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Kernels of Toeplitz operators

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1. Introduction.

Let U be the open unit disc in the complex plane and let ∂U be the boundary of U. If f is analytic in U and $\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta$ is bounded for $0\leqq r<1$, $f(e^{i\theta})$, which we define to be $\lim_{r\to 1}f(re^{i\theta})$, exists almost everywhere on ∂U. If

$$\lim_{r\to 1}\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta=\int_{-\pi}^{\pi}\log^{+}|f(e^{i\theta})|d\theta,$$

then f is said to be of the class N_{+}. The set of all boundary functions in N_{+} is again denoted by N_{+}. For $0<p\leqq\infty$, the Hardy space H^{p} is defined by $N_{+}\cap L^{p}$ where L^{p} denotes $L^{p}(d\theta)$. If $1\leqq p\leqq\infty$, it coincides with the space of functions in L^{p} whose Fourier coefficients with negative indices vanish. Put $H_{0}^{p}=\{f\in H^{p}: f(0)=0\}$. If $f\in L^{p}$ $(1<p<\infty)$ and $f\sim\sum_{n=-\infty}^{\infty}c_{n}e^{in\theta}$, then by a well-known theorem of M. Riesz (cf. [6, p. 54]) the series $\sum_{n=0}^{\infty}c_{n}e^{in\theta}$ is the Fourier series of a function Pf belonging to L^{p} (therefore, to H^{p}), and moreover $\|Pf\|_{p}\leqq A_{p}\|f\|_{p}$ where A_{p} is a constant depending only on p. Thus P is a bounded projection from L^{p} to H^{p}.

Let $\phi\in L^{\infty}$. We define the Toeplitz operator $\mathcal{T}_{\phi}$ on H^{p} by

$$\mathcal{T}_{\phi}f=P(\phi f).$$

Clearly $\mathcal{T}_{\phi}$ is a bounded operator with norm at most $A_{p}\|\phi\|_{\infty}$. We would like to define Toeplitz operators on H^{p} for $p=\infty$ or $0<p\leqq 1$. There we cannot use the projection P. Therefore for $0<p\leqq\infty$ we define the Toeplitz operator T_{ϕ} on H^{p} by

$$T_{\phi}f=\phi f+\bar{H}_{0}^{p}.$$

T_{ϕ} is a bounded operator with norm at most $\|\phi\|_{\infty}$ from H^{p} to $L^{p}/\bar{H}_{0}^{p}$. Denoting the kernel of T_{ϕ} by $\ker T_{\phi}$, we have clearly

$$\ker T_{\phi}=\ker\mathcal{T}_{\phi}$$

for $1<p<\infty$.

In § 4 of this paper, we determine under what conditions $\ker T_{\phi}$ is finite

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dimensional. This was shown independently by Hayashi [8] for $p=2$. For $1\leqq p\leqq\infty$, there is a function ϕ such that $\ker T_{\phi}\neq\{0\}$ and $\ker T_{\phi}$ is finite dimensional. For this purpose, we need special outer functions which we call strong outer functions (see § 3). In § 5, we apply the above result to describe the intersection of past and future of a stationary stochastic process in the case where the intersection is finite dimensional. Bloomfield, Jewell and Hayashi [3] determined the spectral density of the process. To describe the intersection of past and future relates to researches of nonconstant real (or nonnegative) functions in weighted Hardy spaces. Applications to H^1 extremal problems and to Hankel operators are given in §§ 6 and 7, respectively. Recently the author [10] described the solution sets of extremal problems in H^1 when the sets are weak*-compact. Corollary 5 in § 6 implies Theorem 2 in [10].

2. Nontrivial kernels of Toeplitz operators.

Let T_{ϕ} be a Toeplitz operator on H^p $(0<p\leqq\infty)$, then $\ker T_{\phi}\neq\{0\}$ if and only if ϕ has the form $\bar{g}/f$ for some nonzero g in H_0^p and f in H^p. Therefore $\ker T_{\phi}\neq\{0\}$ implies $\log|\phi|\in L^1$; thus $|\phi|=|h|$ for some outer function h in H^{∞}.

PROPOSITION 1. *If* $|\phi|=|h|$ *for some outer function* h *in* H^{∞} *and* $\Phi=\phi h/|\phi h|$, *then* $\ker T_{\phi}=\ker T_{\Phi}$.

