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The fixed point subvarieties of unipotent transformations on the flag varieties

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Introduction.

Let V be an n-dimensional vector space over a field K. Let μ be an ordered partition of n, i.e., an ordered sequence $(\mu_1, \cdots, \mu_s)$ of positive integers such that $\mu_1+\cdots+\mu_s=n$. For μ, let F_μ be the partial flag variety of type μ. Let u be a unipotent transformation of V. In [**12**], we proved that the fixed point subvariety $F_\mu^u=\{(W_i)\in F_\mu\ ;\ uW_i=W_i\ (1\leqq i\leqq s-1)\}$ has a partition into a finite number of locally closed affine spaces and this partition is determined by the Young diagram associated with u. In this paper, we study further the variety F_μ^u. Let $\mathcal{A}$ be the set of all minimal semistandard μ-tableaus of type λ (defined precisely in 4.1), where λ is the Jordan type of u. For $\alpha\in\mathcal{A}$, let λ_α^i $(1\leqq i\leqq s-1)$ be the Young diagram with $\mu_1+\cdots+\mu_i$ squares (defined precisely in 4.9). For $\alpha\in\mathcal{A}$, put $Y_\alpha=\{(W_i)\in F_\mu^u;$ the Jordan type of the restriction of u to W_i is λ_α^i $(1\leqq i\leqq s-1)\}$. Then we have $F_\mu^u=\coprod_{\alpha\in\mathcal{A}}Y_\alpha$ (disjoint union). The main results of the paper are:

(1) For $\alpha\in\mathcal{A}$, the variety Y_α is an irreducible locally closed subvariety of F_μ^u.

(2) For $\alpha\in\mathcal{A}$, the variety Y_α has a partition $Y_\alpha=\coprod_{\beta\in X_\alpha}S_\beta^u$, where X_α is the set of semistandard μ-tableaus determined by α (defined precisely in 4.9) and the varieties S_β^u are the fixed point subvarieties of the Schubert (Bruhat) cells S_β. The variety S_β^u is isomorphic to an affine space.

(3) For β, γ in X_α, we have

$$\beta\leq\gamma \Longleftrightarrow \mathrm{cl}S_\beta^u\supseteqq\mathrm{cl}S_\gamma^u,$$

where "$\leq$" is a partial order (Bruhat order) defined in 1.1, 4.10 and $\mathrm{cl}S_\beta^u$ (resp. $\mathrm{cl}S_\gamma^u$) is the Zariski closure of S_β^u (resp. S_γ^u) in F_μ. In particular, S_α^u is an open dense subvariety of Y_α.

N. Spaltenstein proved these results in the case of the full flag variety, i.e., $\mu=(1, 1, \cdots, 1)$ ([**13**], Chapitre II, 5; [8], p. 92, Example). The above (1), (2) and (3) are stated and proved in §4. The crucial points of the proofs are the proofs in the case of Grassmann variety and are given in §1, §2. The contents of §3 are supplements to §2.

In the appendix, we study the homogeneous coordinate ring of the fixed point subvariety Ω^u of the Schubert variety Ω in the Grassmann variety $G_d(V)$ $(=F_{(d, n-d)})$. If $\dim \Omega^u = \dim \Omega - 1$, we determine the defining ideal of Ω^u and we prove that the homogeneous coordinate ring of Ω^u is normal and Cohen-Macaulay. In the proof of these, we use some results on the homogeneous coordinate ring of Ω ([**1**], [**2**], [**6**], [**10**] and [**11**]). Our results give an alternating proof to the fact that the minimal unipotent variety over a field K is normal and Cohen-Macaulay ([**4**], [**5**] and [**14**]).

The author expresses his hearty thanks to H. Doi and K. Matsui for a number of interesting discussions and for valuable suggestions.

NOTATIONS. For a transformation u of a set X, X^u denotes the set of all u-fixed elements of X. Let V be an n-dimensional vector space over a field K. If u is a linear transformation of V, for a u-stable subspace W of V, $u|_W$ is the restriction of u to W. For an integer d such that $1 \leqq d < n$, $\wedge^d V$ denotes the d-th alternating product of V. Let $\mathbf{P} = \mathbf{P}(\wedge^d V)$ be the projective space associated with $\wedge^d V$. For a subvariety X of $\mathbf{P}$, $\mathrm{cl}X$ denotes the Zariski closure of X in $\mathbf{P}$. Let μ be an ordered partition of n, i.e., an ordered sequence of positive integers $(\mu_1, \cdots, \mu_s)$ such that $\mu_1 + \cdots + \mu_s = n$. If F_μ is a Grassmann variety, i.e., $\mu = (\mu_1, \mu_2)$, we write $G_{\mu_1}(V)$ instead of F_μ. The Young diagrams in the paper are as in [8].

§1. The fixed point subvarieties of the Schubert cells.

1.1. Let K be a fixed algebraically closed field. Let V be a vector space over K of dimension n $(n \geqq 2)$. Fix an integer d such that $1 \leqq d < n$. Let V^d be the vector space $V \oplus \cdots \oplus V$ (d copies). Let $\wedge^d V$ be the d-th alternating product of V. Let

$$\pi : V^d \longrightarrow \wedge^d V$$

be the morphism defined by $(v_1, \cdots, v_d) \mapsto v_1 \wedge \cdots \wedge v_d$. Fix a basis $\{e_1, \cdots, e_n\}$ of V. Then we can identify V^d with the set of all $d \times n$-matrices over K by

$$(v_1, \cdots, v_d) \longmapsto (x_i(j))_{1 \leq i \leq d, 1 \leq j \leq n},$$

where $v_i = \sum_{1 \leq j \leq n} x_i(j) e_j$, $x_i(j) \in K$. Let $\mathbf{P}(\wedge^d V)$ be the projective space associated with $\wedge^d V$. Let

$$p : \wedge^d V - \{0\} \longrightarrow \mathbf{P}(\wedge^d V)$$

be the natural projection. We denote by $G_d(V)$ the *Grassmann variety* of all d-dimensional linear subspaces in V. Then $G_d(V) = p(\pi V^d - \{0\})$. Put

$$I = \{\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbf{Z}^d \ ; \ 1 \leqq \alpha_1 < \cdots < \alpha_d \leqq n\}.$$

For $\alpha=(\alpha_1, \cdots, \alpha_d)$ in I, put

$$D_\alpha=\{(x_i(j))\in V^d \ ; \ x_i(j)=0 \text{ for } j<\alpha_i \ (1\leqq i\leqq d)\},$$

$$C_\alpha=\{(x_i(j))\in D_\alpha \ ; \ x_i(\alpha_j)=\delta_{ij} \ (1\leqq i, j\leqq d)\},$$

where $\delta_{ij}=1$ (if $i=j$) and $\delta_{ij}=0$ (if $i\neq j$). Then $S_\alpha=p\pi C_\alpha$ (resp. $\Omega_\alpha=p(\pi D_\alpha-\{0\})$) is the *Schubert cell* (resp. *Schubert variety*) corresponding to α. The Zariski closure of S_α in $G_d(V)$ is Ω_α. We define a partial order "$\leq$" on I by

$$\alpha\leq\beta \qquad \text{if } \alpha_i\leqq\beta_i \text{ for all } i=1, \cdots, d.$$

Then $(1, 2, \cdots, d)$ (resp. $(n-d+1, n-d+2, \cdots, n)$) is the minimum (resp. maximum) element of I with respect to this ordering.

1.2. LEMMA. *For α and β in I, the following three conditions are equivalent:*

(1) $\alpha\leq\beta$.

(2) $\Omega_\alpha\cap S_\beta$ *is not empty.*

(3) $\Omega_\alpha\supseteqq\Omega_\beta$.

For the proof, see [**10**].

1.3. We fix a positive integer n and a partition λ of n. We write $\lambda=(\lambda_1, \cdots, \lambda_r)$ if $\lambda_1+\cdots+\lambda_r=n$ $(\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_r>0)$ and represent λ by a *Young diagram* with rows consisting of $\lambda_1, \lambda_2, \cdots, \lambda_r$ squares respectively.

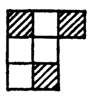
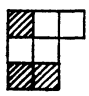
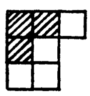
1.4. DEFINITION. Fix a Young diagram λ with n squares. Let d be an integer such that $1\leqq d<n$.

