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On abstract parabolic fundamental solutions

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Introduction.

It is the object of this paper to construct the evolution operator associated with the abstract evolution equation

$$\dot{u}+A(t)u=f(t)\,,\qquad 0<t\leqq T\,, \tag{1}$$

in a Banach space E and to prove some theorems concerning the existence and uniqueness of solutions to the corresponding Cauchy problem. The given function f and the unknown u map $[0, T]$ into E, each $-A(t)$ is the infinitesimal generator of a strongly continuous analytic semigroup on E, and the dot denotes the derivative with respect to t. Thus we consider evolution equations of parabolic type and we are interested in the case where the domain $D(A(t))$ of $A(t)$ varies with t.

This problem has already been studied by several authors. In particular Kato and Tanabe [**11**] established the existence of a fundamental solution (an evolution operator) U for (1) under the assumption that the resolvent of $-A(t)$ has a Hölder continuous derivative and satisfies an estimate of the form

$$\|([\lambda+A(t)]^{-1})^{\cdot}\|\leqq N/|\lambda|^{\rho} \tag{2}$$

for some constant $\rho\in(0, 1]$. More recently Yagi [**25**] has shown that it suffices to assume that

$$[\lambda+A(\cdot)]^{-1}\in C^{1}([0, T], \mathcal{L}(E))\,, \tag{3}$$

where $\mathcal{L}(E)$ is the Banach algebra of all continuous linear operators on E, provided condition (2) is somewhat strengthened (cf. also [**26**]).

Consider now quasilinear parabolic evolution equations of the form

$$\dot{u}+A(t, u)u=f(t, u)\,,\qquad 0<t\leqq T\,. \tag{4}$$

A natural way to solve this equation consists in trying to find fixed points of the map $v\mapsto u(v)$, where v is a function from $[0, T]$ into E and $u(v)$ is the solution of the linearized problem

$$\dot{u}+A(t, v(t))u = f(t, v(t)), \qquad 0<t\leqq T$$

In trying to apply this method to problems coming from concrete parabolic differential equations, it turns out that assumption (3) is so restrictive that it does not give a well defined fixed point map $v\mapsto u(v)$. Hence, if one wants to study the quasilinear equation (4) by this fixed point method, one has to establish the existence of an evolution operator under weaker continuity requirements than (3).

This has been achieved by Sobolevskii [**20**] and Kato [**10**] under the assumption that $D([A(t)]^{\beta})$ is constant for some $\beta\in(0, 1)$, where Kato assumes that $1/\beta$ is a positive integer. Due to results of Seeley [**17, 18**] this condition is satisfied for parabolic differential equations under rather general conditions.

In this paper we give an alternative proof for the existence of an evolution operator for (1), provided $D([A(t)]^{\beta})$ is constant for some $\beta\in(0, 1)$, $A(\cdot)$ is Hölder continuous with exponent $\rho\in(1-\beta, 1)$ in an appropriate sense, and certain additional requirements are satisfied. The precise hypotheses and results are given in Section 4. In the last section of this paper it is shown that our hypotheses are satisfied by large classes of parabolic equations and systems.

Our proof is quite different from the methods of Kato and Sobolevskii. The principal idea is to construct an appropriate extension $\tilde{A}(t)$ of the operator $A(t)$, defined on some Banach space $\tilde{E}\supset E$, such that $D(\tilde{A}(t))$ is constant. Then, by using the results of Sobolevskii [**19**] and Tanabe [**21, 22**] for evolution equations *with constant domain*, we obtain an evolution operator $\tilde{U}$ on $\tilde{E}$ for the extended evolution equation. It is then shown that $\tilde{U}$ restricts to an evolution operator on E for (1).

This construction has been motivated by a result of Tanabe [**22**, Section 5.4], who used such a restriction argument in the case where $A(t)$ is a regularly accretive operator in a Hilbert space. However in that case the superspace $\tilde{E}$ and the extension $\tilde{A}(t)$ are given quite naturally, whereas in our general setting we have to employ an abstract construction to find $\tilde{E}$ and $\tilde{A}(t)$.

In Sections 1 to 3 we present a general abstract method to construct natural extensions of strongly continuous semigroups. These results are basic for our construction of the evolution operator, which is achieved in Section 5. The results of Sections 1-3 are also of independent interest and have further applications which are not discussed in this paper. In Section 6 we give two general existence theorems for the Cauchy problem corresponding to (1). Although we indicate in Section 7 the applicability of our abstract results to parabolic initial boundary value problems, in order to keep this paper in a reasonable length, we do not discuss specific applications to (nonlinear) parabolic equations, which are of our primary interest.

As already mentioned above, the general methods of this paper have further applications. For example, in a forthcoming paper they will be used to study semilinear parabolic systems under nonlinear boundary conditions. Moreover, it is relatively easy to carry out a precise analysis of the dependence of the evolution operator upon the family $\{A(t) \mid 0 \leqq t \leqq T\}$ if these operators have constant domain (cf. [5]). Hence the techniques of this paper allow quite easily to extend those estimates to the general situation considered in the present paper. This fact is important for the study of quasilinear parabolic systems under "moving" boundary conditions (of Neumann type, for example), as will be shown in another publication.

In a recent preprint Acquistapace and Terreni [1] consider also the Cauchy problem for linear time-dependent parabolic evolution equations under the assumption that some (real) interpolation space between E and $D(A(t))$ is independent of t. They do not construct a fundamental solution but derive existence and ("maximal") regularity results by means of representation formulas.

Throughout this paper we use standard notation. All abstract Banach spaces are complex spaces. The real case can be handled by complexification. If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from X into Y, and $\mathcal{L}(X) := \mathcal{L}(X, X)$. Moreover, $\mathrm{Isom}(X, Y)$ is the open set of all isomorphisms in $\mathcal{L}(X, Y)$. Finally we refer to [**8**, **9**, **16**] for the basic facts about semigroups of linear operators which we use freely throughout.

1. Fractional power spaces.

Let $(E, \|\cdot\|)$ be a normed vector space. Recall that E is isometrically isomorphic (that is, norm isomorphic) to a dense linear subspace of a Banach space $\tilde{E}$, which is unique up to norm isomorphisms. More precisely, $\tilde{E}$ can be constructed as a Banach space, whose elements are equivalence classes $\widetilde{(x_j)}$ of Cauchy sequences in E, where two Cauchy sequences (x_j) and (y_j) are equivalent if $\|x_j - y_j\| \to 0$, and where $\|\widetilde{(x_j)}\| := \lim \|x_j\|$. Then E is norm isomorphic to the linear subspace consisting of all constant sequences in E (cf. [**27**] for details). *We identify E with this subspace* and call $\tilde{E}$ *the completion* of E, so that E is dense in $\tilde{E}$.

Suppose that $\|\cdot\|_1$ is another norm on E such that $\|\cdot\|$ is weaker than $\|\cdot\|_1$, that is,

$$E_1 \hookrightarrow E,$$

where $E_1 := (E, \|\cdot\|_1)$ and $\hookrightarrow$ means that the natural injection is continuous. Then it follows from the above that

$$E \overset{d}{\hookrightarrow} \tilde{E}_1 \overset{d}{\hookrightarrow} \tilde{E},$$

where $\tilde{E}_1$ is the completion of E_1 and the letter d indicates dense imbedding.

Let now E be a Banach space. We write $A \in \mathcal{G}(E, M, \omega)$ if $-A$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{-tA} \mid t \geqq 0\}$ on E (that is, in $\mathcal{L}(E)$) such that

$$\|e^{-tA}\| \leqq Me^{\omega t} \qquad \forall t \geqq 0 .$$

Let $A \in \mathcal{G}(E, M, \omega)$ with $\omega < 0$ be given. Then we define the *scale of fractional power spaces* $E^\alpha := E^\alpha(A)$, $\alpha \in \boldsymbol{R}$, *of* A as follows:

$$\|x\|^{(\alpha)} := \|A^\alpha x\| \qquad \forall x \in D(A^\alpha),\ \alpha \in \boldsymbol{R},$$

and

$$E^\alpha := (D(A^\alpha), \|\cdot\|^{(\alpha)}) \qquad \text{if } \alpha \geqq 0 ,$$

whereas

$$E^\alpha \text{ is the completion of } (E, \|\cdot\|^{(\alpha)}) \quad \text{if } \alpha < 0 .$$

Observe that $E^0 = E$ and that $\|\cdot\|^{(\alpha)}$ is equivalent to the graph norm of A^α, if $\alpha > 0$, which implies the completeness of E^α, if $\alpha > 0$. The following proposition is an easy consequence of the properties of the fractional powers (for which we refer to [**12**, **13**, **14**, **16**]).

PROPOSITION 1.1. (i) *If* $\alpha > \beta$, *then* $E^\alpha \overset{d}{\hookrightarrow} E^\beta$.

(ii) A^α *induces naturally* (*that is, by restriction, if* $\alpha > 0$, *and by continuous extension, if* $\alpha < 0$) *a norm isomorphism from* $E^{\alpha+\beta}$ *onto* E^β.

(iii) *If* $\beta > \alpha > 0$, *then* E^β *is a core for* A^α (*that is*, A^α *is the closure of* $A^\alpha | E^\beta$).