PROOF. When $f\in H^p$ and $g\in H_0^p$,

$$\phi f=\bar{g} \quad \text{iff} \quad \Phi f=\overline{gh^{-1}}$$

because $\phi=\Phi\bar{h}$.

Let $f\in H^{\infty}$ and $f=bk$ for an inner function b and an outer function k. When $\phi=\bar{f}$, $\ker T_{\phi}=\ker T_{\bar{b}}$ by Proposition 1 and $\ker T_{\phi}\neq\{0\}$ if b is not constant.

PROPOSITION 2. *Let* $0<p\leqq\infty$. *If* $\ker T_{\phi}\neq\{0\}$, *then the range of* T_{ϕ} *contains the set* $\mathcal{P}+\bar{H}_0^p$ *of all analytic trigonometric polynomials.*

PROOF. When $f\in\ker T_{\phi}$ is nonzero, $\phi f=\bar{g}$ for some nonzero $g\in H_0^p$. g has the form $g=\sum_{j=n}^{\infty}a_j z^j$ and $a_n\neq0$. Hence $\phi z^n f=\bar{a}_n+\sum_{j=1}^{\infty}\bar{a}_{j+n}\bar{z}^j$, and $T_{\phi}H^p\ni1+\bar{H}_0^p$. Moreover $\phi z^{n+1}f=\bar{a}_n z+\bar{a}_{n+1}+\sum_{j=2}^{\infty}\bar{a}_{j+n}\bar{z}^{j-1}$. Thus $T_{\phi}H^p\ni z+\bar{H}_0^p$. Proceeding similarly, we obtain $T_{\phi}H^p\ni z^l+\bar{H}_0^p$ for any $l\geqq0$.

Coburn's theorem states that if T_{ϕ} is a Toeplitz operator on H^2 and if $\ker T_{\phi}\neq\{0\}$, then $\ker T_{\phi}^*=\{0\}$ (cf. [5, p. 185]). Proposition 2 gives a new proof of Coburn's theorem for $p=2$ and generalize it for $p\neq2$. For $p=2$, we can show that $\ker T_{\bar{z}\phi}\neq\{0\}$ if and only if $T_{\phi}H^2\supset\mathcal{P}+\bar{H}_0^2$. If T_{ϕ} is a Toeplitz operator on H^p $(1\leqq p\leqq\infty)$ and if $\ker T_{\phi}\neq\{0\}$, then $\ker T_{\bar{\phi}}=\{0\}$ by Proposition 1 because $H^p\cap\overline{zH^p}=\{0\}$. However this is not true for $0<p<1$.

3. Strong outer functions.

Let g be a nonzero function in H^p $(0<p\leqq\infty)$. Then g is an outer function if and only if k is constant whenever $kg\in H^p$ for some $k\in L^\infty$ with $k\geqq 0$ a.e. For if g is not an outer function, then g has the form $g=qh$ where q is a nonconstant inner function and h is an outer function in H^p. Putting $k=q+\bar{q}+1$, $kg\in H^p$ and k is not constant. If g is an outer function, then $g^{-1}\in N_+$ by [**6**, p. 26]. If $kg\in H^p$ for some $k\in L^\infty$ with $k\geqq 0$ a.e., then $k\in N_+\cap L^\infty$, and k is constant.

DEFINITION. Let g be a nonzero function in H^p $(0<p\leqq\infty)$. We say g is a p-strong outer function if it has the following property: If $kg\in H^p$ for some Lebesgue measurable k with $k\geqq 0$ a.e., then k is constant.

A p-strong outer function is an outer function. In [**10**, p. 225], a 1-strong outer function is called a strong outer function. We remark that de Leeuw and Rudin [**4**, p. 477] used a strong outer function in a slightly different meaning. Let $p\geqq 1$ and $g\in H^p$. Then if $g^{-1}\in H^p$ or $\mathrm{Re}\,g(e^{i\theta})\geqq 0$ a.e., then g is a p-strong outer function (cf. [**10**, Proposition 5], [**11**, Theorem 3]). However if $p<1$ this is not true. Choose $g=1-z$ and $k=-z/(1-z)^2$, then $kg\in H^p$. Suppose $p\geqq 1/2$. If $g\in H^p$ and $g^{-1}\in H^\infty$, then g is a p-strong outer function.

