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Stiefel-Whitney homology classes and homotopy type of Euler spaces

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1. Introduction.

In this paper, we construct Euler spaces in fixed homotopy types such that the Stiefel-Whitney homology classes are equal to given homology elements. As a byproduct, we obtain counterexamples to Halperin's conjecture (Fulton-MacPherson [**4**]).

Let X be a locally compact n-dimensional polyhedron. For a point x in X, let $\chi(X, X-x)$ denote the Euler number of the pair $(X, X-x)$. The polyhedron X is called an *integral Euler space* (resp. *mod* 2 *Euler space*) if for each x in X, $\chi(X, X-x)=(-1)^n$ (resp. $\chi(X, X-x)\equiv 1 \pmod 2$) (Halperin and Toledo [**6**]). Sullivan [**9**] has shown that complex analytic spaces (resp. real analytic spaces) are integral Euler spaces (resp. mod 2 Euler spaces).

Let K' denote the barycentric subdivision of a triangulation K of a polyhedron X. If X is a mod 2 Euler space, the sum of all k-simplexes in K' is a mod 2 cycle and defines an element $s_k(X)$ in $H_k(X;\boldsymbol{Z}_2)$ (cf. [**6**]). Note that, if X is not compact, we consider the homology of infinite chains. The element $s_k(X)$ is called the *k-th Stiefel-Whitney homology class* of X. If X is connected and compact, $s_0(X)$ is the mod 2 reduction of the Euler number $\chi(X)$, where we identify $H_0(X;\boldsymbol{Z}_2)$ with $\boldsymbol{Z}_2$. If X is a smooth manifold, PL-manifold, or $\boldsymbol{Z}_2$-homology manifold, the class $s_k(X)$ is known to be equal to the Poincaré dual of the Stiefel-Whitney cohomology class $w^{n-k}(X)$ (Cheeger [**3**], Halperin-Toledo [**6**], Taylor [**10**], Blanton-McCrory [**2**], Veljan [**11**], Matsui [**8**]). Consequently, for such spaces, the Stiefel-Whitney homology classes $s_*(X)$ are homotopy type invariant. For further properties of Stiefel-Whitney homology classes, see [**1**], [**7**].

A polyhedron X is called purely n-dimensional if the union of all n-simplexes in a triangulation of X is dense in X. We have the following concerning mod 2 Euler spaces:

THEOREM 1. *Let X be a purely n-dimensional* mod 2 *Euler space and let a_i, for $i=1, 2, \cdots, n-1$, be elements in $H_i(X;\boldsymbol{Z}_2)$. Then there exist a purely n-dimensional* mod 2 *Euler space Y and a homotopy equivalence $h: X\to Y$ such that $h_*(a_i)=s_i(Y)$ for $i=1, 2, \cdots, n-1$ and $h_*s_n(X)=s_n(Y)$.*

Let $\beta: H_i(X;\boldsymbol{Z}_2)\to H_{i-1}(X;\boldsymbol{Z}_2)$ be the Bockstein homomorphism associated with the exact sequence $0\to\boldsymbol{Z}_2\to\boldsymbol{Z}_4\to\boldsymbol{Z}_2\to 0$. For an integral Euler space X, we have $s_{i-1}(X)=\beta s_i(X)$ if $n-i$ is even (Halperin-Toledo [6]). Thus we have relations among the Stiefel-Whitney homology classes of an integral Euler space. In particular $\chi(X)=0$ for a compact integral Euler space X of odd dimension.

The following holds for integral Euler spaces:

THEOREM 2. *Let X be a purely n-dimensional integral Euler space and let a_i, for $i=1, 2, \cdots, n-2$, be elements in $H_i(X;\boldsymbol{Z}_2)$ such that $a_{i-1}=\beta a_i$ if $n-i$ is even. Then there exist a purely n-dimensional integral Euler space Y and a homotopy equivalence $h: X\to Y$ such that $h_*(a_i)=s_i(Y)$ for $i=1, 2, \cdots, n-2$, $h_*(s_{n-1}(X))=s_{n-1}(Y)$ and $h_*(s_n(X))=s_n(Y)$.*

Note that, if X is an integral n-dimensional Euler space, and if $n-k$ is odd, then we have the integral Stiefel-Whitney homology class $S_k(X)$ in $H_k(X;\boldsymbol{Z})$ such that $s_k(X)$ is the mod 2 reduction of $S_k(X)$. Let $\tilde{\beta}: H_k(X;\boldsymbol{Z}_2)\to H_{k-1}(X;\boldsymbol{Z})$ be the Bockstein homomorphism associated with the exact sequence $0\to\boldsymbol{Z}\to\boldsymbol{Z}\to\boldsymbol{Z}_2\to 0$. Then $S_{k-1}(X)=\tilde{\beta}s_k(X)$ ([4]). Thus, in Theorem 2, we have also the relation $h_*(\tilde{\beta}a_{i-1})=S_i(Y)$ if $n-i$ is even.