Suppose now that E is reflexive. Then

$$A' \in \mathcal{G}(E', M, \omega) \quad \text{and} \quad e^{-tA'} = (e^{-tA})' , \tag{1}$$

where $'$ denotes the "duality functor". Hence the *dual scale*

$$(E')^\alpha := (E')^\alpha(A') , \qquad \alpha \in \boldsymbol{R} ,$$

is well defined.

LEMMA 1.2. $(A^\alpha)' = (A')^\alpha$.

PROOF. Let $\alpha > 0$. Then $(A^{-\alpha})' = (A')^{-\alpha}$ follows easily from

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA} dt$$

and (1). Since $A^\alpha = (A^{-\alpha})^{-1}$, we see now that $(A^\alpha)' = [(A^{-\alpha})^{-1}]' = [(A^{-\alpha})']^{-1} = [(A')^{-\alpha}]^{-1} = (A')^\alpha$. $\square$

In the following we denote by $\langle \cdot, \cdot \rangle : E' \times E \to \boldsymbol{K}$ the duality pairing and by $\boldsymbol{B}_X$ the open unit ball in the normed vector space X.

Suppose that $\alpha > 0$. Since $(A')^{-\alpha}$ is a norm isomorphism of E' onto $(E')^\alpha$, it follows from $(E')^{\alpha}\hookrightarrow E'$ and Lemma 1.2 that

$$\sup\{|\langle x', x\rangle| \mid x'\in \boldsymbol{B}_{(E')^{\alpha}}\}=\sup\{|\langle (A')^{-\alpha}y', x\rangle| \mid y'\in \boldsymbol{B}_{E'}\}$$
$$=\sup\{|\langle y', A^{-\alpha}x\rangle| \mid y'\in \boldsymbol{B}_{E'}\}=\|x\|^{(-\alpha)}$$

for all $x\in E$. If $\alpha<0$, we obtain analogously that

$$\sup\{|\langle x', x\rangle| \mid x'\in \boldsymbol{B}_{(E')^{\alpha}}\cap E'\}=\sup\{|\langle (A')^{-\alpha}y', x\rangle| \mid y'\in \boldsymbol{B}_{E'}\cap(E')^{-\alpha}\}$$
$$=\sup\{|\langle y', A^{-\alpha}x\rangle| \mid y'\in \boldsymbol{B}_{E'}\cap(E')^{-\alpha}\}=\|x\|^{(-\alpha)}$$

for all $x\in E^{-\alpha}$, since $(E')^{-\alpha}$ is dense in E'. This implies that

$$|\langle x', x\rangle|\leqq\|x'\|_{(E')^{\alpha}}\|x\|^{(-\alpha)}\qquad \forall x'\in(E')^{\alpha}\cap E',\quad x\in E^{-\alpha}\cap E\,. \tag{2}$$

Since $(E')^{\alpha}\cap E'$ is dense in $(E')^{\alpha}$ and $E^{-\alpha}\cap E$ is dense in $E^{-\alpha}$ for each $\alpha\in\boldsymbol{R}$, we see from (2) that the bilinear form $\langle\cdot,\cdot\rangle$ extends continuously to a bilinear form on $(E')^{\alpha}\times E^{-\alpha}$, which we denote again by $\langle\cdot,\cdot\rangle$. Thus

$$|\langle x', x\rangle|\leqq\|x'\|_{(E')^{\alpha}}\|x\|^{(-\alpha)}\qquad \forall x'\in(E')^{\alpha},\quad x\in E^{-\alpha} \tag{3}$$

and

$$\|x\|^{(-\alpha)}=\sup\left\{\frac{|\langle x', x\rangle|}{\|x'\|_{(E')^{\alpha}}}\;\middle|\; x\in(E')^{\alpha}\setminus\{0\}\right\}\qquad \forall x\in E^{-\alpha} \tag{4}$$

for each $\alpha\in\boldsymbol{R}$. In particular we deduce from (3) that

$$(E')^{\alpha}\hookrightarrow(E^{-\alpha})'\qquad \forall\alpha\in\boldsymbol{R}\,. \tag{5}$$

In fact, more is true.

THEOREM 1.3. *Let E be reflexive. Then E^{α} is reflexive and $(E')^{\alpha}=(E^{-\alpha})'$ for every $\alpha\in\boldsymbol{R}$.*

PROOF. Since E^{α} is isomorphic to E, the reflexivity of E^{α} is a consequence of the reflexivity of E.

Similarly as we obtained (4) we deduce from (5) and Lemma 1.2 that

$$\|x'\|_{(E^{-\alpha})'}=\sup\{|\langle x', x\rangle| \mid x\in\boldsymbol{B}_{E^{-\alpha}}\}=\sup\{|\langle x', A^{\alpha}y\rangle| \mid y\in\boldsymbol{B}_{E}\}$$
$$=\sup\{|\langle (A^{\alpha})'x', y\rangle| \mid y\in\boldsymbol{B}_{E}\}=\|(A^{\alpha})'x'\|_{E'}=\|x'\|_{(E')^{\alpha}}$$

for all $x'\in(E')^{\alpha}$. Hence $(E')^{\alpha}$ is a closed linear subspace of $(E^{-\alpha})'$.

Suppose now that $z\in((E^{-\alpha})')'$ vanishes on $(E')^{\alpha}$. Then $z\in E^{-\alpha}$, by the reflexivity of $E^{-\alpha}$, and $\langle y', z\rangle=0$ for all $y'\in(E')^{\alpha}$. Hence $z=0$ by (4), which shows that $(E')^{\alpha}$ is dense in $(E^{-\alpha})'$. $\square$

2. The induced semigroups.

Let X and Y be Banach spaces such that $Y \hookrightarrow X$, and let $A: D(A) \subset X \to X$ be a linear operator in X. Then the *Y-realization* A_Y *of* A (the "part" of A in Y, or the "maximal restriction" of A to Y) is defined by

$$D(A_Y) := \{y \in Y \cap D(A) \mid Ay \in Y\}, \qquad A_Y y := Ay .$$

Clearly A_Y is closed in Y if A is closed in X.

Suppose now that E is a Banach space and $A \in \mathcal{G}(E, M, \omega)$ for some $\omega < 0$. Then we define A_α by:

$$A_\alpha \text{ is the } E^\alpha\text{-realization of } A, \text{ if } \alpha \geqq 0 ,$$

and

$$A_\alpha \text{ is the closure of } A \text{ in } E^\alpha, \text{ if } \alpha < 0 .$$

The following theorem implies in particular that A_α is well defined if $\alpha < 0$.

THEOREM 2.1. *$A_\alpha \in \mathcal{G}(E^\alpha, M, \omega)$ and*

$$e^{-tA_\alpha} = e^{-tA} | E^\alpha , \qquad \text{if } \alpha > 0 ,$$

and

$$e^{-tA_\alpha} \text{ is the continuous extension of } e^{-tA} \text{ over } E^\alpha, \text{ if } \alpha < 0.$$

PROOF. Since

$$A^\alpha e^{-tA} \supset e^{-tA} A^\alpha \tag{1}$$

and since E is dense in E^α, if $\alpha < 0$, it follows easily that $\{e^{-tA} \mid t \geqq 0\}$ induces naturally a strongly continuous semigroup $\{e^{-tB_\alpha} \mid t \geqq 0\}$ on E^α. Moreover

$$\|t^{-1}(e^{-tA}x - x) + Ax\|^{(\alpha)} = \|t^{-1}(e^{-tA}A^\alpha x - A^\alpha x) + A(A^\alpha x)\| \longrightarrow 0$$

as $t \to 0$, for every $x \in E^\beta$, where $\beta := \max\{1, 1+\alpha\}$. Hence $B_\alpha \supset A | E^\beta$. Since E^β is dense in E^α and invariant under $\{e^{-tB_\alpha} \mid t \geqq 0\}$, it follows from the core theorem (e.g. [8, Theorem 1.9]) that E^β is a core for B_α. If $\alpha > 0$, the fact that A induces an isomorphism from $E^{1+\alpha}$ onto E^α implies easily that $A | E^{1+\alpha}$ is closed in E^α. Hence $B_\alpha = A | E^{1+\alpha}$, if $\alpha > 0$. Since $A | E^{1+\alpha}$ is an isomorphism from $E^{1+\alpha}$ onto E^α, it is clear that B_α is the E^α-realization of A, that is, $B_\alpha = A_\alpha$, if $\alpha > 0$. If $\alpha < 0$, then B_α is the closure of A in E^α, by the core theorem. Hence $B_\alpha = A_\alpha$ for each $\alpha \in \boldsymbol{R}$. Finally (1) implies trivially that $A_\alpha = B_\alpha \in \mathcal{G}(E^\alpha, M, \omega)$. $\square$

COROLLARY 2.2. *$D(A_\alpha) = E^{1+\alpha}$ and $A_\alpha \in \mathrm{Isom}(E^{1+\alpha}, E^\alpha)$. Moreover $E^{1+\beta}$ is a core for A_α if $\beta > \alpha$.*

PROOF. The first two assertions follow from the above proof. The last one is an easy consequence of $E^{1+\beta}\overset{d}{\hookrightarrow}E^{1+\alpha}$ and $A_\alpha\in\mathrm{Isom}(E^{1+\alpha}, E^\alpha)$. $\square$

LEMMA 2.3. *Let $\sigma>\omega$ and suppose that*

$$\|(\lambda+A)^{-1}\|_{\mathcal{L}(E)}\leqq N/(1+|\lambda-\sigma|) \qquad \textit{for } \mathrm{Re}\,\lambda\geqq\sigma\,.$$

