J. Math. Soc. Japan
Vol. 38, No. 2, 1986

Induction theorems for equivariant K-theory and J-theory

Dedicated to Professor Nobuo Shimada on his sixtieth birthday

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(Received May 4, 1984)

§ 0. Introduction.

Let G be a group and Λ a G-ring. Then we introduce notions of a family F of ΛG-modules and of F-projective modules. For each family F, we define two kinds of equivariant algebraic K-theories $K^G(\Lambda; F)_d$ and $K^G(\Lambda; F)_e$. We introduced these notions to get equivariant Swan isomorphisms [**10**].

Equivariant algebraic K-theory is studied along the line of Quillen [**17**] by Fiedorowicz, Hauschild and May [**5**], while our approach is along the line of the classical algebraic K-theory [**3**], [**14**] for the purpose of geometric applications.

The purpose of the present paper is to establish induction theorems for our equivariant algebraic and topological K-theories and for equivariant J-theory as promised in [**11**].

We will first show that our equivariant algebraic K-theory is a G-functor in the sense of Green [**6**] in general (see also [**22**]). Accordingly the Dress induction theorem [**4**] is applicable. By a different approach, we have the Swan type induction theorems [**20**] for the equivariant algebraic K-theory associated with the largest family F_a (for the definition of families see § 1).

Next we study the relation between the Grothendieck group of representations over G-rings and the cohomology of groups with coefficients in non-abelian groups in the sense of Serre [**18**]. Consequently we can express the equivariant algebraic K-theory in terms of the cohomology in some special cases. An interesting example is provided by Serre [**18**]. In fact, the example was a starting point of the present investigation. Moreover the observation of the relation above will be employed to prove the Swan type induction theorems for the equivariant algebraic K-theory associated with the family F_t.

On the other hand we define an induction homomorphism for equivariant topological K-theory which corresponds to that for equivariant algebraic K-theory

This work was partially supported by Grant-in-Aid for Scientific Research (No. 59460001), Ministry of Education, Science and Culture.

via the equivariant Swan isomorphism [**10**], [**21**]. Hence we have the Swan type induction theorems for equivariant topological K-theories $KO_G(X)$, $K_G(X)$ and $KSp_G(X)$ where X is a compact G-space.

By showing a relative Frobenius reciprocity formula, we have that the Atiyah-Singer index homomorphisms [**2**] commute with our induction homomorphisms.

Lastly we have a Dress type hyperelementary induction theorem for the equivariant J-theory [**8**]. One of its applications is provided by T. Petrie.

In [**20**], Swan obtained induction theorems for some Grothendieck group $G(\Lambda\pi)$ where a group π acts trivially on Λ. In our case, a group G acts non trivially on Λ in general. According to Swan [**20**], an induction theorem for a Frobenius functor will automatically imply induction and restriction theorems for a Frobenius module over the Frobenius functor (see also [**12**]). Moreover he had an induction theorem for some Frobenius functor. Hence our task for the proof of the Swan type induction theorems is to show that our equivariant algebraic K-theories are Frobenius modules over the Frobenius functor due to Swan. However the multiplication of the module structure is not well-defined unfortunately for a general family F. Namely the key step is to show that the multiplication is well-defined and the consideration of cohomology of groups answers the purpose.

Once we conjecture the present results and become aware of the formulations, the proofs are somewhat easy. So we omit the proofs occasionally.

The author wishes to thank Professors T. Petrie, T. Yoshida and A. Gyōja for stimulating discussions.

§1. Families and equivariant algebraic K-theory.

The word *ring* will always mean associative ring with an identity element 1. Let G be a group. A *G-ring* is a ring Λ together with a G-action on Λ preserving the ring structure. When Λ is a G-ring, a *ΛG-module* is a module M over Λ together with a G-action on M such that

$$(*)\qquad g(\lambda_1 m_1+\lambda_2 m_2)=(g\lambda_1)(gm_1)+(g\lambda_2)(gm_2) \qquad \text{for any } g\in G,\ \lambda_i\in\Lambda,\ m_i\in M.$$

A collection F of ΛG-modules which are finitely generated over Λ is called a *family* if the following holds:

"if $M_1, M_2\in F$, then there exists an element $N\in F$ such that $M_1\oplus M_2$ is a direct summand of N".

When Λ is a commutative G-ring, we can consider a product of two ΛG-modules as follows. If M_1 and M_2 are ΛG-modules, define $M_1\otimes M_2$ to be $M_1\otimes_\Lambda M_2$ as a Λ-module with G-action by $g(m_1\otimes m_2)=gm_1\otimes gm_2$ for $g\in G$, $m_i\in M_i$. Then a collection F of ΛG-modules which are finitely generated over Λ is called a *multiplicative family* if in addition to the above condition the following holds;

"if $M_1, M_2 \in F$, then there exists an element $N \in F$ such that $M_1 \otimes M_2$ is a direct summand of N".

Each element of F is called *F-free*. A ΛG-module M is called *F-projective*, if there exists a ΛG-module N so that $M \oplus N$ is F-free.

We now introduce two kinds of equivariant algebraic K-groups as follows. For each family F, $K^G(\Lambda; F)_d$ (resp. $K^G(\Lambda; F)_e$) is defined to be the abelian group given by generators $[P]$ where P is an F-projective ΛG-module, with relations

$$[P]=[P']+[P'']$$

whenever $P \cong P' \oplus P''$ (resp. $0 \to P' \to P \to P'' \to 0$ is an exact sequence of ΛG-modules).

If Λ is a commutative G-ring and if F is a multiplicative family of ΛG-modules, the product above induces a structure of commutative ring in $K^G(\Lambda; F)_d$ (not in $K^G(\Lambda; F)_e$ in general).

$K^G(\ ;\ )_d$ is a covariant functor from pairs of G-rings and families to abelian groups, while $K^G(\ ;\ )_e$ is not a functor in general, since the tensor product $\Lambda' \otimes_\Lambda$ will not preserve the exactness in general.

Next we introduce a *twisted group ring* $\widetilde{\Lambda G}$. As an additive group, $\widetilde{\Lambda G}$ is the ordinary group ring and the multiplication is given by

$$(\sum_g \lambda_g g) \circ (\sum_{g'} \lambda_{g'} g') = \sum_{g, g'} \lambda_g (g \lambda_{g'}) g g'$$

for $g, g' \in G$, $\lambda_g, \lambda_{g'} \in \Lambda$. It is quite easy to see the following

LEMMA 1.1. *The notion of ΛG-modules coincides with that of $\widetilde{\Lambda G}$-modules. In particular, $\widetilde{\Lambda G}$ is a ΛG-module.*

Hereafter we omit $\sim$ from $\widetilde{\Lambda G}$ for notational convenience.

Let H be a subgroup of G of finite index and Λ be a G-ring, which is also regarded as an H-ring by restriction. Since ΛH is a subring of ΛG, ΛG can be regarded as a right ΛH-module. For a ΛH-module M, we define an *induced ΛG-module* $\mathrm{Ind}_H^G M$ by

$$\mathrm{Ind}_H^G M = \Lambda G \underset{\Lambda H}{\otimes} M.$$

On the other hand, any ΛG-module M can be regarded as a ΛH-module $\mathrm{Res}_H M$ by restriction. Let $i: H \to G$ be the inclusion map. Then we sometimes denote $\mathrm{Ind}_H^G M$ (resp. $\mathrm{Res}_H M$) by $i_* M$ (resp. $i^* M$) for convenience' sake.

For each subgroup H of G of finite index, we consider a family $F(H)$ of ΛH-modules which are finitely generated over Λ. The collection $\{F(H)\}$ of such families is denoted by F and is also called a family. Then we set

$$K^H(\Lambda; F)_\varepsilon = K^H(\Lambda; F(H))_\varepsilon \qquad \text{for } \varepsilon = d \text{ or } e.$$

We call F a *closed family* if for any $M\in F(H)$ (resp. $M\in F(G)$), $\mathrm{Ind}_H^G M$ (resp. $\mathrm{Res}_H M$) is $F(G)$-projective (resp. $F(H)$-projective). Since ΛG is a finitely generated free ΛH-module, we have induction and restriction homomorphisms:

$$i_*=\mathrm{Ind}_H^G : K^H(\Lambda; F)_\varepsilon \longrightarrow K^G(\Lambda; F)_\varepsilon$$

$$i^*=\mathrm{Res}_H : K^G(\Lambda; F)_\varepsilon \longrightarrow K^H(\Lambda; F)_\varepsilon$$

for a closed family F where $\varepsilon=d$ or e.

We now give examples of closed families of a finite group G which will be used in the sequel:

$$F_a=\{F_a(H):\text{ all }\Lambda H\text{-modules} \mid H\leqq G\}$$

$$F_t=\{F_t(H)=\{(\Lambda H)^n \mid n=1, 2, \cdots\} \mid H\leqq G\}$$

$$F_{tf}=\{F_{tf}(H):\text{ all torsion free }\Lambda H\text{-modules} \mid H\leqq G\}$$

$$F_f=\{F_f(H):\text{ all }\Lambda H\text{-modules which are free over }\Lambda \mid H\leqq G\}.$$

Here all ΛH-modules are assumed to be finitely generated.