A special case of our result has been given by Goldstein [5].

In the book [4], Fulton and MacPherson defined the notion of a homologically normally nonsingular map. As an analogy to the Riemann-Roch formula for singular algebraic spaces, they introduced Halperin's conjecture ([4, p. 112]):

If $f: X\to Y$ is a homologically normally nonsingular map of mod 2 *Euler spaces, then*

$$s_*(X)=[wN_f]^{-1}\cap f^! s_*(Y),$$

where $[wN_f]^{-1}$ is the inverse of the Stiefel-Whitney cohomology class of the normal space of f defined by Thom's formula using the Steenrod squares.

Our construction gives examples where the relation does not hold.

Although the proofs of Theorems 1 and 2 are similar, we first give the proof of Theorem 1 in Section 2, since it is much simpler. In Section 3, we give the proof of Theorem 2 assuming Proposition 3.1. We prepare some elementary lemmas in Section 4 and the proof of Proposition 3.1 is given in Section 5. In Section 6, we explain Halperin's conjecture and give concrete counterexamples to this conjecture.

NOTATION AND PRELIMINARIES. If K is a simplicial complex, the underlying topological space $|K|$ is called a polyhedron, and K is said to be a triangulation of $X=|K|$. We denote by $K^{(i)}$ the set of i-simplexes in K and put $K^i=\bigcup_{j\leq i}K^{(j)}$. By a simplex, we mean the closed one. We write $\operatorname{Int}\sigma$ for the interior of a simplex σ and put $\partial\sigma=\sigma-\operatorname{Int}\sigma$. We write $K(\sigma)$ for the simplicial complex consisting of all faces of σ. For two simplexes σ and τ, the relation $\sigma\leq\tau$ means that σ is a face of τ and $\sigma<\tau$ means that σ is a proper face of τ. We write $K(\partial\sigma)$ for the simplicial complex $\{\tau|\tau<\sigma\}$. For a simplex σ in K, we define $\mathrm{st}(\sigma, K)$, $\mathrm{lk}(\sigma, K)$, and $\partial\,\mathrm{st}(\sigma, K)$ as follows;

$$\mathrm{st}(\sigma, K)=\{\mu\in K \mid \exists\tau\geq\sigma,\ \mu\leq\tau\},$$

$$\mathrm{lk}(\sigma, K)=\{\mu\in K \mid \exists\tau>\sigma,\ \mu<\tau,\ \mu\cap\sigma=\emptyset\},$$

$$\partial\,\mathrm{st}(\sigma, K)=\{\mu\in\mathrm{st}(\sigma, K) \mid \sigma \text{ is not a face of } \mu\}.$$

Let x be a point in the polyhedron $X=|K|$. There exists a simplex σ in K such that $x\in\mathrm{Int}\,\sigma$. Put p=dimension of σ. Then $\partial\,\mathrm{st}(\sigma, K)$ is homeomorphic to the join $\partial\sigma*\mathrm{lk}(\sigma, K)$ and, for any ring A, we have the natural isomorphisms

$$\begin{aligned} H_i(X, X-x\,; A)&=H_i(\mathrm{st}(\sigma, K), \partial\,\mathrm{st}(\sigma, K)\,; A)\\ &=\widetilde{H}_{i-1}(\partial\,\mathrm{st}(\sigma, K)\,; A)\\ &=\widetilde{H}_{i-p-1}(\mathrm{lk}(\sigma, K)\,; A).\end{aligned}$$

LEMMA 1.1. *Let σ be a p-simplex of a simplicial complex K and let x be a point in $X=|K|$ such that $x\in\mathrm{Int}\,\sigma$. Then*

$$\chi(X, X-x)=(-1)^p(1-\chi(\mathrm{lk}(\sigma, K))),$$

$$\chi(X, X-x)\equiv1+\chi(\mathrm{lk}(\sigma, K)) \mod 2.$$

In particular, X is a mod 2 *Euler space if and only if* $\chi(\mathrm{lk}(\sigma, K))\equiv0 \mod 2$ *for any $x\in X$ such that $x\in\mathrm{Int}\,\sigma$.*

By a p-disc, we mean a polyhedron homeomorphic to a p-simplex. If D is a p-disc, the boundary ∂D is homeomorphic to S^{p-1}. For a simplex σ, we denote by b_σ the barycenter of σ.

