\ll c_1t^{-r+a_0}\int_0^t\tau^{r-a_0-1}\{\varphi(\tau, h\rho)+B\theta_s(k\rho)(1+\tau^{\mu}\rho\partial_{\rho})\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v(\tau)\|\}d\tau\,.
\end{aligned}
\tag{6.14}
$$

Here, we denote by $\Psi(t, \rho)$ the right hand side of (6.14). Then, $\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v(t)\|\ll\Psi(t, \rho)$ and

$$
(t\partial_t+r-a_0)\Psi(t, \rho) \ll c\varphi(t, h\rho)+cB\theta_s(k\rho)(1+t^{\mu}\rho\partial_{\rho})\Psi(t, \rho) \tag{6.15}
$$

for any $c\geqq c_1$. Therefore, if $a>0$, $b\geqq 2cB$, $c\geqq c_1$ and $r\geqq a_0+a+2cB$, we can obtain $(6.3)_1$ by applying (5) in Lemma 7 to (6.15). Since

$$
\theta(t) = t^{-r}\int_0^t\tau^{r-1}(\tau\partial_\tau+r)\theta(\tau)d\tau
$$

holds for any $\theta(t)\in C^0([0, T])\cap C^1((0, T))$ and $r>0$, we have

$$
\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-j}v(t)\| \ll c_2t^{-r}\int_0^t\tau^{r-1}\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-j+1}v(\tau)\|d\tau \tag{6.16}
$$

for some $c_2>0$. Hence, if $c \geqq c_2$, we can obtain $(6.3)_j$ $(j \geqq 2)$ by induction on j as follows: by applying $(6.3)_{j-1}$ to the right hand side of (6.16) and then using (6) in Lemma 7, we have

$$\begin{aligned}\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-j}v(t)\| &\ll c_2 c^{j-1} t^{-r}\int_0^t \tau^{r-1}(R_{a,b,\mu})^{j-1}[\varphi](\tau, h\rho)d\tau \\ &\ll c^j (R_{a,b,\mu})^j[\varphi](t, h\rho). \qquad \text{Q.E.D.}\end{aligned}$$

COROLLARY TO PROPOSITION 2. *Let $l>0$, and let $(j, \alpha)\in \boldsymbol{Z}_+\times \boldsymbol{Z}_+^n$ such that $j+|\alpha|<m$ and $|\alpha|>0$. Then, under the situation in Proposition 2 and the condition that $\varphi(t, \rho)=t^A\varphi_0(t, \rho)$ for some $A\geqq 0$ and $\varphi_0(t, \rho)\in \mathfrak{P}(\rho; C^0([0, T]))$ we have*

$$\|\nabla_r^{p,\infty}t^l(t\partial_t+r)^j\partial_x^\alpha v(t)\| \ll C\, t^{l+A-\langle\kappa,\beta\rangle}\partial_\rho^{|\alpha|-|\beta|}(R_{a+A,b,\mu})^{m-j-|\beta|}[\varphi_0](t, h\rho) \tag{6.17}$$

on $[0, T]$ for any $\beta\in \boldsymbol{Z}_+^n$ such that $(0, \cdots, 0)\leqq\beta\leqq\alpha$ and $\langle\kappa, \beta\rangle\leqq l+A$, where $C>0$ is a constant independent of r, $v(t, x)$, $g(t, x)$, A and $\varphi_0(t, \rho)$.

PROOF. Let $\beta\in \boldsymbol{Z}_+^n$ such that $(0, \cdots, 0)\leqq\beta\leqq\alpha$ and $\langle\kappa, \beta\rangle\leqq l+A$. Then, by (3) in Lemma 3 we have

$$\begin{aligned}\|\nabla_r^{p,\infty}t^l(t\partial_t+r)^j\partial_x^\alpha v(t)\| &= \|\nabla_r^{p,\infty}t^{l-\langle\kappa,\beta\rangle}\partial_x^{\alpha-\beta}(t^{\langle\kappa,\beta\rangle}(t\partial_t+r)^j\partial_x^\beta v(t))\| \\ &\ll c_3 t^{l-\langle\kappa,\beta\rangle}\partial_\rho^{|\alpha|-|\beta|}\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{j+|\beta|}v(t)\|\end{aligned} \tag{6.18}$$

for some $c_3>0$. Therefore, by applying $(6.3)_{m-j-|\beta|}$ (with $\varphi(t, \rho)$ replaced by $t^A\varphi_0(t, \rho)$) to (6.18) and using (1) in Lemma 7, we can easily obtain (6.17) on $[0, T]$. Q.E.D.

Next, let us show our second a priori estimate, which is the one for

$$\begin{cases} L(t, x, t\partial_t, \partial_x)u=f, \\ \partial_t^i u|_{t=T}=0 \quad \text{for } i=0, 1, \cdots, m-1 \end{cases} \tag{6.19}$$

on $(0, T]\times \boldsymbol{R}^n$. For $a\in \boldsymbol{R}$, $b\geqq 0$, $\mu>0$, $q\in \boldsymbol{Z}_+$ and $\theta(t)\in C^0((0, T])$, we define $S_{a,b,\mu}^{(q)}[\theta](t)\in C^1((0, T])$ by

$$S_{a,b,\mu}^{(q)}[\theta](t) = t^{-a}e^{-bqt^\mu/\mu}\int_t^T \tau^{a-1}e^{bq\tau^\mu/\mu}\theta(\tau)d\tau.$$

For $\varphi(t, \rho)=\sum_{q=0}^\infty \varphi_q(t)\rho^q\in \mathfrak{P}(\rho; C^0((0, T]))$, we define $S_{a,b,\mu}[\varphi](t, \rho)\in \mathfrak{P}(\rho; C^1((0, T]))$ by

$$S_{a,b,\mu}[\varphi](t, \rho) = \sum_{q=0}^\infty S_{a,b,\mu}^{(q)}[\varphi_q](t)\rho^q.$$