Then

$$\|(\lambda+A_\alpha)^{-1}\|_{\mathcal{L}(E^\alpha)}\leqq N/(1+|\lambda-\sigma|) \qquad \textit{for } \mathrm{Re}\,\lambda\geqq\sigma\,.$$

PROOF. Theorem 2.1 and the Hille-Yosida theorem imply that

$$\{\lambda\in\boldsymbol{C}\mid\mathrm{Re}\,\lambda\geqq\sigma\}\subset\rho(-A_\alpha) \qquad \forall\alpha\in\boldsymbol{R}\,.$$

Moreover we obtain from Theorem 2.1 that

(2) $$(\lambda+A_\alpha)^{-1}=(\lambda+A)^{-1}|E^\alpha \qquad \text{for } \alpha>0\,.$$

Hence, by using the density of E in E^α, if $\alpha<0$, it follows that

$$\|(\lambda+A_\alpha)^{-1}x\|^{(\alpha)}=\|A^\alpha(\lambda+A_\alpha)^{-1}x\|=\|(\lambda+A_\alpha)^{-1}A^\alpha x\|\leqq\|(\lambda+A)^{-1}\|_{\mathcal{L}(E)}\|x\|^{(\alpha)}$$

for all $x\in E^\alpha$, which implies the assertion. $\square$

THEOREM 2.4. *If $-A$ generates an analytic semigroup on E, then so does $-A_\alpha$ on E^α for each $\alpha\in\boldsymbol{R}$. Moreover, $e^{-tA_\alpha}(E^\alpha)\subset E^\beta$ and*

$$\|e^{-tA_\alpha}\|_{\mathcal{L}(E^\alpha,E^\beta)}\leqq c(\alpha,\beta,\sigma)t^{\alpha-\beta}e^{\sigma t}$$

for $t>0$, $\alpha<\beta$, and $\sigma>\omega$.

PROOF. The first part follows from Lemma 2.3 and the well known characterization of strongly continuous analytic semigroups. It is well known that

$$\|A^k e^{-tA}\|_{\mathcal{L}(E)}\leqq c(k,\sigma)t^{-k}e^{\sigma t}$$

for $k\in\boldsymbol{N}$, $\sigma>\omega$, and $t>0$. Now the second assertion is an easy consequence of Theorem 2.1 and the moment inequality (e.g. [**14**, Theorem 1.5.2]). $\square$

Our next theorem, which we include for completeness, implies that the semigroup $\{e^{-t(A')_\alpha}\mid t\geqq0\}$ on $(E')^\alpha$ is the dual semigroup of the semigroup $\{e^{-tA_{-\alpha}}\mid t\geqq0\}$ on $E^{-\alpha}$, provided E is reflexive.

THEOREM 2.5. *Let E be reflexive. Then $(A')_\alpha=(A_{-\alpha})'$ for each $\alpha\in\boldsymbol{R}$.*

PROOF. From Theorem 1.3 we know that $(E^{-\alpha})'=(E')^\alpha$ and that $E^{-\alpha}$ is reflexive. Consequently $(E')^\alpha$ is also reflexive. Thus, by using repeatedly (1.1) and Theorem 2.1, it follows that

$$\begin{aligned}\langle e^{-t(A_{-\alpha})'}x',x\rangle&=\langle[e^{-tA_{-\alpha}}]'x',x\rangle=\langle x',e^{-tA_{-\alpha}}x\rangle=\langle x',e^{-tA}x\rangle\\&=\langle[e^{-tA}]'x',x\rangle=\langle e^{-tA'}x',x\rangle=\langle e^{-t(A')_\alpha}x',x\rangle\end{aligned}$$

for all $x\in E^{-\alpha}\cap E$ and all $x'\in(E')^{\alpha}\cap E'$. Hence, by a density argument, $e^{-t(A_{-\alpha})'}=e^{-t(A')_\alpha}$ for each $t\geqq 0$, which implies the assertion. □

REMARK 2.6. Suppose that $A\in\mathcal{G}(E, M, \omega)$ with $\omega\in\boldsymbol{R}$. Then $\lambda+A\in\mathcal{G}(E, M, \omega-\lambda)$ and $e^{-(\lambda+A)t}=e^{-\lambda t}e^{-At}$ for $t\geqq 0$ and each $\lambda\in\boldsymbol{R}$. Hence we can construct the scale of fractional power spaces $E^{\alpha}(\lambda+A)$, $\alpha\in\boldsymbol{R}$, for each $\lambda>\omega$. Then it can be shown that $E^{\alpha}(\lambda+A)=E^{\alpha}(\mu+A)$, up to equivalent norms, for λ, $\mu>\omega$. Thus the assumption that $\omega<0$ is no real restriction. □

It should be noted that Tanabe [**23**] showed recently in a very particular concrete situation that $-A_{-1/2}$ generates an analytic semigroup on $E^{-1/2}$. He considered second order elliptic operators (in an L_p-setting) and defined $A_{-1/2}$ directly by means of a Dirichlet form and a duality argument (cf. the considerations in Section 7 below). In particular in [**23**] there is no general abstract construction of A_α for $\alpha<0$.

3. Fractional power spaces and complex interpolation.

In this section we characterize the fractional power spaces as complex interpolation spaces, provided a certain additional assumption is satisfied.

We suppose throughout that E is a Banach space and that $A\in\mathcal{G}(E, M, \omega)$ for some $\omega<0$.

Let $\alpha\in\boldsymbol{R}$ be fixed and put $F:=E^{\alpha}$ and $B:=A_\alpha$. Then $B\in\mathcal{G}(F, M, \omega)$ by Theorem 2.1 and we can define the scale of fractional power spaces

$$F^{\beta}:=F^{\beta}(B)\,, \qquad \beta\in\boldsymbol{R}\,.$$

The following proposition relates the scale F^{β}, $\beta\in\boldsymbol{R}$, to the scale E^{α}, $\alpha\in\boldsymbol{R}$.

PROPOSITION 3.1. *$F^{\beta}(A_\alpha)=E^{\alpha+\beta}$ and $(A_\alpha)_\beta=A_{\alpha+\beta}$ for all α, $\beta\in\boldsymbol{R}$.*

PROOF. According to Corollary 2.2 $B=A_\alpha\in\mathrm{Isom}(E^{\alpha+1}, E^{\alpha})$. Clearly B is the isomorphism induced by A according to Proposition 1.1 (ii). This implies that $F^{k}=E^{\alpha+k}$ and $B^{k}\in\mathrm{Isom}(E^{\alpha+k}, E^{\alpha})$ is the isomorphism induced by A^{k} for every $k\in\boldsymbol{N}$. Let $\gamma:=\max\{k, k+\alpha\}$, so that $E^{\gamma}=E^{k}\cap E^{k+\alpha}$, where $k\in\boldsymbol{N}$ satisfies $k>\beta$. Then $B^{k}x=A^{k}x$ for each $x\in E^{\gamma}$ and

$$\begin{aligned} B^{\beta}x=B^{\beta-k}B^{k}x&=\frac{1}{\Gamma(k-\beta)}\int_0^\infty t^{k-\beta-1}e^{-tB}B^{k}x\,dt\\ &=\frac{1}{\Gamma(k-\beta)}\int_0^\infty t^{k-\beta-1}e^{-tA}A^{k}x\,dt=A^{\beta}x\,, \end{aligned}$$

due to Theorem 2.1. Hence

$$\|x\|_{F^{\beta}}=\|B^{\beta}x\|^{(\alpha)}=\|A^{\beta}x\|^{(\alpha)}=\|A^{\alpha+\beta}x\|=\|x\|^{(\alpha+\beta)} \qquad \forall x\in E^{\gamma}\,.$$

Since E^{γ} is dense in $E^{\alpha+\beta}$ and in F^{β} by Proposition 1.1 (i), it follows that $F^{\beta}=E^{\alpha+\beta}$. It is now clear that $(A_{\alpha})_{\beta}=A_{\alpha+\beta}$. $\square$

Observe that Proposition 3.1 implies in particular that A is the E-realization of A_{α} for every $\alpha<0$.

It is well known that A^{z} can be defined for every $z\in\boldsymbol{C}$. In particular,

$$A^{it}x=\frac{1}{\Gamma(1+it)\Gamma(1-it)}\int_{0}^{\infty}\lambda^{it}A(\lambda+A)^{-2}xd\lambda \qquad \forall x\in E^{1},\quad t\in\boldsymbol{R} \tag{1}$$

(e.g. [**13**, **24**]). Similarly we can define $(A_{\alpha})^{it}$ for each $t\in\boldsymbol{R}$ by replacing A in the above integral by A_{α} and E^{1} by $E^{\alpha+1}$. Hence it follows from Theorem 2.1 (cf. in particular (2.2)), that

$$A^{it}x=(A_{\alpha})^{it}x \qquad \forall x\in E^{1}\cap E^{\alpha+1},\quad t\in\boldsymbol{R}. \tag{2}$$

Using this fact we can prove the following

LEMMA 3.2. *Suppose that there are constants $\varepsilon>0$ and a such that $A^{it}\in\mathcal{L}(E)$ and $\|A^{it}\|\leqq a$ for $|t|\leqq\varepsilon$. Then $(A_{\alpha})^{it}\in\mathcal{L}(E^{\alpha})$ and $\|(A_{\alpha})^{it}\|_{\mathcal{L}(E^{\alpha})}\leqq a$ for $|t|\leqq\varepsilon$ and every $\alpha\in\boldsymbol{R}$.*

PROOF. It follows from (1) that $A^{\alpha}(A^{it}x)=A^{it}(A^{\alpha}x)$ for each $x\in E^{\alpha}\cap E^{1}$. Hence (2) implies

$$\|(A_{\alpha})^{it}x\|^{(\alpha)}=\|A^{\alpha}(A_{\alpha})^{it}x\|=\|A^{it}(A^{\alpha}x)\|\leqq a\|x\|^{(\alpha)}$$

for $|t|\leqq\varepsilon$ and all $x\in E^{\alpha+1}\cap E^{1}$. Now the assertion follows from the density of $E^{\alpha+1}\cap E^{1}$ in E^{α}. $\square$

We denote by $[\cdot,\cdot]_{\theta}$, $0<\theta<1$, the complex interpolation functor and we refer to [**6**, **24**] for the basic facts about interpolation theory which we shall use below.

