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Anosov maps on closed topological manifolds

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Introduction.

The notion of Anosov maps given in Definition 4 is a strict generalization of expanding maps and expansive homeomorphisms with pseudo-orbit tracing property (abbrev. POTP). If in particular an Anosov map is bijective, then it has expansivity and POTP (see Remark 2 (i)). But we remark that the notion of expanding maps is not defined for homeomorphisms of compact connected metric spaces which are not one point (see Remark 2 (ii) (b)). It is known (cf. A. Morimoto [**10**]) that every homeomorphism with expansivity and POTP is topologically stable in the class of homeomorphisms. However it is impossible that a homeomorphism is topologically stable in the class of continuous surjective maps (see Remark 1). By using the same technique in [**10**] we can see that every expanding map satisfies topological stability in the class of continuous surjective maps. Thus it is natural to ask whether every Anosov map which is not bijective satisfies it in the class of continuous surjective maps.

We prove the following

THEOREM 1. *Let M be a closed topological manifold and $f: M\to M$ be a local homeomorphism but not bijective. If f is an Anosov map which satisfies topological stability in the class of continuous surjective maps, then f must be expanding.*

Let (X, d) be a compact metric space and $f: X\to X$ be a continuous surjective map. Denote by Ω the non-wandering set of f ($\Omega\neq\emptyset$ since X is compact). We obtain Smale-Bowen's decomposition theorem for an Anosov map of X as follows.

THEOREM 2. *Every Anosov map f of X has the following properties.*

(i) *$f(\Omega)=\Omega$ and $f: \Omega\to\Omega$ is an Anosov map,*

(ii) *Ω contains a finite sequence B_i $(1\leqq i\leqq l)$ of f-invariant (i.e. $f(B_i)=B_i$) closed subsets such that $\Omega=\bigcup_{i=1}^{l}B_i$ and $f: B_i\to B_i$ is topologically transitive,*

(iii) *for $1\leqq i\leqq l$, there exists $a>0$, and B_i contains a finite sequence C_{i_j} $(0\leqq j\leqq a-1)$ such that $f^a(C_{i_j})=C_{i_j}$, $C_{i_j}\cap C_{i_{j'}}=\emptyset$, $f(C_{i_j})=C_{i_{j+1}}$ for $0\leqq j\neq j'\leqq a-1$ $(C_{i_a}=C_{i_0})$, $f^a: C_{i_j}\to C_{i_j}$ is topologically mixing and $B_i=\bigcup_{j=0}^{a-1}C_{i_j}$.*

Theorem 2 was described in [1] for an expansive homeomorphism with POTP of X.

The following is obtained in proving Theorem 1.

COROLLARY. (i) *If X is connected and f is an expanding map, then $X=\Omega$.*

(ii) *Let M be a closed topological manifold and $f: M\to M$ be a local homeomorphism. If f is an Anosov map, then Ω is an infinite set.*

First of all we show in §1 that the definition of an Anosov endomorphism given by Mañé and Pugh is equivalent to that of Przytycki (see Proposition 1). After that we introduce, in topological setting, the notions of an Anosov map and a special Anosov map of X. In §2 we show that every Anosov map has POTP (see Lemma 3), and by using this property, we give the proof of Theorem 1. In §3 we give the proof of Theorem 2.

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§1. Anosov maps.

Let M be a closed C^∞-manifold and $f: M\to M$ be a C^r-map ($r\geqq 1$). Denote by $S_f(M)$ the set of all f-orbits $\{x_i\}$ ($f(x_i)=x_{i+1}$ for all $i\in \boldsymbol{Z}$) of M.

DEFINITION 1 (Mañé-Pugh [9]). Let $f: M\to M$ be an immersion. Then we say that f is a *weakly Anosov endomorphism* if there is a continuous subbundle $E^s=Df(E^s)\subset TM$ and if there are constants C, $C'>0$, $0<\mu<1<\lambda$ and a Riemannian metric $\|\cdot\|$ on TM such that for all $n>0$,

$$\|Df^n(v)\|\leqq C\mu^n\|v\| \qquad \text{for } v\in E^s,$$

$$(*) \qquad |||\overline{Df}^n(v+E^s)|||\geqq C'\lambda^n|||v+E^s||| \qquad \text{for } v+E^s\in TM/E^s$$

where $\overline{Df}$ is the induced map on TM/E^s and $|||\cdot|||$ is the induced metric on TM/E^s.

DEFINITION 2 (Przytycki [12]). Let $f: M\to M$ be an immersion. Then we say that f is an *Anosov endomorphism* if there exist constants $C>0$, $0<\nu<1$ and a Riemannian metric $\|\cdot\|$ on TM such that for all $\{x_i\}\in S_f(M)$, there is a splitting

$$T_{\{x_i\}}M=\bigcup_{i=-\infty}^{\infty}\{E^s_{x_i}\oplus E^u_{x_i}\}$$

which is preserved by Df and for all $n>0$,

$$\|Df^n_{x_i}(v)\|\leqq C\nu^n\|v\| \qquad \text{for } v\in E^s_{x_i},$$

$$\|Df^n_{x_i}(v)\|\geqq C^{-1}\nu^{-n}\|v\| \qquad \text{for } v\in E^u_{x_i}.$$

We remark that $E^u_{x_0}$ depends on f-orbit $\{x_i\}$ when f is not bijective. It is known (see [**12**, Theorem 2.15]) that $E^u_{x_0} \neq E^u_{y_0}$ when $x_0 = y_0$ but $\{x_i\} \neq \{y_i\}$. But such a phenomenon does not happen for $E^s_{x_0}$ (see [**12**, p. 250]).

For a special type of Anosov endomorphisms the following is defined.

DEFINITION 3 (Przytycki [**12**]). Let $f: M \to M$ be a C^r-map. We say that f is a *special Anosov endomorphism* if f is an Anosov endomorphism and if $E^u_{x_0} = E^u_{y_0}$ for every $\{x_i\}$, $\{y_i\} \in S_f(M)$ with $x_0 = y_0$.

In order to give the notion of Anosov maps of compact metric spaces, we first check the following

PROPOSITION 1. *Definitions* 1 *and* 2 *are equivalent.*

PROOF. Let $f: M \to M$ be a weakly Anosov endomorphism and put

$$T_{S_f(M)}M = \bigcup_{\{x_i\} \in S_f(M)} T_{\{x_i\}}M .$$

Then we see that $T_{S_f(M)}M$ is a bundle over $S_f(M)$. Let $Df^{\#}: T_{S_f(M)}M \to T_{S_f(M)}M$ be an automorphism defined by

$$Df^{\#}_{\{x_i\}}(\{v_i\}_{i=-\infty}^{\infty}) = \{Df_{x_i}(v_i)\}_{i=-\infty}^{\infty} \in T_{\{f(x_i)\}}M$$

for every $\{v_i\}_{i=-\infty}^{\infty} \in T_{\{x_i\}}M$ and every $\{x_i\} \in S_f(M)$. Define the norm of $Df^{\#}$ by

$$\|Df^{\#}\| = \sup_{\{x_i\} \in S_f(M)} \sup_{\{v_i\} \in T_{\{x_i\}}M} \sup_{i \in Z} \|Df_{x_i}(v_i)\| / \|v_i\|$$