Then, we have

PROPOSITION 3. *Let $1\leqq s<\infty$ and $p\in \boldsymbol{N}$. Assume that $\|\nabla^{p,\infty}L\|_\infty \ll B\theta_s(k\rho)$ for some $B>0$ and $k>0$. Then, we have the following condition. If $u(t, x)$,*

$f(t, x)\in C^{\infty}((0, T], H^{\infty}(\boldsymbol{R}^{n}))$ satisfy (6.19) *and $\partial_t^i u|_{t=T}=0$ for $i=0, 1, \cdots, m+p-1$, and if $\|\nabla_0^{p,\infty}f(t)\|\ll\varphi(t, h\rho)$ for some $h\geqq 2k$ and some $\varphi(t, \rho)\in\mathfrak{P}(\rho ; C^0((0, T]))$ satisfying* (M_s), *then we have*

$$\|\nabla_0^{p,\infty}\nabla_{\kappa,0}^{m-j}u(t)\| \ll c^j(S_{a,b,\mu})^j[\varphi](t, h\rho) \tag{6.20$_j$}$$

on $(0, T]$ for $j=1, 2, \cdots, m$, where $\mu=\min\{\kappa_1, \cdots, \kappa_n\}$ (>0), and $a>0$, $b>0$, $c>0$ are constants independent of $u(t, x)$, $f(t, x)$, $\varphi(t, \rho)$ and h.

The proof of Proposition 3 is quite similar to that of Proposition 2. So, we give only a sketch.

LEMMA 9. *If $u(t, x)$, $f(t, x)\in C^{\infty}((0, T], H^{\infty}(\boldsymbol{R}^n))$ satisfy* (6.19), *then we have*

$$\|\nabla_{\kappa,0}^{m-1}u(t)\| \leqq c_0 t^{-a_0}\int_t^T \tau^{a_0-1}\|f(\tau)\|d\tau$$

on $(0, T]$, where $\|\nabla_{\kappa,0}^{m-1}u(t)\|$, $\|f(t)\|$ are the same as in (6.5), *and $a_0>0$, $c_0>0$ are constants independent of $u(t, x)$ and $f(t, x)$.*

LEMMA 10. *Let $a\in\boldsymbol{R}$, $b\geqq 0$, $\mu>0$ and $\varphi(t, \rho)\in\mathfrak{P}(\rho ; C^0((0, T]))$. Then, we have the following results.* (1) *$S_{a,b,\mu}[t^A\varphi](t, \rho)=t^A S_{a+A,b,\mu}[\varphi](t, \rho)$ for any $A\in\boldsymbol{R}$.* (2) *If $\varphi(t, \rho)$ satisfies $\varphi(t, \rho)\gg 0$ and* (M_s), *$S_{a,b,\mu}[\varphi](t, \rho)$ also satisfies* (M_s). (3) *The equation*

$$\begin{cases} (-t\partial_t - a - bt^{\mu}\rho\partial_\rho)\Phi(t, \rho) = \varphi(t, \rho), \\ \Phi(T, \rho) = 0 \end{cases}$$

has a unique solution $\Phi(t, \rho)$ in $\mathfrak{P}(\rho ; C^1((0, T]))$ and it is given by $\Phi(t, \rho)= S_{a,b,\mu}[\varphi](t, \rho)$. (4) *If $\varphi(t, \rho)\gg 0$, then $S_{a,b,\mu}[\partial_\rho\varphi](t, \rho)\ll\partial_\rho S_{a,b,\mu}[\varphi](t, \rho)$.* (5) *If $B\geqq 0$, $h\geqq 2k>0$, $a\geqq r+2B$ and $b\geqq 2B$, if $\varphi(t, \rho)$ satisfies $\varphi(t, \rho)\gg 0$ and* (M_s), *and if $\Psi(t, \rho)\in\mathfrak{P}(\rho ; C^1((0, T]))$ satisfies $\Psi(t, \rho)\gg 0$, $\Psi(T, \rho)=0$ and*

$$(-t\partial_t - r)\Psi(t, \rho) \ll B\theta_s(k\rho)(1+t^{\mu}\rho\partial_\rho)\Psi(t, \rho)+\varphi(t, h\rho),$$

then $\Psi(t, \rho)\ll S_{a,b,\mu}[\varphi](t, h\rho)$. (6) *If $r\geqq a$ and if $\varphi(t, \rho)\gg 0$, then*

$$\int_t^T \tau^{-1}\varphi(\tau, \rho)d\tau \ll S_{a,b,\mu}[\varphi](t, \rho).$$

Lemma 9 is clear from the proof of [**12**, Proposition 5] with $\mu=\min\{\kappa_1, \cdots, \kappa_n\}$ and $Q(t, \xi)=t^{2(\kappa_1-\mu)}\xi_1^2+\cdots+t^{2(\kappa_n-\mu)}\xi_n^2$. Lemma 10 is proved in the same way as Lemma 7.