We can now prove the main result of this section, namely

THEOREM 3.3. *Suppose that there exist constants $\varepsilon>0$ and a such that $A^{it}\in\mathcal{L}(E)$ and $\|A^{it}\|\leqq a$ for $|t|\leqq\varepsilon$. Then, up to equivalent norms,*

$$[E^{\alpha}, E^{\beta}]_{\theta}=E^{\alpha(1-\theta)+\beta\theta}$$

for $0<\theta<1$ and $-\infty<\alpha<\beta<\infty$.

PROOF. Let $F:=E^{\alpha}$ and $B:=A_{\alpha}$. Then it follows from Lemma 3.2 and [**24**, Theorem 1.15.3] that $[F, F^{\beta-\alpha}]_{\theta}=F^{\theta(\beta-\alpha)}$, up to equivalent norms. Hence we obtain the assertion from Proposition 3.1. $\square$

REMARK 3.4. Let $A\in\mathcal{G}(E, M, \omega)$ with $\omega<0$. Then we know that $A_{-1}\in\mathcal{G}(E^{-1}, M, \omega)$, that $A_{1}\in\mathcal{G}(E^{1}, M, \omega)$, and that $E^{1}\overset{d}{\hookrightarrow}E\overset{d}{\hookrightarrow}E^{-1}$. Suppose now that for each $\theta\in(0, 1)$ there is given an interpolation functor $\mathfrak{F}_\theta$ of exponent θ (say the complex interpolation functor $[\cdot, \cdot]_\theta$ or a real interpolation functor $(\cdot, \cdot)_{\theta,p}$, $1\leqq p\leqq\infty$) and let $E_\theta := \mathfrak{F}_\theta(E, E^1)$ and $E_{\theta-1} := \mathfrak{F}_\theta(E^{-1}, E)$. Suppose that E^1 is dense in E_θ and E is dense in $E_{\theta-1}$ (which is the case if $\mathfrak{F}_\theta := [\cdot, \cdot]_\theta$ or $\mathfrak{F}_\theta := (\cdot, \cdot)_{\theta,p}$, $1\leqq p<\infty$). Then

$$E^1 \overset{d}{\hookrightarrow} E_\theta \overset{d}{\hookrightarrow} E \overset{d}{\hookrightarrow} E_{\theta-1} \overset{d}{\hookrightarrow} E^{-1}$$

for $0<\theta<1$. Similarly as in the proof of Theorem 2.1 it is easy to verify that the maximal restriction $A_{[\zeta]}$ of A_{-1} to E_ζ, $\zeta\in\{\theta, \theta-1\}$ belongs to $\mathcal{G}(E_\zeta, M, \omega)$, and that $e^{-tA_{[\zeta]}}$ is the restriction of $e^{-tA_{-1}}$ to E_ζ. Moreover $\{e^{-tA_{[\zeta]}} \mid t\geqq0\}$ is an analytic semigroup on E_ζ if $\{e^{-tA} \mid t\geqq 0\}$ is analytic on E (cf. also [**13**, Theorem 4.3] and the proof of [**3**, Lemma 10.1]). However in general E will not be an interpolation space between $E_{\theta-1}$ and E_θ.

It should be noted that E^{-1} is an "extrapolation space" in the terminology of Da Prato and Grisvard [7] and that our construction is much simpler than the one in [7]. $\square$

4. Parabolic fundamental solutions.

In the following T is a fixed positive number, $\dot{T}_\Delta := \{(t, s)\in\boldsymbol{R}^2 \mid 0\leqq s<t\leqq T\}$, and T_Δ is the closure of $\dot{T}_\Delta$ in $\boldsymbol{R}^2$. Moreover $\Sigma_\vartheta := \{z\in\boldsymbol{C} \mid |\arg z|\leqq\vartheta+\pi/2\}$ for $0\leqq\vartheta\leqq\pi/2$, and $\rho(A)$ denotes the resolvent set of the linear operator A.

We impose the *assumption* (A):

> $\{A(t) \mid 0\leqq t\leqq T\}$ is a family of closed and densely defined linear operators in the Banach space E such that $\rho(-A(t))\supset\Sigma_0$ and that there exists a constant M with
>
> $$\|(\lambda+A(t))^{-1}\| \leqq M/(1+|\lambda|)\,, \qquad \lambda\in\Sigma_0\,,$$
>
> for all $t\in[0, T]$.

Thus each $-A(t)$ is the infinitesimal generator of a strongly continuous analytic semigroup $\{e^{-sA(t)} \mid s\geqq0\}$ on E, and there exist constants $M_0\geqq1$ and $\omega<0$ such that $A(t)\in\mathcal{G}(E, M_0, \omega)$ for all $t\in[0, T]$. Hence the scales of fractional power spaces

$$E^\alpha(t) := E^\alpha(A(t))\,, \qquad \alpha\in\boldsymbol{R}\,,$$

are well defined for each $t\in[0, T]$.

In the following we write $X\doteq Y$ if X and Y are normed linear spaces which coincide as vector spaces and carry equivalent norms (that is, $X\hookrightarrow Y$ and $Y\hookrightarrow X$). Then we can impose *assumption* $(\mathrm{C})_\beta$:

There exists a number $\beta \in (0, 1)$ such that

$$E^{\beta}(t) \doteq E^{\beta}(0) =: E^{\beta}$$

and

$$E^{\beta-1}(t) \doteq E^{\beta-1}(0) =: E^{\beta-1}$$

for all $t \in [0, T]$.

The following proposition contains a useful sufficient condition for condition $(\mathrm{C})_{\beta}$ to be satisfied.

PROPOSITION 4.1. *Let E be a reflexive Banach space and let assumption* (A) *be satisfied. Suppose that there exists a number $\beta \in (0, 1)$ such that*

(1) $$D(A^{\beta}(t)) = D(A^{\beta}(0))$$

and

(2) $$D((A')^{1-\beta}(t)) = D((A')^{1-\beta}(0))$$

for $0 \leqq t \leqq T$. Then assumption $(\mathrm{C})_{\beta}$ is satisfied.

PROOF. It follows from (1) and the closed graph theorem that $E^{\beta}(t) \doteq E^{\beta}$. Letting

$$(E')^{\alpha}(t) := (E')^{\alpha}(A'(t)), \qquad \alpha \in \boldsymbol{R}, \quad t \in [0, T],$$

we deduce from (2) that $(E')^{1-\beta}(t) \doteq (E')^{1-\beta}(0) =: (E')^{1-\beta}$. Hence, by Theorem 1.3,

$$E^{\beta-1}(t) = [(E')^{1-\beta}(t)]' \doteq [(E')^{1-\beta}]' = E^{\beta-1}. \qquad \square$$

Next we impose the *assumption* $(\mathrm{CI})_{1-\beta}$:

$$E \doteq [E^{\beta-1}, E^{\beta}]_{1-\beta}.$$

The following proposition gives an important sufficient condition for $(\mathrm{CI})_{1-\beta}$ to be satisfied.

PROPOSITION 4.2. *Let assumption* (A) *be satisfied. Suppose that there are positive constants ε and a such that $A^{i\tau}(0) \in \mathcal{L}(E)$ and $\|A^{i\tau}(0)\| \leqq a$ for $-\varepsilon \leqq \tau \leqq \varepsilon$. Then assumption $(\mathrm{CI})_{1-\beta}$ is satisfied.*

PROOF. This follows from Theorem 3.3. $\square$

REMARK 4.3. Given the assumptions of Proposition 4.2 and assumption $(\mathrm{C})_{\beta}$, it follows from Theorem 3.3 that $E^{\alpha}(t) \doteq E^{\alpha}(0)$ for $\beta-1 \leqq \alpha \leqq \beta$ and $0 \leqq t \leqq T$. Thus, in particular, $D(A^{\alpha}(t))$ is independent of t for $0 \leqq \alpha \leqq \beta$. If, moreover, E is reflexive, then we deduce from Theorem 1.3 that also $D((A'(t))^{\alpha})$ is independent of t for $0 \leqq \alpha \leqq 1-\beta$. $\square$

Finally we impose a Hölder continuity assumption upon the family $\{A(t) \mid 0 \leqq t \leqq T\}$, namely *assumption* $(\mathrm{H})_\rho$:

$$A_{\beta-1}(\cdot) \in C^\rho([0, T], \mathcal{L}(E^\beta, E^{\beta-1})) \qquad \text{for some } \rho \in (1-\beta, 1).$$

This implies the existence of a constant L such that

$$\|A_{\beta-1}(s) - A_{\beta-1}(t)\|_{\mathcal{L}(E^\beta, E^{\beta-1})} \leqq L|s-t|^\rho \qquad \forall s, t \in [0, T].$$

Lemma 2.3 and the smoothness of the inversion $B \mapsto B^{-1}$ from $\mathrm{Isom}(X, Y)$ onto $\mathrm{Isom}(Y, X)$, where X and Y are Banach spaces, imply

$$[(s, t) \mapsto A_{\beta-1}(s)[A_{\beta-1}(t)]^{-1}] \in C([0, T]^2, \mathcal{L}(E^{\beta-1})).$$

Hence there exists a constant N such that

$$\|A_{\beta-1}(s)[A_{\beta-1}(t)]^{-1}\|_{\mathcal{L}(E^{\beta-1})} \leqq N \qquad \forall s, t \in [0, T].$$

If X and Y are Banach spaces we denote by $\mathcal{L}_s(X, Y)$ the space of all continuous linear operators from X to Y, endowed with the strong topology, that is, the topology of pointwise convergence. Moreover $\mathcal{L}_s(X) := \mathcal{L}_s(X, X)$.