and put

$$F^s = \bigcup_{\{x_i\} \in S_f(M)} \Big(\bigcup_{i=-\infty}^{\infty} E^s_{x_i} \Big) ,$$

$$F^{s\perp} = \bigcup_{\{x_i\} \in S_f(M)} \Big(\bigcup_{i=-\infty}^{\infty} E^{s\perp}_{x_i} \Big)$$

where $E^{s\perp}_x$ is the orthogonal subspace to E^s_x in T_xM. Then as in the proof of Proposition 5 of [**9**], there exists a splitting

$$T_{S_f(M)}M = F^s \oplus F^u$$

and $0 < \nu < 1$ such that $Df^{\#}(F^\sigma) = F^\sigma$ $(\sigma = s, u)$ and $\max\{\|Df^{\#}_{|F^s}\|, \|Df^{\#-1}_{|F^u}\|\} < \nu$. Therefore the fiber

$$T_{\{x_i\}}M = \Big(\bigcup_{i=-\infty}^{\infty} E^s_{x_i} \Big) \oplus \Big(\bigcup_{i=-\infty}^{\infty} E^u_{x_i} \Big)$$

at $\{x_i\} \in S_f(M)$ of $T_{S_f(M)}M = F^s \oplus F^u$ satisfies the condition of an Anosov endomorphism.

Conversely, let $f: M \to M$ be an Anosov endomorphism. In Definition 2 we can omit the constant C (by [**12**, Proposition 1.4]). Thus it follows from Proposition 1.7 of [12] that there is an $\alpha>0$ such that for every $x\in M$,

$$\sup\{\|w_1\|/\|w_2\| : w_1\in E_x^s,\ w_2\in E_x^{s\perp},\ w_1+w_2\in E_x^u-\{0\}\}<\alpha.$$

By an easy calculation we have that for every $x\in M$ and every subspace E_x^u,

$$\frac{1}{\sqrt{1+\alpha^2}}\|v_u\| \leqq |||v_u+E_x^s||| \leqq \|v_u\| \qquad (v_u\in E_x^s).$$

From this it is easily checked that $\overline{Df}: TM/E^s\to TM/E^s$ satisfies (*) of Definition 1. Put $E^s=\{E_x^s : x\in M\}$. Since E^s is a continuous subbundle of TM, f is a weakly Anosov endomorphism.

As before let (X, d) be a compact metric space and $f: X\to X$ be a continuous surjective map. Denote $S_f^-(X)$ by the set of all backward f-orbits of X. Let $\varepsilon>0$. We define the *local stable* set $W_\varepsilon^s(x)$ for $x\in X$ by

$$W_\varepsilon^s(x)=\{z\in X : d(f^n(z), f^n(x))\leqq\varepsilon \text{ for } n\geqq 0\}$$

and the *local unstable set* $W_\varepsilon^u(\{y_i\})$ for $\{y_i\}\in S_f^-(X)$ by

$$W_\varepsilon^u(\{y_i\})=\{z\in X \text{ such that there is } \{z_i\}\in S_f^-(X) \text{ with } z_0=z,\ d(y_i, z_i)\leqq\varepsilon \text{ for } i\leqq 0\}.$$

DEFINITION 4. We say that $f: X\to X$ is an *Anosov map* with constant $c>0$ if there exists c with the property that for every $0<\varepsilon\leqq c$ there is a $\delta>0$ such that for every $x\in X$ and every $\{y_i\}\in S_f^-(X)$, $d(x, y_0)<\delta$ implies that

$$W_\varepsilon^s(x)\cap W_\varepsilon^u(\{y_i\})=\{\text{exactly one point}\}.$$

If an Anosov map f is not bijective, then there is a case such that $W_\varepsilon^u(\{x_i\})\neq W_\varepsilon^u(\{y_i\})$ if $x_0=y_0$ but $\{x_i\}\neq\{y_i\}$. As mentioned in Definition 2, this follows easily when X and f are replaced by M and an Anosov endomorphism respectively. We generalize the type of Definition 3 as follows.

DEFINITION 5. We say that $f: X\to X$ is a *special Anosov map* if f is an Anosov map and if $W_\varepsilon^u(\{x_i\})=W_\varepsilon^u(\{y_i\})$ for every $\{x_i\}$, $\{y_i\}\in S_f^-(X)$ with $x_0=y_0$.

We remark that Definitions 4 and 5 coincide whenever f is bijective, and that these new notions are independent of metrics for X. We can easily check that f is an (special) Anosov map if and only if f^n is an (special) Anosov map for $n\geqq 2$.

An example of special Anosov maps which is not bijective is easily constructed on a solenoidal group which is not a torus. For instance, let Θ be the canonical base of $\boldsymbol{R}^n$ $(n>0)$ and, let γ be an $n\times n$ matrix with entries in $\boldsymbol{Q}$ such that $\det\gamma\neq 0$, the eigenvalues of γ are off the unit circle and $\boldsymbol{Z}^n=\mathrm{gp}\,\Theta\subsetneqq G'=\mathrm{gp}\bigcup_{j=0}^{\infty}\gamma^j\Theta$ (the notation $\mathrm{gp}\,E$ means the subgroup generated by a set E). The algebraic subgroup $G=\mathrm{gp}\bigcup_{j=-\infty}^{\infty}\gamma^j G'\subsetneqq \boldsymbol{R}^n$ is γ-invariant. We assume that G is endowed with the discrete topology. Then the dual group X of G is a solenoidal group with $\dim X=n$. Let σ denote the dual automorphism of γ_G. Then (X, σ) is expansive (see [**2**, Theorem 1]). Hence by [**2**, pp. 88-90] there are a translation invariant metric d for X and constants $\alpha_0>0$, $0<\lambda<1$ such that for $0<\varepsilon<\alpha_0$, $W_\varepsilon(0)=\{x\in X: d(x, 0)\leqq\varepsilon\}$ is expressed as $W_\varepsilon(0)=W_\varepsilon^s(0)\oplus W_\varepsilon^u(0)$ since σ is expansive and $G=\mathrm{gp}\bigcup_{j=-\infty}^{\infty}\gamma^j\Theta$ (see [**2**, P. 4]). Here $W_\varepsilon^\tau(0)$ $(\tau=s, u)$ are subsets of $W_\varepsilon(0)$ with the property:

$$d(\sigma^n(x), 0)\leqq\begin{cases}\lambda^n d(x, 0) & (x\in W_\varepsilon^s(0),\ n\geqq 0),\\ \lambda^{-n}d(x, 0) & (x\in W_\varepsilon^u(0),\ n\leqq 0).\end{cases}$$

Let F be the annihilator of $\mathrm{gp}\,\Theta$ in X and put $F^+=\bigcap_{j=0}^{\infty}\sigma^{-j}F$. Then it is easily obtained that the factor group X/F^+ is the dual group of G', a solenoidal group and not a torus. Denote by $\tilde\sigma$ the dual homomorphism of $\gamma_{G'}$. Then $\tilde\sigma: X/F^+\to X/F^+$ is finite-to-one since $\gamma: G'\to G'$ is injective and $F^+/\sigma F^+$ is finite (see [**2**, p. 3]). Finally, it is not hard to show that $\tilde\sigma$ is a special Anosov map.