Denote by $K_0(\ )$ the ordinary algebraic K_0 group [**14**].

PROPOSITION 1.2. *When G is a finite group, we have the following isomorphisms of abelian groups:*

$$K^G(\Lambda; F_t)_d \underset{(\mathrm{I})}{\cong} K^G(\Lambda; F_t)_e \cong K_0(\Lambda G).$$

If Λ is commutative, (I) *is an isomorphism of rings.*

PROOF. This is an immediate consequence of Lemma 1.1.

REMARK 1.3. Proposition 1.2 implies that our definition of an equivariant algebraic K-group includes $K_0(\Lambda G)$ as a special case. However $K_0(\Lambda G)$ is insufficient as an equivariant algebraic K_0-theory for various reasons. The following is one of them. When G is not a finite group, the notion of "ΛG-projective modules" is unsuitable for the equivariant Swan isomorphism [**10**]. For this reason, we first introduced the notions of families F and F-projective modules. Moreover our definition includes $G(R\pi)$ and $G'(R\pi)$ of Swan [**20**] as special cases as follows. If a group π acts trivially on a ring R, then our definition is related with that of Swan by

$$K^\pi(R; F_a)_e=G(R\pi), \qquad K^\pi(R; F_{tf})_e=G'(R\pi).$$

It will be useful to notice the following;

PROPOSITION 1.4. *If a G-ring Λ is semi-simple and contains $1/|G|$, then we have an isomorphism*

$$K^G(\Lambda; F)_d \cong K^G(\Lambda; F)_e$$

for any family F where $|G|$ denotes the order of G.

PROOF. Since Λ is semi-simple, every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of ΛG-modules is split exact as Λ-modules. Since $\Lambda \ni 1/|G|$, we can change the splitting into that of ΛG-modules by the averaging argument. Hence the short exact sequence relation coincides with the direct sum one. This completes the proof.

In particular, we have

COROLLARY 1.5. *If Λ is a G-field such that the characteristic of Λ is zero or prime to $|G|$, then*

$$K^G(\Lambda; F)_d \cong K^G(\Lambda; F)_e$$

for any family F.

§ 2. Shapiro isomorphism, Mackey and Frobenius properties.

Let H be a subgroup of G of finite index. Fix a set $\{\sigma\}$ of coset representatives for G/H (denoted $\{\sigma\}=G/H$). We now introduce the following notations. For $g \in G$, there exist unique $\sigma(g, \sigma) \in \{\sigma\}$ and $h(g, \sigma) \in H$ such that

$$g\sigma = \sigma(g, \sigma)h(g, \sigma).$$

Given an H-ring Λ, we construct an *induced G-ring* $\mathrm{Ind}_H^G \Lambda$ as follows. Denote by Λ_σ copies of Λ indexed by the set $\{\sigma\}$. As a ring $\mathrm{Ind}_H^G \Lambda$ is the direct sum $\bigoplus_\sigma \Lambda_\sigma$. A G-action is given by

$$g \circ (\bigoplus_\sigma \lambda_\sigma) = \bigoplus_{\sigma(g, \sigma)} h(g, \sigma)\lambda_\sigma$$

for $g \in G$, $\lambda_\sigma \in \Lambda_\sigma$. Here the right hand side means that we put $h(g, \sigma)\lambda_\sigma$ to the $\sigma(g, \sigma)$ factor. It is easy to see that $\mathrm{Ind}_H^G \Lambda$ becomes a G-ring. Note that $\mathrm{Ind}_H^G \Lambda$ is isomorphic to $Z[G] \otimes_{Z[H]} \Lambda$ as additive G-groups. The latter, however, is just an additive G-group (not a G-ring!).

Then we have the following "Shapiro isomorphism".

PROPOSITION 2.1. *There is a one to one correspondence between the set of isomorphism classes of ΛH-modules and the set of isomorphism classes of* ($\mathrm{Ind}_H^G \Lambda$)*G-modules. In particular, we have an isomorphism*

$$\Phi_1 : K^H(\Lambda; F_a)_\varepsilon \longrightarrow K^G(\mathrm{Ind}_H^G \Lambda; F_a)_\varepsilon$$

for $\varepsilon = d$ or e.

PROOF. For a ΛH-module M, we set $\Phi_1(M) = \bigoplus_\sigma M_\sigma$ where M_σ are copies of M indexed by the set $\{\sigma\}$. An $\mathrm{Ind}_H^G \Lambda$-module structure is given by

$$(\bigoplus_\sigma \lambda_\sigma)\circ(\bigoplus_\sigma m_\sigma)=\bigoplus_\sigma \lambda_\sigma m_\sigma \qquad \text{for } m_\sigma\in M_\sigma .$$

A G-action is given by

$$g\circ(\bigoplus_\sigma m_\sigma)=\bigoplus_{\sigma(g,\sigma)} h(g,\sigma)m_\sigma .$$

With these definitions, $\Phi_1(M)$ becomes an $(\mathrm{Ind}_H^G \Lambda)G$-module. It is easily seen that the correspondence $M\mapsto\Phi_1(M)$ gives rise to the required one.

If Λ is a G-ring, we define a homomorphism

$$\Phi_2 : K^G(\mathrm{Ind}_H^G\Lambda ; F_\alpha)_\varepsilon \longrightarrow K^G(\Lambda ; F_\alpha)_\varepsilon$$

as follows. For an $(\mathrm{Ind}_H^G \Lambda)G$-module M, we put a new ΛG-module structure on M by

$$\lambda\circ m=(\bigoplus_\sigma \sigma^{-1}\lambda)\cdot m \qquad \text{for } \lambda\in\Lambda,\ m\in M,$$

$$g\circ m=g\cdot m \qquad \text{for } g\in G,\ m\in M$$

where $\cdot$ denote the old operations, while $\circ$ denote the new ones. The correspondence $[M]\mapsto[M]$ gives rise to the above homomorphism Φ_2.

LEMMA 2.2. *When Λ is a G-ring, the composition $\Phi_2\cdot\Phi_1$ of the above two homomorphisms is nothing but the induction homomorphism* Ind_H^G *in* § 1.

PROOF. Let M be a ΛH-module. Then

$$\Phi_2\cdot\Phi_1(M)=\bigoplus_\sigma M_\sigma$$

and the G-action is given by

$$g\circ(\bigoplus_\sigma m_\sigma)=\bigoplus_{\sigma(g,\sigma)} h(g,\sigma)m_\sigma$$

and the Λ operation is given by

$$\lambda\circ(\bigoplus_\sigma m_\sigma)=\bigoplus_\sigma(\sigma^{-1}\lambda)m_\sigma$$

for $g\in G$, $m_\sigma\in M_\sigma$, $\lambda\in\Lambda$. We now define a map

$$f : \bigoplus_\sigma M_\sigma \longrightarrow \Lambda G\bigotimes_{\Lambda H} M$$

by $f(\bigoplus_\sigma m_\sigma)=\sum_\sigma\sigma\otimes m_\sigma$. It is easy to see that f gives the required isomorphism.

Let Λ be a G-ring. Let H and K be subgroups of G and $\{s\}$ a set of double coset representatives for $K\backslash G/H$ (denoted $\{s\}=K\backslash G/H$). We may assume that $\{s\}$ is a subset of $\{\sigma\}=G/H$. Set $H_s=sHs^{-1}\cap K$. For a ΛH-module M, we construct a ΛH_s-module M_s as follows. As an additive group, M_s is given by M itself and a ΛH_s-module structure is given by

$$g\circ m_s=(s^{-1}gs)\cdot m_s \qquad \text{for } g\in G,\ m_s\in M_s,$$

and

$$\lambda\circ m_s=(s^{-1}\lambda)\cdot m_s \qquad \text{for } \lambda\in\Lambda,\ m_s\in M_s$$

where $\cdot$ denote the old operations, while $\circ$ denote the new ones. With these definitions, M_s becomes a ΛH_s-module and we have

PROPOSITION 2.3 (Mackey decomposition).

$$\mathrm{Res}_K \mathrm{Ind}_H^G M \cong \bigoplus_{s\in K\backslash G/H} \mathrm{Ind}_{H_s}^K M_s .$$

PROOF. Paying attention to the G-action on Λ, we can give an explicit ΛK-module isomorphism by virtue of Lemma 2.2.

REMARK 2.4. It follows from Proposition 2.3 that $K^G(\Lambda; F)_\varepsilon$ is a G-functor in the sense of Green [6] and the Dress induction theorem is applicable to $K^G(\Lambda; F)_\varepsilon$ for any closed family F.

PROPOSITION 2.5 (Frobenius reciprocity). *Let Λ be a commutative G-ring and H a subgroup of G. Let V be a ΛH-module and W a ΛG-module. Then*

$$\mathrm{Ind}_H^G (V\otimes_\Lambda \mathrm{Res}_H W) \cong (\mathrm{Ind}_H^G V)\otimes_\Lambda W$$

as ΛG-modules.