After these preparations we can formulate the following theorem, which, together with the theorems of Section 6 below, constitutes the main result of this paper.

THEOREM 4.4. *Let assumptions* (A), $(\mathrm{C})_\beta$, $(\mathrm{CI})_{1-\beta}$ *and* $(\mathrm{H})_\rho$ *be satisfied. Then there exists a unique function* $U: T_\Delta \to \mathcal{L}(E)$ *possessing the following properties:*

(U1) $U \in C(T_\Delta, \mathcal{L}_s(E)) \cap C(\dot{T}_\Delta, \mathcal{L}(E))$.

(U2) $U(t, t) = \mathrm{id}$ *and* $U(t, s) = U(t, \tau)U(\tau, s)$ *for* $0 \leqq s \leqq \tau \leqq t \leqq T$.

(U3) $R(U(t, s)) \subset D(A(t))$ *for* $(t, s) \in \dot{T}_\Delta$,

$[(t, s) \mapsto A(t)U(t, s)] \in C(\dot{T}_\Delta, \mathcal{L}(E))$ *and*

$\|A(t)U(t, s)\| \leqq c_0/(t-s)$ *for* $(t, s) \in \dot{T}_\Delta$.

Moreover

$U(\cdot, s) \in C^1((s, T], \mathcal{L}(E))$ *for* $0 \leqq s < T$ *and*

$D_1U(t, s) = -A(t)U(t, s)$ *for* $(t, s) \in \dot{T}_\Delta$.

(U4) $(U \mid E^\beta)(t, \cdot) \in C^1([0, t), \mathcal{L}_s(E^\beta, E))$ *for* $0 < t \leqq T$ *and*

$D_2U(t, s)x = U(t, s)A(s)x$ *for* $(t, s) \in \dot{T}_\Delta$ *and* $x \in D(A(s))$.

(U5) $[(t, s) \mapsto A(t)U(t, s)A^{-1}(s)] \in C(T_\Delta, \mathcal{L}_s(E))$.

Finally

$$\|U(t, s)\| \leqq c_1 \quad \textit{and} \quad \|A(t)U(t, s)A^{-1}(s)\| \leqq c_2$$

for all $(t, s)\in T_\Delta$, *and the constants* c_0, c_1 *and* c_2 *depend only upon* L, M, N, T, β *and* ρ, *but not upon the individual operators* $A(t)$, $0\leqq t\leqq T$.

In general a function $U: T_\Delta\to\mathcal{L}(E)$ is said to be a *parabolic fundamental solution for* $\{A(t) \mid 0\leqq t\leqq T\}$ *on* E provided it satisfies (U1)-(U4), where E^β can be replaced by any subspace F of E such that $D(A(t))\subset F$ for $0\leqq t\leqq T$. If $f:[0, T]\to E$ then by a *solution* of the linear evolution equation

$$\dot{u}+A(t)u = f(t)\,, \qquad 0<t\leqq T\,,$$

we mean a function $u\in C([0, T], E)\cap C^1((0, T], E)$ such that $u(t)\in D(A(t))$ and $\dot{u}(t)+A(t)u(t)=f(t)$ for $0<t\leqq T$. If, in addition, $u(0)=x$, then u is said to be a solution of the *(linear) Cauchy problem*

$$(\mathrm{CP})_x \qquad \dot{u}+A(t)u = f(t)\,, \qquad 0<t\leqq T, \quad u(0)=x\,.$$

If u is a solution of the Cauchy problem $(\mathrm{CP})_x$ with $x\in E$, and if $f\in C([0, T], E)$, then it is easily seen that

$$(3) \qquad u(t) = U(t, 0)x+\int_0^t U(t, \tau)f(\tau)d\tau\,, \qquad 0\leqq t\leqq T\,,$$

where U is any function satisfying (U1), (U3) and (U4) (with E^β replaced by F, as above) (cf. [**22**, Theorem 5.2.2]). Thus (U1), (U3) and (U4) imply already the uniqueness of U as well as the fact that $(\mathrm{CP})_x$ has for each $f\in C([0, T], E)$ and each $x\in E$ at most one solution. Moreover since the homogeneous Cauchy problem $\dot{u}+A(t)u=0$, $0<t\leqq T$, $u(0)=x$ has for every $x\in E$ at most one solution, we see that (U2) is a consequence of (U1), (U3) and (U4).

5. Proof of Theorem 4.4.

Let X and Y be Banach spaces. Then we denote, for each $\alpha\in\boldsymbol{R}$, by $\mathcal{K}(X, Y, \alpha)$ the Banach space of all functions $k\in C(\dot{T}_\Delta, \mathcal{L}(X, Y))$ satisfying

$$\|k\|_{(\alpha)} := \sup_{(t,s)\in T_\Delta}(t-s)^\alpha\|k(t, s)\| < \infty\,,$$

endowed with the norm $\|\cdot\|_{(\alpha)}$, and $\mathcal{K}(X, \alpha) := \mathcal{K}(X, X, \alpha)$. It is easily seen that

$$(1) \qquad \mathcal{K}(X, Y, \beta) \hookrightarrow \mathcal{K}(X, Y, \alpha) \qquad \text{for } \beta<\alpha\,,$$

and that

$$(2) \qquad \mathcal{K}(X, Y, \alpha) \hookrightarrow C(T_\Delta, \mathcal{L}(X, Y)) \qquad \text{if } \alpha<0\,,$$

provided each $k\in\mathcal{K}(X, Y, \alpha)$ is extended over T_Δ by letting $k(t, t)=0$ for

$0 \leqq t \leqq T$.

For $k \in \mathcal{K}(X, Y, \alpha)$ and $h \in \mathcal{K}(Y, Z, \beta)$ with $\alpha, \beta < 1$ we let

$$h*k(t, s) := \int_s^t h(t, \tau)k(\tau, s)d\tau, \qquad (t, s) \in \dot{T}_\Delta .$$

Then it is not difficult to see that

$$h*k \in \mathcal{K}(X, Z, \alpha+\beta-1) \tag{3}$$

and that

$$\|h*k\|_{(\alpha+\beta-1)} \leqq B(1-\alpha, 1-\beta)\|h\|_{(\beta)}\|k\|_{(\alpha)}, \tag{4}$$

where $B(\cdot, \cdot)$ is the beta function (cf. [5, Lemma 1.1]).

Throughout this section we use the following simplifying notation: whenever U is a function of two real variables and V is a function of one real variable, we write

$$VU(t, s) := V(t)U(t, s) \qquad \text{and} \qquad UV(t, s) := U(t, s)V(s),$$

provided the right hand sides are meaningful.

We presuppose now the assumptions (A), $(\mathrm{C})_\beta$, $(\mathrm{CI})_{1-\beta}$ and $(\mathrm{H})_\rho$. In the following we denote by c constants, which may be different from formula to formula, but are always independent of the specific independent variables occurring at a given place. These constants can depend upon the constants L, M, N, T, β and ρ, but they do not depend upon the individual operators $A(t)$, $0 \leqq t \leqq T$. Usually this fact will be easy to verify so that we do not give details.