A sequence of points $\{x_i\}_{i=a}^{b}$ $(-\infty\leqq a<b\leqq\infty)$ is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1})<\delta$ for $a\leqq i\leqq b-1$. A sequence of points $\{x_i\}_{i=a}^{b}$ $(a\leqq i\leqq b)$ is called to be *ε-traced* by a f-orbit $\{y_i\}_{i=a}^{b}$ if $d(x_i, y_i)<\varepsilon$ holds for $a\leqq i\leqq b$. We say that f has *pseudo-orbit tracing property* (POTP) if for every $\varepsilon>0$ there is a $\delta>0$ such that every δ-pseudo-orbit of f can be ε-traced by some f-orbit of X. It is easy to check that f^n has POTP $(n\geqq 2)$ if and only if f has POTP (cf. [**10**]).

The following is used in the proof of Theorem 2.

PROPOSITION 2. *Let (X, d) and $f: X\to X$ be as before. If f has POTP, then $f(\Omega)=\Omega$ and $f: \Omega\to\Omega$ has POTP.*

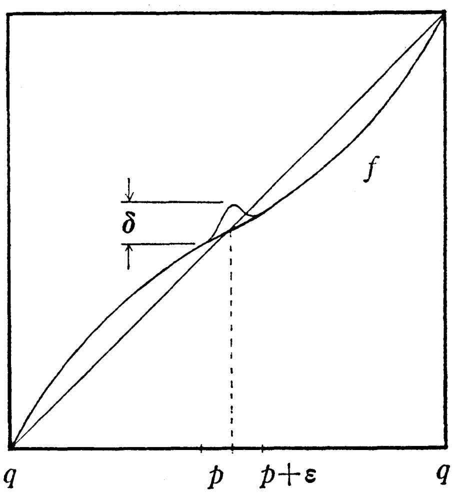
PROOF. When f is bijective, the proposition is proved in [**1**, Theorem 1]. Thus we prove the proposition to the case when f is not bijective. It is clear that $f(\Omega)\subset\Omega$. By POTP we have $f(\Omega)=\Omega$. For, assume that $\Omega\setminus f(\Omega)\neq\emptyset$. Then there are $x\in\Omega\setminus f(\Omega)$ and $\varepsilon>0$ such that $U_\varepsilon'(x)=\{y\in\Omega: d(x, y)<\varepsilon\}\subset\Omega\setminus f(\Omega)$. Let $\delta=\delta(\varepsilon/2)>0$ be as in the definition of POTP of f. Since x is in Ω, there are $n>0$ and an n-cyclic δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ such that $x_{nj}=x$ for all $j\geqq 0$. Since f has POTP, there is $y\in X$ such that $d(f^i(y), x_i)<\varepsilon/2$ for all $i\geqq 0$. Thus $\{f^{nj}(y)\}_{j=0}^{\infty}\subset U_{\varepsilon/2}(x)=\{y\in X: d(x, y)<\varepsilon/2\}$. If there is a subsequence $\{f^{nj'}(y)\}\subset\{f^{nj}(y)\}$ such that $f^{nj'}(y)\to y'\in X$ as $j'\to\infty$, then we have $y', f^n(y')\in U_\varepsilon'(x)$ since $y'\in\omega(y)$, where $\omega(y)$ denotes the ω-limit set of $y\in X$. This is a contradiction. The second statement is proved by the same method of [**1**, Theorem 1] and so we omit the proof.

DEFINITION 6. Let $\mathfrak{C}(X)$ and $\mathfrak{H}(X)$ denote the set of all continuous maps and all homeomorphisms of X onto itself respectively. For $f \in \mathfrak{C}(X)$ we say that f is *topologically stable in* $\mathfrak{C}(X)$ if for every $\varepsilon>0$ there is a $\delta>0$ with the property that for every $g \in \mathfrak{C}(X)$ with $d(f(x), g(x))<\delta$ $(x \in X)$, there is a continuous map $h: X \to X$ such that

$$\text{(i)} \quad h \circ g = f \circ h, \qquad \text{(ii)} \quad d(h(x), x) < \varepsilon \quad (x \in X).$$

Especially for $f \in \mathfrak{H}(X)$, if for every $g \in \mathfrak{H}(X)$ with $d(f(x), g(x))<\delta$ $(x \in X)$, there is a continuous map $h: X \to X$ such that the above (i) and (ii) hold, then the homeomorphism f is said to be *topologically stable in* $\mathfrak{H}(X)$.

REMARK 1. For $f \in \mathfrak{H}(X)$ it is impossible that f is topologically stable in $\mathfrak{C}(X)$. Indeed, consider S^1 as $\boldsymbol{R}/\boldsymbol{Z}$ and denote by d the standard metric on S^1. Let $f: S^1 \to S^1$ be a Morse-Smale diffeomorphism which has a sink and a source fixed points $p, q \in S^1$ respectively. It is well known that f is topologically stable in $\mathfrak{H}(S^1)$ (see [**11**]). But f is not topologically stable in $\mathfrak{C}(S^1)$. To see this, let p and q be as above and fix $0<\varepsilon<d(p, q)/4$. Denote by U an interval $[p-\varepsilon, p+\varepsilon]$ and let δ be a number of topological stability. Then we obtain $g \in \mathfrak{C}(S^1)$ (see the graph).



Clearly $g_{|S^1 \setminus U}=f$, $d(f(x), g(x))<\delta$ $(x \in S^1)$ and there are $p' \neq q' \in U \setminus \{p\}$ such that $g(p')=p'=g(q')$. Put $q'_{-1}=q'$ and define $S_g^-(S^1)$ as in Definition 4. Then we can find $\{q'_{-i}\} \in S_g^-(S^1)$ with $q'_{-i} \to q$ as $i \to \infty$. If there is $h \in \mathfrak{C}(S^1)$ which holds (i) and (ii) of Definition 6, then we have $h(q'_{-i})=h(p')$ for $i \geqq 1$. Hence $d(q'_{-i}, p') <2\varepsilon$ for $i \geqq 1$. Since $q'_{-i} \to q$ $(i \to \infty)$, we have $2\varepsilon < \lim_i d(q'_{-i}, p') \leqq 2\varepsilon$, thus contradiction.

A homeomorphism $f: X \to X$ is called to be *expansive* if there exists an $\alpha>0$ such that $d(f^n(x), f^n(y)) \leqq \alpha$ for every $n \in \boldsymbol{Z}$ implies $x=y$. We say that α is an *expansivity constant* of f.

REMARK 2. Let $f: X\to X$ be an Anosov map with expansivity constant $c>0$.

(i) If in particular f is bijective, then f is expansive and has POTP (see Definition 4 and Lemma 3).

(ii) (a) If there exists $0<e<c$ such that $W_e^s(x)=\{x\}$ for every $x\in X$, then f is an *expanding map*, that is, f is an open map and there are metric ρ for X with constants $\varepsilon>0$ and $\lambda>1$ such that $\rho(x, y)<\varepsilon$ implies $\rho(f(x), f(y))\geqq\lambda\rho(x, y)$.

(b) If in particular an expanding map is bijective, then X is a finite set.