PROOF. Define

$$f\ :\ \Lambda G\otimes_{\Lambda H}(V\otimes_\Lambda \mathrm{Res}_H W) \longrightarrow (\Lambda G\otimes_{\Lambda H} V)\otimes_\Lambda W$$

by

$$f(g\otimes(v\otimes w))=(g\otimes v)\otimes gw$$

for $g\in G$, $v\in V$, $w\in W$. Paying attention to the G-action on Λ, we can prove that f is well-defined and gives the required isomorphism.

§ 3. *GR*-algebras and Frobenius modules.

In this section, we introduce notions of GR-algebras and Frobenius modules for the purpose of induction theorems in § 4 and § 6.

DEFINITION 3.1. Let R be a commutative G-ring. Then a G-ring Λ is called a *GR-algebra* if Λ is an RG-module as well as an R-algebra.

REMARK 3.2. Our GR-algebra is different from an algebra over the twisted group ring RG.

Let Λ be a GR-algebra. Let A (resp. B) be an RG- (resp. ΛG-) module. Define $A\otimes B$ to be $A\otimes_R B$ where R acts on B by

$$r\circ b=(r\cdot 1)b \qquad \text{for } r\in R,\ b\in B,\ 1\in\Lambda .$$

We now set

$$(\sum_g \lambda_g g)\circ(a\otimes b)=\sum_g (ga\otimes\lambda_g gb)$$

for $\lambda_g\in\Lambda$, $g\in G$, $a\in A$, $b\in B$. Since Λ is a GR-algebra, one verifies that the operation $\circ$ is well-defined and gives a ΛG-module structure on $A\otimes B$.

Then we have the following equivariant version of Lemma 3.1 of [**20**].

LEMMA 3.3. *Let R be a Dedekind G-ring and Λ a GR-algebra. Then $K^G(\Lambda; F_a)_\varepsilon$ is a module over $K^G(R; F_a)_\varepsilon$. If $i: H\subset G$, then*

(i) $i^*(x\cdot y)=i^*(x)\cdot i^*(y)$ *for* $x\in K^G(R; F_a)_\varepsilon$, $y\in K^G(\Lambda; F_a)_\varepsilon$,

(ii) $i_*(i^*(x)\cdot y)=x\cdot i_*(y)$ *for* $x\in K^G(R; F_a)_\varepsilon$, $y\in K^H(\Lambda; F_a)_\varepsilon$,

(iii) $i_*(x\cdot i^*(y))=i_*(x)\cdot y$ *for* $x\in K^H(R; F_a)_\varepsilon$, $y\in K^G(\Lambda; F_a)_\varepsilon$.

PROOF. We consider only the case where $\varepsilon=e$, since the proof is easier for $\varepsilon=d$. It is easy to see that the equivariant versions of Propositions 1.1 and 1.2 and Corollaries 1.1 and 1.3 in [**20**] hold for a Dedekind G-ring R. In particular, $K^G(R; F_a)_e$ is a commutative ring for a Dedekind G-ring R and it is sufficient to make $K^G(\Lambda; F_a)_e$ a module over $K^G(R; F_{tf})_e$. This is done by setting $[A]\cdot[B]=[A\otimes_R B]$. Since A is a torsion free RG-module, A is projective over R. Hence the multiplication $[A]\cdot[B]$ is well-defined.

We now prove the assertion (iii). Let A (resp. B) be an RH- (resp. ΛG-) module. Define

$$f: \Lambda G\otimes_{\Lambda H}(A\otimes_R B)\longrightarrow (RG\otimes_{RH} A)\otimes_R B$$

by

$$f((\sum_g \lambda_g g)\otimes(a\otimes b))=\sum_g (g\otimes a)\otimes\lambda_g gb$$

for $g\in G$, $\lambda_g\in\Lambda$, $a\in A$, $b\in B$. Since G acts non trivially on R and on Λ in general, it is not obvious that f is well-defined. In the following, we give a portion of its proof. For $g\in G$, $h\in H$, λ_g, $\lambda_h\in\Lambda$, $a\in A$, $b\in B$, $r\in R$, we have four expressions for an element:

$$(\sum_g \lambda_g g)\cdot(\sum_h \lambda_h h)\otimes(a\otimes(r\cdot 1)b) \quad\cdots\cdots(\mathrm{I})$$

$$(\sum_g \lambda_g g)\otimes(\sum_h \lambda_h h)\cdot(a\otimes(r\cdot 1)b) \quad\cdots\cdots(\mathrm{II})$$

$$(\sum_g \lambda_g g)\cdot(\sum_h \lambda_h h)\otimes(ra\otimes b) \quad\cdots\cdots(\mathrm{III})$$

$$(\sum_g \lambda_g g)\otimes(\sum_h \lambda_h h)\cdot(ra\otimes b) \quad\cdots\cdots(\mathrm{IV}).$$

Then we have

$$\begin{aligned} f(\mathrm{I})&=f(\sum_{g,h}\lambda_g(g\lambda_h)gh\otimes a\otimes(r\cdot 1)b)\\ &=\sum_{g,h} gh\otimes a\otimes\lambda_g(g\lambda_h)gh((r\cdot 1)b)\end{aligned}$$

$$=\sum_{g,h} g\otimes ha\otimes((gh)r)\cdot\lambda_g(g\lambda_h)(gh)b$$
$$=\sum_{g,h}((gh)r)g\otimes ha\otimes\lambda_g(g\lambda_h)(gh)b$$
$$=\sum_{g,h} g((hr)e)\otimes ha\otimes\lambda_g g(\lambda_h hb)$$
$$=\sum_{g,h} g\otimes(hr)(ha)\otimes\lambda_g g(\lambda_h hb)$$
$$=f(\sum_{g,h}\lambda_g g\otimes h(ra)\otimes\lambda_h hb)$$
$$=f((\sum_g\lambda_g g)\otimes\sum_h(h(ra)\otimes\lambda_h hb))$$
$$=f((\sum_g\lambda_g g)\otimes(\sum_h\lambda_h h)\cdot(ra\otimes b))=f(\mathrm{VI}).$$

Similarly we can prove that

$$f(\mathrm{I})=f(\mathrm{II})=f(\mathrm{III}).$$

Thus f is well-defined. Next we show that f is a ΛG-module homomorphism. For $g, g'\in G$, $\lambda_g, \lambda_{g'}\in\Lambda$, $a\in A$, $b\in B$, we have

$$f((\sum_{g'}\lambda_{g'}g')\cdot(\sum_g\lambda_g g\otimes a\otimes b))$$
$$=f(\sum_{g',g}\lambda_{g'}(g'\lambda_g)g'g\otimes a\otimes b)$$
$$=\sum_{g',g} g'g\otimes a\otimes\lambda_{g'}(g'\lambda_g)((g'g)b)$$
$$=\sum_{g',g} g'g\otimes a\otimes\lambda_{g'}g'(\lambda_g gb)$$
$$=(\sum_{g'}\lambda_{g'}g')\cdot(\sum_g g\otimes a\otimes\lambda_g gb)$$
$$=(\sum_{g'}\lambda_{g'}g')\cdot f(\sum_g\lambda_g g\otimes a\otimes b).$$

Define

$$f' : (RG\underset{RH}{\otimes}A)\underset{R}{\otimes}B \longrightarrow \Lambda G\underset{\Lambda H}{\otimes}(A\underset{R}{\otimes}B)$$

by

$$f'((\sum_g r_g g)\otimes a\otimes b)=\sum_g(r_g\cdot 1)g\otimes a\otimes g^{-1}b$$

for $g\in G$, $r_g\in R$, $a\in A$, $b\in B$. One verifies that f is well-defined and satisfies

$$f'\cdot f=\text{identity} \qquad \text{and} \qquad f\cdot f'=\text{identity}.$$

Hence we have the assertion (iii). The assertion (ii) will be shown similarly while (i) is trivial.

This makes the proof of Lemma 3.3 complete.

REMARK 3.4. A module with the property (iii) in Lemma 3.3 is called a Frobenius module [**12**].

Let G be a finite group and Λ a G-ring. Let S be some class of subgroups of G. For a closed family F, we define $K_S^G(\Lambda; F)_\varepsilon$ to be the sum of the images of the maps

$$i_* : K^H(\Lambda; F)_\varepsilon \longrightarrow K^G(\Lambda; F)_\varepsilon \qquad \text{for all } i: H \subset G \text{ with } H \in S.$$

Let k be an integer. Following Swan [20], we say $K_S^G(\Lambda; F)_\varepsilon$ has exponent k in $K^G(\Lambda; F)_\varepsilon$ if

$$k \cdot K^G(\Lambda; F)_\varepsilon \subset K_S^G(\Lambda; F)_\varepsilon .$$

COROLLARY 3.5. $K_S^G(R; F_a)_\varepsilon \cdot K^G(\Lambda; F_a)_\varepsilon \subset K_S^G(\Lambda; F_a)_\varepsilon$.