(1) A *d-tableau* is a Young diagram of type λ whose d squares are distinguished by ▨.

(2) A d-tableau is said to be *semistandard* if every square on the left position to ▨ on the same row is ▨.

(3) A semistandard d-tableau is said to be *minimal* if every square on the upper side to ▨ on the same column is ▨.

1.5. EXAMPLES. For $\lambda=(3, 2, 2)$ and $d=3$, put

(1) , (2) , (3) .

All are 3-tableaus, (1) is not semistandard, (2) is semistandard but not minimal and (3) is minimal.

1.6. We fill in all the squares of λ with the numbers $1, 2, \cdots, n$ in the following way: For a start, we put the integers into the squares from the top square of the λ_1-th column to the bottom square of this column, next, from the top square of the (λ_1-1)-th column to the bottom square of this column, and so on. For example, if $\lambda=(4, 3, 2)$, we have $\begin{array}{|c|c|c|c|}\hline 7&4&2&1\\ \hline 8&5&3\\ \cline{1-3} 9&6\\ \cline{1-2}\end{array}$. These numbers indicate the places of all squares of λ.

Let I be the set defined in 1.1. Fix a Young diagram λ with n squares. We identify $\alpha=(\alpha_1, \cdots, \alpha_d)$ in I with the d-tableau of type λ whose $\alpha_1, \cdots, \alpha_d$-th squares are ▨. Therefore the set I is identified with the set of all d-tableaus of type λ. For example, in 1.5, the 3-tableaus of (1), (2) and (3) are identified with the elements (1, 4, 5), (4, 5, 7) and (2, 5, 6) in $I=\{(\alpha_1, \alpha_2, \alpha_3)\in \boldsymbol{Z}^3 \ ; \ 1\leqq\alpha_1<\alpha_2<\alpha_3\leqq 7\}$ respectively.

1.7. Let V be an n-dimensional vector space over a field K with basis $\{e_1, \cdots, e_n\}$. For a Young diagram λ with all squares numbered as in 1.6, we define a unipotent transformation u of V of Jordan type λ by

$$ue_i=e_i+e_j \qquad \text{if } \lambda \text{ contains } \boxed{j\,|\,i},$$

$$ue_i=e_i \qquad \text{if } \boxed{i} \text{ lies on the first column of } \lambda.$$

Put $N=u-1$, a nilpotent transformation of V of Jordan type λ.

1.8. LEMMA. *In the above notations, for a d-tableau α in I of type λ, put*

$$S_\alpha^u=\{p(v_1\wedge \cdots \wedge v_d)\in S_\alpha \ ; \ uv_1\wedge \cdots \wedge uv_d=v_1\wedge \cdots \wedge v_d\}$$
$$=\{W\in S_\alpha \ ; \ uW=W\},$$

where S_α is the Schubert cell corresponding to α. Then S_α^u is nonempty if and only if α is semistandard.

PROOF. If α is semistandard, we have $p(e_{\alpha_1}\wedge \cdots \wedge e_{\alpha_d})\in S_\alpha^u$. Hence $S_\alpha^u\neq\emptyset$. On the other hand, take $p(v_1\wedge \cdots \wedge v_d)$ in S_α^u. By $uv_1\wedge \cdots \wedge uv_d=v_1\wedge \cdots \wedge v_d$, $u\langle v_1, \cdots, v_d\rangle=\langle v_1, \cdots, v_d\rangle$, where $\langle v_1, \cdots, v_d\rangle$ is the K-vector space generated by $\{v_1, \cdots, v_d\}$; hence we can write

$$Nv_k=\sum_{1\leqq i\leqq d} a_i v_i, \qquad a_i\in K.$$

If the square $\boxed{\alpha_k}$ does not lie on the first column of λ, take a number k' $(\alpha_k<k'\leqq n)$ such that the square $\boxed{k'}$ lies next to the left of the square $\boxed{\alpha_k}$. By the definition of S_α,

$$v_i=\sum_{\alpha_i\leqq j\leqq n} x_i(j)e_j,$$

where $x_i(j)\in K$ and $x_i(\alpha_j)=\delta_{ij}$ $(1\leqq i, j\leqq d)$. By

$$\sum_{\alpha_k\leqq j\leqq n} x_k(j)N(e_j)=\sum_{1\leqq i\leqq d} a_i \sum_{\alpha_i\leqq j\leqq n} x_i(j)e_j,$$

we have

$$e_{k'}+\cdots=\sum_{1\leqq i\leqq d} a_i e_{\alpha_i}+\cdots.$$

By this formula, we see that $a_i=0$ for all i such that $\alpha_i<k'$ and there is a number m such that $k'=\alpha_m$ and $a_m=1$. This means that the square $\boxed{k'}$ is ▨. Thus α is semistandard and the proof of the lemma is completed.

1.9. For a semistandard d-tableau $\alpha=(\alpha_1, \cdots, \alpha_i, \cdots, \alpha_d)$ of type λ, we say i $(1\leqq i\leqq d)$ is an *initial number* of α if the square on the right side of $\boxed{\alpha_i}$ (if any) is not ▨. For example, in 1.5 (3), 1 and 3 are the initial numbers of α.

1.10. LEMMA. *For a semistandard d-tableau α, let C_α be the subvariety of V^d defined in* 1.1. *Put*

$$C_\alpha^u=\{(v_1, \cdots, v_d)\in C_\alpha\ ;\ Nv_i=v_j \text{ if } \alpha \text{ contains } \overline{\alpha_j\alpha_i}\ (1\leqq i<j\leqq d)\}.$$

Then the isomorphism $p\pi: C_\alpha \xrightarrow{\sim} S_\alpha$ induces an isomorphism

$$C_\alpha^u \xrightarrow{\sim} S_\alpha^u.$$

PROOF. The injectivity of this morphism is clear. Take an element $p(w_1\wedge \cdots \wedge w_d)\in S_\alpha^u$, where $(w_1, \cdots, w_d)\in C_\alpha$. Let $(v_1, \cdots, v_d)$ be an element in C_α^u such that $\{v_1, \cdots, v_d\}=\{N^h w_i\ ;\ i\text{'s are the initial numbers of } \alpha \text{ and } h\geqq 0\}-\{0\}$. Then $v_1\wedge \cdots \wedge v_d=w_1\wedge \cdots \wedge w_d$. Hence the lemma.

1.11. PROPOSITION. *For a semistandard d-tableau $\alpha=(\alpha_1, \cdots, \alpha_d)$, put $\alpha'=\{1, \cdots, n\}-\{\alpha_1, \cdots, \alpha_d\}$. For $i\in\alpha'$, let $\alpha[i]$ be the cardinality of the set $\{\alpha_j$; j runs through all initial numbers of α such that $\alpha_j<i\}$. Put $d(\alpha)=\sum_{i\in\alpha'}\alpha[i]$. Then we have:*

(1) *The subvariety S_α^u of S_α is isomorphic to the $d(\alpha)$-dimensional affine space $\mathbf{A}^{d(\alpha)}$ over K.*

(2) *If α is a minimal semistandard d-tableau, $\alpha[i]$ is the number of all squares ▨ on the upper side to the square $\boxed{i}$ on the same column.*

PROOF. (1) The variety C_α^u in 1.10 is isomorphic to $\mathbf{A}^{d(\alpha)}$. Thus the assertion follows from 1.10.

(2) If α is minimal, there is no square $\boxed{\alpha_j}$ (i.e., ▨) on the lower side to the square $\boxed{i}$ on any column. Hence the proposition.

For example, in 1.5 (2) and (3), we have $d((4, 5, 7))=2$ and $d((2, 5, 6))=4$ respectively.

1.12. REMARK. 1.11 is an essential part of the proof of the theorem in [**12**] (see 4.7 and 4.8). The proof in this paper is simpler than that of [**12**].

§2. Inclusion relations.

We use the notations in §1.