(ii) are checked as follows. To see (a), it is enough to prove that f is open for the case when f is not bijective since f is positively expansive with an expansivity constant e (see [**13**, Theorem 1]). Fix $0<\varepsilon\leqq e$ and let $\delta>0$ be as in Definition 4. Take $y\in X$. Then for every $x\in U_\delta(f(y))$ and every $\{y_i\}\in S_f^-(X)$ with $y_0=f(y)$ and $y_{-1}=y$, we have $W_\varepsilon^u(\{y_i\})\cap W_\varepsilon^s(x)=\{x\}$. Hence there is $x_{-1}\in X$ such that $d(x_{-1}, y)<\varepsilon$ and $f(x_{-1})=x$, and so $U_\delta(f(y))\subset f(U_\varepsilon(y))$. Since ε is arbitrary, f is open.

Following the proof of [**5**, Theorem 10. 30] we obtain (b). In fact let f be a bijective expanding map and put $\Phi^-=\{f^{-i}: i\geqq 0\}$ and $\Phi^+=\{f^i: i\geqq 0\}\subset\mathfrak{H}(X)$, where $\mathfrak{H}(X)$ is endowed with metric $d(f, g)=\max\{d(f(x), g(x)): x\in X\}$ $(f, g\in\mathfrak{H}(X))$. Since f is expanding, Φ^- is uniformly equicontinuous. Define a map $G: \Phi^-\to\Phi^+$ by $G(f^{-i})=f^i$ for $f^{-i}\in\Phi^-$. Then it is easy to see that for $\alpha>0$ there is $\beta>0$ such that for every $f^{-i}\in\Phi^-$ we have $G(U_\beta(f^{-i})\cap\Phi^-)\subset U_\alpha(f^i)\cap\Phi^+$ since Φ^- is uniformly equicontinuous. Here $U_\delta(f)=\{g\in\mathfrak{H}(X): d(f, g)<\delta\}$. Since Φ^- is totally bounded (by Ascoli's theorem), there is an integer $k>0$ such that $\Phi^-=\bigcup_{n=1}^k(U_\beta(f^{-i_n})\cap\Phi^-)$. Thus we have $\Phi^+=\bigcup_{n=1}^k(U_\alpha(f^{i_n})\cap\Phi^+)$, and so Φ^+ is uniformly equicontinuous. Thus for an expansivity constant e of f, there is a $\nu>0$ such that $d(x, y)<\nu$ $(x, y\in X)$ implies $d(f^i(x), f^i(y))\leqq e$ for all $i\geqq 0$. Therefore $U_\nu(x)=\{x\}$ for $x\in X$.

§2. Proof of Theorem 1.

Let (X, d) be a compact metric space and $f: X\to X$ be an Anosov map with constant $c>0$.

LEMMA 1. *Let* $\{x_i\}, \{y_i\}\in S_f(X)$. *If* $d(x_i, y_i)\leqq c$ *for every* $i\in\boldsymbol{Z}$, *then* $x_0=y_0$.

PROOF. This follows from Definition 4.

The following is a slight extension of a result of Mañé [8].

LEMMA 2. *For all $r>0$, there is some $N_r>0$ such that*

(i) *$f^n(W_c^s(x))\subset W_r^s(f^n(x))$ for all $x\in X$ and $n\geqq N_r$,*

(ii) *if $d(x_i, y_i)\leqq c$ ($\{x_i\}, \{y_i\}\in S_f^-(X)$) for all $i\leqq 0$, then $d(x_{-n}, y_{-n})\leqq r$ for all $n\geqq N_r$.*

PROOF. If (i) is false, then we can find sequences $x^n, y^n\in X$ and $m_n\geqq n$ such that $y^n\in W_c^s(x^n)$ and $d(f^{m_n}(x^n), f^{m_n}(y^n))\geqq r$. If $f^{m_n}(x^n)\to x_0$ and $f^{m_n}(y^n)\to y_0$ when $n\to\infty$, then $d(x_0, y_0)\geqq r$. If $f^{m_n-1}(x^n)\to x_{-1}$ and $f^{m_n-1}(y^n)\to y_{-1}$ when $n\to\infty$, then $d(x_{-1}, y_{-1})\leqq c$ since $d(f^j(x^n), f^j(y^n))\leqq c$ for $0\leqq j\leqq m_n$. Clearly $f(x_{-1})=x_0$ and $f(y_{-1})=y_0$. By induction we get $\{x_i\}, \{y_i\}\in S_f^-(X)$ such that $d(x_i, y_i)\leqq c$ for all $i\leqq 0$. Therefore $y_0\in W_c^u(\{x_i\})$. On the other hand, since $y^n\in W_c^s(x^n)$, we have $d(f^j(f^{m_n}(x^n)), f^j(f^{m_n}(y^n))\leqq c$ for all $j\geqq -m_n$. Thus $y_0\in W_c^s(x_0)$ as $n\to\infty$. By Lemma 1 we have $x_0=y_0$. This is a contradiction.

Assume that (ii) is false. Then we can find sequences $\{x_i^n\}_{n=0}^\infty, \{y_i^n\}_{n=0}^\infty\in S_f^-(X)$ and $m_n\geqq n$ such that $d(x_{-m_n}^n, y_{-m_n}^n)\geqq r$ and $d(x_i^n, y_i^n)\leqq c$ for all $i\leqq 0$. As above we can derive a contradiction.

LEMMA 3. *f has POTP.*

PROOF. For $0<\mu\leqq c/2$, let $0<\nu<\mu$ be as in Definition 4. Take and fix $N_{\nu/2}>0$ as in Lemma 2 corresponding to $\nu/2$. To simplify the proof, we put $g=f^{N_{\nu/2}}$. To get the conclusion, it is enough to see that every $\nu/2$-pseudo-orbit of g is 2μ-traced by some g-orbit. Denote by $W_{\varepsilon,g}^s(x)$ the local stable set for g at $x\in X$ and by $W_{\varepsilon,g}^u(\{x_i\})$ the local unstable set for $\{x_i\}\in S_g^-(X)$. Fix $l>0$ and choose a $\nu/2$-pseudo-orbit $\{x^j\}_{j=0}^l$ of g. Since $d(g(x^1), x^2)<\nu$, there is a

$$z_0^2=W_{\mu,g}^u(g(x^1)\cup\{x_i^1\})\cap W_{\mu,g}^s(x^2),$$

where $\{x_i^1\}$ is a g-orbit with $x_0=x^1$ fixed at your will. By Lemma 2 (i) $g(z_0^2)\in W_{\nu/2,g}^s(g(x^2))$, and so $d(g(z_0^2), x^3)<\nu$. Thus there is a

$$z_0^3=W_{\mu,g}^u(g(z_0^2)\cup\{z_i^2\})\cap W_{\mu,g}^s(x^3),$$

where $\{z_i^2\}$ is a g-orbit such that $d(z_{i-1}^2, x_i)<\mu$ for $i\leqq 0$. Since $d(g(z_0^{j-1}), x^j)<\nu$ for $4\leqq j\leqq l$, there is a