2. Proof of Theorem 1.

The construction of the mod 2 Euler space Y is given by the repetition of the following simple procedure. Let X be a polyhedron and let σ be a k-simplex of a triangulation K of X. Choose a k-simplex τ in K such that $\sigma\cap\tau$ is a vertex. Let $f:\sigma\to\tau$ be a linear homeomorphism such that $f|\sigma\cap\tau$ is the identity. We give an equivalence relation in X by $f(x)\sim x$ and denote by X_f the quotient space. We denote by π_f the projection $X\to X_f$. Obviously we have the following:

LEMMA 2.1. *The projection $\pi_f: X\to X_f$ is a homotopy equivalence.*

By subdividing sufficiently finely, we may assume that there exists a triangulation K_f of X_f and $\pi_f: K\to K_f$ is a simplicial map such that σ and τ are simplexes in K. For a simplex μ such that $\mu\leq\sigma$, we denote by $[\mu]$ the simplex

$\pi_f(\mu)$ in K_f.

LEMMA 2.2. *The following holds:*

(1) $\chi(\mathrm{lk}([\sigma], K_f))=\chi(\mathrm{lk}(\sigma, K))+\chi(\mathrm{lk}(\tau, K))$,

(2) $\chi(\mathrm{lk}([\mu], K_f))=\chi(\mathrm{lk}(\mu, K))+\chi(\mathrm{lk}(f(\mu), K))-1$ *if* $\mu<\sigma$, $\mu\neq\sigma\cap\tau$,

(3) $\chi(\mathrm{lk}([\sigma\cap\tau], K_f))=\chi(\mathrm{lk}(\sigma\cap\tau, K))-1$.

PROOF. This is easy by counting the number of simplexes in the links.

LEMMA 2.3. *Let K be a purely k-dimensional locally finite simplicial complex. For an i-simplex τ in the barycentric subdivision K' of K with $i\leqq k-2$, the number of k-simplexes σ in K' such that $\sigma>\tau$ is even.*

PROOF. It is sufficient to prove the case where K is a k-simplex Δ. We prove it by an induction on the dimension of Δ. If $\dim\Delta=1$, there is nothing to prove. Suppose that the lemma holds for $\dim\Delta<k$. Let Δ be a k-simplex and let b_Δ be the barycenter of Δ. The barycentric subdivision $K(\Delta)'$ of $K(\Delta)$ is equal to the cone $b_\Delta * K(\partial\Delta)'$. Note that, if $\sigma>\tau$, $\sigma\in K'$, then $\sigma\in\mathrm{st}(\tau, K')$. If τ is not contained in $K(\partial\Delta)'$, then there exists an $(i-1)$-simplex μ in $K(\partial\Delta)'$ such that $\tau=b_\Delta*\mu$. The number of k-simplexes in $\mathrm{st}(\tau, K(\Delta)')$ is equal to the number of $(k-1)$-simplexes in $\mathrm{st}(\mu, K(\partial\Delta)')$, which is even by the induction hypothesis. Now suppose that τ is a simplex in $K(\partial\Delta)'$. The number of k-simplexes in $\mathrm{st}(\tau, K(\Delta)')$ is equal to that of $(k-1)$-simplexes in $\mathrm{st}(\tau, K(\partial\Delta)')$. If $i=k-2$, then the number of $(k-1)$-simplexes in $\mathrm{st}(\tau, K(\partial\Delta)')$ is equal to two, since $K(\partial\Delta)'$ is a $(k-1)$-dimensional manifold. If $i<k-2$, then, by the induction hypothesis, we know that the number of $(k-1)$-simplexes in $\mathrm{st}(\tau, K(\partial\Delta)')$ is even. This completes the proof.

LEMMA 2.4. *Let $h: X\to Y$ be a PL-map of* mod 2 *Euler spaces X and Y. Let K and L be triangulations of X and Y such that h is a simplicial map. Let α be a homology class in $H_k(X; \boldsymbol{Z}_2)$ represented by a* mod 2 *cycle $c=\sum_p\sigma_p$, where σ_p are k-simplexes in K. Suppose that $h|(X-\bigcup_p \mathrm{Int}\,\sigma_p-|K^{k-1}|)$: $X-\bigcup_p \mathrm{Int}\,\sigma_p-|K^{k-1}|\to Y-|L^{k-1}|$ is a bijection. Then $h_*: H_*(X; \boldsymbol{Z}_2)\to H_*(Y; \boldsymbol{Z}_2)$ satisfies the relations $h_*(s_i(X))=s_i(Y)$ for $i>k$ and $h_*((s_k(X)-\alpha)=s_k(Y)$.*