2.1. LEMMA. *For a semistandard d-tableau α, let D_α be the subvariety of V^d defined in* 1.1. *Put*

$$D^u_\alpha=\{(v_1, \cdots, v_d)\in D_\alpha\ ;\ Nv_i=v_j \text{ if } \alpha \text{ contains } \boxed{\alpha_j\alpha_i}\ (1\leqq i<j\leqq d)\}.$$

Let $\mathrm{cl}S^u_\alpha$ *be the Zariski closure of* S^u_α *in* $\mathbf{P}(\wedge^d V)$. *Then*

$$S^u_\alpha\subseteqq p(\pi D^u_\alpha-\{0\})\subseteqq \mathrm{cl}S^u_\alpha,$$

where p *and* π *are morphisms defined in* 1.1.

PROOF. Let C'_α be the dense subvariety of D^u_α consisting of all $(v_1, \cdots, v_d)$ in D^u_α, $v_i=\sum_{\alpha_i\leqq j\leqq n} x_i(j)e_j$, which satisfy the condition $x_i(\alpha_i)\neq 0$ $(1\leqq i\leqq d)$. By 1.10, we have

$$p\pi C'_\alpha=S^u_\alpha.$$

Hence

$$S^u_\alpha\subseteqq p(\pi D^u_\alpha-\{0\}).$$

By the continuity of p and π, we have

$$\begin{aligned}\mathrm{cl}S^u_\alpha&=\mathrm{cl}\,p(\pi C'_\alpha)\\ &\supseteqq p((\text{the closure of } \pi C'_\alpha \text{ in } \wedge^d V)-\{0\})\\ &\supseteqq p(\pi D^u_\alpha-\{0\}).\end{aligned}$$

Thus the lemma.

2.2. LEMMA. *For a semistandard d-tableau* α *of type* $\lambda=(\lambda_1, \cdots, \lambda_r)$, *let* $\lambda_i(\alpha)$ $(i=1, 2, \cdots, r)$ *be the number of all* ▨*-squares on the i-th row of* α. *For integers* h_1 *and* h_2 $(1\leqq h_1<h_2\leqq r)$, *we assume that*

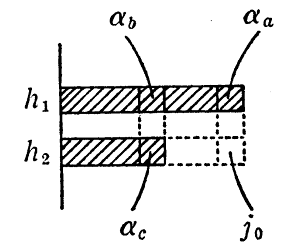
$$\lambda_{h_2}\geqq\lambda_{h_1}(\alpha)>\lambda_{h_2}(\alpha).$$

Let β *be the semistandard d-tableau of type* λ *obtained by exchanging the number of* ▨*-squares on the* h_1*-th row of* α *for the number of* ▨*-squares on the* h_2*-th row of* α. *Then we have*

$$\mathrm{cl}S^u_\alpha\supset \mathrm{cl}S^u_\beta.$$

PROOF. Let $\boxed{\alpha_a}$ be the square on the h_1-th row of α such that the square on the right side to $\boxed{\alpha_a}$ (if any) is not ▨. Let $\boxed{j_0}$ be the square which lies on the h_2-th row and on the same column as that of $\boxed{\alpha_a}$.

First, we assume that $\lambda_{h_2}(\alpha)\neq 0$. Let $\boxed{\alpha_c}$ be the square on the h_2-th row of λ such that the square on the right side to $\boxed{\alpha_c}$ (if any) is not ▨. Let $\boxed{\alpha_b}$ be the square which lies on the h_1-th row and on the same column as that of $\boxed{\alpha_c}$. Thus we get the picture of α as below:



Let E_α be the subvariety of V^d consisting of all $(v_1, \cdots, v_d)$ in V^d, $v_i=\sum_{1\leq j\leq n}x_i(j)e_j$ $(1\leqq i\leqq d)$, which satisfy the following conditions:

(1) If i is an initial number (see 1.9) of α and $i\neq c$, $x_i(j)=0$ for $j<\alpha_i$.

(2) $Nv_i=v_j$ if α contains $\boxed{\alpha_j\alpha_i}$.

(3) We can write $v_a=kx+y$ $(k\in K)$ and $v_c=N^m x+z$ $(m=\lambda_{h_1}(\alpha)-\lambda_{h_2}(\alpha))$, where x, y and z are defined by

$$x=\sum x(j)e_j \qquad (\alpha_a\leqq j\leqq j_0),$$

$$y=\sum y(j)e_j \qquad (j_0\leqq j\leqq n)$$

and

$$z=\sum z(j)e_j \qquad (\alpha_c\leqq j\leqq n).$$

Then we have

$$kv_c=N^m v_a+(kz-N^m y).$$

Hence, for $(v_1, \cdots, v_d)\in E_\alpha$, we have

$$v_1\wedge \cdots \wedge kv_c\wedge \cdots \wedge v_d=v_1\wedge \cdots \wedge(kz-N^m y)\wedge \cdots \wedge v_d.$$

Therefore, for the dense subvariety E'_α of E_α defined by $x_i(\alpha_i)\neq 0$ $(1\leqq i\leqq d)$ and $kz(\alpha_c)\neq y(i_0)$, we have

$$p\pi E'_\alpha\subseteqq G_d(V)^u\cap S_\alpha=S^u_\alpha.$$

Hence

$$p(\pi E_\alpha-\{0\})\subseteqq \mathrm{cl}S^u_\alpha.$$

Let E^0_α be the closed subvariety of E_α defined by $k=0$. Then $\pi E^0_\alpha=\pi D^u_\beta$, where D^u_β is the variety defined in 2.1. Therefore we have $\mathrm{cl}S^u_\alpha\supseteqq \mathrm{cl}S^u_\beta$ by 2.1.

Next, we assume that $\lambda_{h_2}(\alpha)=0$. Let D_a be the subvariety of D^u_α defined by $x_a(j)=0$ for $j<j_0$. Then we have $\pi D_a=\pi D^u_\beta$. Therefore we have $\mathrm{cl}S^u_\alpha\supseteqq \mathrm{cl}S^u_\beta$ by 2.1. Thus the proof of the lemma is completed.

2.3. Let α and β be d-tableaus like the ones in 2.2. Then we say that these α and β $(\alpha<\beta)$ are in the *elementary relation*, or shortly, that $\alpha<\beta$ is *elementary*. If a sequence of d-tableaus $\alpha^1<\alpha^2<\cdots<\alpha^p$ satisfies the condition that $\alpha^i<\alpha^{i+1}$ is elementary for all $i=1, \cdots, p-1$, then we say that the sequence $\alpha^1<\alpha^2<\cdots<\alpha^p$ is an *elementary sequence* from α^1 to α^p. The following lemma and its proof were communicated by H. Doi.

2.4. LEMMA. *For a minimal semistandard d-tableau* α, *let* X_α *be the set of all semistandard d-tableaus* β *such that* $\lambda_i'(\alpha)=\lambda_i'(\beta)$ *for* $i=1, 2, \cdots$, *where* $\lambda_i'(\alpha)$ (*resp.* $\lambda_i'(\beta)$) *is the number of all* ▨*-squares on the i-th column of* α (*resp.* β). *For* β *and* γ *in* X_α, *assume that* $\beta<\gamma$. *Then there is an elementary sequence* $\alpha^1<\alpha^2<\cdots<\alpha^p$ *such that* $\alpha^i\in X_\alpha$ $(i=1, \cdots, p)$, $\beta=\alpha^1$ *and* $\gamma=\alpha^p$.

PROOF. It suffices to show that there is a d-tableau β' in X_α such that $\beta<\beta'$ is elementary and $\beta'\leq\gamma$. We prove this by induction on d. If $d=1$, it is trivial. So we assume that $d>1$. Suppose that the square $\boxed{\gamma_1}$ lies on the p-th row of λ. Since $\beta<\gamma$, there is a number k $(1\leqq k)$ such that $\beta_k\leqq\gamma_1<\beta_{k+1}$. We assume that the square $\boxed{\beta_k}$ lies on the h-th row of λ. Then $h\leqq p$. If $h=p$, we see that

$$\beta-(h\text{-th row})<\gamma-(h\text{-th row}).$$

Then the assertion follows from the induction hypothesis. From now on, we assume that $h\neq p$.

Case 1. We assume that, for any square $\boxed{\beta_i}$ on the h-th row, the square $\boxed{\gamma_i}$ lies on the p-th row or a lower row than the p-th row. Let β' be the d-tableau of type λ obtained by exchanging the number of ▨-squares on the h-th row of β for the number of ▨-squares on the $(h+1)$-th row of β. Then $\beta<\beta'$ is elementary and $\beta'\leq\gamma$.