$$z_0^j=W_{\mu,g}^u(g(z_0^{j-1})\cup\{z_i^{j-1}\})\cap W_{\mu,g}^s(x^j),$$

where $\{z_i^{j-1}\}$ is a g-orbit such that $d(z_{i-1}^{j-1}, z_i^{j-2})<\mu$ for $i\leqq 0$. Clearly $d(z_0^l, x^l)<2\mu$. By the choice of z_0^l there is a g-orbit $\{z_i^l\}$ such that $d(z_{-1+k}^l, z_k^{l-1})<\mu$ for all $k\leqq 0$. Hence $d(z_{-1}^l, g(z_0^{l-2}))\leqq 2\mu$ and $d(z_{-2+k}^l, z_k^{l-2})\leqq 2\mu$ for all $k\leqq 0$ since $z_0^{l-1}\in W_{\mu,g}^u(g(z_0^{l-2})\cup\{z_i^{l-2}\})$. By Lemma 2 (ii) it follows that $d(z_{-2+k}^l, z_k^{l-2})\leqq\nu/2$ for all $k\leqq 0$, and so $d(z_{-2}^l, x^{l-2})<2\mu$. In this manner we see that $\{g^j(y_l)\}_{j=0}^l$ 2μ-traces $\{x^j\}_{j=0}^l$, where $y_l=z_{-(l-1)}^l$. Thus for an arbitrary large $l>0$, every $\nu/2$-pseudo-orbit $\{x^j\}_{j=-l}^l$ is 2μ-traced by some g-orbit. Since X is compact, it is easy to see that g has POTP.

LEMMA 4. *The set of all periodic points of f, $\mathrm{per}(f)$, is dense in Ω.*

PROOF. Recall that $\Omega=\{x\in X$: for every neighborhood U of x, $f^{-n}(U)\cap U\neq\emptyset$ for some $n\geqq 1\}$. For $0<\mu\leqq c/2$, let $0<\nu<\mu$ be as in the definition of POTP of f (see Lemma 3). Take and fix $x\in\Omega$. Then there are $l>0$ and $y\in X$ such that $\{x=x_0, x_i=f^i(y)\ (1\leqq i\leqq l-1), x_l=x\}$ is a ν-pseudo-orbit of f. Put $y_{nl+i}=x_i$ $(n\geqq 0, 0\leqq i\leqq l-1)$ and $y_{nl-i}=x_{l-i}$ $(n\leqq 0, 0\leqq i\leqq l-1)$. Then $\{y_j\}_{j=-\infty}^{\infty}$ $(y_0=x)$ is a cyclic ν-pseudo-orbit of f. By Lemma 3 there is $\{z_j\}\in S_f(X)$ such that $d(y_{i+j}, f^i(z_j))<\mu$ for all $i\geqq 0$ and $j\in\boldsymbol{Z}$. Hence $d(z_{l+i}, z_i)\leqq c$ for all $i\in\boldsymbol{Z}$ and so by Lemma 1, $f^l(z_0)=z_0\in U_\mu(x)$.

Let (Y, ρ) be a metric space and g be a homeomorphism from Y onto itself. For $\varepsilon>0$ and $x\in Y$, define the local stable and unstable sets $W_\varepsilon^s(x)$, $W_\varepsilon^u(x)$ by

$$W_\varepsilon^s(x)=\{y\in Y : \rho(g^n(x), g^n(y))\leqq\varepsilon, n\geqq 0\},$$
$$W_\varepsilon^u(x)=\{y\in Y : \rho(g^n(x), g^n(y))\leqq\varepsilon, n\leqq 0\}.$$

We say that g has the *canonical coordinates* if there exists a constant $c'>0$ with the property that for each $0<\varepsilon\leqq c'$ there is a $\delta>0$ such that $\rho(x, y)<\delta$ $(x, y\in Y)$ implies

$$W_\varepsilon^s(x)\cap W_\varepsilon^u(y)=\{\text{exactly one point}\}.$$

In this case g is expansive and c' is an expansivity constant of g.

Let M be a closed topological manifold with metric d. Denote by $(\tilde{M}, \pi)$ the universal covering space of M. By Theorem 1 of [**4**] there are a complete metric ρ and an $\alpha_0>0$ such that for every $0<\alpha\leqq\alpha_0$ and every $x\in M$ an open ball $U_\alpha(x)$ is evenly covered by π and for every $u\in\tilde{M}$ the restriction to $B_{\alpha_0}(u)$ of π is an isometry, where $B_\delta(u)=\{v\in\tilde{M}: \rho(u, v)<\delta\}$. We note that there is an $\alpha_1>0$ $(\alpha_1<\alpha_0)$ such that $u\neq v$ $(u, v\in\tilde{M})$ and $\pi(u)=\pi(v)$ implies $\rho(u, v)<\alpha_1$ (cf. [**10**, Lemma 14]). If $f: M\to M$ is a local homeomorphism, then a lift $g:\tilde{M}\to\tilde{M}$ of f under π is bijective and biuniformly continuous (cf. [**10**, Lemma 14]).

LEMMA 5. *If f is an Anosov map with constant c and a local homeomorphism, then its lift $g:\tilde{M}\to\tilde{M}$ has the canonical coordinates.*

PROOF. Since M is compact and $f: M\to M$ is a local homeomorphism, there is a $\beta>0$ such that if $x\neq y$ $(x, y\in M)$ and $f(x)=f(y)$, then $d(x, y)\geqq\beta$. Let α_1 be as above and put $\alpha_2=\min\{\alpha_1, \beta\}$. Choose $0<c'\leqq\min\{c, \alpha_2/2\}$ such that $\rho(u, v)<c'$ $(u, v\in\tilde{M})$ implies $\max\{\rho(g(u), g(v)), \rho(g^{-1}(u), g^{-1}(v))\}<\alpha_2/2$. For $0<\varepsilon\leqq c'$, let $0<\delta<\varepsilon$ be as in Definition 4. If $\rho(u, v)<\delta$ $(u, v\in\tilde{M})$, then there is $z=W_\varepsilon^s(\pi(u))\cap W_\varepsilon^u(\{\pi g^i(v)\})$ (since $d(\pi(u), \pi(v))<\delta$ and $\{\pi g^i(v)\}\in S_f^-(M)$). So we take $w\in\tilde{M}$ such that $\pi(w)=z$, $\rho(w, u)\leqq\varepsilon$ and $\rho(w, v)\leqq\varepsilon$. Then $\rho(w, u)\leqq\varepsilon$ implies $\rho(g(w), g(u))\leqq\varepsilon$ (since $\rho(g(w), g(u))<\alpha_2/2$ and the restriction to $B_{\alpha_1}(g(w))$ of π is an isometry). Since $\rho(g(w), g(u))\leqq\varepsilon$, in the above manner we have $\rho(g^2(w), g^2(u))\leqq\varepsilon$ and by induction $\rho(g^n(w), g^n(u))\leqq\varepsilon$ for $n\geqq 0$. Therefore $w\in W^s_\varepsilon(u)$. We next show that $w\in W^u_\varepsilon(v)$. Since $\rho(w, u)\leqq\varepsilon$, we have $\rho(g^{-1}(w), g^{-1}(u))<\alpha_2/2$. On the other hand, since there is $z_{-1}\in M$ such that $f(z_{-1})=z$ and $d(z_{-1}, \pi g^{-1}(v))\leqq\varepsilon$, we have $\pi(w_{-1})=z_{-1}$ and $\rho(w_{-1}, g^{-1}(v))\leqq\varepsilon$ for some $w_{-1}\in\tilde{M}$. Since $d(\pi(w_{-1}), \pi g^{-1}(w))<\beta$ and $f\pi(w_{-1})=f\pi(g^{-1}(w))$, we have $\pi(w_{-1})=\pi g^{-1}(w)$ and so $w_{-1}=g^{-1}(w)$. Hence $\rho(g^{-1}(w), g^{-1}(v))\leqq\varepsilon$. By induction we see that $\rho(g^{-n}(w), g^{-n}(v))\leqq\varepsilon$ for all $n\geqq 0$. Therefore $w\in W^u_\varepsilon(v)$.