Let w be a nonnegative function in L^1. The weighted Hardy space $H^p(w)=H^p(wd\theta)$, $0<p\leqq\infty$, is defined as follows. For $0<p<\infty$, $H^p(w)$ is the closure of all analytic polynomials in $L^p(wd\theta)$, while $H^\infty(w)$ is the weak*-closure of all analytic polynomials in $L^\infty(wd\theta)$. We assume that $w=|g|^p$ for some outer function g in H^p. Then $H^p(w)=g^{-1}H^p$ for $p\neq\infty$ and $H^\infty(w)=H^\infty$. $H^p(w)_+$ denotes the set of all nonnegative functions in $H^p(w)$.

PROPOSITION 3. *Let* $0<p<\infty$. *A function g is a p-strong outer function in H^p if and only if $H^p(|g|^p)_+$ consists of nonnegative constants.*

PROOF. Let k be a Lebesgue measurable function. Then $kg\in H^p$ if and only if $k\in H^p(|g|^p)$, from which the proposition follows.

PROPOSITION 4. *Let* $0<p\leqq\infty$. *Suppose g is a p-strong outer function and h is an outer function in H^p. If $|g|\leqq\gamma|h|$ and γ is a positive constant then h is a p-strong outer function.*

The proof follows easily from the definition of a p-strong outer function.

Let $0<p<1/2$. $(H^p)_+$ contains nonconstant functions. Hence 1 is not a p-strong outer function. It is reasonable to guess that we do not have any p-

strong outer functions. Unfortunately we could not prove it.

PROPOSITION 5. *Let* $0<p<1/2$. *If* g *is an outer function in* H^p *that is bounded on some open set, then it is not a* p*-strong outer function.*

PROOF. Suppose $g(e^{i\theta})$ is bounded by γ_1 on an open interval (c, d). Put $k=z/((z-a)(1-\bar{a}z))$ with $a\in(c, d)$, then $k\in H^p$, $k\geqq 0$ a.e. and k is bounded by γ_2 on $(-\pi, c]\cup[d, \pi]$.

$$\int_{-\pi}^{\pi}|kg|^p d\theta/2\pi \leqq \gamma_1\int_c^d |k|^p d\theta/2\pi+\gamma_2\Big(\int_{-\pi}^{c}|g|^p d\theta/2\pi+\int_d^{\pi}|g|^p d\theta/2\pi\Big) < \infty$$

and hence $kg\in L^p$. This implies $kg\in H^p$ because $kg\in N_+$.

PROPOSITION 6. *Let* $0<q\leqq\infty$ *and* $0<p<q/(2q+1)<q$ *where if* $q=\infty$ *we assume* $q/(2q+1)=1/2$. *If* g *is a function in* H^q *then it is not a* p*-strong outer function.*

PROOF. Put $k=-z/(1-z)^2$. Then $k\geqq 0$ a.e. and $k\in\bigcup\{H^\gamma;\ 0<\gamma<1/2\}$. Let $1/s+1/t=1$ ($s\geqq 1$ and $t\geqq 1$), then

$$\int|kg|^p d\theta/2\pi \leqq \Big(\int k^{ps}d\theta/2\pi\Big)^{1/s}\Big(\int|g|^{pt}d\theta/2\pi\Big)^{1/t}.$$

If $t=q/p$ then $s=q/(q-p)$, and $ps=pq/(q-p)<1/2$. Thus kg belongs to H^p because $k\in H^{ps}$ and $g\in H^{pt}$. This implies that g is not a p-strong outer function.

Strong outer functions are defined for the Hardy class H^p on a polydisc and are studied in [7], [11].

§4. Finite dimensional kernels of Toeplitz operators.

For $0<p\leqq\infty$, $T_\phi=T_\phi^p$ denotes a Toeplitz operator on H^p. Let $\phi=\bar{z}^l$ and $l\in \boldsymbol{Z}_+$ where $\boldsymbol{Z}_+$ denotes the set of all nonnegative integers. When $1\leqq p\leqq\infty$, $\ker T_\phi^p=H^2\ominus z^l H^2$ and $\dim\ker T_\phi^p=l$. When $0<p<1$, $\ker T_\phi^p=H^p\cap z^{l+1}\bar{H}^p$ and $\dim\ker T_\phi^p=\infty$. We denote by $\mathcal{P}_n$ the set of all analytic polynomials with degree $\leqq n$.

LEMMA 1. *Let* $T_\phi=T_\phi^p$ *be a Toeplitz operator on* H^p $(0<p\leqq\infty)$. *If* $f\in H^p$ *is a nonzero function and* $z^n f\in\ker T_\phi$ *for some* $n\in\boldsymbol{Z}_+$, *then* $pf\in\ker T_\phi$ *for any* $p\in\mathcal{P}_n$ *and thus* $\dim\ker T_\phi\geqq n+1$.