COROLLARY 3.6. *If $K_S^G(R; F_a)_\varepsilon$ has exponent k in $K^G(R; F_a)_\varepsilon$, then $K_S^G(\Lambda; F_a)_\varepsilon$ has exponent k in $K^G(\Lambda; F_a)_\varepsilon$.*

REMARK 3.7. For a general family F, the multiplication above does not induce a multiplication

$$K^G(R; F)_\varepsilon \times K^G(\Lambda; F)_\varepsilon \longrightarrow K^G(\Lambda; F)_\varepsilon$$

in general, even if R is a Dedekind G-ring. In § 6, we deal with two special families F_t and F_f in which case the multiplication above is well-defined.

The following lemma is well-known for a Frobenius module (see Theorem 9.2 of [20]).

LEMMA 3.8. *Suppose that $K_S^G(\Lambda; F)_\varepsilon$ has exponent k in $K^G(\Lambda; F)_\varepsilon$. If $i^*(x) = 0$ for all $i: H \subset G$ with $H \in S$, then $kx = 0$.*

§ 4. Induction and restriction theorems for $K^G(\Lambda; F_a)_\varepsilon$.

We recall the following terminology. A finite group is called *elementary* if it is the direct product of a p-group and a cyclic group. A finite group is called *hyperelementary* if it has a cyclic normal subgroup such that the quotient of the group by this subgroup is a p-group.

If G is any finite group, C will denote the class of all cyclic subgroups of G, E will denote the class of all elementary subgroups of G, while HE will denote the class of all hyperelementary subgroups of G.

Let n be the order of G. Denote by $a(G)$ the *Artin exponent* for G in the sense of Lam [12]. Note that $a(G)$ divides n. We write $d = (a(G), \phi(n))$ where ϕ is the Euler function.

Denote by Q, Z or Z_p (p: prime) the field of rational numbers, the ring of integers, the field of integers modulo p respectively. Let G act trivially on them.

DEFINITION 4.1. I will say a G-ring Λ contains a *primitive n-th root of unity in the centre* if there is an element x in the intersection of the fixed point set Λ^G and the centre of Λ such that $\Phi_n(x) = 0$, Φ_n being the n-th cyclotomic polynomial.

THEOREM 4.2. *For any G-ring Λ, we have*

(a) $K_C^G(\Lambda; F_a)_e$ *has exponent* $a(G)^2$ *in* $K^G(\Lambda; F_a)_e$,

(b) $K_E^G(\Lambda; F_a)_e$ *has exponent* d^2 *in* $K^G(\Lambda; F_a)_e$,

(c) $K_{HE}^G(\Lambda; F_a)_e = K^G(\Lambda; F_a)_e$.

If Λ is a GQ- or GZ_p-algebra, we can replace $a(G)^2$ and d^2 in (a) *and* (b) *by $a(G)$ and d. If Λ contains a primitive n-th root of unity in the center, we can replace d^2 in* (b) *by* 1.

PROOF. Let G act trivially on $Z[\zeta]$ ($\zeta = \exp 2\pi i/n$). It follows from [**12**] and [**20**] that Theorem 4.2 holds for $\Lambda = Z$, Q, Z_p and $Z[\zeta]$. Note that any G-ring is a GZ-algebra. If Λ contains a primitive n-th root of unity in the center, then Λ is a $GZ[\zeta]$-algebra. Hence Theorem 4.2 follows from Corollary 3.6.

For a subgroup H of G, let $i_H: H \to G$ be the inclusion map. Then for a class S of subgroups of G, we set

$$\mathrm{Res}_S = \prod_{H \in S} i_H^* : \; K^G(\Lambda; F)_\varepsilon \longrightarrow \prod_{H \in S} K^H(\Lambda; F)_\varepsilon .$$

By combining Lemma 3.8 with Theorem 4.2, we have

THEOREM 4.3. *For $F = F_a$ and $\varepsilon = e$, we have*

(a) $a(G)^2 \mathrm{Ker}\, \mathrm{Res}_C = 0$,

(b) $d^2 \mathrm{Ker}\, \mathrm{Res}_E = 0$,

(c) $\mathrm{Ker}\, \mathrm{Res}_{HE} = 0$.

If Λ is a GQ- or GZ_p-algebra, we can replace $a(G)^2$ and d^2 in (a) *and* (b) *by $a(G)$ and d. Moreover if Λ contains a primitive n-th root of unity in the centre, then* $\mathrm{Ker}\, \mathrm{Res}_E = 0$.

§ 5. Representations over G-rings and Galois cohomology.

In this section, we introduce a group $R(G, \Lambda)$ which is a generalization of the representation ring $R(G)$ and we express $R(G, \Lambda)$ in terms of the cohomology $H^1(G; \Gamma)$ of a group with coefficients in a non-abelian group (see [**11**] and [**18**]). This observation will be used to prove induction theorems for the equivariant K-theory associated with the family F_t in § 6.

$R(G, \Lambda)$ is defined to be the abelian group given by generators $[M]$ where M is a finitely generated ΛG-module which is free as a Λ-module, with relations $[M] = [M'] + [M'']$ whenever $M \cong M' \oplus M''$.

Let us recall the cohomology $H^1(G;\Gamma)$ of Serre [18]. A *G-group* is a group Γ together with a G-action preserving the group structure. Then a map $A: G\to\Gamma$ is called a *cocycle* if $A(g'g)=a(g')\cdot(g'A(g))$ for any $g, g'\in G$. Set

$$Z^1(G;\Gamma)=\{A: G\to\Gamma \text{ cocycle}\}.$$

Two elements $A, B\in Z^1(G;\Gamma)$ are *cohomologous* (denoted by $A\sim B$) if and only if there exists $C\in\Gamma$ such that

$$B(g)=C^{-1}\cdot A(g)\cdot(gC) \qquad \text{for any } g\in G.$$

Then $H^1(G;\Gamma)$ is defined to be the quotient set $Z^1(G;\Gamma)/\sim$.

Let $GL(n, \Lambda)$ be the group of invertible $n\times n$ matrices over Λ. The G-action on each entry of a matrix induces a G-action on $GL(n, \Lambda)$, which makes $GL(n, \Lambda)$ a G-group.

If the ring Λ is such that, given m, $n>0$, $\Lambda^m\cong\Lambda^n$ (forgetting G-action) only if $m=n$, we say that Λ has *invariant basis number* (IBN).

THEOREM 5.1. *Suppose that Λ has IBN. Let M be a free Λ-module of rank n. Then the isomorphism classes of ΛG-module structures on M are in one to one correspondence with $H^1(G; GL(n, \Lambda))$.*

PROOF. Choose a basis $\{e_i\}$ for M over Λ. Given a ΛG-module structure on M, the G-action is completely described by the matrix

$$A(g)=(\alpha_{ij}(g))$$

over Λ, where

$$ge_i=\sum_j \alpha_{ij}(g)e_j.$$

Following our definition of a ΛG-module, we have

$$\begin{aligned} g'(ge_i)&=g'\sum_j \alpha_{ij}(g)e_j=\sum_j (g'\alpha_{ij}(g))g'e_j \\ &=\sum_j (g'\alpha_{ij}(g))\sum_k \alpha_{jk}(g')e_k \\ &=\sum_k\{\sum_j (g'\alpha_{ij}(g))\alpha_{jk}(g')\}e_k. \end{aligned}$$

On the other hand, we have

$$(g'g)e_i=\sum_k \alpha_{ik}(g'g)e_k.$$

Since they must coincide, we have the following equality

$$(1) \qquad A(g'g)=(g'A(g))\cdot A(g') \qquad \text{for any } g, g'\in G.$$

In particular, we have

$$\begin{aligned} I=gI=gA(g^{-1}g)&=g\{(g^{-1}A(g))\cdot A(g^{-1})\} \\ &=A(g)\cdot(gA(g^{-1})) \end{aligned}$$

and

$$I=A(gg^{-1})=(gA(g^{-1}))\cdot A(g).$$

Namely $A(g)$ is an invertible matrix with the two sided inverse matrix

$$A(g)^{-1}=gA(g^{-1}).$$

Thus we have a map

$$A : G \longrightarrow GL(n, \Lambda)$$

with the property (1) above.