REMARK 3. Let g, ε and δ be as in Lemma 5. Since $\tilde{M}$ is the universal covering space of M and π is a local isometry, it is easily checked that for all $r>0$ there is an $N>0$ such that $g^n(W^s_\varepsilon(v))\subset W^s_r(g^n(v))$ and $g^{-n}(W^u_\varepsilon(v))\subset W^u_r(g^{-n}(v))$ for all $v\in\tilde{M}$ and $n\geqq N$. Since $\tilde{M}$ is relatively compact, by using the proof of Lemma 3 we can see that g has POTP.

Hereafter we assume that $f: M\to M$ is an Anosov map and a local homeomorphism, and $g:\tilde{M}\to\tilde{M}$ is a lift of f. Since g has the canonical coordinates (by Lemma 5), there is a c' such that for $\varepsilon_0=c'/3$ there is $0<\delta_0<\varepsilon_0$ such that $\rho(u, v)<\delta_0$ $(u, v\in\tilde{M})$ implies $W^s_{\varepsilon_0}(u)\cap W^u_{\varepsilon_0}(v)=\{\text{one point}\}$. Put

$$\Delta(\delta_0)=\{(u, v)\in\tilde{M}\times\tilde{M} : \rho(u, v)<\delta_0\}$$

and define a map $[\ ,\ ]:\Delta(\delta_0)\to\tilde{M}$ by

$$[u, v]=W^s_{\varepsilon_0}(u)\cap W^u_{\varepsilon_0}(v) \qquad \text{to } (u, v)\in\Delta(\delta_0).$$

LEMMA 6. *The map* $[\ ,\ ]:\Delta(\delta_0)\to\tilde{M}$ *is continuous and satisfies* $[[u, v], w]=[u, w]$ *and* $[u, [v, w]]=[u, w]$ *whenever the two sides of the relations are defined.*

PROOF. This follows easily from the proof of Lemma 2 of [**6**] since every bounded subset of $\tilde{M}$ is relatively compact.

LEMMA 7. *There is* $0<\delta_1<\delta_0/2$ *such that for all* $u\in\tilde{M}$, *putting*

$$W^\sigma_{\text{loc}}(u)=W^\sigma_{\varepsilon_0}(u)\cap B_{\delta_1}(u) \qquad (\sigma=s, u),$$

$$N_u=[W^u_{\text{loc}}(u), W^s_{\text{loc}}(u)],$$

(i) N_u *is an open subset of* $\tilde{M}$ *and*
(ii) $[\ ,\ ]: W^u_{\text{loc}}(u)\times W^s_{\text{loc}}(u)\to N_u$ *is a homeomorphism.*

PROOF. The conclusion is easily obtained by Proposition 3 of [**6**] and Lemma 6.

Let δ_1 be the number as in Lemma 7. For all $u\in\tilde{M}$ we denote by $W^{\sigma}_{\mathrm{com}}(u)$ the connected component of u in $W^{\sigma}_{\mathrm{loc}}(u)$ $(\sigma=s, u)$.

LEMMA 8. *Both sets* $\tilde{M}^{\sigma}=\{u\in\tilde{M}: W^{\sigma}_{\mathrm{com}}(u)=\{u\}\}$ $(\sigma=s, u)$ *are open and closed in* $\tilde{M}$.

PROOF. We prove only the case $\sigma=s$ of the lemma. The case $\sigma=u$ follows in a similar way. Let $\delta_1>0$ be as in Lemma 7. For $\delta_1/2>0$, choose $\delta_2>0$ $(2\delta_2<\delta_0)$ as in the definition of the canonical coordinates of g. Take and fix $u\in\tilde{M}$. Then we have $[u, B_{\delta_2}(u)]\subset W^{s}_{\mathrm{com}}(u)=\{u\}$. If there is $v\in B_{\delta_2}(u)$ such that $W^{s}_{\mathrm{com}}(v)\neq\{v\}$, then we choose $w\in W^{s}_{\mathrm{com}}(v)\cap B_{\delta_2}(u)\setminus\{v\}$. It is easy to see that $[u, w]=u=[u, v]$ and so $w\in W^{u}_{\varepsilon_0}(v)$. This is a contradiction because $w\in W^{s}_{\varepsilon_0}(v)$ and $w\neq v$. Therefore $\tilde{M}^{s}$ is open in $\tilde{M}$.

Next we show that $\tilde{M}^{s}$ is closed in $\tilde{M}$. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence of $\tilde{M}^{s}$ such that $u_n\to u\in\tilde{M}$ as $n\to\infty$. Since $u_n\in\tilde{M}^{s}$, we have $\{u_n\}=W^{s}_{\mathrm{com}}(u_n)\supset[u_n, B_{\delta_2}(u_n)]$ for $n\geqq 0$. To prove $u\in\tilde{M}^{s}$, if $W^{s}_{\mathrm{com}}(u)\neq\{u\}$, then for a sufficiently large $n>0$ such that $u_n\in B_{\delta_2}(u)$ there is a $w\in W^{s}_{\mathrm{com}}(u)\cap B_{\delta_2}(u_n)\setminus\{u\}$. Thus $w\in W^{s}_{\varepsilon_0}(u)$. On the other hand, $[u_n, w]=u_n=[u_n, u]$ (since w, $u\in B_{\delta_2}(u_n)$) and so $w\in W^{u}_{\varepsilon_0}(u)$. This is a contradiction.

We denote by $W^{s}_{\mathrm{com}}(\pi(u))$ the subset $\pi(W^{s}_{\mathrm{com}}(u))$ of M for $u\in\tilde{M}$.

LEMMA 9. *If f is not expanding, then* $W^{s}_{\mathrm{com}}(x)\neq\{x\}$ *for all* $x\in M$.