PROOF. If $T_\phi(z^n f)=0$, then there is a $g\in H_0^p$ such that $\phi z^n f=\bar{g}$. If $p\in\mathcal{P}_n$, then we can write $p=\gamma(z-a_1)\cdots(z-a_l)$ where $l\leqq n$. Thus

$$\phi pf=\gamma(\bar{z})^{n-l}\bar{g}(1-a_1\bar{z})\cdots(1-a_l\bar{z}^l);$$

hence $T_\phi(pf)=0$.

LEMMA 2. *Let $T_\phi=T_\phi^p$ be a Toeplitz operator on H^p $(0<p\leqq\infty)$. If* $\dim\ker T_\phi\geqq n+1$, *then there exists a nonzero $f\in H^p$ such that $z^n f\in\ker T_\phi$.*

PROOF. Since $\ker T_\phi$ is a subspace of H^p and has at least $n+1$ linearly independent functions, we can find a function $z^n f\in\ker T_\phi$ for some nonzero $f\in H^p$.

THEOREM 7. *Let $0<p\leqq\infty$ and $n\in\boldsymbol{Z}_+\setminus\{0\}$. Suppose $T_\phi=T_\phi^p$ is a Toeplitz operator on H^p. Then the following conditions* (1), (2) *and* (3) *are equivalent:*

(1) $\dim\ker T_\phi=n<\infty$,

(2) *There is a $p/2$-strong outer function g^2 in $H^{p/2}$ such that*

$$\ker T_\phi=\{pg\,;\,p\in\mathcal{P}_{n-1}\},$$

(3) *There is an outer function h in H^∞ with $|\phi|=|h|$ and a $p/2$-strong outer function g^2 in $H^{p/2}$ such that*

$$\Phi=\frac{\phi}{|\phi|}\frac{h}{|h|}=\bar{z}^n\frac{|g|^2}{g^2}.$$

PROOF. (1)$\Rightarrow$(2). By Lemma 2, there exists a nonzero $g\in H^p$ such that $z^{n-1}g\in\ker T_\phi$. By Lemma 1, $pg\in\ker T_\phi$ for any $p\in\mathcal{P}_{n-1}$. Thus each $f\in\ker T_\phi$ has the form pg because $\dim\ker T_\phi=n$. If $g=q_1g_1$ for some nonconstant inner function q_1 and $g_1\in H^p$, that is, g is not an outer function, then $z^{n-1}(q_1-q_1(0))g_1$ belongs to $\ker T_\phi$ and so $z^n k\in\ker T_\phi$ for some nonzero $k\in H^p$. This contradicts $\dim\ker T_\phi=n$; hence g is an outer function. We shall show that g^2 is a $p/2$-strong outer function. Since $\ker T_\phi\neq\{0\}$, there is an outer function h in H^∞ with $|\phi|=|h|$. Setting $\Phi=\phi h/|\phi h|$, one has a nonzero $k\in H_0^p$ such that $\Phi z^{n-1}g=\bar{k}$, because $\ker T_\Phi=\ker T_\phi$ by Proposition 1. Since $|\Phi|=1$, k has the form $k=zq_2g$ where q_2 is an inner function; thus $z^{n-1}q_2g\in\ker T_\Phi=\ker T_\phi$. By what was just proved above, $z^{n-1}q_2g=pg$ for some $p\in\mathcal{P}_{n-1}$, and q_2 is a constant function c_2 with $|c_2|=1$. Thus $\Phi=\bar{c}_2\bar{z}^n|g|^2/g^2$. If g^2 is not a $p/2$-strong outer function, then there exists a nonzero $f\in H^{p/2}$ such that f is not a positive scalar multiple of g^2 and $\arg f=\arg g^2$. Suppose $f=q_3l^2$ where q_3 is an inner function and l^2 is an outer function. Then $\Phi=\bar{c}_2\bar{z}^n\bar{q}_3|l|^2/l^2$, and $\Phi z^{n-1}q_3l=\bar{c}_2\bar{z}\bar{l}$ and $z^{n-1}q_3l\in\ker T_\Phi=\ker T_\phi$. Hence $z^{n-1}q_3l=pg$ for some $p\in\mathcal{P}_{n-1}$. Since g is an outer function, $p=cz^{n-1}$ for some nonzero constant c and so q_3 is a constant function c_3. Thus $l=\bar{c}_3cg$ and so $f=|c_3|^2\bar{c}_3c^2g^2$. This implies that g^2 is a $p/2$-strong outer function.