PROOF OF PROPOSITION 3. By operating $(t\partial_t)^j\partial_x^{\alpha+\beta}$ on both sides of (6.19), we have

$$\begin{cases} L((t\partial_t)^j\partial_x^{\alpha+\beta}u)=(t\partial_t)^j\partial_x^{\alpha+\beta}f-[(t\partial_t)^j\partial_x^{\alpha+\beta},\ L]u\,, \\ \partial_t^i(t\partial_t)^j\partial_x^{\alpha+\beta}u|_{t=T}=0 \quad \text{for } i=0,1,\cdots,m-1\,. \end{cases} \tag{6.21}$$

Therefore, by applying Lemma 9 to (6.21) we have

$$\begin{aligned} &\|\nabla_0^{p,\infty}\nabla_{\kappa,0}^{m-1}u(t)\| \\ &\ll c_1t^{-a_0}\int_t^T \tau^{a_0-1}\{\varphi(\tau,\ h\rho)+B\theta_s(k\rho)(1+\tau^{\mu}\rho\partial_\rho)\|\nabla_0^{p,\infty}\nabla_{\kappa,0}^{m-1}u(\tau)\|\}d\tau \end{aligned} \tag{6.22}$$

for some $c_1>0$. Here, we denote by $\Psi(t,\rho)$ the right hand side of (6.22). Then, $\|\nabla_0^{p,\infty}\nabla_{\kappa,0}^{m-1}u(t)\|\ll\Psi(t,\rho)$, $\Psi(T,\rho)=0$ and

$$(-t\partial_t-a_0)\Psi(t,\rho)\ll c\varphi(t,\ h\rho)+cB\theta_s(k\rho)(1+t^{\mu}\rho\partial_\rho)\Psi(t,\rho)$$

for any $c\geqq c_1$. Hence, if $a\geqq a_0+2cB$, $b\geqq 2cB$ and $c\geqq c_1$. we can obtain $(6.20)_1$ by (5) in Lemma 10. Since

$$\|\nabla_0^{p,\infty}\nabla_{\kappa,0}^{m-j}u(t)\|\ll c_2\int_t^T\tau^{-1}\|\nabla_0^{p,\infty}\nabla_{\kappa,0}^{m-j+1}u(\tau)\|d\tau$$

for some $c_2>0$, we can also obtain $(6.20)_j$ $(j\geqq 2)$ by induction on j. Q. E. D.

COROLLARY TO PROPOSITION 3. *Let $l>0$, and let $(j,\alpha)\in \boldsymbol{Z}_+\times\boldsymbol{Z}_+^n$ such that $j+|\alpha|<m$ and $|\alpha|>0$. Then, under the situation in Proposition 3 and the condition that $\varphi(t,\rho)=t^A\varphi_0(t,\rho)$ for some $A\in\boldsymbol{R}$ and $\varphi_0(t,\rho)\in\mathfrak{P}(\rho;C^0((0,T]))$ we have*

$$\|\nabla_0^{p,\infty}t^l(t\partial_t)^j\partial_x^\alpha u(t)\|\ll Ct^{l+A-\langle\kappa,\beta\rangle}\partial_\rho^{|\alpha|-|\beta|}(S_{a+A,b,\mu})^{m-j-|\beta|}[\varphi_0](t,\ h\rho)$$

on $(0,T]$ for any $\beta\in\boldsymbol{Z}_+^n$ such that $(0,\cdots,0)\leqq\beta\leqq\alpha$, where $C>0$ is a constant independent of $u(t,x)$, $f(t,x)$, A and $\varphi_0(t,\rho)$.

§ 7. Proof of Theorem 1.

Seventhly, we prove Theorem 1 (in § 2). Our plan is as follows: first we establish an L^2 version of Theorem 1 by treating (S) in the L^2 framework, and then we obtain Theorem 1 by the cut-off argument.

Let $P(t,x,t\partial_t,\partial_x)$ $(=P)$ be the operator in (1.1), and let us consider

$$P(t,x,t\partial_t,\partial_x)(t^rv)=t^rg\,, \tag{7.1$_r$}$$

where r is a real parameter. Note that $(7.1)_r$ is equivalent to (S) under $u=t^rv$ and $f=t^rg$. In order to treat $(7.1)_r$ in the L^2 framework, we impose the following condition on the coefficients:

(D_p) $a_{j,\alpha}(t,x)\in C^\infty((0,T)\times\boldsymbol{R}^n)$ $(j+|\alpha|\leqq m$ and $j<m)$ satisfy $(t\partial_t)^l\partial_x^\beta a_{j,\alpha}(t,x)\in B^0([0,T]\times\boldsymbol{R}^n)$ for any $(l,\beta)\in\boldsymbol{Z}_+\times\boldsymbol{Z}_+^n$ and

$$\|\nabla^{p,\infty} a_{j,\alpha}\|_\infty \in \mathcal{E}^{\{s\}}.$$

Then, we have

PROPOSITION 4. *Let $p \in \boldsymbol{N}$. Assume that P and s satisfy* (A_κ), (B′), (C′) *and* (D_p). *Then, there is an $a_3>0$ which satisfies the following condition. If $r>a_3$, and if $g(t, x) \in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ satisfies $(t\partial_t)^l g(t, x) \in C^0([0, T], H^\infty(\boldsymbol{R}^n))$ for any $l \in \boldsymbol{Z}_+$ and $\|\nabla_r^{p,\infty} g(t)\| \in \mathcal{E}^{\{s\}}$ uniformly on $[0, T]$, $(7.1)_r$ has a unique solution $v(t, x) \in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ such that $(t\partial_t)^l v(t, x) \in C^0([0, T], H^\infty(\boldsymbol{R}^n))$ for any $l \in \boldsymbol{Z}_+$ and that $\|\nabla_r^{p,\infty} \nabla_{\kappa,r}^{m-1} v(t)\| \in \mathcal{E}^{\{s\}}$ uniformly on $[0, T]$. In addition, if $g(t, x)$ satisfies* $\mathrm{supp}(g) \subset C_\mu(0, K)$ *for some $K \Subset \boldsymbol{R}^n$, $v(t, x)$ also satisfies* $\mathrm{supp}(v) \subset C_\mu(0, K)$.

Here, $C_\mu(0, K)$ is defined by the case $t_0=0$ of

$$C_\mu(t_0, K) = \left\{(t, x) \in [0, T] \times \boldsymbol{R}^n \ ; \ \min_{y \in K} |x-y| \leqq \frac{\lambda_{\max} T^{k-\mu}}{\mu} |t^\mu - t_0^\mu| \right\}, \quad (7.2)$$

where $\mu = \min\{\kappa_1, \cdots, \kappa_n\}$, $k = \max\{\kappa_1, \cdots, \kappa_n\}$ and $\lambda_{\max}$ is the least upper bound of $|\lambda_i(t, x, \xi)|$ $(1 \leqq i \leqq m)$ on $\{(t, x, \xi) \in [0, T] \times \boldsymbol{R}^n \times \boldsymbol{R}^n \ ; \ |\xi|=1\}$.