PROOF. Let $\tilde{M}^{s}$ be as in Lemma 8. Then $\tilde{M}^{s}$ is open and closed in $\tilde{M}$. Thus if $\tilde{M}\neq\tilde{M}^{s}$, then $\tilde{M}^{s}=\emptyset$ since $\tilde{M}$ is connected. Therefore the conclusion of the lemma is obtained. From now on we prove that $\tilde{M}^{s}=\tilde{M}$ can not happen. To do this, assume that $\tilde{M}^{s}=\tilde{M}$. For $u\in\tilde{M}$, N_u is open in $\tilde{M}$ by Lemma 7 and so locally connected. Denote by $N_{u,\mathrm{com}}$ the connected component of u in N_u. Then we have that $N_{u,\mathrm{com}}$ is open in $\tilde{M}$. Thus there is $l>0$ such that $M=\bigcup_{i=1}^{l}\pi(N_{u_i,\mathrm{com}})$. By the assumption we note that $N_{u,\mathrm{com}}=W^{u}_{\mathrm{com}}(u)$ for $u\in\tilde{M}$. Let $0<\nu'<c$ be a Lebesgue number for $\{\pi(N_{u_i,\mathrm{com}})\}_{i=1}^{l}$. Since f is not expanding, $W^{s}_{\nu'}(x)\neq\{x\}$ for some $x\in M$. So we choose $y\in W^{s}_{\nu'}(x)\setminus\{x\}$. Then there are v, $w\in N_{u_i,\mathrm{com}}$ such that $\pi(v)=x$ and $\pi(w)=y$. Therefore $y\in W^{u}_{c}(\{\pi g^{i}(v)\})\cap W^{s}_{c}(x)=\{x\}$, thus contradiction.

PROOF OF THEOREM 1. Let f and M be as in Theorem 1. For the proof we assume that an Anosov map f with constant c is topologically stable but not expanding. By Lemma 4 there is a periodic point $p_0\in M$ with period $n>0$. As before let β be a number such that $x\neq y$ and $f(x)=f(y)$ implies $d(x, y)\geqq\beta$, and take $0<\varepsilon<\min\{\beta/2, c\}$ such that $d(x, y)<\varepsilon$ $(x, y\in M)$ implies $d(f^{i}(x), f^{i}(y))<c$ for $0\leqq i\leqq n$. Let $\delta=\delta(\varepsilon)>0$ be a number with property of topological stability. For an n-cyclic f-orbit $\{p_i\}_{i=-\infty}^{\infty}$, there are $p'_{-1}\in f^{-1}(p_0)\setminus\{p_{-1}\}$ and a neighborhood U of p'_{-1} such that $\operatorname{diam} U<\beta/2$, $\operatorname{diam} f(U)<\delta$, $\{p_i\}_{i=-\infty}^{\infty}\cap U=\emptyset$ and $f_{|U}$ is a homeomorphism. By Lemma 9 there is $q_{-1}\in W_c^s(p'_{-1})\cap U\setminus\{p'_{-1}\}$.

We consider a perturbation g of f such that $f=g$ on $M\setminus U$ and $g(q_{-1})=p_0$. It is clear that $d(f(x), g(x))<\delta$ $(x\in M)$. Thus there is a continuous map $h: M\to M$ which holds (i) and (ii) of Definition 6. Since $\{p_i\}_{i=-\infty}^{\infty}\cap U=\emptyset$, it follows that $f^n(h(p_0))=h(g^n(p_0))=h(p_0)$ and so $\varepsilon>d(p_0, h(p_0))=d(f^{nj}(p_0), f^{nj}(h(p_0)))$ for all $j\geqq 0$. Thus $h(p_0)=p_0$ by Lemma 1, and so $h(q_{-1})=p'_{-1}$ (since $f(h(q_{-1}))=h(g(q_{-1}))=h(p_0)$, $f(p'_{-1})=p_0$ and $d(h(q_{-1}), p'_{-1})<\beta$). Next take $q_{-2}\in f^{-1}(q_{-1})\cap(M\setminus U)$ and put $p'_{-2}=h(q_{-2})$. Then $f(p'_{-2})=h(q_{-1})=p'_{-1}$ and $d(p'_{-2}, q_{-2})<\varepsilon$. We can construct inductively $\{q_{-i}\}_{i=1}^{\infty}$, $\{p'_{-i}\}_{i=1}^{\infty}\in S_f^-(M)$ such that $d(p'_{-i}, q_{-i})<\varepsilon$ for all $i\geqq 1$. Since $q_{-1}\in W_c^s(p'_{-1})$, we must have $p'_{-1}=q_{-1}$ by Lemma 1. This is a contradiction.

It remains to check Corollary. Let $f: X\to X$ be an expanding map of a compact connected metric space X. Then by Lemma 2 of [**14**] we have $X=\Omega$, and so (i) was proved.

Let f and M be as before. To see (ii), let f be an Anosov map with constant $c>0$. If f is expanding, then $M=\Omega$ (by (i)). When f is not expanding, assume that $\operatorname{card}\Omega$ is finite. To simplify the proof, let us put $\Omega=\{x, y\}$ (i.e. $\operatorname{card}\Omega=2$) and $g=f^2$. Thus $g(x)=x$ and $g(y)=y$. Take and fix $0<\varepsilon<\min\{c/2, d(x, y)/2\}$. Let $\delta>0$ $(\delta<\varepsilon)$ be as in the definition of POTP of g. By Lemma 9 there is $x'\in W_c^s(x)\cap U_{\delta/2}(x)\setminus\{x\}$. Fix $\{x'_i\}\in S_g^-(M)$ with $x'_0=x'$. Then by Lemma 1, it is easy to see that there is an integer $I>0$ such that $d(x'_{-I}, x)>c$. Since M is compact, there is a subsequence $\{x'_{i_j}\}_{j=0}^{\infty}\subset\{x'_i\}$ such that $x'_{i_j}\to x'_{\infty}\in\Omega$ $(j\to\infty)$. If $x'_{\infty}=x$, then there is a $k>I$ such that $d(x'_{-k}, x)<\delta/2$. Hence

$$\{\cdots, x'_{-1}, x, x'_{-k}, x'_{-k+1}, \cdots, x'_{-1}, x, x'_{-k}, \cdots\}$$

is a cyclic δ-pseudo-orbit of g. Since g has POTP, there is $z\in U_\varepsilon(x)$ such that $g^{k+1}(z)=z$ and $d(g^m(z), x'_{-k-1+m})<\varepsilon$ for $1\leqq m\leqq k$. Obviously $z=x$. Since $d(g^{k+1-I}(z), x'_{-I})<c/2$ and $d(x'_{-I}, x)>c$, we have $x\neq z$. But this can not happen and so $x'_{\infty}=y$. By the same reason, there are $y'\in W_c^s(y)\cap U_{\delta/2}(y)\setminus\{y\}$, $\{y'_i\}\in S_g^-(M)$ with $y'_0=y'$ and $l>0$ such that $d(y'_{-l}, x)<\delta/2$. Therefore

$$\{\cdots, x'_{-1}, x, y'_{-l}, y'_{-l+1}, \cdots, y'_{-1}, y, x'_{-k}, \cdots, x'_{-1}, x, y'_{-l}, y'_{-l+1}, \cdots\}$$

is a cyclic δ-pseudo-orbit of g. By using POTP of g there is $w\in U_\varepsilon(x)$ such that $g^{l+k+2}(w)=w$ and $d(g^{l+1}(w), y)<\varepsilon$. Hence $d(x, w)<\varepsilon$ implies $w=x$ and so $g^{l+1}(w)=x$. On the other hand, by the choice of $\varepsilon>0$ we have $d(x, y)>\varepsilon$. This is a contradiction, and so (ii) was proved.

§ 3. Proof of Theorem 2.