Conversely given a map

$$A : G \longrightarrow GL(n, \Lambda)$$

with the property (1) above, we give a G-action on M by

$$g(\sum_i \lambda_i e_i)=\sum_j (\sum_i (g\lambda_i)\alpha_{ij}(g))e_j.$$

It is easy to see that with this definition M becomes a ΛG-module, which is denoted by the pair (M, A). Let (M, B) be another ΛG-module with $B(g)=(\beta_{ij}(g))$. Suppose that we are given a ΛG-module isomorphism

$$f : (M, A) \longrightarrow (M, B),$$

which is completely described by the matrix

$$C=(\gamma_{ij})$$

over Λ, where

$$f(e_i)=\sum_j \gamma_{ij}e_j.$$

Since f is a ΛG-map, we have

$$\begin{aligned}f(ge_i)&=f(\sum_j \alpha_{ij}(g)e_j)=\sum_j \alpha_{ij}(g)f(e_j)\\&=\sum_j \alpha_{ij}(g)\sum_k \gamma_{jk}e_k=\sum_k (\sum_j \alpha_{ij}(g)\gamma_{jk})e_k\\&=gf(e_i)=g(\sum_j \gamma_{ij}e_j)=\sum_j (g\gamma_{ij})ge_j\\&=\sum_j (g\gamma_{ij})\sum_k \beta_{jk}(g)e_k=\sum_k (\sum_j (g\gamma_{ij})\beta_{jk}(g))e_k.\end{aligned}$$

Thus we have

$$A(g)\cdot C=(gC)\cdot B(g) \qquad \text{for any } g\in G.$$

Since C is invertible, we can express it as

$$(2) \qquad A(g)=(gC)\cdot B(g)\cdot C^{-1} \qquad \text{for any } g\in G.$$

Conversely given an invertible matrix C over Λ with the property (2) above, we set

$$f(\sum_i \lambda_i e_i)=\sum_j (\sum_i \lambda_i\gamma_{ij})e_j.$$

It is easy to see that with this definition f gives a ΛG-module isomorphism between (M, A) and (M, B).

We now introduce a new G-group $GL(n, \Lambda)^{\circ}$ as follows. As a G-set, $GL(n, \Lambda)^{\circ}$ is given by $GL(n, \Lambda)$. A new multiplication $A \circ B$ is given by the reversed multiplication $B \cdot A$. Clearly $GL(n, \Lambda)^{\circ}$ becomes a G-group with this definition.

In the above, we have shown that the isomorphism classes of ΛG-module structures on M are in one to one correspondence with $H^1(G; GL(n, \Lambda)^{\circ})$.

Since the correspondence $A \mapsto A^{-1}$ gives rise to an isomorphism

$$f : GL(n, \Lambda)^{\circ} \longrightarrow GL(n, \Lambda)$$

of G-groups, there is a one to one correspondence between

$$H^1(G; GL(n, \Lambda)^{\circ}) \quad \text{and} \quad H^1(G; GL(n, \Lambda)).$$

This completes the proof of Theorem 5.1.

Next we put an abelian semi-group structure on the set

$$\coprod_{n \geq 0} H^1(G; GL(n, \Lambda))$$

where $\coprod_{n \geq 0}$ denotes the disjoint union and we set $H^1(G; GL(0, \Lambda)) = \{0\}$.

Let $A: G \to GL(m, \Lambda)$ and $B: G \to GL(n, \Lambda)$ be cocycles. A summation $A+B: G \to GL(m+n, \Lambda)$ is defined by

$$(A+B)(g) = \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix},$$

and a multiplication $A \times B: G \to GL(mn, \Lambda)$ is defined by

$$(A \times B)(g) = A(g) \otimes B(g)$$

where $\otimes$ denotes the tensor product of matrices. It is easy to see that $A+B$ is again a cocycle and that $A+B \sim B+A$. Moreover if $A \sim A'$ and $B \sim B'$, then we have $A+B \sim A'+B'$. Hence $\coprod_{n \geq 0} H^1(G; GL(n, \Lambda))$ becomes an abelian semi-group. When Λ is commutative, $A \times B$ is again a cocycle and $\coprod_{n \geq 0} H^1(G; GL(n, \Lambda))$ becomes a semi-ring. The Grothendieck group associated with the abelian semi-group above is denoted by

$$K(\coprod_{n \geq 0} H^1(G; GL(n, \Lambda))).$$

PROPOSITION 5.2. *If Λ has IBN, then we have*

$$R(G, \Lambda) \cong K(\coprod_{n \geq 0} H^1(G; GL(n, \Lambda))).$$

When Λ is commutative, both terms have ring structures and $\cong$ stands for a ring isomorphism.

PROOF. Easy and omitted.

Let Λ be a G-ring such that any projective module over Λ is stably free. Then we have easily that

$$K^G(\Lambda; F_f)_d \cong R(G, \Lambda).$$

Hence, in view of [14], we have in particular,

PROPOSITION 5.3. *If a G-ring Λ is a field, a skew field, a principal ideal domain, or a local ring, then we have*

$$K^G(\Lambda; F_f)_d \cong K(\coprod_{n\geq 0} H^1(G; GL(n, \Lambda))).$$

When Λ is commutative, both terms have ring structures and $\cong$ stands for a ring isomorphism.

Let $\boldsymbol{K}/\boldsymbol{k}$ be a Galois extension and G the Galois group of $\boldsymbol{K}/\boldsymbol{k}$. Then $\boldsymbol{K}$ is a G-ring in our sense. According to Serre [18], the first cohomology $H^1(G; GL(n, \boldsymbol{K}))$ vanishes for all n and hence we have

COROLLARY 5.4. *Under the condition above, we have*

$$K^G(\boldsymbol{K}; F_a)_d \cong K^G(\boldsymbol{K}; F_f)_d \cong K^G(\boldsymbol{K}; F_{tf})_d \cong Z.$$

If the characteristic of $\boldsymbol{K}$ is zero or prime to $|G|$, then d in the formula can be replaced by e.

§ 6. Induction theorems for $K^G(\Lambda; F_t)_\varepsilon$ and $K^G(\Lambda; F_f)_e$.

In this section, we shall deal with two special families F_t and F_f, and have induction theorems for $K^G(\Lambda; F_t)_\varepsilon$ and $K^G(\Lambda; F_f)_d$.

As an application, we shall have induction theorems for equivariant topological K-theory *via* the equivariant Swan isomorphism, which will be dealt with in the next section.

First we show the following lemma on which the induction theorem is based.

LEMMA 6.1. *Let R be a commutative G-ring and Λ a GR-algebra. Let A be an RG-module which is free as an R-module. Then $A\otimes_R \Lambda G$ is an F_t-free ΛG-module. Here a ΛG-module structure on $A\otimes_R \Lambda G$ is given by*

$$(\sum_{g'} \lambda_{g'} g')\circ(a\otimes \sum_g \lambda_g g) = \sum_{g'} \{g'a \otimes \sum_g \lambda_{g'}(g'\lambda_g)g'g\}$$

for $g, g' \in G$, $\lambda_g, \lambda_{g'} \in \Lambda$, $a \in A$.

PROOF. Choose a basis $\{e_i | i=1, 2, \cdots, m\}$ for A over R. Then the G-action on A is completely described by the matrix

$$A(g) = (\alpha_{ij}(g)), \qquad \alpha_{ij}(g) \in R$$

over R, where

$$ge_i=\sum_j \alpha_{ij}(g)e_j$$

Define a map

$$\phi : \Lambda\bigotimes_R \Lambda G \longrightarrow (\Lambda G)^m$$

by the correspondence:

$$\sum_i r_i e_i \otimes \sum_g \lambda_g g \longmapsto \bigoplus_j \sum_{g,i} r_i(g\alpha_{ij}(g^{-1}))\lambda_g g$$

for $r_i \in R$, $\lambda_g \in \Lambda$, $g \in G$. It is easy to see that ϕ is well-defined. We now show that ϕ is a ΛG-module homomorphism. By definition, we have

$$\begin{aligned}&(\sum_{g'} \lambda_{g'} g')\circ(\sum_i r_i e_i \otimes \sum_g \lambda_g g)\\ &=\sum_{g'}\{\sum_i (g'r_i)(g'e_i)\otimes\sum_g \lambda_{g'}(g'\lambda_g)g'g\}\\ &=\sum_{g'}\{\sum_j (\sum_i (g'r_i)\alpha_{ij}(g'))e_j\otimes\sum_g \lambda_{g'}(g'\lambda_g)g'g\}\end{aligned}$$

which is mapped by ϕ to

$$\bigoplus_k \sum_{g',g,j,i} (g'r_i)\alpha_{ij}(g')(g'g\alpha_{jk}((g'g)^{-1}))\lambda_{g'}(g'\lambda_g)g'g,$$

which is computed by the observations in §5 as:

$$\begin{aligned}&\bigoplus_k \sum_{g',g,i,l} (g'r_i)\sum_j \alpha_{ij}(g')(g'\alpha_{jl}(g'^{-1}))(g'g\alpha_{lk}(g^{-1}))\lambda_{g'}(g'\lambda_g)g'g\\ &=\bigoplus_k \sum_{g',g,i} (g'r_i)(g'g\alpha_{ik}(g^{-1}))\lambda_{g'}(g'\lambda_g)g'g.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&(\sum_{g'} \lambda_{g'} g')\circ\phi(\sum_i r_i e_i \otimes \sum_g \lambda_g g)\\ &=\bigoplus_j \sum_{g',g,i} \lambda_{g'}[g'\{r_i(g\alpha_{ij}(g^{-1}))\}](g'\lambda_g)g'g\\ &=\bigoplus_j \sum_{g',g,i} \lambda_{g'}(g'r_i)(g'g\alpha_{ij}(g^{-1}))(g'\lambda_g)g'g.\end{aligned}$$

Since Λ is an R-algebra, we may conclude that ϕ is a ΛG-module homomorphism.