Put $Q_{j,\alpha}(t, x, t\partial_t, \partial_x)$ $(j+|\alpha|<m$ and $|\alpha|>0)$ and $L(t, x, t\partial_t, \partial_x)$ as follows:

$$\begin{aligned} Q_{j,\alpha}(t, x, t\partial_t, \partial_x) &= (-1) t^{l(j,\alpha)} a_{j,\alpha}(t, x)(t\partial_t)^j \partial_x^\alpha, \\ L(t, x, t\partial_t, \partial_x) &= P(t, x, t\partial_t, \partial_x) + \sum_{\substack{j+|\alpha|<m \\ |\alpha|>0}} Q_{j,\alpha}(t, x, t\partial_t, \partial_x). \end{aligned} \quad (7.3)$$

Then, $L(t, x, t\partial_t, \partial_x)$ has the form (6.1) and $(7.1)_r$ is equivalent to

$$L(t, x, t\partial_t, \partial_x)(t^r v) = t^r\Big(g + \sum_{\substack{j+|\alpha|<m \\ |\alpha|>0}} Q_{j,\alpha}(t, x, t\partial_t + r, \partial_x) v\Big).$$

Therefore, to solve $(7.1)_r$ we can use the method of successive approximations: first we solve

$$\begin{cases} L(t, x, t\partial_t, \partial_x)(t^r v_0) = t^r g, \\ L(t, x, t\partial_t, \partial_x)(t^r v_k) = t^r\Big(\sum_{\substack{j+|\alpha|<m \\ |\alpha|>0}} Q_{j,\alpha}(t, x, t\partial_t + r, \partial_x) v_{k-1}\Big), \quad k=1, 2, \cdots, \end{cases} \quad (7.4)$$

and then we show the convergence of $\sum_{k=0}^\infty v_k(t, x)$ by using the a priori estimates in §6. The existence of $\{v_k(t, x)\}_{k=0}^\infty$ is guaranteed by

LEMMA 11. *Let $L(t, x, t\partial_t, \partial_x)$ be as above. Then, there is an $a_2>0$ which satisfies the following condition. If $r>a_2$, and if $g(t, x) \in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ satisfies $(t\partial_t)^l g(t, x) \in C^0([0, T], H^\infty(\boldsymbol{R}^n))$ for any $l \in \boldsymbol{Z}_+$, $L(t, x, t\partial_t, \partial_x)(t^r v) = t^r g$ has a unique solution $v(t, x) \in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ such that $(t\partial_t)^l g(t, x) \in C^0([0, T], H^\infty(\boldsymbol{R}^n))$ for any $l \in \boldsymbol{Z}_+$. In addition, if $g(t, x)$ satisfies* $\mathrm{supp}(g) \subset C_\mu(0, K)$ *for some $K \Subset \boldsymbol{R}^n$, $v(t, x)$ also satisfies* $\mathrm{supp}(v) \subset C_\mu(0, K)$.

PROOF. Since $L(t, x, t\partial_t, \partial_x)$ has the form (6.1), we can apply [**11**, Theorem 2.3] with $a=\sigma=1$, $\mu=\min\{\kappa_1, \cdots, \kappa_n\}$ and $Q(t, \xi)=t^{2(\kappa_1-\mu)}\xi_1^2+\cdots+t^{2(\kappa_n-\mu)}\xi_n^2$. Q.E.D.

PROOF OF PROPOSITION 4. Let $a_1>0$ be as in Proposition 2, let $a_2>0$ be as in Lemma 11, and put $a_3=\max\{a_1, a_2\}$. Then, for any $r>a_3$ we can apply Proposition 2 and Lemma 11 to $L(t, x, t\partial_t, \partial_x)$. Take any $r>a_3$ and fix it hereafter. For simplicity, we write $L=L(t, x, t\partial_t, \partial_x)$ and $Q_r(j, \alpha)=Q_{j,\alpha}(t, x, t\partial_t+r, \partial_x)$.

Let $g(t, x)$ be as in Proposition 4. Then, by Lemma 11 we can solve (7.4) successively and obtain $v_k(t, x)\in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ $(k=0, 1, 2, \cdots)$ such that $(t\partial_t)^l v_k(t, x)\in C^0([0, T], H^\infty(\boldsymbol{R}^n))$ for any $l\in\boldsymbol{Z}_+$. If we have

$$\sum_{k=0}^{\infty}\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v_k(t)\|\in\mathcal{E}^{\{s\}}\quad\text{uniformly on } [0, T],$$

$v(t, x)=\sum_{k=0}^{\infty}v_k(t, x)$ becomes a desired solution in Proposition 4. Therefore, to obtain the existence part of Proposition 4 it is sufficient to show that

$$\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v_k(t)\|\ll K^{k+1}\frac{1}{(k!)^\nu}\theta_s(H\rho)\quad\text{on } [0, T],\qquad k=0, 1, 2, \cdots \tag{7.5}$$

for some $K>0$, $\nu>0$ and $H>0$.