(2)$\Rightarrow$(3). As in the proof of (1)$\Rightarrow$(2), there is an outer function h in H^∞ with $|\phi|=|h|$. Thus for $\Phi=\phi h/|\phi h|$, there holds $\Phi=c\bar{z}^n|g_1|^2/g_1^2$ where c is a constant function with $|c|=1$ because $z^{n-1}g_1\in\ker T_\phi$. $g=c^{1/2}g_1$ satisfies the condition of (3).

(3)$\Rightarrow$(1). It is sufficient to show that $\dim\ker T_{\phi}=n$ by Proposition 1. If $\dim\ker T_{\Phi}\geqq n+1$, then $z^n f\in\ker T_{\Phi}$ for some nonzero $f\in H^p$ by Lemma 2. Hence $\Phi z^n f=\bar{k}$ for some nonzero $k\in H_0^p$, and

$$\Phi=\bar{z}^n\frac{|fk|}{fk}=z^{-n}\frac{|g|^2}{g^2}.$$

This contradicts the fact that g^2 is a $p/2$-strong outer function because $fk\in H_0^{p/2}$. Thus $\dim\ker T_{\Phi}\leqq n$. On the other hand, $\Phi z^{n-1}g=\bar{z}\bar{g}$ so that $z^{n-1}g\in\ker T_{\Phi}$. By Lemma 1, $\dim\ker T_{\Phi}\geqq n$.

COROLLARY 1. *Suppose* $0<p<1$ *and* T_{ϕ} *is a Toeplitz operator on* H^p. *If there is a nonzero function* f *in* $\ker T_{\phi}$ *which is bounded on some open set, then* $\dim\ker T_{\phi}=\infty$.

PROOF. If $\dim\ker T_{\phi}=n<\infty$ for $n\neq 0$ then $f=pg$ for some $p\in\mathcal{P}_{n-1}$ and some $p/2$-strong outer function g^2 by Theorem 7. Hence g^2 is bounded on some open set. This contradicts Proposition 5.

COROLLARY 2. *Let* g *be a nonzero function in* $H^{p/2}$ $(0<p\leqq\infty)$. *Suppose* $\phi=|g|/g$ *and* T_{ϕ} *is a Toeplitz operator on* H^p. g *is a* $p/2$*-strong outer function if and only if* $\ker T_{\phi}=\{0\}$.

PROOF. If $\ker T_{\phi}=\{0\}$, then g is an outer function; hence $g^{1/2}\in\ker T_{\bar{z}\phi}$. If $\dim\ker T_{\bar{z}\phi}\geqq 2$, then $zf\in\ker T_{\bar{z}\phi}$ for some nonzero $f\in H^p$ by Lemma 2, hence $f\in\ker T_{\phi}$. This contradiction implies $\dim\ker T_{\bar{z}\phi}=1$, therefore $\bar{z}\phi=\bar{z}|k|^2/k^2$ for some $p/2$-strong outer function k^2 by Theorem 7. Hence $|g|/g=|k|^2/k^2$ and $g=\gamma k^2$ for some positive constant γ. Thus g is a $p/2$-strong outer function. Conversely if g is a $p/2$-strong outer function then it is easy to see that $\dim\ker T_{\phi}=\{0\}$.

If $0<p<q\leqq\infty$, then $\ker T_{\phi}^q\subset\ker T_{\phi}^p$ and it may happen that $\ker T_{\phi}^q\subsetneqq\ker T_{\phi}^p$.

COROLLARY 3. *Let* $0<p<q\leqq\infty$. *If* $\dim\ker T_{\phi}^p=n$ *and* $n\in\boldsymbol{Z}_+\setminus\{0\}$, *then* $\ker T_{\phi}^q=\{0\}$ *or* $\ker T_{\phi}^q=\ker T_{\phi}^p$.

PROOF. By Proposition 1 and Theorem 7 we can write $\phi=\bar{z}^n|g|^2/g^2$ for some $p/2$-strong outer function g^2. If $\ker T_{\phi}^q\neq\{0\}$, then $\phi=\bar{z}^l|k|^2/k^2$ for some $q/2$-strong outer function k^2 and $l\leqq n$ by Theorem 7. Thus $k^2\in\ker T_{\phi}^p$ because $k^2\in\ker T_{\phi}^q$, and $k^2=\gamma g^2$ for some positive constant γ. Thus $l=n$, and $\ker T_{\phi}^q=\ker T_{\phi}^p$ by Theorem 7.