For $(t, s) \in \dot{T}_\Delta$ we put $B(t) := A_{\beta-1}(t)$ and

$$a(t, s) := e^{-(t-s)B(s)}, \qquad k(t, s) := -[B(t)-B(s)]a(t, s).$$

Then

$$a \in C(T_\Delta, \mathcal{L}_s(E^{\beta-1})) \cap \mathcal{K}(E^{\beta-1}, 0) \tag{5}$$

and $k \in \mathcal{K}(E^{\beta-1}, 1-\rho)$ (cf. [5, Lemma 2.1]). Hence, by [5, Theorem 1.2] there exists a unique solution $U \in \mathcal{K}(E^{\beta-1}, 0)$ of the "convolution type equation" $U = a + U*k$, which is given by

$$U = a + a*w, \tag{6}$$

where the "resolvent kernel"

$$w \in \mathcal{K}(E^{\beta-1}, 1-\rho) \tag{7}$$

is the unique solution of

$$w = k + k*w \ (= k + w*k). \tag{8}$$

The function U is precisely the unique parabolic fundamental solution for $\{B(t) \mid 0 \leqq t \leqq T\}$ on $E^{\beta-1}$, constructed by Sobolevskii [**19**] and Tanabe [**21, 22**] (cf. [**5**, Section 3]). Hence

$$U \in C(T_\Delta, \mathcal{L}_s(E^{\beta-1})) \cap \mathcal{K}(E^{\beta-1}, 0) \cap \mathcal{K}(E^\beta, 0) \cap \mathcal{K}(E^{\beta-1}, E^\beta, 1). \tag{9}$$

Thus, by interpolation,

$$U \in \mathcal{K}(E, 0) \cap \mathcal{K}(E^{\beta-1}, E, 1-\beta). \tag{10}$$

Assumption $(\mathrm{CI})_{1-\beta}$ implies the "moment inequality"

$$\|x\|_E \leqq c\|x\|_{E^{\beta-1}}^{\beta}\|x\|_{E^\beta}^{1-\beta}, \qquad x \in E^\beta, \tag{11}$$

(e.g. [**24**, Theorem 1.9.3]). Hence we deduce from (9)-(11) and the density of E^β in E that

$$U \in C(T_\Delta, \mathcal{L}_s(E)). \tag{12}$$

In the following we let $Z := \{\beta-1, 0, \beta\}$.

LEMMA 5.1. $\|(\lambda+B(t))^{-1}\|_{\mathcal{L}(E^\zeta, E^\eta)} \leqq c|\lambda|^{\eta-\zeta-1}$ *for* $0 \leqq t \leqq T$ *and* $\eta, \zeta \in Z$ *with* $\zeta \leqq \eta$, *and for* $\lambda \in \Sigma_\vartheta$.

PROOF. If $\eta=\zeta$, the assertion follows from (A) and Lemma 2.3. Since $B(t)(\lambda+B(t))^{-1} = 1-\lambda(\lambda+B(t))^{-1}$ we see that

$$\begin{aligned}\|(\lambda+B(t))^{-1}\|_{\mathcal{L}(E^{\beta-1}, E^\beta)} &\leqq \|B(0)(\lambda+B(t))^{-1}\|_{\mathcal{L}(E^{\beta-1})} \\ &\leqq N\|B(t)(\lambda+B(t))^{-1}\|_{\mathcal{L}(E^{\beta-1})} \leqq c,\end{aligned}$$

due to Lemma 2.3. The remaining cases are obtained by interpolation. □

LEMMA 5.2. $\|B^j(t)e^{-sB(t)}\|_{\mathcal{L}(E^\zeta, E^\eta)} \leqq cs^{\zeta-\eta-j}$ *for* $s>0$, $0 \leqq t \leqq T$, $j=0, 1$, *and* $\eta, \zeta \in Z$ *with* $\zeta \leqq \eta$.

PROOF. Since

$$B^j(t)e^{-sB(t)} = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^j e^{\lambda s}(\lambda+B(t))^{-1} d\lambda$$

for $j=0, 1$, the assertion is an easy consequence of Lemma 5.1. □

Lemma 5.2 implies in particular that

$$a \in \mathcal{K}(E^{\beta-1}, E, 1-\beta) \cap \mathcal{K}(E, E^\beta, \beta) \cap \mathcal{K}(E, 0).$$

Hence we deduce from $(\mathrm{H})_\rho$ that $k \in \mathcal{K}(E, E^{\beta-1}, \beta-\rho)$. Now it follows from (8), (1) and (3) that

$$w \in \mathcal{K}(E, E^{\beta-1}, \beta-\rho). \tag{13}$$

Since U is the parabolic fundamental solution for $\{B(t) \mid 0 \leqq t \leqq T\}$ on $E^{\beta-1}$, we know that $U(\cdot, s) \in C^1((s, T], \mathcal{L}(E^{\beta-1}))$, that

$$D_1 U = -BU = D_1 a + D_1(a*w)\,, \tag{14}$$

and that

$$\begin{aligned} D_1(a*w)(t, s) \\ = e^{-(t-s)B(t)} w(t, s) + \int_s^t [B(t)e^{-(t-\tau)B(\tau)} - B(\tau)e^{-(t-\tau)B(\tau)}] w(\tau, s) d\tau \\ + \int_s^t B(t) e^{-(t-\tau)B(t)} [w(t, s) - w(\tau, s)] d\tau \end{aligned} \tag{15}$$

for $(t, s) \in \dot{T}_\Delta$ (cf. [**22**, formula (5.29)]). In order to estimate the last term in (15) we need the following

LEMMA 5.3. *Suppose that* $0 < \gamma < \rho$. *Then*

$$\begin{aligned} &\|w(t, s) - w(\tau, s)\|_{\mathcal{L}(E, E^{\beta-1})} \\ &\leqq c(\gamma) \Big\{ (t-\tau)^\rho (\tau - s)^{-\beta} + \int_\tau^t (t-\sigma)^{\rho-1} (\sigma - s)^{\rho-\beta} d\sigma + (t-\tau)^\gamma (\tau - s)^{2\rho-\gamma-\beta} \Big\} \end{aligned}$$

for $0 \leqq s < \tau < t \leqq T$.

PROOF. This follows from the estimates of Lemma 5.2 by obvious modifications of the proof of [**22**, Lemma 5.4.2] (where R corresponds to w and R_1 to k). □

Observe that, due to (13) and Lemma 5.2, the first summand in (15) belongs to $\mathcal{K}(E, 1-\rho)$. Since

$$B(t)e^{-(t-s)B(t)} - B(s)e^{-(t-s)B(s)} = \frac{1}{2\pi i} \int_\Gamma \lambda e^{\lambda(t-s)} [(\lambda + B(t))^{-1} - (\lambda + B(s))^{-1}] d\lambda\,,$$

and since

$$\begin{aligned} &\|(\lambda + B(t))^{-1} - (\lambda + B(s))^{-1}\|_{\mathcal{L}(E^{\beta-1}, E)} \\ &\leqq \|(\lambda + B(t))^{-1}\|_{\mathcal{L}(E^{\beta-1}, E)} \|B(s) - B(t)\|_{\mathcal{L}(E^\beta, E^{\beta-1})} \|(\lambda + B(s))^{-1}\|_{\mathcal{L}(E^{\beta-1}, E^\beta)} \\ &\leqq c|t-s|^\rho |\lambda|^{-\beta} \end{aligned}$$

by Lemma 5.1, we see that the second summand in (15) belongs to $\mathcal{K}(E, 1-\rho)$. Finally, by means of Lemma 5.3 it is easily verified that the last summand in (15) belongs also to $\mathcal{K}(E, 1-\rho)$. Thus, since $D_1 a(t, s) = -B(s)e^{-(t-s)B(s)}$, we deduce from (14) and Lemma 5.2 that

$$D_1 U \in \mathcal{K}(E, 1)\,, \tag{16}$$

and, due to the fact that $A(t)$ is the E-realization of $B(t)$, that

(17) $$R((U\,|\,E)(t,s)) \subset D(A(t)) \qquad \text{for } (t,s)\in \dot{T}_\Delta\,,$$

and that

(18) $$D_1 U = -AU\,.$$

Consequently

(19) $$U(\cdot, s) \in C^1((s,T], \mathcal{L}(E))\,, \qquad 0\leqq t<T\,.$$

We put now

$$b(t,s) := e^{-(t-s)B(t)} \qquad \text{and} \qquad h(t,s) := b(t,s)[B(t)-B(s)]$$

for $(t,s)\in T_\Delta$. Then it is not difficult to see that

(20) $$b \in C(T_\Delta, \mathcal{L}_s(E^\beta)) \cap \mathcal{K}(E^\beta, 0)\,,$$

(21) $$h \in \mathcal{K}(E^\beta, 1-\rho)\,,$$

and $Bh \in \mathcal{K}(E^\beta, E, 2-\beta-\rho)$ (cf. [**5**, Lemmas 2.1 and 3.1] and Lemma 5.2). Consequently,

(22) $$BhB^{-1} \in \mathcal{K}(E^{\beta-1}, E, 2-\beta-\rho)\,.$$

Moreover, by integrating the identity

$$\frac{\partial}{\partial \tau}[e^{-(t-\tau)B(t)}U(\tau,s)] = h(t,\tau)U(\tau,s)\,, \qquad 0\leqq s<\tau<t\leqq T\,,$$

it follows that U satisfies the equation

(23) $$U = b + h * U\,.$$

Since $BbB^{-1} \in \mathcal{K}(E^{\beta-1}, 0)$, we deduce from (23) that

(24) $$BUB^{-1} = BbB^{-1} + B(h*U)B^{-1} = BbB^{-1} + (BhB^{-1})*(BUB^{-1})\,,$$

where the last "convolution" is meaningful due to (22) and $\rho>1-\beta$. But

(25) $$BbB^{-1}(t,s) = e^{-(t-s)B(s)} + [B(t)e^{-(t-s)B(t)} - B(s)e^{-(t-s)B(s)}]B^{-1}(s)$$

and

$$\begin{aligned}&[B(t)e^{-(t-s)B(t)} - B(s)e^{-(t-s)B(s)}]B^{-1}(s)\\ &= \frac{-1}{2\pi i}\int_\Gamma \lambda e^{\lambda(t-s)}(\lambda+B(t))^{-1}[B(s)-B(t)](\lambda+B(s))^{-1}B^{-1}(s)d\lambda\,.\end{aligned}$$

By Lemma 5.1 the norm in $\mathcal{L}(E)$ of the last term can be estimated by

$$c\int_\Gamma |\lambda|\,|e^{\lambda(t-s)}|\,|\lambda|^{-\beta}|s-t|^\rho|\lambda|^{-1}|d\lambda| = c|t-s|^{\rho+\beta-1}\,.$$

Since $\rho>1-\beta$, we see that the last term in (25) belongs to $C(T_\Delta, \mathcal{L}(E))$. Hence we obtain

$$BbB^{-1} \in C(T_{\Delta}, \mathcal{L}_s(E)) \tag{26}$$

from (25). Now it follows from $E \hookrightarrow E^{\beta-1}$, (22), (24), (26) and [**5**, Theorem 1.2] that $BUB^{-1} \in C(T_{\Delta}, \mathcal{L}_s(E))$, which implies

$$AUA^{-1} \in C(T_{\Delta}, \mathcal{L}_s(E)) \cap \mathcal{K}(E, 0)\,, \tag{27}$$

due to the fact that $A(t)$ is the E-realization of $B(t)$ and due to (17).