Let us reduce the problem to a simpler case. For any sequence $(j_i, \alpha_{(i)})\in\boldsymbol{Z}_+\times\boldsymbol{Z}_+^n$ $(i=1, 2, \cdots, k)$ such that $j_i+|\alpha_{(i)}|<m$ and $|\alpha_{(i)}|>0$, we define $v((j_1, \alpha_{(1)}), \cdots, (j_i, \alpha_{(i)}))(t, x)\in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ $(i=1, 2, \cdots, k)$ by the solution of

$$\begin{cases} L(t^r v((j_1, \alpha_{(1)})))=t^r Q_r(j_1, \alpha_{(1)})v_0, \\ L(t^r v((j_1, \alpha_{(1)}), \cdots, (j_i, \alpha_{(i)}))) \\ \quad =t^r Q_r(j_i, \alpha_{(i)})v((j_1, \alpha_{(1)}), \cdots, (j_{i-1}, \alpha_{(i-1)})),\quad i=2, \cdots, k \end{cases} \tag{7.6}$$

such that $(t\partial_t)^l v((j_1, \alpha_{(1)}), \cdots, (j_i, \alpha_{(i)}))(t, x)\in C^0([0, T], H^\infty(\boldsymbol{R}^n))$ for any $l\in\boldsymbol{Z}_+$ Then, by the uniqueness part of Lemma 11 we have

$$v_k=\sum_{\substack{j_1+|\alpha_{(1)}|<m\\ |\alpha_{(1)}|>0}}\cdots\sum_{\substack{j_k+|\alpha_{(k)}|<m\\ |\alpha_{(k)}|>0}}v((j_1, \alpha_{(1)}), \cdots, (j_k, \alpha_{(k)})).$$

Hence, our problem is reduced to proving the following: there are $K_1>0$, $\nu>0$ and $H>0$ such that

$$\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v((j_1, \alpha_{(1)}), \cdots, (j_k, \alpha_{(k)}))(t)\|\ll K_1^{k+1}\frac{1}{(k!)^\nu}\theta_s(H\rho) \tag{7.7}$$

for any $(j_1, \alpha_{(1)}), \cdots, (j_k, \alpha_{(k)})$ and $k=1, 2, \cdots$. We will show this from now on.

By the conditions in Proposition 4, we may assume that

$$\|\nabla^{p,\infty}L\|_\infty,\ \|\nabla^{p,\infty}a_{j,\alpha}\|_\infty \ll B\theta_s(k\rho),$$

$$\|\nabla_r^{p,\infty}g(t)\| \ll A\theta_s(h\rho) \quad \text{on } [0, T]$$

for some $A>0$, $B>0$ and $h\geqq 2k>0$. Therefore, by applying Corollary to Proposition 2 $(k+1)$-times to (7.6) and by using (1) in Lemma 3, Lemma 5 and (1), (2), (4) in Lemma 7, we have

$$\begin{aligned}(7.8)\quad &\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v((j_1, \alpha_{(1)}), \cdots, (j_k, \alpha_{(k)}))(t)\| \\ &\ll A(2B)^k C^{k+1}t^{l_1+\cdots+l_k-\langle\kappa,\beta_{(1)}+\cdots+\beta_{(k)}\rangle}\times\partial_\rho^{(|\alpha_{(1)}|-|\beta_{(1)}|)+\cdots+(|\alpha_{(k)}|-|\beta_{(k)}|)} \\ &\quad\times R_{a_{k+1},b,\mu}\times(R_{a_k,b,\mu})^{m-j_k-|\beta_{(k)}|}\times\cdots\times(R_{a_1,b,\mu})^{m-j_1-|\beta_{(1)}|}[\theta_s](t, h\rho)\end{aligned}$$

on $[0, T]$ for any $\beta_{(i)}\in \boldsymbol{Z}_+^n$ $(i=1, 2, \cdots, k)$ such that $(0, \cdots, 0)\leqq\beta_{(i)}\leqq\alpha_{(i)}$ and $\langle\kappa, \beta_{(1)}+\cdots+\beta_{(i)}\rangle\leqq l_1+\cdots+l_i$, where we put $l_i=l(j_i, \alpha_{(i)})$, $a_1=a$,

$$a_i = a+l_1+\cdots+l_{i-1}-\langle\kappa, \beta_{(1)}+\cdots+\beta_{(i-1)}\rangle$$

$(i=2, \cdots, k+1)$, and $\mu>0$, $a>0$, $b>0$, $C>0$ are the same as in Corollary to Proposition 2. Here, we notice the following lemma (the proof will be given later).

LEMMA 12. *Let $c_i>0$ $(i=1, 2, \cdots, k)$, $b\geqq 0$ and $\mu>0$. Then, we have*

$$R_{c_k,b,\mu}\times\cdots\times R_{c_1,b,\mu}[\theta_s](t, \rho) \ll \frac{1}{c_1c_2\cdots c_k}\theta_s(e^{bT^\mu/\mu}\rho). \tag{7.9}$$

Hence, by applying Lemma 12 to (7.8) we obtain

$$\begin{aligned}(7.10)\quad &\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v((j_1, \alpha_{(1)}), \cdots, (j_k, \alpha_{(k)}))(t)\| \\ &\ll C_1^{k+1}t^{l_1+\cdots+l_k-\langle\kappa,\beta_{(1)}+\cdots+\beta_{(k)}\rangle}\times\frac{1}{a_{k+1}} \\ &\quad\times\frac{[((|\alpha_{(1)}|-|\beta_{(1)}|)+\cdots+(|\alpha_{(k)}|-|\beta_{(k)}|))!]^s}{a_1^{m-j_1-|\beta_{(1)}|}\times\cdots\times a_k^{m-j_k-|\beta_{(k)}|}}\theta_s(H\rho)\end{aligned}$$

for some $C_1>0$ and $H>0$.