COROLLARY 4. *Let* $0<p<q\leqq\infty$. *If* $\dim\ker T_{\phi}^q=n$ *and* $n\in\boldsymbol{Z}_+$, *then* $\dim\ker T_{\phi}^p=\infty$ *or* $\ker T_{\phi}^p=\ker T_{\phi}^q$.

PROOF. If $\log|\phi|\notin L^1$, then $\ker T^p_\phi=\ker T^q_\phi=\{0\}$. We may assume $\log|\phi|\in L^1$ so that $\phi=\bar{z}^n|g|^2/g^2$ for some $p/2$-strong outer function g^2 by Theorem 7. If $0\neq\dim\ker T^p_\phi<\infty$, then $\ker T^q_\phi=\{0\}$ or $\ker T^q_\phi=\ker T^p_\phi$ by Corollary 3. Hence if $\ker T^p_\phi\neq\ker T^q_\phi$, then $\dim\ker T^p_\phi=\infty$.

Let $1\leqq p<q\leqq\infty$ and $n>0$. Then there exists ϕ in L^∞ such that $\ker T^q_\phi=\{0\}$ and $\dim\ker T^p_\phi=n$. For put $g=(1+z)^{-1/q}$ and $\phi=\bar{z}^n|g|^2/g^2$, then $\dim\ker T^p_\phi=n$ by Theorem 7. While $\ker T^q_\phi=\{0\}$ by Corollary 3 because $g\notin H^q$.

5. The intersection of past and future.

Let w be a nonnegative function in L^1 and $H^p(w)$ the weighted Hardy space, $0<p\leqq\infty$. Levinson and McKean [9] showed essentially that $\dim\overline{H^2(w)}\cap zH^2(w)=1$ if and only if $w=|h|^2$ for some 1-strong outer function h^2. From the view point of probability theory, $zH^2(w)$ denotes the future of a discrete stationary stochastic process and $\overline{H^2(w)}$ denotes its past. In this section, we consider $\overline{H^p(w)}\cap zH^p(w)$ in general. If $p=\infty$ and $w>0$ a.e., $H^\infty(w)=H^\infty$; hence $\overline{H^\infty(w)}\cap zH^\infty(w)=\{0\}$. For $p\neq\infty$ we can assume that $w=|h|^p$ for some outer function h in H^p. Otherwise $\overline{H^p(w)}\cap zH^p(w)=L^p(wd\theta)$. Note that $hH^p(w)=H^p$.

PROPOSITION 8. *Let $0<p<\infty$. Suppose $w=|h|^p$ for some outer function h in H^p and $\phi=|h|^2/h^2$. Then*

$$\overline{H^p(w)}\cap zH^p(w) = zh^{-1}\ker T^p_\phi\,.$$

PROOF. If $f\in\overline{H^p(w)}\cap zH^p(w)$ is nonzero, then $f=\overline{h^{-1}}\bar{g}=zh^{-1}k$ for some g and k in H^p. Hence $\phi k=\bar{z}\bar{g}$ and $k\in\ker T^p_\phi$. This implies $\overline{H^p(w)}\cap zH^p(w)\subset zh^{-1}\ker T^p_\phi$. Conversely, if $k\in\ker T^p_\phi$, then $\phi k=\bar{z}\bar{g}$ for some $g\in H^p_0$. Thus $zh^{-1}k=\overline{h^{-1}}\bar{g}$ belongs to $\overline{H^p(w)}\cap zH^p(w)$. Hence $zh^{-1}\ker T^p_\phi\subset\overline{H^p(w)}\cap zH^p(w)$.

THEOREM 9. *Let $0<p<\infty$. Let w be a nonnegative function in L^1 such that $\log w\in L^1$ and $n\in\boldsymbol{Z}_+$. Then the following are equivalent:*

(1) $\dim\overline{H^p(w)}\cap zH^p(w)=n$.