Case 2. We assume that, for some square $\boxed{\beta_i}$ on the h-th row, the square $\boxed{\gamma_i}$ lies on the strictly upper row than the p-th row. We take the $\boxed{\beta_i}$ which lies on the extreme right of these. Assume that this $\boxed{\beta_i}$ lies on the s-th column from the left. Put $m=\lambda_1(\beta)$ (see 2.2). Then $m\geqq 2$ and $s<m$ by the choice of h. Take a number h' which is either the number $h+1$, if $\lambda_{h+1}(\beta)\geqq s$, or a number which satisfies $\lambda_j(\beta)<s$ $(h+1\leqq j<h')$ and $\lambda_{h'}(\beta)\geqq s$. Then $h'\leqq p$. Let β' be the d-tableau of type λ obtained by exchanging the number of ▨-squares on the h-th row of β for the number of ▨-squares on the h'-th row of β. For a $\boxed{\beta_j}$, which lies on a strictly right part from the s-th column and between the h-th and p-th rows, the $\boxed{\gamma_j}$ lies on the p-th row or a lower row than the p-th row. Therefore $\beta<\beta'$ is elementary and $\beta'\leq\gamma$. Thus the proof of the lemma is completed.

2.5. PROPOSITION. *For a minimal semistandard d-tableau* α, *let* X_α *be the set defined in* 2.4. *Then, for* β *and* γ *in* X_α, *the following two conditions are equivalent:*

(1) $\beta\leq\gamma$,

(2) $\mathrm{cl}S_\beta^u\supseteqq\mathrm{cl}S_\gamma^u$.

In particular for any β *in* X_α, *we have* $\mathrm{cl}S_\alpha^u\supseteqq\mathrm{cl}S_\beta^u$.

PROOF. The implication (1)$\Rightarrow$(2) follows from 2.2 and 2.4. On the other hand, (2) implies $\Omega_\beta\supseteqq\Omega_\gamma$. Therefore, the implication (2)$\Rightarrow$(1) follows from 1.2.

Thus the proposition.

2.6. PROPOSITION. *For a minimal semistandard d-tableau α of type λ, let λ_α be the Young diagram consisting of the squares ▨ of α. For the α, put*

$$Y_\alpha=\{W\in G_d(V)^u\ ;\ \text{the Jordan type of } u|_W \text{ is } \lambda_\alpha\}.$$

Then, we have

$$Y_\alpha=\bigcup_{\beta\in X_\alpha} S^u_\beta.$$

And the variety Y_α is an irreducible locally closed subvariety of $G_d(V)^u$.

PROOF. The first assertion follows from the definition of S^u_β. Then, we have

$$Y_\alpha=\Omega^u_\alpha-\bigcup_{\substack{\beta;\text{minimal},\\ \beta>\alpha}}\Omega^u_\beta.$$

Hence Y_α is a locally closed subvariety of $G_d(V)^u$. For any β in X_α, we have $\beta\geq\alpha$. Then the proposition follows from 2.5.

§ 3. Inclusion relations in some particular cases.

We use the notations in the preceding sections.

3.1. LEMMA. *Let $\lambda=(\lambda_1, \cdots, \lambda_r)$ be a Young diagram with n squares. For a semistandard d-tableau α, let $\lambda_i(\alpha)$ be the number of all ▨-squares on the i-th row of α. Fix two numbers h_1 and h_2 such that $h_1\neq h_2$ and $1\leqq h_1, h_2\leqq r$. Assume that there is an integer p such that $\lambda_{h_1}(\alpha)\gneqq p\gneqq\lambda_{h_2}(\alpha)$. Let β be a semistandard d-tableau such that*

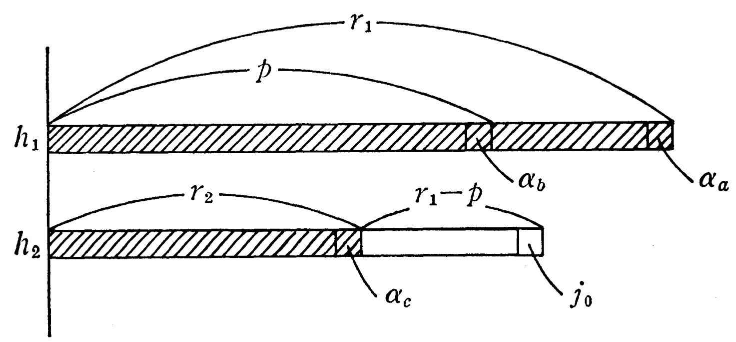
$$\lambda_i(\beta)=\begin{cases}\lambda_i(\alpha) & \text{if } i\neq h_1, h_2,\\ p & \text{if } i=h_1,\\ \lambda_{h_2}(\alpha)+\lambda_{h_1}(\alpha)-p & \text{if } i=h_2.\end{cases}$$

Then:

(1) *The intersection of* $\mathrm{cl}S^u_\alpha$ *and* S^u_β *is not empty.*

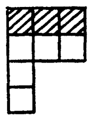
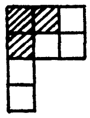
(2) *If $\lambda_r\geqq\lambda_i(\alpha)$ for any i $(1\leqq i\leqq r)$,* $\mathrm{cl}S^u_\alpha$ *contains* S^u_β.

PROOF. For $i=1, 2$, put $r_i=\lambda_{h_i}(\alpha)$. First, we assume that $r_2>0$. Take the numbers α_a, α_b, α_c and j_0 as in the following figure:



Put $m=r_1-p$ and construct the varieties E_α and E_α^0 as in the proof of 2.2. Then the intersection $\pi E_\alpha^0 \cap \pi C'_\beta$ is not empty, where C'_β is the dense subvariety of D_β^u defined in the proof of 2.1. This implies (1). If $\lambda_r \geqq \lambda_i(\alpha)$ for any i $(1 \leqq i \leqq r)$, we have $\pi E_\alpha^0 = \pi D_\beta^u$. Hence we get (2). If $r_2=0$, the proof is similar to the above case. Thus the lemma.

3.2. REMARK. In 3.1 (1), we can not expect an inclusion like $\mathrm{cl}S_\alpha^u \supseteqq S_\beta^u$. For example, put

$\alpha =$  and $\beta =$  .

Then α and β satisfy the conditions of 3.1. We see that $\mathrm{cl}S_\alpha^u \not\supset S_\beta^u$, since $\dim S_\alpha^u = \dim S_\beta^u = 5$.

3.3. PROPOSITION. *For a partition $\lambda=(\lambda_1, \cdots, \lambda_r)$ of n, let R be the set of all d-tableaus α of type λ such that $\lambda_r \geqq \lambda_i(\alpha)$ for $i=1, \cdots, r$. Then we have:*

(1) *For α, β in R, we assume that α and β are minimal, then $\mathrm{cl}S_\alpha^u \supseteqq \mathrm{cl}S_\beta^u$ if and only if $\alpha \leq \beta$.*

(2) *For a minimal d-tableau α in R, the fixed point subvariety $\Omega_\alpha^u = \{L \in \Omega_\alpha \ ; \ uL=L\}$ equals $\mathrm{cl}S_\alpha^u$, and Ω_α^u is irreducible.*

PROOF. (1) If $\mathrm{cl}S_\alpha^u \supseteqq \mathrm{cl}S_\beta^u$, we see that $\alpha \leq \beta$ by 1.2. Let $(\lambda_1(\alpha), \lambda_2(\alpha), \cdots)$ and $(\lambda_1(\beta), \lambda_2(\beta), \cdots)$ be partitions of d corresponding to λ_α and λ_β as in 2.6 respectively. Assume that $\alpha \leq \beta$. Then, for $i=1, 2, \cdots$, we have

$$\lambda_1(\alpha)+ \cdots +\lambda_i(\alpha) \geqq \lambda_1(\beta)+ \cdots +\lambda_i(\beta).$$

By a property of an ordering of the partitions of d ([8], 1.2) and 3.1 (2), we have $\mathrm{cl}S_\alpha^u \supseteqq \mathrm{cl}S_\beta^u$.