Now, let $s<s_0<s_1$, $\varepsilon>0$ and $d\in\boldsymbol{N}$ be as in Corollary to Proposition 1. Choose $\beta_{(i)}\in\boldsymbol{Z}_+^n$ $(i=1, 2, \cdots, k)$ in (7.10) so that (i)~(iv) in Corollary to Proposition 1 are satisfied, and put p_i, $h_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, [k/d]$ $(=\max\{q\in\boldsymbol{Z}; q\leqq k/d\}))$ as follows:

$$p_i = (|\alpha_{((i-1)d+1)}|-|\beta_{((i-1)d+1)}|)+\cdots+(|\alpha_{(id)}|-|\beta_{(id)}|),$$

$$h_i = (m-j_{(i-1)d+1}-|\beta_{((i-1)d+1)}|)+\cdots+(m-j_{id}-|\beta_{(id)}|).$$

Then, we have $1\leqq p_i\leqq md$, $d\leqq h_i\leqq md$ and $(h_i/p_i)>s_0$ for any i. Since $l_1+\cdots+l_i-\langle\kappa, \beta_{(1)}+\cdots+\beta_{(i)}\rangle\geqq\varepsilon i$ and $a_i\geqq a+\varepsilon i$ for any i, by (7.10) we have

$$\begin{aligned}
&\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}v((j_1,\alpha_{(1)}),\cdots,(j_k,\alpha_{(k)}))(t)\| \\
&\ll C_2^{k+1}t^{\varepsilon k}\frac{[(p_1+p_2+\cdots+p_{[k/d]})!]^s}{(a)^{h_1}(a+\varepsilon)^{h_2}\cdots(a+([k/d]-1)\varepsilon)^{h_{[k/d]}}}\theta_s(H\rho) \qquad (7.11)\\
&\ll C_3^{k+1}\frac{[(p_1+p_2+\cdots+p_{[k/d]})!]^s}{(1)^{h_1}(2)^{h_2}\cdots([k/d])^{h_{[k/d]}}}\theta_s(H\rho)
\end{aligned}$$

for some $C_2>0$ and $C_3>0$. In addition, we have

$$\begin{aligned}
(p_1+p_2+\cdots+p_{[k/d]})! &\leqq (p_1)^{p_1}(p_1+p_2)^{p_2}\cdots(p_1+p_2+\cdots+p_{[k/d]})^{p_{[k/d]}}\\
&\leqq (md)^{p_1}(2md)^{p_2}\cdots([k/d]md)^{p_{[k/d]}}\\
&\leqq (md)^{mk}(1)^{p_1}(2)^{p_2}\cdots([k/d])^{p_{[k/d]}},
\end{aligned}$$

$h_i-sp_i=p_i((h_i/p_i)-s)\geqq p_i(s_0-s)\geqq(s_0-s)$ for any i, and hence

$$\begin{aligned}
&\frac{[(p_1+p_2+\cdots+p_{[k/d]})!]^s}{(1)^{h_1}(2)^{h_2}\cdots([k/d])^{h_{[k/d]}}}\\
&\leqq (md)^{smk}\frac{1}{(1)^{h_1-sp_1}(2)^{h_2-sp_2}\cdots([k/d])^{h_{[k/d]}-sp_{[k/d]}}} \qquad (7.12)\\
&\leqq (md)^{smk}\frac{1}{([k/d]!)^{(s_0-s)}}.
\end{aligned}$$

Therefore, by (7.11) and (7.12) we obtain (7.7). Thus, the existence part of Proposition 4 is proved.

Let $v^{(1)}(t,x)$, $v^{(2)}(t,x)$ be two solutions of $(7.1)_r$ in Proposition 4. Then, $P(t,x,t\partial_t,\partial_x)(t^r(v^{(1)}-v^{(2)}))=0$. Therefore, by the same argument as above we can show that

$$\|\nabla_r^{p,\infty}\nabla_{\kappa,r}^{m-1}(v^{(1)}-v^{(2)})(t)\| \ll K^{k+1}\frac{1}{(k!)^\nu}\theta_s(H\rho), \qquad \text{for } k=0,1,2,\cdots \qquad (7.13)$$

for some $K>0$, $\nu>0$ and $H>0$. Hence, by letting $k\to\infty$ in (7.13) we obtain $(v^{(1)}-v^{(2)})(t,x)=0$, that is, $v^{(1)}(t,x)=v^{(2)}(t,x)$ on $[0,T]\times\boldsymbol{R}^n$. Thus, the uniqueness part is also proved.

The support condition is verified by the above construction of the solution $v(t,x)=\sum_{k=0}^{\infty}v_k(t,x)$. Assume that $\mathrm{supp}(g)\subset C_\mu(0,K)$ for some $K\in\boldsymbol{R}^n$. Then, by Lemma 11 we have $\mathrm{supp}(v_k)\subset C_\mu(0,K)$ $(k=0,1,2,\cdots)$. Hence, we obtain $\mathrm{supp}(v)\subset C_\mu(0,K)$. Q.E.D.

PROOF OF LEMMA 12. This is verified by the following:

$$\begin{aligned}
&t^{c_k}e^{-bqt^\mu/\mu}R_{c_k,b,\mu}^{(q)}\times\cdots\times R_{c_1,b,\mu}^{(q)}[1](t)\\
&\leqq\int_0^t\tau_k^{c_k-c_{k-1}-1}d\tau_k\times\cdots\times\int_0^{\tau_3}\tau_2^{c_2-c_1-1}d\tau_2\int_0^{\tau_2}\tau_1^{c_1-1}d\tau_1\\
&=\frac{1}{c_1c_2\cdots c_k}t^{c_k}.
\end{aligned}$$

Q.E.D.

COROLLARY TO PROPOSITION 4. *Assume that P and s satisfy* (A_κ), (B), (C) *and* (D). *Then, for any $K \Subset \boldsymbol{R}^n$ there is an $a_K>0$ which satisfies the following condition. If $f(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ satisfies* $\mathrm{supp}(f) \subset C_\mu(0, M)$ *for some $M \subset K$ and $f(t, x)=o(t^A; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for some $A>a_K$,* (S) *(in* § 2) *has a unique solution $u(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ such that* $\mathrm{supp}(u) \subset C_\mu(0, M)$ *and $u(t, x)=o(t^B; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for any $B<A$.*

PROOF. Let $f(t, x)$ be as above and put $g(t, x)=t^{-A}f(t, x)$. Then, we can apply Proposition 4 to

$$P(t, x, t\partial_t, \partial_x)(t^A v) = t^A g \tag{7.14}$$

and obtain a solution $v(t, x)$ of (7.14). By putting $u(t, x)=t^A v(t, x)$, we have a desired solution of (S). The uniqueness is proved in the same way. Q. E. D.