Since U is the parabolic fundamental solution for $\{B(t) \mid 0 \leqq t \leqq T\}$ on $E^{\beta-1}$ we know that $U(t, \cdot) \in C^1([0, t), \mathcal{L}_s(E^{\beta}, E^{\beta-1}))$ and $D_2U = UB$. Hence we deduce from (10) that $D_2U(t, \cdot) \in C([0, t), \mathcal{L}_s(E^{\beta}, E))$. Thus

$$U(t, \cdot) \in C^1([0, t), \mathcal{L}_s(E^{\beta}, E)) \qquad \text{for } 0 < t \leqq T\,, \tag{28}$$

and

$$D_2U(t, s)x = U(t, s)A(s)x\,, \qquad (t, s) \in \dot{T}_{\Delta}\,, \tag{29}$$

provided $x \in D(A(s))$, where we used again the fact that $A(s)$ is the E-realization of $B(s)$.

Now the assertions of Theorem 4.4 follow from (10), (12), (16)-(19), (27)-(29), and the remarks following the statement of Theorem 4.4. □

The author is grateful to the referee of this paper for suggesting the use of the "moment inequality" (11) for simplifying the original proof of (12). Moreover the same referee pointed out that the evolution operators constructed by Kato [**10**] and Sobolevskii [**20**] possess property (U5), although it is not mentioned in their papers. Property (U5) is important for the study of semilinear evolution equations of the form

$$\dot{u} + A(t)u = f(t, u)\,, \qquad 0 < t \leqq T\,,$$

since it allows the use of continuation arguments to construct maximal solutions from local ones (cf. [**3**]).

6. The linear Cauchy problem.

Throughout this section we presuppose the assumptions (A), $(\mathrm{C})_{\beta}$, $(\mathrm{CI})_{1-\beta}$ and $(\mathrm{H})_{\rho}$, and we consider the linear Cauchy problem

$$\dot{u} + A(t)u = f(t)\,, \qquad 0 < t \leqq T, \quad u(0) = x\,, \tag{CP$_x$}$$

where $f : [0, T] \to E$.

THEOREM 6.1. *Suppose that $f \in C^{\alpha}([0, T], E)$ for some $\alpha \in (0, 1)$. Then* $(\mathrm{CP})_x$ *has for each $x \in E$ a unique solution.*

PROOF. It follows from (U3) and the remarks at the end of Section 4 that it suffices to show that the function

(1) $$v(t):=\int_0^t U(t, \tau)f(\tau)d\tau\,, \qquad 0\leqq t\leqq T\,,$$

is a solution of $(CP)_0$. Since U is a parabolic fundamental solution for $\{B(t) \mid 0\leqq t\leqq T\}$ on $E^{\beta-1}$, we know from the results of Sobolevskii and Tanabe (e. g. [22, Theorem 5.2.3]), that v is a solution of the evolution equation $\dot{v}+B(t)v=f(t)$, $0<t\leqq T$, in $E^{\beta-1}$. Hence it suffices to verify that

$$\left[t \mapsto B(t)v(t)=B(t)\int_0^t U(t, \tau)f(\tau)d\tau\right] \in C((0, T], E)\,.$$

Observe that

$$B(t)v(t) = \int_0^t A(t)U(t, \tau)[f(\tau)-f(t)]d\tau+A(t)\int_0^t U(t, \tau)f(t)d\tau$$

for $0<t\leqq T$, and that the first summand is continuous as a function from $[0, T]$ to E, due to

$$\|A(t)U(t, \tau)[f(\tau)-f(t)]\| \leqq c(t-\tau)^{\alpha-1}\,, \qquad (t, \tau)\in\dot{T}_\Delta\,,$$

as follows from (U3). Hence it remains to show that

$$\left[t \mapsto w(t):=A(t)\int_0^t U(t, \tau)f(t)d\tau\right] \in C((0, T], E)\,.$$

Let $t\in(0, T]$ be fixed and observe that

$$g_\varepsilon(t):=\int_0^{t-\varepsilon} U(t, \tau)f(t)d\tau \longrightarrow g(t):=\int_0^t U(t, \tau)f(t)d\tau$$

in E^β as $\varepsilon\to 0$ in $(0, t)$, due to the fact that $U\in\mathcal{K}(E, E^\beta, \beta)$, as follows from (5.6), Lemma 5.2 and (5.14). Hence, since $B(t)\in\mathcal{L}(E^\beta, E^{\beta-1})$,

$$B(t)g_\varepsilon(t) \longrightarrow B(t)g(t) = w(t) \qquad \text{in } E^{\beta-1}\,,$$

as $\varepsilon\to 0$ in $(0, t)$. But

$$B(t)g_\varepsilon(t) = \int_0^{t-\varepsilon} B(t)U(t, \tau)f(t)d\tau = U(t, 0)f(t)-U(t, t-\varepsilon)f(t)\,,$$

which shows that $B(t)g_\varepsilon(t) \to U(t, 0)f(t)-f(t)$ in $E^{\beta-1}$. Thus $w(t)=U(t, 0)f(t)-f(t)$ for $0<t\leqq T$, which implies $w\in C((0, T], E)$ by (U1). $\square$

We prove also a second existence theorem for $(CP)_x$, where we impose more "regularity in space" instead of imposing time regularity as in Theorem 6.1.

THEOREM 6.2. *Let* $0<\theta<1$ *and* $E_\theta:=(E, E^\beta)_\theta$, *where* $(\cdot, \cdot)_\theta$ *denotes either the complex interpolation functor* $[\cdot, \cdot]_\theta$ *or any one of the real interpolation functors* $(\cdot, \cdot)_{\theta,p}$, $1\leqq p\leqq\infty$, *respectively. Moreover suppose that* $f\in C([0, T], E_\theta)$.

Then $(\mathrm{CP})_x$ *has for each* $x\in E$ *a unique solution.*

PROOF. It suffices again to show that the function v defined by (1) is a solution of $(\mathrm{CP})_0$. Since U is a fundamental solution for $\{B(t)\mid 0\leqq t\leqq T\}$ on $E^{\beta-1}$ and since $E_\theta\hookrightarrow E=[E^{\beta-1}, E^\beta]_{1-\beta}$, it follows from [**5**, Theorem 4.1] that v is a solution of the evolution equation $\dot{v}+B(t)v=f(t)$, $0<t\leqq T$, in $E^{\beta-1}$. Hence

$$\dot{v}(t)=f(t)-B(t)\int_0^t U(t,\tau)f(\tau)d\tau\,,\qquad 0<t\leqq T\,,$$

in $E^{\beta-1}$, and it remains to show that

$$\left[t\mapsto B(t)\int_0^t U(t,\tau)f(\tau)d\tau\right]\in C((0,T],E)\,.$$

For this it suffices to prove that $BU\in\mathcal{K}(E_\theta, E, \alpha)$ for some $\alpha<1$.

It follows from the reiteration theorem for the complex interpolation method [**6**, Theorem 4.6.1] and the commutativity theorem [**24**, Theorem 1.10.2] (cf. also [**6**, Theorem 4.7.2]) that

$$E_\theta=([E^{\beta-1}, E^\beta]_{1-\beta}, E^\beta)_\theta\doteq(E^{\beta-1}, E^\beta)_\eta\,,$$

where $\eta:=1-\beta(1-\theta)$. Hence we obtain from [**5**, Theorem 3.2 (iii)] that $BU\in\mathcal{K}(E_\theta, E, 1-\beta\theta)$. $\square$

7. Remarks on parabolic differential operators.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ of class C^2 and suppose that $\partial\Omega=\Gamma_0\cup\Gamma_1$, where $\Gamma_0\cap\Gamma_1=\emptyset$ and Γ_0 is open and closed in $\partial\Omega$.

Let M denote either $\bar{\Omega}$ or Γ_1. Then we write $a\in C^{r,s}(M\times[0,T])$, where $r\in\{0,1\}$ and $0<s<1$, provided $a(\cdot,t)\in C^r(M,\boldsymbol{R})$ for $0\leqq t\leqq T$, and $a(x,\cdot)\in C^s([0,T],\boldsymbol{R})$ uniformly with respect to $x\in M$. Moreover we use the summation convention throughout.