Define a map

$$\psi : (\Lambda G)^m \longrightarrow \Lambda\bigotimes_R \Lambda G$$

by the correspondence

$$\bigoplus_i \sum_g \lambda_g^i g \longmapsto \sum_{g,i} g e_i \otimes \lambda_g^i g$$

for $\lambda_g^i \in \Lambda$, $g \in G$. Then it is easily verified that

$$\psi\phi=\text{identity} \qquad \text{and} \qquad \phi\psi=\text{identity}.$$

This completes the proof of Lemma 6.1.

PROPOSITION 6.2. *Let R be a commutative ring with trivial G-action and Λ*

a GR-algebra. Let A (resp. P) be an RG- (resp. ΛG-) module which is R-projective (resp. F_t-projective). Then $A\otimes_R P$ is an F_t-projective ΛG-module.

PROOF. Let A' be an R-module such that $A\oplus A'$ is R-free. Make A' into an RG-module by making G act trivially on A'. Let P' be a ΛG-module such that $P\oplus P'$ is F_t-free. Then it follows from Lemma 6.1 that

$$A\underset{R}{\otimes}P\oplus A\underset{R}{\otimes}P'\oplus A'\underset{R}{\otimes}P\oplus A'\underset{R}{\otimes}P' \cong (A\oplus A')\underset{R}{\otimes}(P\oplus P')$$

is F_t-free. Therefore, $A\otimes_R P$ is F_t-projective.

COROLLARY 6.3. *If R is a Dedekind ring with trivial G-action and if Λ is a GR-algebra, then $K^G(\Lambda;F_t)_e$ is a module over $K^G(R;F_a)_e$. If $i:H\subset G$, i_* and i^* satisfy the equalities* (i), (ii) *and* (iii) *in Lemma* 3.3.

PROOF. According to [20], $K^G(R;F_a)_e\cong K^G(R;F_{tf})_e$. Define

$$K^G(R;F_{tf})_e\otimes K^G(\Lambda;F_t)_e \longrightarrow K^G(\Lambda;F_t)_e$$

by $[A]\otimes[P]\mapsto[A\otimes_R P]$. Since A is torsion free, it is R-projective. Therefore, $A\otimes_R P$ is F_t-projective by Proposition 6.2. The rest of the proof is the same as that of Lemma 3.3.

Hence we deduce the following induction theorem as in the manner of the proofs of Theorems 4.2 and 4.3.

THEOREM 6.4. *The statements in Theorems* 4.2 *and* 4.3 *hold for the families F_t and F_f in place of F_a.*

PROOF. For the family F_f, a similar proof works.

REMARK 6.5. Since $K^G(\Lambda;F_t)_e$ is isomorphic to $K^G(\Lambda;F_t)_d$ for a finite group G, we have a similar induction theorem for $K^G(\Lambda;F_t)_d$.

§7. Induction theorems for equivariant topological K-theories.

In this section, we define an induction homomorphism for equivariant topological K-theory and show that it corresponds to the induction homomorphism for equivariant algebraic K-theory *via* the equivariant Swan isomorphism in [10]. Accordingly induction theorems for equivariant topological K-theories follow from that for equivariant algebraic K-theory in §6.

Let Δ be one of the classical fields $\boldsymbol{R}$ (the real numbers), $\boldsymbol{C}$ (the complex numbers) or $\boldsymbol{H}$ (the quaternions). Let X be a compact Hausdorff G-space. A ΔG-vector bundle ξ on X is a Δ-vector bundle together with a G-action on ξ preserving the Δ-vector bundle structure [1]. The set of isomorphism classes of ΔG-vector bundles on X forms an abelian semi-group under the Whitney sum. The associated abelian group is denoted by $K\Delta_G(X)$. The tensor product of

G-vector bundles induces a structure of commutative ring in $K\Delta_G(X)$ for $\Delta=\boldsymbol{R}$ or $\boldsymbol{C}$.

Let H be a subgroup of G of finite index and ξ a ΔH-vector bundle on X. Then an induced ΔG-vector bundle $\mathrm{Ind}_H^G \xi$ is defined as follows. In the following, we employ the notations in § 2. We assume that the coset H is represented by the identity element e of G for simplicity.

As a Δ-vector bundle, we set

$$\mathrm{Ind}_H^G \xi = \bigoplus_\sigma (\sigma^{-1})^* \xi$$

where $(\sigma^{-1})^*\xi$ denotes the induced bundle of ξ by the map $\sigma^{-1} : X \to X$ of G-action. For $x \in X$, we denote by ξ_x the fiber over x of the bundle ξ. Since the fiber over a point $x \in X$ of the bundle $\mathrm{Ind}_H^G \xi$ is the direct sum

$$(\mathrm{Ind}_H^G \xi)_x = \bigoplus_\sigma \xi_{\sigma^{-1}x},$$

a point y in the fiber is expressed uniquely as

$$y = \bigoplus_\sigma y_\sigma \qquad \text{for } y_\sigma \in \xi_{\sigma^{-1}x}.$$

Then a G-action is defined by

$$g\circ y = g\circ(\bigoplus_\sigma y_\sigma) = \bigoplus_{\sigma(g,\sigma)} h(g, \sigma) y_\sigma$$

where $h(g, \sigma)y_\sigma$ is in the fiber over

$$h(g, \sigma)\sigma^{-1}x = \sigma(g, \sigma)^{-1}gx$$

of the vector bundle ξ. Hence $g\circ y$ is in the fiber over gx of the vector bundle $\mathrm{Ind}_H^G \xi$.

It is easy to see that with these definitions, $\mathrm{Ind}_H^G \xi$ becomes a ΔG-vector bundle.

REMARK 7.1. Note that our definition of $\mathrm{Ind}_H^G \xi$ is different from that of $\mathrm{tr}_H^G \xi$ in McClure [**13**].

One verifies the following

LEMMA 7.2. *$\mathrm{Ind}_H^G \xi$ does not depend on the choice of the set of coset representatives for G/H.*

We now show a universal property of $\mathrm{Ind}_H^G \xi$. Let

$$i : \xi \longrightarrow \mathrm{Ind}_H^G \xi = \bigoplus_\sigma (\sigma^{-1})^* \xi$$

be the inclusion map onto the direct summand $e^*\xi \cong \xi$ of $\mathrm{Ind}_H^G \xi$. Then i is a ΔH-vector bundle homomorphism.

PROPOSITION 7.3 (Universal property). *For an arbitrary ΔG-vector bundle η*

over X and for an arbitrary ΔH-vector bundle homomorphism $f:\xi\rightarrow\eta$, there exists a unique ΔG-vector bundle homomorphism

$$F:\mathrm{Ind}_H^G\xi\longrightarrow\eta$$

such that $f=F\cdot i$.

PROOF. As before, write an arbitrary point y of the total space as

$$y=\bigoplus_{\sigma}y_{\sigma}\qquad\text{for } y_{\sigma}\in\xi_{\sigma^{-1}x}.$$

Then define the map $F:\mathrm{Ind}_H^G\xi\rightarrow\eta$ by

$$F(y)=F(\bigoplus_{\sigma}y_{\sigma})=\sum_{\sigma}\sigma f(y_{\sigma}).$$

Since $\sigma f(y_{\sigma})$ is in the fiber η_x over x, the summation makes sense. By definition, we compute;

$$\begin{aligned}F(g\circ y)&=F(\bigoplus_{\sigma(g,\sigma)}h(g,\sigma)y_{\sigma})\\&=\sum_{\sigma}\sigma(g,\sigma)f(h(g,\sigma)y_{\sigma})\\&=\sum_{\sigma}\sigma(g,\sigma)h(g,\sigma)f(y_{\sigma})\\&=\sum_{\sigma}g\sigma f(y_{\sigma})=g(\sum_{\sigma}\sigma f(y_{\sigma}))\\&=gF(y),\end{aligned}$$

which shows that F is a G-map. The rest of the proof is routine.

REMARK 7.5. It is a routine work to see that such $\mathrm{Ind}_H^G\xi$ with the universal property is unique.

The correspondence $\xi\mapsto\mathrm{Ind}_H^G\xi$ gives rise to a homomorphism

$$\mathrm{Ind}_H^G:K\Delta_H(X)\longrightarrow K\Delta_G(X)$$

which we call an *induction homomorphism.*

Let $C_{\Delta}(X)$ be the ring of continuous Δ-valued functions on X. Then G acts on $C_{\Delta}(X)$ by $(g\circ a)(x)=a(g^{-1}x)$ for $g\in G$, $a\in C_{\Delta}(X)$. With these definitions, $C_{\Delta}(X)$ becomes a G-ring. Then Δ is a G-subring of $C_{\Delta}(X)$ by regarding each element $a\in\Delta$ as the constant function of value a. We now introduce a new family F_r of $C_{\Delta}(X)G$-modules as follows. Let V be a finite dimensional G-representation space over Δ. Regarding $C_{\Delta}(X)$ as a right Δ-module, we form a finitely generated $C_{\Delta}(X)G$-module $C_{\Delta}(X)\otimes_{\Delta}V$. Define F_r to be the family consisting of such modules $C_{\Delta}(X)\otimes_{\Delta}V$.