PROOF OF THEOREM 1. Note that it is sufficient to show the case $A=0$. Therefore, by using Corollary to Proposition 4, we can prove Theorem 1 (with $A=0$) in the same way as Tahara [**13**, Theorem 1]. In other words, the proof of [**13**, Theorem 1] becomes a proof of Theorem 1 (with $A=0$), if we use Corollary to Proposition 4 instead of [**13**, Lemma 1] and if we replace $C^\infty(\boldsymbol{R}^n)$, $C^\infty((0, T) \times \boldsymbol{R}^n)$, $u=o(t^{\rho(x)}; \nabla^\infty)$ on $\boldsymbol{R}^n$ (as $t \to +0$), $\cdots$ by $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$, $C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$, $u=o(t^{\rho(x)}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$), $\cdots$, respectively. So, we may omit the details. Q. E. D.

§ 8. Proof of Theorem 2.

Lastly, we prove Theorem 2 (in § 2) and complete the proof of Main Theorem in the true sense.

Let $P(t, x, t\partial_t, \partial_x)$ $(=P)$ be the operator in (1.1), and let us consider

$$\begin{cases} P(t, x, t\partial_t, \partial_x)u = f, \\ \partial_t^i u|_{t=T} = 0 \quad \text{for } i=0, 1, \cdots, m-1 \end{cases} \tag{8.1}$$

in the L^2 framework. Then, we have

PROPOSITION 5. *Let $p \in \boldsymbol{N}$. Assume that P and s satisfy* (A_κ), (B′), (C′) *and* (D_p). *Then, if $f(t, x) \in C^\infty((0, T], H^\infty(\boldsymbol{R}^n))$ satisfies $\partial_t^i f|_{t=T}=0$ for $i=0, 1, \cdots, p-1$ and $t^A \|\nabla_0^{p,\infty} f(t)\| \in \mathcal{E}^{\{s\}}$ uniformly on $(0, T]$ for some $A>0$,* (8.1) *has a unique solution $u(t, x) \in C^\infty((0, T], H^\infty(\boldsymbol{R}^n))$ such that $\partial_t^i u|_{t=T}=0$ for $i=0, 1, \cdots, m+p-1$ and that $t^B \|\nabla_0^{p,\infty} \nabla_{\kappa,0}^{m-1} u(t)\| \in \mathcal{E}^{\{s\}}$ uniformly on $(0, T]$ for some $B>0$. In addition, if $f(t, x)$ satisfies* $\mathrm{supp}(f) \subset C_\mu(T, K)$ *for some $K \Subset \boldsymbol{R}^n$, $u(t, x)$ also satisfies* $\mathrm{supp}(u) \subset C_\mu(T, K)$. *Here, $C_\mu(T, K)$ is defined by the case $t_0=T$ of* (7.2).

PROOF. Let $L(t, x, t\partial_t, \partial_x)$ and $Q_{j,\alpha}(t, x, t\partial_t, \partial_x)$ ($j+|\alpha|<m$ and $|\alpha|>0$) be the same as in (7.3). Then, we can solve (8.1) by the method of successive approximations as follows. Let $u_0(t, x)\in C^\infty((0, T], H^\infty(\boldsymbol{R}^n))$ be the solution of

$$\begin{cases} L(t, x, t\partial_t, \partial_x)u_0 = f, \\ \partial_t^i u_0|_{t=T} = 0 \quad \text{for } i=0, 1, \cdots, m+p-1, \end{cases} \tag{8.2)$_0$}$$

and let $u_k(t, x)\in C^\infty((0, T], H^\infty(\boldsymbol{R}^n))$ ($k=1, 2, \cdots$) be the solution of

$$\begin{cases} L(t, x, t\partial_t, \partial_x)u_k = \sum_{\substack{j+|\alpha|<m \\ |\alpha|>0}} Q_{j,\alpha}(t, x, t\partial_t, \partial_x)u_{k-1}, \\ \partial_t^i u_k|_{t=T} = 0 \quad \text{for } i=0, 1, \cdots, m+p-1 \end{cases} \tag{8.2)$_k$}$$

($k=1, 2, \cdots$). The unique solvability of $(8.2)_k$ ($k=0, 1, 2, \cdots$) is guaranteed by the fact that $L(t, x, t\partial_t, \partial_x)$ is a regularly hyperbolic operator on $[\varepsilon, T]\times\boldsymbol{R}^n$ for any $\varepsilon>0$ (see Mizohata [**9**]). Then, by the same argument as in the proof of (7.5) we can show that

$$\|\nabla_0^{p,\infty}\nabla_{K,0}^{m-1}u_k(t)\| \ll t^{-B}K^{k+1}\frac{1}{(k!)^\nu}\theta_s(H\rho) \quad \text{on } (0, T], \quad k=0, 1, 2, \cdots \tag{8.3}$$

for some $B>0$, $K>0$, $\nu>0$ and $H>0$, by using Corollary to Proposition 3 and Lemma 13 (given below) instead of Corollary to Proposition 2 and Lemma 12, respectively. Hence, by putting $u(t, x)=\sum_{k=0}^\infty u_k(t, x)$ we obtain a desired solution $u(t, x)$ of (8.1) in Proposition 5. The other part may be proved in the same way. Q.E.D.