We let

$$\mathcal{A}(t)u:=-D_j(a_{jk}(\cdot,t)D_ku)+a_j(\cdot,t)D_ju+a_0(\cdot,t)u\,,$$

where

$$a_{jk}=a_{kj}\,,\quad a_j\in C^{1,\rho}(\bar{\Omega}\times[0,T])\,,\qquad j,k=1,\cdots,n\,,$$

and $a_0\in C^{0,\rho}(\bar{\Omega}\times[0,T])$ for some $\rho\in(1/2,1)$, and where

$$a_{jk}(x,t)\xi^j\xi^k>0\qquad\forall(x,t)\in\bar{\Omega}\times[0,T],\quad \xi:=(\xi^1,\cdots,\xi^n)\in\boldsymbol{R}^n\setminus\{0\}.$$

We let

$$\mathcal{B}(t)u:=\begin{cases} u & \text{on } \Gamma_0\,,\\ \dfrac{\partial u}{\partial\nu_a(t)}+\beta_0(\cdot,t)u & \text{on } \Gamma_1\,,\end{cases}$$

where $\beta_0 \in C^{1,\rho}(\Gamma_1 \times [0, T])$ and $\nu_a(t)$ is the outer conormal with respect to the matrix $(a_{jk}(\cdot, t))$. Finally we assume that $a_0 \geqq 0$, $\beta_0 \geqq 0$ and that either $\Gamma_0 \neq \emptyset$ or $a_0 \neq 0$, if $\beta_0 = 0$.

Let $1 < p < \infty$ be fixed and put $E := L_p(\Omega, \boldsymbol{R})$, $W_p^s := W_p^s(\Omega, \boldsymbol{R})$, $0 < s \leqq 2$, and

$$W_{p, \mathcal{B}(t)}^2 := \{u \in W_p^2 \mid \mathcal{B}(t)u = 0\}.$$

Moreover let

$$A(t)u := \mathcal{A}(t)u \qquad \forall u \in W_{p, \mathcal{B}(t)}^2 . \tag{1}$$

Then it is well known that $\{A(t) \mid 0 \leqq t \leqq T\}$ satisfies assumption (A). It follows from the results of Seeley [**17**, **18**] that $A^{i\tau}(t) \in \mathcal{L}(E)$, that $\|A^{i\tau}(t)\| \leqq c(\varepsilon)$ for $\varepsilon > 0$, $|\tau| \leqq \varepsilon$ and $t \in [0, T]$, and that

$$D(A^{1/2}(t)) = W_{p,0}^1 := \{u \in W_p^1 \mid u | \Gamma_0 = 0\}.$$

It is known (e.g. [**2**, Theorem 7.1]) that $A'(t)$ is induced by the formally adjoint elliptic boundary value problem $(\mathcal{A}^{\#}(t), \mathcal{B}^{\#}(t))$, which is of the same form as $(\mathcal{A}(t), \mathcal{B}(t))$. Hence Seeley's results imply

$$D((A')^{1/2}(t)) = W_{p',0}^1 , \qquad \text{where } p' := p/(p-1) .$$

Consequently we deduce from Propositions 4.1 and 4.2 the validity of assumptions $(\mathrm{C})_{1/2}$ and $(\mathrm{CI})_{1/2}$.

For $u \in W_{p,0}^1$ and $v \in W_{p',0}^1$ let

$$a(t, u, v) := \int_\Omega [a_{jk}(\cdot, t) D_j u D_k v + v a_j(\cdot, t) D_j u + a_0(\cdot, t) uv] dx + \int_{\Gamma_1} \beta_0(\cdot, t) uv d\sigma$$

and observe that

$$[t \mapsto a(t, \cdot, \cdot)] \in C^\rho([0, T], \mathcal{L}^2(W_{p,0}^1, W_{p',0}^1 ; \boldsymbol{R})) ,$$

where $\mathcal{L}^2(\cdots)$ is the Banach space of all continuous bilinear forms on $W_{p,0}^1 \times W_{p',0}^1$. Since

$$a(t, u, v) = \int_\Omega v A(t) u dx \qquad \forall u \in W_{p, \mathcal{B}(t)}^2, \quad v \in W_{p',0}^1$$

by Gauss' theorem, it follows from Theorem 1.3 that

$$a(t, u, v) = \langle v, A_{1/2}(t) u \rangle$$

for all $u \in W_{p,0}^1 \doteq E^{1/2}$ and $v \in W_{p',0}^1 \doteq (E^{-1/2})'$. This implies the validity of assumption $(\mathrm{H})_\rho$. Hence *Theorems* 4.4, 6.1 *and* 6.2 *are applicable to the family* $\{A(t) \mid 0 \leqq t \leqq T\}$ *defined by* (1) *in* $L_p(\Omega, \boldsymbol{R})$ *for* $1 < p < \infty$.

Suppose now that Ω is a bounded domain of class C^{2m} and

$$\mathcal{A}(t)u = (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha(\cdot, t) D^\alpha u$$

is a differential operator of order $2m$ acting on N-vector valued functions $u: \Omega \rightarrow C^N$. Moreover suppose that

$$\mathcal{B}(t) := \{\mathcal{B}^\sigma(t) \mid 1 \leqq \sigma \leqq mN\}$$

is a system of boundary operators of the form

$$\mathcal{B}^\sigma(t)u = \sum_{|\alpha| \leqq m_\sigma} b_\alpha^\sigma(\cdot, t) D^\alpha u$$

such that $(\mathcal{A}(t), \mathcal{B}(t), \Omega)$, $0 \leqq t \leqq T$, is for each t a strongly 0-regular elliptic boundary value problem of order $2m$ in the sense of [**3**, Section 13], uniformly with respect to $t \in [0, T]$. Then, by adding a sufficiently large constant $\omega_0(p)$ to a_0, it follows from [**3**, Theorems 12.2 and 13.1] that we can assume that $\{A(t) \mid 0 \leqq t \leqq T\}$ satisfies assumption (A) in $E := L_p(\Omega, C^N)$, $1 < p < \infty$, where $A(t)u := \mathcal{A}(t)u$ for all

$$u \in W_{p, \mathcal{B}(t)}^{2m} := \{u \in W_p^{2m}(\Omega, C^N) \mid \mathcal{B}(t)u = 0\}.$$

Moreover, Seeley's results are again applicable to give $A^{i\tau}(t) \in \mathcal{L}(E)$, $\|A^{i\tau}(t)\| \leqq c(\varepsilon)$ for $\varepsilon > 0$, $|\tau| \leqq \varepsilon$, and $t \in [0, T]$ and

$$D(A^{k/2m}(t)) = W_{p, \mathcal{B}(t)}^k, \qquad k = 1, 2, \cdots, 2m, \tag{2}$$

where

$$W_{p, \mathcal{B}(t)}^s := \{u \in W_p^s(\Omega, C^N) \mid \mathcal{B}^\sigma(t)u = 0 \text{ for } m_\sigma < s - 1/p\} \tag{3}$$

for $0 < s \leqq 2m$ (cf. [**3**, Theorem 13.3]).

Suppose now that there exists a "formally adjoint" system $(\mathcal{A}^\#(t), \mathcal{B}^\#(t), \Omega)$, $0 \leqq t \leqq T$, such that the corresponding $L_{p'}$-realization $A^\#(t)$, given by

$$A^\#(t)v = \mathcal{A}^\#(t)v \qquad \forall v \in W_{p', \mathcal{B}^\#(t)}^{2m},$$

is also a strongly 0-regular elliptic boundary value problem of order $2m$, uniformly with respect to $t \in [0, T]$, and such that $A'(t) = A^\#(t)$. Then it follows that

$$D((A')^{k/2m}(t)) = W_{p', \mathcal{B}^\#(t)}^k, \qquad k = 1, 2, \cdots, 2m. \tag{4}$$

Observe that this is always the case if $N=1$ (cf. [**15**, Theorem II.8.4] for the case $p=2$. A similar result holds for $p \neq 2$.). It is also easily seen that this is the case if $N>1$ and $m=1$, provided $(\mathcal{A}(t), \mathcal{B}(t), \Omega)$ is a second order system of the form treated in [**4**, Section 6].

By using (2), (3) and (4) we deduce from Propositions 4.1 and 4.2 *the validity of the assumptions* $(\mathrm{C})_\beta$ *and* $(\mathrm{CI})_{1-\beta}$, *provided* $\beta = k/2m$ *for some* $k \in \{1, 2, \cdots, 2m-1\}$ *and the boundary operators* $\mathcal{B}^\sigma(t)$ *having orders* $m_\sigma < k$, *and* $(\mathcal{B}^\#)^\tau(t)$ *having orders* $m_\tau^\# < 2m - k$, *are independent of* $t \in [0, T]$.

Finally suppose that there exists a function

$$[t \mapsto a(t, \cdot, \cdot)] \in C^{\rho}([0, T], \mathcal{L}^2(W^k_{p, \mathcal{B}(0)}, W^{2m-k}_{p', \mathcal{B}^\sharp(0)}, \boldsymbol{C}))$$

for some $\rho \in (1-k/2m, 1)$ such that

$$a(t, u, v) = \langle v, A(t)u \rangle \qquad \forall u \in W^{2m}_{p, \mathcal{B}(t)}, \quad v \in W^{2m-k}_{p', \mathcal{B}(t)} . \tag{5}$$

Then condition $(\mathrm{H})_\rho$ is satisfied. Clearly (5) is deduced in practical cases from an appropriate "Green's formula".

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