(2) *There is a $p/2$-strong outer function g^2 and an analytic polynomial s_0 of degree n with all of its zeros on ∂U such that $w=|s_0g|^p$, leading thus to $\overline{H^p(w)}\cap zH^p(w)=\{zss_0^{-1}\,;\,s\in\mathcal{P}_{n-1}\}$.*

PROOF. We may assume $w=|h|^p$ for some outer function in H^p. Put $\phi=|h|^2/h^2$. (1)⇒(2). By Proposition 8 $\dim\ker T^p_\phi=n$ and $\phi=\bar{z}^n|g|^2/g^2$ for some $p/2$-strong outer function g^2 by Theorem 7. Since $h\in\ker T^p_{\bar{z}\phi}$ and $\dim\ker T_{\bar{z}\phi}=n+1$ by Theorem 7, $h=s_0g$ for some $s_0\in\mathcal{P}_n$ by Theorem 7. Since h is an outer function, all zeros of s_0 are on ∂U. s_0 is an analytic polynomial of degree n exactly because $\phi=|s_0g|^2/s_0g^2$. By Theorem 7, $\ker T^p_\phi=\{sg\,;\,s\in\mathcal{P}_{n-1}\}$ and by Proposition 5, (2) follows. (2)$\Rightarrow$(1) is clear.

Bloomfield, Jewell and Hayashi [**3**] determined w such that $\overline{H^2(w)}\cap z^k H^2(w)=\{0\}$ but $\overline{H^2(w)}\cap z^{k-1}H^2(w)\neq\{0\}$. This result follows from Theorem 9 which we obtained independently of them. Similarly we can study $H^p(w)\cap\overline{H^p(w)}$ and $H^p(w)_+$ in the special weights w as in Theorem 9. However we do not know their structures in general. For $0<p<1$ (resp. $0<p<1/2$), $H^p\cap\overline{H^p}$ (resp. H^p_+) is not well understood even for $w=1$.

6. Extremal problems.

For $\phi\in L^\infty$, we define the functional K_ϕ on the Hardy space H^1 by

$$K_\phi(f)=\int_{-\pi}^{\pi}f(e^{i\theta})\phi(e^{i\theta})d\theta/2\pi\,.$$

The norm of K_ϕ is $\|K_\phi\|=\sup\{|K_\phi(f)|\,;\,f\in S\}$, where $S=\{f\in H^1\,;\,\|f\|_1\leqq 1\}$. Let S_ϕ denote the set of all $f\in S$ for which $K_\phi(f)=\|K_\phi\|$. There is always an extremal kernel ψ of K_ϕ, that is, $\|\psi\|_\infty=\|K_\phi\|$. Let $S^1=\{f\in H^1\,;\,\|f\|_1=1\}$.

PROPOSITION 10. *Suppose S_ϕ is not empty and ψ is an extremal kernel of K_ϕ. Then $S_\phi=\{\bar{\psi}|f|^2\in S^1\,;\,f\in\ker T^2_{\bar{z}\psi}\}$.*

PROOF. If $f\in\ker T^2_{\bar{z}\psi}$ and $|f|^2\in S^1$, then $\bar{z}\psi f=\bar{z}\bar{g}$ for some $g\in H^2$. Thus $|f|^2=\psi fg$ and $fg\in H^1$ because $|\psi|=\|K_\phi\|$ a.e. (cf. [**6**, p. 133]). Hence $\bar{\psi}|f|^2\in S_\phi$ and $S_\phi\supset\{\bar{\psi}|f|^2\in S^1\,;\,f\in\ker T^2_{\bar{z}\psi}\}$. If $F\in S_\phi$, then $\psi F\geqq 0$ (cf. [**4**, p. 133]) and $\psi F=\psi qh^2=h\bar{h}$ where $F=qh^2$ denotes an inner outer factorization. Hence $\bar{z}\psi h=\bar{z}\bar{q}\bar{h}$ and so $h\in\ker T^2_{\bar{z}\psi}$. F has the form $F=\bar{\psi}|h|^2$ and this implies the proposition.

Let ψ be an extremal kernel of K_ϕ. By Proposition 8 $S_\phi\neq\emptyset$ if and only if $\ker T_{\bar{z}\psi}\neq\{0\}$. Hence by Proposition 1 $\ker T_\phi\neq\{0\}$ if and only if $S_{z\Phi}\neq\emptyset$ and $z\Phi$ is an extremal kernel, where $|\phi|=|h|$ for some outer function h in H^∞ and $\Phi=\phi h/|\phi h|$. If q is an inner function, then $\bar{q}$ is an extremal kernel and $S_{\bar{q}}\neq\emptyset$. By Proposition 10, $S_{\bar{q}}=\{q|f|^2\in S^1\,;\,f\in\ker T_{\bar{z}\bar{q}}\}$. Hence $S_{\bar{q}}=\{q|f|^2\in S^1\,;\,f\in H^2\ominus zqH^2\}$ because $\ker T_{\bar{z}\bar{q}}=H^2\ominus zqH^2$. If $\phi=\bar{z}^n$ for some $n\in\boldsymbol{Z}_+$, then

$$S_\phi=\{z^n|p|^2\in S^1\,;\,p\in\mathcal{P}_n\}=\{\gamma\prod_{j=1}^{n}(z-a_j)(1-\bar{a}_jz)\in S^1\,;\,\gamma>0\text{ and }|a_j|\leqq 1\}.$$