(2) We have, by the results of § 2,

$$\Omega_\alpha^u = \bigcup \mathrm{cl}S_\gamma^u,$$

where the union is taken over all minimal semistandard d-tableaus γ such that $\gamma \geq \alpha$. Then we have

$$\Omega_\alpha^u = \mathrm{cl}S_\alpha^u,$$

by (1). Since S_α^u is isomorphic to an affine space, by 1.11, Ω_α^u is irreducible. Thus the proposition.

3.4. REMARKS. Let R be as in 3.3.

(1) If $\lambda=(\lambda_1, \cdots, \lambda_r)$ is a rectangular Young diagram, i.e., $\lambda_1= \cdots =\lambda_r$, we see that any semistandard d-tableau of type λ is in R.

(2) For α, β in R, we conjecture that the following two conditions are

equivalent:

(a) $\mathrm{cl}S_\alpha^u \supseteqq \mathrm{cl}S_\beta^u$,

(b) $\alpha \leq \beta$.

As a corollary of 3.3, we consider a compactification of a unipotent conjugacy class of a general linear group $GL_d(K)$, defined in [7].

3.5. COROLLARY. *Let $\lambda=(d, \cdots, d)$ be a partition of $n=d^2$. For a partition μ of d, let C_μ be the unipotent conjugacy class in $GL_d(K)$ of Jordan type μ. Let u be a unipotent transformation of V of Jordan type λ. Then $\Omega_\alpha^u=\mathrm{cl}S_\alpha^u$ is a projective closure of C_μ, where α is a minimal semistandard d-tableau such that $\mu_i=\lambda_i(\alpha)$ for $i=1, 2, \cdots$.*

PROOF. Let Y_α be the subvariety of $G_d(V)^u$ defined in 2.6. Then the compactification of C_μ defined in [7] is the Zariski closure of Y_α in $\mathbf{P}(\wedge^d V)$. Thus the corollary follows from 2.6, 3.3 and 3.4 (1).

3.6. REMARK. Let $\bar{C}_\mu$ be the Zariski closure of a unipotent conjugacy class C_μ in $GL_d(K)$. For a partition μ (resp. ν) of d, let α (resp. β) be a minimal semistandard d-tableau such that $\mu_i=\lambda_i(\alpha)$ (resp. $\nu_i=\lambda_i(\beta)$). By 3.5, we have $\bar{C}_\mu \supseteqq \bar{C}_\nu \Leftrightarrow \mathrm{cl}S_\alpha^u \supseteqq \mathrm{cl}S_\beta^u \Leftrightarrow \alpha \leq \beta \Leftrightarrow \mu_1+\cdots+\mu_i \geqq \nu_1+\cdots+\nu_i$ for $i=1, 2, \cdots$.

§ 4. The fixed point subvarieties of the flag varieties.

We use the notations in the preceding sections. An *ordered partition* μ of n is an ordered sequence of positive integers $(\mu_1, \cdots, \mu_s)$ such that $\mu_1+\cdots+\mu_s=n$, where the μ_i are not necessarily in decreasing order.

4.1. DEFINITION. Let λ be a partition of n. Let $\mu=(\mu_1, \cdots, \mu_s)$ be an ordered partition of n.

(1) A *μ-tableau of type λ* is a Young diagram of type λ whose squares are numbered with the figures from 1 to s such that the cardinality of the squares with figure i is μ_i.

(2) A μ-tableau is said to be *semistandard* if, on each row, the sequence of the figures in the squares increases (may be stationary).

(3) A semistandard μ-tableau is said to be *minimal* if, on each column, the sequence of the figures in the squares increases (may be stationary).

(4) A minimal semistandard μ-tableau is said to be *standard* if, on each column, the sequence of the figures in the squares strictly increases. [If $\mu=(1, \cdots, 1)$, a minimal semistandard tableau is a standard tableau.]

4.2. REMARK. By [8], 5.14, the number of all standard μ-tableaus of type λ is given by the *Kostka coefficient* $K_\lambda^{\bar{\mu}}$ where $\bar{\mu}$ is the unique partition of n determined by μ.

4.3. EXAMPLES. (1) For $\lambda=(3, 3, 1)$ and $\mu=(2, 3, 2)$, put

$$\text{(a)}\ \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 3&2&2\\\hline 1\\\cline{1-1}\end{array}\ ,\quad \text{(b)}\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&2&3\\\hline 1\\\cline{1-1}\end{array}\ ,\quad \text{(c)}\ \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2&3\\\hline 2\\\cline{1-1}\end{array}\ ,\quad \text{(d)}\ \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&2&3\\\hline 3\\\cline{1-1}\end{array}\ .$$

All are μ-tableaus of type λ. (a) is not semistandard, (b) is semistandard but not minimal, (c) is minimal but not standard and (d) is standard.

(2) If d is an integer such that $1\leqq d<n$, we can consider the d-tableaus defined in 1.4 as the $(d, n-d)$-tableaus by changing the squares ▨ into $\boxed{1}$ and the squares □ into $\boxed{2}$.

4.4. For an ordered partition $\mu=(\mu_1, \cdots, \mu_s)$ of n, put $d_i=\mu_1+\cdots+\mu_i$ $(i=1, 2, \cdots, s-1)$. For a μ-tableau α of type λ, let α^i $(i=1, 2, \cdots, s-1)$ be a d_i-tableau of type λ obtained by changing the squares $\boxed{k}$ $(k\leqq i)$ into ▨ and the squares $\boxed{j}$ $(j\geqq i+1)$ into □. Then the following two conditions are equivalent:

(1) α is a semistandard (resp. minimal semistandard) μ-tableau of type λ.

(2) For each i $(1\leqq i\leqq s-1)$, α^i is a semistandard (resp. minimal semistandard) d_i-tableau (1.4) of type λ.

Let $F_\mu=F_\mu(V)$ be the *partial flag variety of type* μ defined by

$$\{(W_1, \cdots, W_{s-1})\in \prod_{1\leq i\leq s-1} G_{d_i}(V)\ ;\ W_i\subset W_{i+1}\,(1\leqq i\leqq s-2)\}.$$

For a μ-tableau α, put

$$S_\alpha=\{(W_i)\in F_\mu\ ;\ W_i\in S_{\alpha^i}\,(1\leqq i\leqq s-1)\}$$

$$(\text{resp. } \Omega_\alpha=\{(W_i)\in F_\mu\ ;\ W_i\in \Omega_{\alpha^i}\,(1\leqq i\leqq s-1)\}),$$

where S_{α^i} (resp. Ω_{α^i}) is a Schubert cell (resp. variety) corresponding to a d_i-tableau α^i of type λ. Then Ω_α is the Zariski closure of S_α in F_μ and $F_\mu=\coprod_\alpha S_\alpha$, where the (disjoint) union is taken over the μ-tableaus α of type λ.

4.5. PROPOSITION. *Let u be a unipotent transformation of V of Jordan type λ defined in* 1.7. *For a μ-tableau α of type λ, put*

$$S_\alpha^u=\{(W_i)\in S_\alpha\ ;\ uW_i=W_i\,(1\leqq i\leqq s-1)\}.$$

Then S_α^u is nonempty if and only if α is semistandard.

PROOF. If $S_\alpha^u\neq\emptyset$, we have $S_{\alpha^i}^u\neq\emptyset$ $(1\leqq i\leqq s-1)$. Then α^i is semistandard by 1.8. Hence α is semistandard by 4.4. On the other hand, assume that α is semistandard. Let $\{e_i\ ;\ 1\leqq i\leqq n\}$ be the basis of V defined as in 1.7. For i $(1\leqq i\leqq s-1)$, let W_i be the d_i-dimensional subspace of V spanned by the vectors e_k which correspond to the squares $\boxed{j}$ of α such that $j\leqq i$ (see 1.6). Then $(W_i)\in S_\alpha$ and $uW_i=W_i$. Hence $S_\alpha^u\neq\emptyset$ and the proof of the proposition is completed.

4.6. DEFINITION. For a semistandard μ-tableau α of type λ, let $d(\alpha)$ be a non-negative integer defined by the following recurrence rule:

(1) If $\mu=(n)$, put $d(\alpha)=0$.

(2) If $\mu=(\mu_1, \mu_2)$, by 4.3 (2), let $d(\alpha)$ be the number defined in 1.11.