PROPOSITION 7.5. *The following diagram is commutative:*

$$\begin{array}{ccccc} KΔ_H(X) & \xrightarrow{\mathrm{Ind}_H^G} & KΔ_G(X) & \xrightarrow{\mathrm{Res}_H} & KΔ_H(X) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ K^H(C_Δ(X); F_r)_d & \xrightarrow{\mathrm{Ind}_H^G} & K^G(C_Δ(X); F_r)_d & \xrightarrow{\mathrm{Res}_H} & K^H(C_Δ(X); F_r)_d . \end{array}$$

Here the vertical arrows denote the equivariant Swan isomorphism [**10**].

PROOF. Easy and omitted.

For a class S of subgroups of G, $KΔ_G^S(X)$ is defined similarly to $K_S^G(A; F)_ε$ and the notion of exponent is defined similarly.

THEOREM 7.6. *For a finite group* G, *we have*

(a) $KΔ_G^C(X)$ *has exponent* $a(G)$ *in* $KΔ_G(X)$ *for* $Δ=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$,

(b) $KΔ_G^E(X)$ *has exponent* d *in* $KΔ_G(X)$ *for* $Δ=\boldsymbol{R}$ *or* $\boldsymbol{H}$,

(c) $K\boldsymbol{C}_G^E(X)=K\boldsymbol{C}_G(X)$,

(d) $KΔ_G^{HE}(X)=KΔ_G(X)$ *for* $Δ=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$.

PROOF. Since G is a finite group, we have isomorphisms

$$K^G(C_Δ(X); F_r)_d \cong K^G(C_Δ(X); F_t)_d \cong K^G(C_Δ(X); F_t)_e$$

by Theorem 4.3 in [**10**]. It is trivial to see that these isomorphisms commute with Ind_H^G and Res_H. Hence it follows from Proposition 7.5 that showing the formulae in Theorem 7.6 is equivalent to showing the corresponding formulae for $K^G(C_Δ(X); F_t)_e$. Since $C_Δ(X)$ is a GQ-algebra, (a), (b) and (d) follow from Theorem 6.4. Since $C_C(X)$ contains a primitive n-th root of unity in the centre, (c) follows from Theorem 6.4 again.

REMARK 7.7. Theorem 7.6 will be proved differently as follows. We first prove that concerning our induction homomorphism, $KΔ_G(X)$ is a Frobenius module over the real representation ring $RO(G)$ for $Δ=\boldsymbol{R}$ or $\boldsymbol{H}$ and that $K\boldsymbol{C}_G(X)$ is a Frobenius module over the complex representation ring $R(G)$. It follows from Swan [**20**] and Lam [**12**] that (a), (b) and (d) hold for $RO(G)$. On the other hand, it is well-known that (c) holds for $R(G)$ (see for example Serre [**19**]). Hence the rest of the proof will be given similarly to that of Theorem 6.4.

The restriction homomorphism

$$\mathrm{Res}_S : KΔ_G(X) \longrightarrow \prod_{H \in S} KΔ_H(X)$$

is defined as before and we have

THEOREM 7.8. *We have*

(a) $a(G)\operatorname{Ker}\operatorname{Res}_C=0$ *for* $\varDelta=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$,

(b) $d\operatorname{Ker}\operatorname{Res}_E=0$ *for* $\varDelta=\boldsymbol{R}, \boldsymbol{H}$,

(c) $\operatorname{Ker}\operatorname{Res}_E=0$ *for* $\varDelta=\boldsymbol{C}$,

(d) $\operatorname{Ker}\operatorname{Res}_{HE}=0$ *for* $\varDelta=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$.

Let $f: X\to Y$ be a G-map between compact G-spaces. Let S be a class of subgroups of a finite group G and k a positive integer. Concerning the pair (S, k), we consider the following statement:

"if $f_H^*: K\varDelta_H(Y)\to K\varDelta_H(X)$ is injective, surjective or an isomorphism for every $H\in S$, then $k\cdot\operatorname{Ker} f_G^*=0$, $k\cdot\operatorname{Coker} f_G^*=0$ or $k\cdot\operatorname{Ker} f_G^*=k\cdot\operatorname{Coker} f_G^*=0$ respectively".

Then as an application of our induction and restriction theorems, we have

COROLLARY 7.9. *The statement above is true for the pairs* $(C, a(G))$, (E, d), $(HE, 1)$ *where* $\varDelta=\boldsymbol{R}$, $\boldsymbol{C}$ *or* $\boldsymbol{H}$. *When* $\varDelta=\boldsymbol{C}$, *it is true for* $(E, 1)$.

PROOF. One verifies the commutativity of the following diagrams:

$$\begin{array}{ccc} K\varDelta_G(Y) & \xrightarrow{f_G^*} & K\varDelta_G(X) \\ \Big\downarrow{\scriptstyle \operatorname{Res}_H} & & \Big\downarrow{\scriptstyle \operatorname{Res}_H} \\ K\varDelta_H(Y) & \xrightarrow{f_H^*} & K\varDelta_H(X), \end{array} \qquad \begin{array}{ccc} K\varDelta_G(Y) & \xrightarrow{f_G^*} & K\varDelta_G(X) \\ \Big\uparrow{\scriptstyle \operatorname{Ind}_H^G} & & \Big\uparrow{\scriptstyle \operatorname{Ind}_H^G} \\ K\varDelta_H(Y) & \xrightarrow{f_H^*} & K\varDelta_H(X). \end{array}$$

Hence Corollary 7.9 follows from Theorems 7.6 and 7.8.

REMARK 7.10. Corollary 7.9 holds for a G-map between compact G-space pairs. When $\varDelta=\boldsymbol{C}$, a stronger result is obtained in the category of G-CW complexes by Jackowski as follows. He showed that if f_H^* is an isomorphism for every $H\in C$, then f_G^* is an isomorphism [**7**].

The following proposition enables us to show that the Atiyah-Singer index homomorphisms [**2**] and our induction homomorphisms commute.

PROPOSITION 7.11 (Relative Frobenius reciprocity). *Let A be a G-invariant closed subset of a compact G-space X. If* $x\in K\varDelta_H(X)$, $y\in K\varDelta_G(X, A)$, *then*

$$\operatorname{Ind}_H^G(x\otimes\operatorname{Res}_H y)=(\operatorname{Ind}_H^G x)\otimes y$$

where $\varDelta=\boldsymbol{R}$ *or* $\boldsymbol{C}$. *Similar for* $x\in K\boldsymbol{R}_H(X)$, $y\in K\boldsymbol{H}_G(X, A)$.

PROOF. Easy and omitted.

REMARK 7.12. For a finite G-covering $p: \tilde{X}\to X$, there are two kinds of homomorphisms: $K\varDelta_G(\tilde{X})\to K\varDelta_G(X)$. One is the homomorphism defined by the direct image construction [23] and the other is the transfer homomorphism induced by the Becker-Gottlieb stable map [25]. According to [24], these two homomorphisms agree with each other when $\Delta=\boldsymbol{C}$. This fact is kindly informed to the author by Professor A. Kono.

For a subgroup H of G of finite index and for a finite G-CW complex X, the map $G\times_H X\to X$ defined by $[g, x]\mapsto gx$ $(g\in G,\ x\in X)$ gives a finite G-covering. Then our induction homomorphism

$$\mathrm{Ind}_H^G : \boldsymbol{KC}_H(X) \longrightarrow \boldsymbol{KC}_G(X)$$

coincides with the composition of the Shapiro isomorphism $\boldsymbol{KC}_H(X)\cong \boldsymbol{KC}_G(G\times_H X)$ and the Becker-Gottlieb transfer homomorphism $\boldsymbol{KC}_G(G\times_H X)\to \boldsymbol{KC}_G(X)$.

§ 8. Induction theorems for equivariant *J*-theory.

We first recall the definition of the equivariant J-group [8], [9]. Let X be a compact Hausdorff G-space. Let ξ and η be orthogonal G-vector bundles over X. Denote by $S(\xi)$ (resp. $S(\eta)$) the sphere bundle associated with ξ (resp. η). Then $S(\xi)$ and $S(\eta)$ are said to be of the same G-fiber homotopy type if there exist fiber preserving G-maps:

$$f : S(\xi) \longrightarrow S(\eta), \qquad f' : S(\eta) \longrightarrow S(\xi)$$

and fiber preserving G-homotopies:

$$h : S(\xi)\times I \longrightarrow S(\xi), \qquad h' : S(\eta)\times I \longrightarrow S(\eta)$$

with

$$h\,|\,S(\xi)\times 0=f'\cdot f, \qquad h\,|\,S(\xi)\times 1=\text{identity}$$

$$h'\,|\,S(\eta)\times 0=f\cdot f', \qquad h'\,|\,S(\eta)\times 1=\text{identity}.$$

We write $\xi\sim\eta$ if $S(\xi)$ and $S(\eta)$ are of the same G-fiber homotopy type.