LEMMA 13. *Let* $c_k>0$, $c_k>c_i$ ($i=1, \cdots, k-1$), $b\geqq 0$ *and* $\mu>0$. *Then, for any* $A\geqq 0$ *we have*

$$\begin{aligned} &S_{c_k-A,b,\mu}\times\cdots\times S_{c_1-A,b,\mu}[\theta_s](t, \rho) \\ &\ll \frac{1}{(c_k-c_1)\cdots(c_k-c_{k-1})c_k}\left(\frac{T}{t}\right)^{c_k}\theta_s(e^{bT^\mu/\mu}\rho). \end{aligned}$$

PROOF. Since $S^{(q)}_{c_i-A,b,\mu}[\theta](t)\leqq S^{(q)}_{c_i,b,\mu}[\theta](t)$ for any $\theta(t)\geqq 0$, Lemma 13 is verified by the following:

$$\begin{aligned} &t^{c_k}e^{-bq(T^\mu-t^\mu)/\mu}S^{(q)}_{c_k-A,b,\mu}\times\cdots\times S^{(q)}_{c_1-A,b,\mu}[1](t) \\ &\leqq t^{c_k}e^{-bq(T^\mu-t^\mu)/\mu}S^{(q)}_{c_k,b,\mu}\times\cdots\times S^{(q)}_{c_1,b,\mu}[1](t) \\ &\leqq \int_t^T \tau_k^{c_k-c_{k-1}-1}d\tau_k\times\cdots\times\int_{\tau_3}^T \tau_2^{c_2-c_1-1}d\tau_2\int_{\tau_2}^T \tau_1^{c_1-1}d\tau_1 \\ &\leqq \int_0^T \tau_k^{c_k-c_{k-1}-1}d\tau_k\times\cdots\times\int_{\tau_3}^T \tau_2^{c_2-c_1-1}d\tau_2\int_{\tau_2}^T \tau_1^{c_1-1}d\tau_1 \end{aligned}$$

$$= \int_0^T \tau_1^{c_1-1} d\tau_1 \int_0^{\tau_1} \tau_2^{c_2-c_1-1} d\tau_2 \times \cdots \times \int_0^{\tau_{k-1}} \tau_k^{c_k-c_{k-1}-1} d\tau_k$$

$$= \frac{1}{(c_k-c_1)\cdots(c_k-c_{k-1})c_k} T^{c_k}. \qquad \text{Q.E.D.}$$

COROLLARY TO PROPOSITION 5. *Assume that P and s satisfy* (A_κ), (B), (C) *and* (D). *Then, if $f(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ satisfies* $\mathrm{supp}(f) \subset C_\mu(T, K) \cap \{t \leqq T/2\}$ *for some $K \Subset \boldsymbol{R}^n$ and $f(t, x) = o(t^{-A}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for some $A > 0$,* (S) *(in* §2) *has a unique solution $u(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ such that* $\mathrm{supp}(u) \subset C_\mu(T, K) \cap \{t \leqq T/2\}$ *and $u(t, x) = o(t^{-B}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for some $B > 0$.*

PROOF OF THEOREM 2. Assume that $u(t, x)$, $f(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ satisfy (S) and that $f(t, x)$ is tempered in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ (as $t \to +0$). Let $\rho(t) \in C^\infty(\boldsymbol{R})$ such that $\rho(t) = 1$ for $t \leqq T/4$ and $\rho(t) = 0$ for $t \geqq T/2$, and let $\phi_i(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n) \cap C_0^\infty(\boldsymbol{R}^n)$ ($i = 1, 2, \cdots$) such that $\sum_{i=1}^\infty \phi_i(x)$ is a locally finite sum and that $\sum_{i=1}^\infty \phi_i(x) = 1$ on $\boldsymbol{R}^n$. Put $(\rho u)(t, x) = \rho(t) u(t, x)$, $g_i(t, x) = \phi_i(x)(P\rho u)(t, x)$ and $K_i = \mathrm{supp}(\phi_i)$. Then, $\mathrm{supp}(g_i) \subset C_\mu(T, K_i) \cap \{t \leqq T/2\}$ and $g_i(t, x) = o(t^{-A_i}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for some $A_i > 0$. Therefore, by Corollary to Proposition 5 we have a solution $v_i(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ of $P(t, x, t\partial_t, \partial_x) v_i = g_i$ such that $\mathrm{supp}(v_i) \subset C_\mu(T, K_i) \cap \{t \leqq T/2\}$ and $v_i(t, x) = o(t^{-B_i}; \nabla^\infty, \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ (as $t \to +0$) for some $B_i > 0$. By putting $v(t, x) = \sum_{i=1}^\infty v_i(t, x)$ we obtain a solution $v(t, x) \in C^\infty((0, T), \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ of

$$\begin{cases} P(t, x, t\partial_t, \partial_x) v = (P\rho u), \\ \mathrm{supp}(v) \subset \{t \leqq T/2\}. \end{cases} \tag{8.4}$$

Here, we note that $(\rho u)(t, x)$ also satisfies (8.4) and that $P(t, x, t\partial_t, \partial_x)$ is a strictly hyperbolic operator on $(0, T) \times \boldsymbol{R}^n$. Hence, we have $(\rho u)(t, x) = v(t, x)$ on $(0, T) \times \boldsymbol{R}^n$. This immediately leads us to the conclusion of Theorem 2, because we know from the construction that $v(t, x)$ is tempered in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ (as $t \to +0$). Q.E.D.

CORRECTION. There is a serious misprint in [**12**, p. 471, $l \downarrow 4$]: $Q = \xi_1^{2(\kappa_1-\mu)} + \cdots + \xi_n^{2(\kappa_n-\mu)}$ should be read as $Q = t^{2(\kappa_1-\mu)} \xi_1^2 + \cdots + t^{2(\kappa_n-\mu)} \xi_n^2$.

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Hidetoshi TAHARA
Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku
Tokyo 102
Japan