(3) If $\mu=(\mu_1, \cdots, \mu_{s-1}, \mu_s)$ and $s>2$, put $\mu'=(\mu_1, \cdots, \mu_{s-1})$. Let α' be the μ'-tableau obtained by extracting the squares with figure s from α and by rearranging the rows in the appropriate order. Thus α' is a semistandard μ'-tableau of type λ', where λ' is a Young diagram with $n-\mu_s$ squares. Then one defines

$$d(\alpha)=d(\alpha')+d(\alpha^{s-1}),$$

where α^{s-1} is the semistandard $(n-\mu_s, \mu_s)$-tableau of type λ defined in 4.4.

4.7. THEOREM ([**12**]). *Let u be a unipotent transformation of V of Jordan type λ. Let α be a semistandard μ-tableau of type λ. Then the variety S_α^u is isomorphic to the $d(\alpha)$-dimensional affine space $\mathbf{A}^{d(\alpha)}$.*

The proof follows from 1.11 and is given in [**12**], p. 64.

4.8. COROLLARY ([**3**], [**12**]). *Put*

$$F_\mu^u=\{(W_i)\in F_\mu \ ; \ uW_i=W_i \ (1\leqq i\leqq s-1)\}.$$

Then the variety F_μ^u has a partition into locally closed affine spaces $\mathbf{A}^{d(\alpha)}$ as α runs through the semistandard μ-tableaus of type λ.

The proof follows from 4.4, 4.5 and 4.7.

4.9. For a minimal semistandard μ-tableau α of type λ, let X_α be the set of all semistandard μ-tableaus β of type λ such that the tableaus β are obtained by rearranging, on each column, the figures in the squares of α. For i $(1\leqq i\leqq s-1)$, let λ_α^i be the Young diagram consisting of the squares $\boxed{j}$ of α such that $j\leqq i$. For the α, put

$$Y_\alpha=\{(W_i)\in F_\mu^u \ ; \text{ the Jordan type of } u|_{W_i} \text{ is } \lambda_\alpha^i \ (1\leqq i\leqq s-1)\}.$$

Then, by the definition of S_β^u, we have

$$Y_\alpha=\bigcup_{\beta\in X_\alpha} S_\beta^u.$$

4.10. DEFINITION. Let "$\leq$" be a partial order on the set of all μ-tableaus of type λ defined by

$$\alpha\leq\beta \iff \alpha^i\leq\beta^i \text{ for all } i \ (1\leqq i\leqq s-1),$$

where α^i (resp. β^i) is a d_i-tableau of type λ determined by α (resp. β) as in 4.4 and the order "$\alpha^i\leq\beta^i$" is the partial order defined in 1.1.

4.11. LEMMA. *For two μ-tableaus α and β, the following three conditions* *are equivalent:*

(1) $\alpha \leq \beta$.

(2) $\Omega_\alpha \cap S_\beta$ *is not empty.*

(3) $\Omega_\alpha \supseteqq \Omega_\beta$.

The proof follows from 1.2.

4.12. THEOREM. *Let* α *be a minimal semistandard* μ*-tableau of type* λ. *Then* Y_α *is an irreducible locally closed subvariety of* F_μ^u *and* S_α^u *is an open subvariety of* Y_α.

PROOF. By 4.4, 4.9 and 4.11, we have

$$Y_\alpha = \Omega_\alpha^u - \bigcup_{\substack{\beta;\text{minimal},\\ \beta>\alpha}} \Omega_\beta^u .$$

Hence Y_α is a locally closed subvariety of F_μ^u. Let

$$p : Y_\alpha \longrightarrow Y_{\alpha^{s-1}}$$

be the projection defined by $p((W_i)) = W_{s-1}$. By 2.7, $p^{-1}(S_{\alpha^{s-1}}^u)$ is an open dense subvariety of Y_α. Let V' be the $n-\mu_s$ dimensional subspace of V spanned by the vectors e_k which correspond to the squares $\boxed{i}$ of α such that $1 \leqq i \lneqq s$ (see 1.6). Let $f : V \rightarrow V'$ be the projection defined by

$$f(e_i) = \begin{cases} 0 & \text{if } e_i \notin V', \\ e_i & \text{if } e_i \in V'. \end{cases}$$

By $(W_i) \mapsto (W_{s-1}, (f(W_i))_{1\leqq i \leqq s-2})$, we have two isomorphisms

$$p^{-1}(S_{\alpha^{s-1}}^u) \xrightarrow{\sim} S_{\alpha^{s-1}}^u \times Y_{\alpha'},$$

$$S_\alpha^u \xrightarrow{\sim} S_{\alpha^{s-1}}^u \times S_{\alpha'}^{u'},$$

where $\lambda' = \lambda_\alpha^{s-1}$ (4.9), α' is a minimal semistandard μ'-tableau of type λ' (4.6 (3)) and u' is the restriction of u to V'. By the induction argument, we see that S_α^u is open dense in $p^{-1}(S_{\alpha^{s-1}}^u)$. Hence S_α^u is open dense in Y_α and Y_α is irreducible by 4.7. Thus the proof of the theorem is completed.

4.13. COROLLARY. *For* β, γ *in* X_α, *we have*

$$\beta \leq \gamma \iff \mathrm{cl} S_\beta^u \subseteqq \mathrm{cl} S_\gamma^u .$$

PROOF. If $\mathrm{cl}S_\beta^u \supseteqq \mathrm{cl}S_\gamma^u$, we have $\beta \leq \gamma$ by 4.11. Assume that $\beta \leq \gamma$. Similarly to the proof of the theorem, we see that the variety S_β^u is open dense in $\bigcup_{\delta\in X_\alpha, \delta\geq\beta} S_\delta^u$. Hence $\mathrm{cl}S_\beta^u \supseteqq \mathrm{cl}S_\gamma^u$ and the proof of the corollary is completed.

4.14. Let α be a minimal semistandard μ-tableau of type λ. For a square $\boxed{i}$ of α, let $\alpha(i)$ be the number of all squares $\boxed{j}$ of α, on the upper side and on the same column to the square $\boxed{i}$, such that $j \lneqq i$. Then, by 1.11 (2), we have

$$d(\alpha)=\sum\alpha(i),$$

where the summation is taken over the squares of α. Then we have

$$d(\alpha)\leqq\sum_{1\leqq i\leqq p}\{(\lambda_i'-1)+(\lambda_i'-2)+\cdots+2+1\},$$

where $(\lambda_1',\cdots,\lambda_p')$ is the dual partition of λ and the equality holds if and only if α is standard. By this formula, we have a proof of the following theorem due to Steinberg.

4.15. THEOREM ([**3**], [**13**]). *Put* $n_\lambda=\sum_{1\leqq i\leqq p}\lambda_i'(\lambda_i'-1)/2$. *Let* S *be the set of all standard* μ*-tableaus of type* λ. *Then we have:*

(1) $\dim F_\mu^u\leqq n_\lambda$.

(2) *The irreducible components of dimension* n_λ *of* F_μ^u *are the closures* $\mathrm{cl}S_\alpha^u$ *of* S_α^u $(\alpha\in S)$.

PROOF. By 4.12, we have

$$F_\mu^u=\bigcup_{\alpha;\text{minimal}}\mathrm{cl}S_\alpha^u.$$

Hence (1) and (2) follow from 4.14.

Appendix: Homogeneous coordinate rings.

We use the notations in § 1, § 2 and § 3. The purpose of this appendix is to study the homogeneous coordinate ring of Ω_α^u, when α is a minimal semi-standard μ-tableau of type λ such that $\dim\Omega_\alpha^u=\dim\Omega_\alpha-1$. First, we recall some basic facts on the homogeneous coordinate rings of the Schubert varieties.