Let $T_G(X)$ be the additive subgroup of $\boldsymbol{KR}_G(X)$ generated by elements of the form $[\xi]-[\eta]$ where $\xi\sim\eta$. We define

$$J_G(X)=\boldsymbol{KR}_G(X)/T_G(X),$$

which is called an *equivariant J-group*. The natural epimorphism $\boldsymbol{KR}_G(X)\to J_G(X)$ is denoted by J_G.

LEMMA 8.1. $\mathrm{Ind}_H^G(T_H(X))\subset T_G(X)$.

PROOF. Let ξ and η be orthogonal H-vector bundles with $\xi\sim\eta$. Let f, f', h, h' be as above. Then we construct a fiber preserving G-map

$$\bar{f}=\mathrm{Ind}_H^G(f) : S(\mathrm{Ind}_H^G\,\xi) \longrightarrow S(\mathrm{Ind}_H^G\,\eta)$$

as follows. For a point $x\in X$, an arbitrary point of the fiber over x can be written as $\bigoplus_\sigma a_\sigma y_\sigma$ where $y_\sigma \in S(\xi_{\sigma^{-1}x})$ and $\sum_\sigma a_\sigma^2=1$. Then the correspondence

$$\bigoplus_\sigma a_\sigma y_\sigma \longmapsto \bigoplus_\sigma a_\sigma f(y_\sigma)$$

defines a fiber preserving map

$$\bar{f}=\mathrm{Ind}_H^G(f) : \quad S(\mathrm{Ind}_H^G \xi) \longrightarrow S(\mathrm{Ind}_H^G \eta).$$

It is easy to see that $\bar{f}$ is a well-defined continuous map. We now show that $\bar{f}$ is a G-map as follows:

$$\begin{aligned}
\bar{f}(g\circ(\bigoplus_\sigma a_\sigma y_\sigma)) &= \bar{f}(\bigominus_{\sigma(g,\sigma)} a_\sigma h(g,\sigma) y_\sigma) \\
&= \bigoplus_{\sigma(g,\sigma)} a_\sigma f(h(g,\sigma) y_\sigma) \\
&= \bigoplus_{\sigma(g,\sigma)} a_\sigma h(g,\sigma) f(y_\sigma) \\
&= \bigoplus_{\sigma(g,\sigma)} h(g,\sigma) a_\sigma f(y_\sigma) \\
&= g\circ(\bigoplus_\sigma a_\sigma f(y_\sigma)) = g\circ \bar{f}(\bigoplus_\sigma a_\sigma y_\sigma).
\end{aligned}$$

Similarly we have a fiber preserving G-map

$$\bar{f}'=\mathrm{Ind}_H^G(f') : \quad S(\mathrm{Ind}_H^G \eta) \longrightarrow S(\mathrm{Ind}_H^G \xi)$$

and fiber preserving G-homotopies:

$$\bar{h}=\mathrm{Ind}_H^G(h) : \quad S(\mathrm{Ind}_H^G \xi)\times I \longrightarrow S(\mathrm{Ind}_H^G \xi)$$

and

$$\bar{h}'=\mathrm{Ind}_H^G(h') : \quad S(\mathrm{Ind}_H^G \eta)\times I \longrightarrow S(\mathrm{Ind}_H^G \eta).$$

It is easy to see that the maps $\bar{f}$, $\bar{f}'$, $\bar{h}$, $\bar{h}'$ give the relation $\mathrm{Ind}_H^G \xi \sim \mathrm{Ind}_H^G \eta$.

COROLLARY 8.2. $\mathrm{Ind}_H^G : K\boldsymbol{R}_H(X) \to K\boldsymbol{R}_G(X)$ *induces* $\mathrm{Ind}_H^G : J_H(X) \to J_G(X)$. *They are connected by the following commutative diagram:*

$$\begin{array}{ccc}
K\boldsymbol{R}_H(X) & \xrightarrow{J_H} & J_H(X) \\
\Big\downarrow \mathrm{Ind}_H^G & & \Big\downarrow \mathrm{Ind}_H^G \\
K\boldsymbol{R}_G(X) & \xrightarrow{J_G} & J_G(X).
\end{array}$$

Denote by $A(G)$ the Burnside ring of G and by $\pi : A(G) \to RO(G)$ the natural ring homomorphism. Namely for a subgroup H of G, $\pi(G/H)$ is the permutation representation over the G-set G/H. Denote by $\underline{\underline{\pi(G/H)}}$ the G-vector bundle

$$X\times \pi(G/H) \longrightarrow X.$$

LEMMA 8.3. *Let ξ and η be G-vector bundles over X. If $\xi \sim \eta$, then we have*

$$\underline{\pi(G/H)}\otimes\xi \sim \underline{\pi(G/H)}\otimes\eta .$$

PROOF. Let L be the trivial line bundle $X\times \boldsymbol{R}\rightarrow X$ where G acts trivially on $\boldsymbol{R}$. Then L is a G-vector bundle and satisfies

$$\mathrm{Ind}_H^G L \cong \underline{\pi(G/H)} .$$

Since Frobenius reciprocity holds for G-vector bundles (cf. Lemma 7.9), we have

$$\begin{aligned}\mathrm{Ind}_H^G \mathrm{Res}_H \xi &\cong \mathrm{Ind}_H^G (L\otimes \mathrm{Res}_H \xi) \\ &\cong (\mathrm{Ind}_H^G L)\otimes\xi \cong \underline{\pi(G/H)}\otimes\xi .\end{aligned}$$

Similarly we have

$$\mathrm{Ind}_H^G \mathrm{Res}_H \eta \cong \underline{\pi(G/H)}\otimes\eta .$$

Since $\xi\sim\eta$, we have $\mathrm{Res}_H \xi \sim \mathrm{Res}_H \eta$. It follows from Lemma 8.1 that

$$\mathrm{Ind}_H^G \mathrm{Res}_H \xi \sim \mathrm{Ind}_H^G \mathrm{Res}_H \eta .$$

This completes the proof of Lemma 8.3.

THEOREM 8.4. *$J_G(X)$ is a Frobenius module over the subring $\pi(A(G))$ of $RO(G)$.*

PROOF. By making use of Lemma 8.3, one verifies that $J_G(X)$ is a module over $\pi(A(G))$. Since $K\boldsymbol{R}_G(X)$ is a Frobenius module over $RO(G)$, $J_G(X)$ is a Frobenius module over $\pi(A(G))$.

THEOREM 8.5. *The homomorphism*

$$\sum_{H\in HE} \mathrm{Ind}_H^G : \bigoplus_{H\in HE} J_H(X) \longrightarrow J_G(X)$$

is surjective and the homomorphism

$$\mathrm{Res}_{HE} = \prod_{H\in HE} \mathrm{Res}_H : J_G(X) \longrightarrow \prod_{H\in HE} J_H(X)$$

is injective.

PROOF. Let x be an element of $\pi(A(G))\subset RO(G)$. If $i^*(x)=0$ for all $i: H\subset G$ with $H\in C$, then $x=0$. It follows from the Dress induction theorem [**4**] that

$$\sum_{H\in HE} \mathrm{Ind}_H^G : \bigoplus_{H\in HE} \pi(A(H)) \longrightarrow \pi(A(G))$$

is surjective. Since $J_G(X)$ is a Frobenius module over $\pi(A(G))$, the proof proceeds as in that of Corollary 3.6.

COROLLARY 8.6. *Let $f: X \to Y$ be a G-map between compact G-spaces X, Y. If $f^*: J_H(Y) \to J_H(X)$ are isomorphisms for all $H \in HE$, then $f^*: J_G(Y) \to J_G(X)$ is an isomorphism.*

The following application of Theorem 8.5 was suggested to the author by T. Petrie. Denote by Ψ^p the p-th Adams operation.

COROLLARY 8.7. *Let G be a finite group of order n such that every hyper-elementary subgroup of G is abelian. Let p, q be integers with $(p, n)=(q, n)=(p, q)=1$. Then for any G-vector bundle ξ,*

$$(\Psi^p-1)(\Psi^q-1)\xi \in \mathrm{Ker}\, J_G.$$

PROOF. Note that $\mathrm{Res}_H J_G((\Psi^p-1)(\Psi^q-1)\xi)=J_H((\Psi^p-1)(\Psi^q-1)\mathrm{Res}_H \xi)$. According to Petrie **[15]** (see also **[16]**), $J_H((\Psi^p-1)(\Psi^q-1)\mathrm{Res}_H \xi)=0$ for every abelian subgroup H. Hence $J_H((\Psi^p-1)(\Psi^q-1)\mathrm{Res}_H \xi)=0$ for every hyperelementary subgroup H by assumption. It follows from Theorem 8.5 that

$$J_G((\Psi^p-1)(\Psi^q-1)\xi)=0.$$

This completes the proof of Corollary 8.7.

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