For a general inner function q, we know the structure of $H^2\ominus zqH^2$ by Ahern and Clark [**2**] and hence that of $S_{\bar{q}}$.

COROLLARY 5. *If $\phi=\bar{z}^n|k|/k$ for some $n\in\boldsymbol{Z}_+$ and some 1-strong outer function k in H^1, then*

$$S_\phi=(\{\gamma\}\times S_{\bar{z}^n}\times k)\cap S^1$$

where $\{\gamma\}$ *denotes the set of all positive numbers.*

PROOF. It is clear that S_{ϕ} is not empty and ϕ is an extremal kernel. So by Proposition 10, $S_{\phi}=\{z^{n}(k/|k|)|f|^{2}\in S^{1};\ f\in\ker T_{\bar{z}\phi}\}$. On the other hand, by Theorem 7 $\ker T_{\bar{z}\phi}^{2}=\{pg;\ p\in\mathcal{P}_{n}\}$, where $g^{2}=k$. Thus $S_{\phi}=\{z^{n}|p|^{2}k\in S^{1};\ p\in\mathcal{P}_{n}\}$ and the corollary follows.

Corollary 5 is known ([**10**]) and it implies that if S_{ϕ} is weak*-compact and nonempty, then S_{ϕ} has the form in Corollary 5.

7. Hankel operators.

Let Q be an orthogonal projection from L^{2} to $\overline{H_{0}^{2}}$. Let $\phi\in L^{\infty}$. We define the Hankel operator H_{ϕ} on H^{2} by

$$H_{\phi}f=Q(\phi f)\,.$$

We investigate the set $\max H_{\phi}=\{f\in H^{2};\ \|H_{\phi}f\|_{2}=\|H_{\phi}\|\|f\|_{2}\}$.

PROPOSITION 11. *Let* ϕ *be in* L^{∞} *and* $\psi\in\phi+H^{\infty}$ *and* $\|\phi+H^{\infty}\|=\|\psi\|_{\infty}$. *Suppose* $\max H_{\phi}\neq\emptyset$. *Then*

$$\max H_{\phi}=\ker T_{\psi}^{2}\,.$$

PROOF. It is well known that $\|H_{\phi}\|=\|\phi+H^{\infty}\|$ and $H_{\phi}=H_{\psi}$, and if $\max H_{\phi}\neq\emptyset$, then $|\psi|=\|H_{\phi}\|$ a.e. (cf. [**1**]) so that $\|T_{\psi}f\|_{2}^{2}+\|H_{\psi}f\|_{2}^{2}=\|H_{\phi}\|\|f\|_{2}^{2}$. Hence $\max H_{\phi}=\ker T_{\psi}$.

REMARK. We note that Theorem 2.2 in [**1**] implies Corollary 5 easily. If $\phi=\bar{z}^{n}|k|/k$ for some $n\in\boldsymbol{Z}_{+}$ and some 1-strong outer function k in H^{1}, then the Hankel operator $H_{\bar{z}\phi}$ has an s-number $\|H_{\bar{z}\phi}\|$ of multiplicity exactly $n+1$, that is, the dimension of the set of eigenvectors of the operator $H_{\bar{z}\phi}{}^{*}H_{\bar{z}\phi}$ corresponding to the eigenvalue $\|H_{\bar{z}\phi}\|^{2}$. For if the s-number of multiplicity is more than $n+2$, then $\bar{z}\phi=\bar{z}^{l}|F|/F$ for some 1-strong outer function $F\in H^{1}$ with $l\geqq n+2$ by Theorem 2.2 in [**1**] because $\|\bar{z}\phi+H^{\infty}\|=1$. Therefore $\phi=\bar{z}^{l-1}|F|/F$ and this contradicts the definition of ϕ. Now Theorem 2.2 in [**1**] implies Corollary 5.

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