Let V^d and $\wedge^d V$ be as in 1.1. Let $K[X_i(j)]=K[X_i(j)\ ;\ 1\leqq i\leqq d,\ 1\leqq j\leqq n]$ be the coordinate ring of the affine space V^d. Using the basis $\{e_\alpha=e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_d}\ ;\ \alpha\in I\}$ of $\wedge^d V$, let $K[X_\alpha\ ;\ \alpha\in I]$ be the coordinate ring of the affine space $\wedge^d V$, where I is the set defined in 1.1. The comorphism $\pi^*:K[X_\alpha\ ;\ \alpha\in I]\to K[X_i(j)]$ associated with a canonical morphism $\pi:V^d\to\wedge^d V$ is defined by $\pi^*(X_\alpha)=p_\alpha$, where $p_\alpha=\det(X_i(\alpha_j))_{1\leqq i,j\leqq d}$. Then the homogeneous coordinate ring R of $G_d(V)$ is identified with the subalgebra $K[p_\alpha\ ;\ \alpha\in I]$ of $K[X_i(j)]$. For α in I, let I_α be the homogeneous ideal of $R=K[p_\beta\ ;\ \beta\in I]$ generated by the set $\{p_\beta\ ;\ \beta\in I,\ \beta\not\geqq\alpha\}$. Then I_α is a prime ideal and R/I_α is isomorphic to the homogeneous coordinate ring R_α of a Schubert variety Ω_α. Let $K[X_i(j)]_\alpha=K[X_i(j)\ ;\ X_i(j)=0 \text{ if } j<\alpha_i\,(1\leqq i\leqq d)]$ be the coordinate ring of the affine space D_α (see 1.1). Then we identify R_α with the subalgebra $K[p_\beta\ ;\ \beta\in I,\ \beta\geqq\alpha]$ of $K[X_i(j)]_\alpha$, where $p_\beta=\det(X_i(\beta_j))_{1\leqq i,j\leqq d}$.

A.1. LEMMA. (1) *For* β *in* I *such that* $\beta\geqq\alpha$, *let* $I_{\alpha\beta}$ *be the homogeneous ideal of* $R_\alpha=K[p_\gamma\ ;\ \gamma\geqq\alpha]$ *generated by the set* $\{p_\gamma\ ;\ \gamma\in I,\ \gamma\not\geqq\beta\}$. *Then* $I_{\alpha\beta}$ *is a prime ideal of* R_α *and* $R_\alpha/I_{\alpha\beta}$ *is isomorphic to the homogeneous coordinate ring* R_β *of the Schubert variety* Ω_β.

(2) (Pieri's formula) *Let* (p_α) *be the ideal of* R_α *generated by* p_α. *Then*

$$(p_\alpha)=\bigcap I_{\alpha\beta},$$

where β*'s are all the smallest elements of* I *such that* $\alpha \lneqq \beta$.

For the proof, see [6], [10].

A.2. LEMMA. *For an integer* i *satisfying* $0 \leqq i \leqq \dim \Omega_\alpha$, *put*

$$f_i=\sum a_\gamma p_\gamma \qquad (a_\gamma \in K-\{0\}),$$

where the sum is taken over all γ *in* I *such that* $\gamma \geq \alpha$ *and* $\dim \Omega_\gamma = i$. *Then* f_0, $f_1, \cdots, f_q$ $(q=\dim \Omega_\alpha)$ *is a regular sequence in the irrelevant maximal ideal* $(R_\alpha)_+$ *generated by* $\{p_\beta \,;\, \beta \in I, \beta \geq \alpha\}$.

For the proof, see [1], Theorem 8.1 and [11], Theorem 4.1.

For α in I, let C'_α be the subvariety of V^d consisting of all $(v_1, \cdots, v_d)$ in V^d, $v_i=\sum x_i(j)e_j$ $(1 \leqq i \leqq d,\ 1 \leqq j \leqq n)$, which satisfy the following conditions:

(1) $x_i(j)=0$ for $j<\alpha_i$.

(2) $x_1(\alpha_1) \neq 0$.

(3) $x_i(\alpha_j)=\delta_{ij}$ for $1 \leqq i, j \leqq d$, $(i, j) \neq (1, 1)$.

Then the coordinate ring of C'_α can be written as $B_\alpha[1/t]$, where t is a variable over K and B_α is the K-algebra generated by $\{Y_i(j) \,;\, 1 \leqq i \leqq d,\ 1 \leqq j \leqq n\}$ which satisfy the following conditions:

(1′) $Y_i(j)=0$ for $j<\alpha_i$.

(2′) $Y_1(\alpha_1)=t$.

(3′) $Y_i(\alpha_j)=\delta_{ij}$ for $1 \leqq i, j \leqq d$, $(i, j) \neq (1, 1)$ and the other $Y_i(j)$'s are variables over K.

A.3. LEMMA. *In the above notations, let* $\varphi: R_\alpha \to B_\alpha[1/t]$ *be the homomorphism defined by* $\varphi(p_\beta)=\det(Y_i(\beta_j))_{1 \leqq i, j \leqq d}$. *Then, for all* α *belonging to* I, *the* φ *induces an isomorphism*

$$\varphi' : R_\alpha[1/p_\alpha] \xrightarrow{\sim} B_\alpha[1/t].$$

PROOF. Let C_α be the variety defined in 1.1. Since $C'_\alpha \supset C_\alpha$, $\pi C'_\alpha$ is dense in the cone over Ω_α. Therefore the φ is injective. Since R_α is an integral domain, p_α is not a zero-divisor. Hence φ' is injective. We get several images of p_β's as follows:

$$\varphi(p_\alpha)=t.$$

$$\varphi(p_{(j, \alpha_2, \cdots, \alpha_d)})=\pm Y_1(j)$$

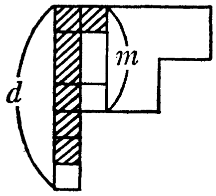
for j such that $\alpha_1 < j \leqq n$ and $j \neq \alpha_k$ $(2 \leqq k \leqq d)$.

$$\varphi(p_{(\alpha_1, \cdots, \alpha_{i-1}, j, \alpha_{i+1}, \cdots, \alpha_d)})=\pm t Y_i(j)$$

for j such that $\alpha_i < j \leqq n$ $(2 \leqq i \leqq d)$ and $j \neq \alpha_k$ $(2 \leqq k \leqq d)$.

Therefore φ' is surjective. Hence the lemma.

If α is minimal and $\dim \Omega_\alpha^u = \dim \Omega - 1$, α has the picture below:



where d (resp. m) is the number of all squares on the first (resp. second) column of λ and $d \geqq 2$. Hence, we may assume that $\lambda = (\underbrace{2, \cdots, 2}_{m}, 1, \cdots, 1)$ and $d+m=n$. Then the unipotent transformation u of V defined in 1.7 is given by

$$ue_i = e_i + e_{i+m} \quad \text{for} \quad 1 \leqq i \leqq m,$$

$$ue_j = e_j \quad \text{for} \quad m+1 \leqq j \leqq n.$$

In the notations of 1.1, we can write $\alpha = (1, m+1, \cdots, n-1)$. For i $(1 \leqq i \leqq m)$ and j $(m+1 \leqq j \leqq n)$, we denote by $(i\,;j)$ the d-tableau $(i, m+1, \cdots, \check{j}, \cdots, n)$, where $\check{j}$ means that the integer j has been removed from the sequence. We have

$$\{\beta \in I\ ;\ \beta \geq \alpha\} = \{(i\,;j)\ ;\ 1 \leqq i \leqq m,\ m+1 \leqq j \leqq n\} \cup \{\mu\},$$

where I is the set of all d-tableaus of type λ and $\mu = (m+1, \cdots, n)$. We have

$$\dim \Omega_{(i;j)} = j - i, \qquad \dim \Omega_\mu = 0.$$

Therefore we have

$$\begin{aligned} &\{\beta \in I\ ;\ \beta \geq \alpha,\ \dim \Omega_\beta = m\} \\ &= \{\beta \in I\ ;\ \beta \geq \alpha \text{ and } \beta \text{ is not semistandard}\} \\ &= \{(i\,;m+i)\ ;\ 1 \leqq i \leqq m\}. \end{aligned}$$

Let R_α be the homogeneous coordinate ring of Ω_α. Put

$$f_m = \sum_{1 \leqq i \leqq m} (-1)^i p_{(i;m+i)}.$$

For $i = 0, \cdots, \check{m}, \cdots, n-1$ $(n-1 = \dim \Omega_\alpha)$, put

$$f_i = \sum p_\gamma,$$

where the sum is taken over all $\gamma \geq \alpha$ such that $\dim \Omega_\gamma = i$.