As before let (X, d) be a compact metric space and $f: X \to X$ be a continuous surjective map. Let $x, y \in X$. For $\alpha > 0$, we say that x is *α-related* to y (written $x \overset{\alpha}{\sim} y$) if there are α-pseudo-orbits of f such that $x_0 = x, x_1, \cdots, x_k = y$ and $y_0 = y, y_1, \cdots, y_l = x$ $(k, l \geqq 1)$. If $x \overset{\alpha}{\sim} y$ for every $\alpha > 0$, then we say that x is *related* to y (written $x \sim y$). If x is in Ω, then $x \sim f(x)$ (see [**1**, L. 1]). The *chain recurrent set* of f, R, is $\{x \in X : x \sim x\}$. Obviously $\Omega \subset R$. If f has POTP, then $\Omega = R$.

Hereafter let f be an Anosov map with constant c. When f is bijective, the theorem is proved in [**1**]. Thus we prove the theorem to the case when f is not bijective. Note that $\overline{\mathrm{per}(f)} = \Omega$ (by Lemma 4) and $f_{|\Omega}$ is Anosov (by Lemma 3 and Proposition 2). Thus (i) was proved. The statements (ii) and (iii) are proved by replacing the homeomorphism in the proof of [**3**, pp. 72-74] and [**1**, Theorems 2 and 3] with our map. More precisely we state the proof as follows.

Since $\Omega = R$, we split Ω into the equivalence classes B_λ under the relation $\sim$ (i.e. $\Omega = \bigcup_\lambda B_\lambda$). Then each B_λ is closed and $f(B_\lambda) = B_\lambda$ (see [**1**, p. 330]).

LEMMA 10. *Each B_λ is open in Ω.*

PROOF. For $0 < \varepsilon < c$, let $\delta > 0$ be as in the definition of POTP of $f_{|\Omega}$. Then for $p \in U_\delta'(B_\lambda) \cap \mathrm{per}(f)$, there is $y \in B_\lambda$ such that $d(y, p) < \delta$, where $U_\delta'(B_\lambda) = \{y \in \Omega : d(y, B_\lambda) < \delta\}$. Since $f_{|\Omega}$ has POTP, we have $W_\varepsilon^u(\{p_i\}) \cap W_\varepsilon^s(y) \cap \Omega \neq \emptyset$ for a cyclic f-orbit $\{p_i\}_{i=0}^\infty$ with $p = p_0$. Thus $p \in B_\lambda$ (since $x \sim f(x)$ for all $x \in \Omega$), and so B_λ is open in Ω (see [**1**, L. 4]).

Since Ω is compact and each B_λ is open in Ω, there is a $k > 0$ such that $\Omega = \bigcup_{i=1}^k B_i$. It is easily checked that $f(B_i) = B_i$ and $f_{|B_i}: B_i \to B_i$ is topologically transitive (see [**1**, L. 5]). We remark that $f_{|B_i}$ has POTP since B_i is open and closed in Ω. Fix B_i and put

$$V^s(x) = \{y \in B_i : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$$

for $x \in B_i$. Notice that $f(V^s(x)) = V^s(f(x))$.

LEMMA 11. *Put $C_p = \overline{V^s(p)}$ for $p \in \mathrm{per}(f) \cap B_i$, then C_p is open in B_i.*

PROOF. Let $\varepsilon > 0$ and $\delta > 0$ be as in Lemma 10. We assume that $f^m(p) = p$ for some $m > 0$. Take and fix $q \in U_\delta'(\overline{V^s(p)}) \cap B_i \cap \mathrm{per}(f)$ with $f^n(q) = q$ for some $n > 0$. Then there is $x \in V^s(p)$ such that $d(x, q) < \delta$. Hence for an n-cyclic f-orbit $\{q_i\}_{i=0}^{-\infty}$ $(q_0 = q)$ we have $W_\varepsilon^u(\{q_i\}) \cap W_\varepsilon^s(x) \neq \emptyset$. Since $\{q_i\}$ is cyclic, $q \in \overline{V^s(p)}$. Therefore C_p is open in B_i.

LEMMA 12. *If* $q\in C_p\cap\mathrm{per}(f)$, *then* $C_p=C_q$.

PROOF. We assume that $f(p)=p$ and $f(q)=q$. For $r>0$, let $N_r>0$ be as in Lemma 2. Fix $0<\varepsilon\leqq c$ and let $\delta>0$ be as in the definition of POTP of $f_{|B_i}$. For every $x\in V^s(q)$ there is $J_r\geqq N_r$ such that $d(q, f^{J_r}(x))<\delta/2$. Then for $y\in U'_{\delta/2}(q)\cap V^s(p)\cap B_i$ we have

$$W_\varepsilon^u(\{\cdots, x, f(x), \cdots, f^{J_r}(x)\})\cap W_\varepsilon^s(y)\cap B_i\neq\emptyset .$$

Since $r>0$ is arbitrary, $x\in\overline{V^s(p)}$; i.e. $C_p\subset C_q$. On the other hand, by [1, L. 7] we have $p\in C_q$. Therefore $C_p=C_q$.

Let C_p be as in Lemma 11. Since $f^m(p)=p$ for some $m>0$, we have $C_{f^m(p)}=C_p$. If $a>0$ is the smallest integer such that $C_{f^a(p)}=C_p$, then $f^i(C_p)\cap f^j(C_p)=\emptyset$ for $0\leqq i\neq j\leqq a-1$. For, if $f^{i'}(C_p)\cap f^{j'}(C_p)\neq\emptyset$ for some $0\leqq i'<j'\leqq a-1$, then $f^{a-j'}(f^{i'}(C_p)\cap f^{j'}(C_p))\subset f^{a-j'+i'}(C_p)\cap C_p\subset C_{f^{a-j'+i'}(p)}\cap C_p$ and have $C_p=C_{f^{a-j'+i'}(p)}$ (by Lemmas 11 and 12). This is inconsistent with the choice of a. Thus we have $B_i=C_p\cup f(C_p)\cup\cdots\cup f^{a-1}(C_p)$.

LEMMA 13. *A map* $f^a: C_p\to C_p$ *is topologically mixing.*

PROOF. Let U, V be nonempty open subsets of C_p. Then by Lemma 11 there is $q\in V\cap\mathrm{per}(f)$ with $f^n(q)=q$. For $\varepsilon>0$ with $U'_\varepsilon(q)\subset V$, let $0<\delta<\varepsilon$ be a number such that $d(f^{aj}(x), f^{aj}(y))<\varepsilon$ for $1\leqq j\leqq n-1$ whenever $d(x, y)<\delta$ $(x, y\in C_p)$. According to Lemma 11, for every $1\leqq j\leqq n-1$ there are $z_j\in U\cap V^s(f^{aj}(q))$ and $N_j>0$ such that $t\geqq N_j$ implies $d(f^{ant}(z_j), f^{ant}(q))<\delta$. Hence $f^{a(nt+n-j)}(U)\cap V\neq\emptyset$ for every $t\geqq N_j$ and $0\leqq j\leqq n-1$. Put $N=\max\{N_j: 0\leqq j\leqq n-1\}$. Then $s\geqq nN$ implies $f^{as}(U)\cap V\neq\emptyset$.

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