A.4. LEMMA. *For an element x in R_α, let x' be the image of x under a natural homomorphism $R_\alpha \to R_\alpha/(f_m)$. Then $f'_0, \cdots, f'_{m-1}, f'_{m+1}, \cdots, f'_{n-1}$ is a regular sequence in the irrelevant maximal ideal $(R_\alpha/(f_m))_+$ and the ring $R_\alpha/(f_m)$ is Cohen-Macaulay. In particular, $p'_\alpha = f'_{n-1}$ is not a zero-divisor of $R_\alpha/(f_m)$.*

This lemma follows from A.2, [**6**], Lemma 11 and [**11**], Lemma 4.2.

A.5. LEMMA. *The homogeneous coordinate ring of* Ω_α^u *is isomorphic to* $R_\alpha/\sqrt{(f_m)}$.

PROOF. We have $ue_\mu=e_\mu$ and

$$\begin{aligned} ue_{(i;j)} &= ue_i\wedge u(e_{m+1}\wedge\cdots\wedge\check{e}_j\wedge\cdots\wedge e_n) \\ &= (e_i+e_{m+i})\wedge e_{m+1}\wedge\cdots\wedge\check{e}_j\wedge\cdots\wedge e_n \\ &= \begin{cases} e_{(i;j)} & \text{if } j\neq m+i, \\ e_{(i;j)}-(-1)^i e_\mu & \text{if } j=m+i. \end{cases} \end{aligned}$$

For β in I, let X_β be a coordinate function of $\wedge^d V$ corresponding to the vector $e_\beta=e_{\beta_1}\wedge\cdots\wedge e_{\beta_d}$. Put

$$F_m=\sum_{1\leq i\leq m}(-1)^i X_{(i;m+i)}.$$

Then, for $x=\sum x_\beta e_\beta$ in $\wedge^d V$, we have

$$x-ux=\Big(\sum_{1\leq i\leq m}(-1)^i x_{(i;m+i)}\Big)e_\mu=F_m(x)e_\mu.$$

Hence

$$\Omega_\alpha^u=\Omega_\alpha\cap\{x\in\mathbf{P}(\wedge^d V)\ ;\ F_m(x)=0\}.$$

By 3.1(1), Ω_α^u is irreducible. Thus the lemma.

A.6. LEMMA. *Let* p'_α *be the image of* p_α *in the ring* $R_\alpha/(f_m)$. *Then*

$$(R_\alpha/(f_m))[1/p'_\alpha]\simeq K[X_1,\cdots,X_{n-1}][1/X_1],$$

where $K[X_1,\cdots,X_{n-1}]$ *is a polynomial ring in* $n-1$ *variables over a field* K.

PROOF. For $\alpha=(1;n)$, let the notations be as in A.3 and its proof: For example

$$(Y_i(j))=\begin{pmatrix} t & Y_1(2) & \cdots & Y_1(i) & \cdots & Y_1(m) & 0 & \cdot & \cdot & \cdot & \cdot & 0 & Y_1(n) \\ & & & & & & 1 & 0 & \cdot & \cdot & \cdot & 0 & Y_2(n) \\ & & & & & & & \cdot & \cdot & & & \cdot & \cdot \\ & & & \mathbf{0} & & & & & \cdot & \cdot & & \cdot & \cdot \\ & & & & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & & & & \cdot & 0 & Y_{d-1}(n) \\ & & & & & & & & & & & 1 & Y_d(n) \end{pmatrix}$$

and $\varphi: R_\alpha\to B_\alpha[1/t]$ is defined by $\varphi(p_\beta)=\det(Y_i(\beta_j))_{1\leq i,j\leq d}$. Then we see that

$$\varphi(p_{(i;m+1)})=(-1)^d tY_2(n),$$

$$\varphi(p_{(i;m+i)})=(-1)^{d-i+1}Y_1(i)Y_{i+1}(n)$$

for $2\leqq i\leqq m$. Hence, for $f_m=\sum_{1\leq i\leq m}(-1)^i p_{(i;m+i)}$, we have

$$\varphi(f_m)=(-1)^{d+1}\Big(tY_2(n)+\sum_{2\leq i\leq n}Y_1(i)Y_{i+1}(n)\Big).$$

Hence, by A. 3, we have

$$(R_\alpha/(f_m))[1/p'_\alpha] \simeq R_\alpha[1/p_\alpha]/f_m R_\alpha[1/p_\alpha]$$
$$\simeq B_\alpha[1/t]\Big/\Big(Y_2(n)+\sum_{2\leq i\leq m}\frac{1}{t}Y_1(i)Y_{i+1}(n)\Big)B_\alpha[1/t]$$
$$\simeq K[t,\ Y_1(i),\ Y_j(n)\ ;\ 2\leqq i\leqq m,\ 1\leqq j\leqq d \text{ and } j\neq 2][1/t].$$

This ring is isomorphic to $K[X_1, \cdots, X_{n-1}][1/X_1]$. Thus the lemma.

A.7. THEOREM. *The ideal (f_m) is a prime ideal of R_α and the homogeneous coordinate ring of Ω_α^u is isomorphic to $R_\alpha/(f_m)$.*

PROOF. By A. 4, we have an injection

$$R_\alpha/(f_m) \hookrightarrow (R_\alpha/(f_m))[1/p'_\alpha].$$

By A. 6, $(R_\alpha/(f_m))[1/p'_\alpha]$ is an integral domain. Hence, (f_m) is a prime ideal. Then the second statement follows from A. 5. Thus the proof of the theorem is completed.

A.8. REMARKS. (1) Take β in I so that $\alpha\lneqq\beta\lneqq(i\,;m+i)$ for some i $(1\leqq i\leqq m)$. Let f_m^* be the image of f_m under a homomorphism $R_\alpha\rightarrow R_\beta$ (A. 1(1)). Then (f_m^*) is a prime ideal of R_β and $R_\beta/(f_m^*)$ is isomorphic to the homogeneous coordinate ring of Ω_β^u. The proof is similar to the case of the α.

(2) Take β in I so that $(i\,;m+i)\lneqq\beta\leq\mu$ for some i $(1\leqq i\leqq m)$. Then we have $\Omega_\beta^u=\Omega_\beta$.

A.9. PROPOSITION. *The ring $R_\alpha/(f_m)$ is normal.*

PROOF. First, assume that $d>2$ and $m\geqq 2$. Put $\beta=(2\,;n)$ and $\gamma=(1\,;n-1)$. By A. 1 (1), we have $R_\alpha/(p_\alpha,\ p_\gamma,\ f_m)\simeq R_\beta/(f_m^*)$. Hence $(p_\alpha,\ p_\gamma,\ f_m)$ is a prime ideal of R_α by A. 8 (1). Similarly, $(p_\alpha,\ p_\beta,\ f_m)$ is a prime ideal of R_α. Then we have $(p_\alpha,\ f_m)=(p_\alpha,\ p_\beta,\ f_m)\cap(p_\alpha,\ p_\gamma,\ f_m)$, by A. 1. Hence

$$(p'_\alpha)=(p'_\alpha,\ p'_\beta)\cap(p'_\alpha,\ p'_\gamma),$$

where $(p'_\alpha,\ p'_\beta)$ and $(p'_\alpha,\ p'_\gamma)$ are prime ideals of $R_\alpha/(f_m)$. By A. 6, $(R_\alpha/(f_m))[1/p'_\alpha]$ is a regular ring. Hence $R_\alpha/(f_m)$ satisfies the condition (R_1) ([**9**], 17. I). Therefore $R_\alpha/(f_m)$ is normal by A. 4. Next, assume that $d>2$ and $m=1$. By A. 1, (p'_α) is a prime ideal of $R_\alpha/(f_m)$. Then $R_\alpha/(f_m)$ is a UFD by A. 6 and [**9**], 19. B. The remaining is the case $d=m=2$. Then we see that

$$R_\alpha/(f_m)\simeq K[X_1,\ X_2,\ X_3,\ X_4]/(X_1X_3-X_2^2),$$

where X_i's are variables over K. This is a normal ring. Thus the proposition.

A.10. COROLLARY. *Let the notations be as in* 3.5. *Let U be the Zariski closure of the unipotent conjugacy class $C_{(2,1,\cdots,1)}$ in $GL_d(K)$. Then U is a normal* *and Cohen-Macaulay variety.*

PROOF. Let α be the standard (2, 1, $\cdots$, 1)-tableau of type (d, $\cdots$, d). Then

$$\dim \Omega_{\alpha}^{u} = \dim \Omega_{\alpha} - 1.$$

Thus the corollary follows from 3.5.

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