PROOF. Consider the mod 2 chain map $h_\#: C_i(K')\to C_i(L')$ associated with $h: K'\to L'$. Then

$$h_\#(\sum_{\sigma\in K'^{(i)}}\sigma)=\sum_{\tau\in L'^{(i)}}\tau$$

for $i>k$ and

$$h_\#(\sum_{\sigma\in L'^{(k)}}\sigma-\sum_q\sigma'_q)=\sum_{\tau\in L'^{(k)}}\tau,$$

where $\sum_q\sigma'_q$ is the mod 2 k-chain in K' consisting of all k-simplexes in $|c|=\bigcup_p|\sigma_p|$. Hence $h_*(s_i(X))=s_i(Y)$ for $i>k$ and $h_*(s_k(X)-\alpha)=s_k(Y)$.

RROPOSITION 2.5. *Let X be a purely n-dimensional* mod 2 *Euler space and let a be an element in $H_k(X;\boldsymbol{Z}_2)$ for $0<k<n$. Then there exist a purely n-dimensional* mod 2 *Euler space Y and a homotopy equivalence $h:X\to Y$ such that $h_*(a)=s_k(Y)$ and $h_*(s_i(X))=s_i(Y)$ for $i>k$.*

PROOF. Let T be a triangulation of X. Let $\sum\alpha_p$ be a mod 2 cycle which represents $s_k(X)-a$, where α_p are k-simplexes in T. Let K be the simplicial complex consisting of all faces of all α_p. We subdivide T' as follows. To each p-simplex μ of T' ($p\geqq k+1$), star μ at the barycenter b_μ. Then K' remains unchanged under the subdivision. Repeating this subdivision twice, we get a subdivision T_0 of T' which satisfies the following. Corresponding to each k-simplexes σ_i in K', we can choose a k-simplex τ_i in T_0 such that

(1) $\sigma_i\cap\tau_i$ is a vertex, say v_i,

(2) $(\tau_i-v_i)\cap(\sigma_j-v_j)=\emptyset$ for all i, j,

(3) $(\tau_i-v_i)\cap(\tau_j-v_j)=\emptyset$ for $i\neq j$.

Let $f_i:\sigma_i\to\tau_i$ be a linear homeomorphism such that $f_i(v_i)=v_i$. Give an equivalence relation in X by $f_i(x)\sim x$ for some i. Let X_α be the quotient space and let $\pi_\alpha:X\to X_\alpha$ be the projection. We may assume that there exist a subdivision L of T_0 and a triangulation L_α of X_α such that $\pi_\alpha:L\to L_\alpha$ is a simplicial map. Since L is locally finite, the map $\pi_\alpha:X\to X_\alpha$ is a homotopy equivalence by Lemma 2.1. Now we see that X_α is a mod 2 Euler space. Recall that X_α is a mod 2 Euler space if $\chi(\mathrm{lk}(\sigma, L_\alpha))\equiv 0 \pmod 2$ for any simplex σ in L_α. From Lemma 2.3 and the fact that $\sum\alpha_p$ is a mod 2 cycle, we infer that the number of k-simplexes τ in K' such that $\tau>\sigma$ is even for any i-simplex σ in K' if $i<k$. If $0<\dim\sigma<k$, from (2) of Lemma 2.2, we see that

$$\chi(\mathrm{lk}([\sigma], L_\alpha))\equiv\#\{\tau\in K' \mid \dim\tau=k,\ \tau>\sigma\} \pmod 2.$$

If $\dim\sigma=0$, from (2) and (3) of Lemma 2.2, we also have the equation

$$\chi(\mathrm{lk}([\sigma], L_\alpha))\equiv\#\{\tau\in K' \mid \dim\tau=k,\ \tau>\sigma\} \pmod 2.$$

From (1) of Lemma 2.2, we have $\chi(\mathrm{lk}[\sigma], L_\alpha)\equiv 0 \pmod 2$ if $\dim\sigma=k$. Consequently, we obtain that X_α is a mod 2 Euler space. By Lemma 2.4, $h_*(s_i(X))=s_i(X_\alpha)$ for $i<k$ and $h_*(s_k(X)-\alpha)=s_k(X_\alpha)$. Since $s_k(X)-\alpha=a$, putting $Y=X_\alpha$ and $h=\pi_\alpha$, we get a mod 2 Euler space Y and a homotopy equivalence $h:X\to Y$ satisfying the required properties. The proof is complete.

Using Proposition 2.5, we easily get the proof of Theorem 1.

PROOF OF THEOREM 1. By Proposition 2.5, we have a purely n-dimensional mod 2 Euler space Y_1 and a homotopy equivalence $h_1:X\to Y_1$ such that $(h_1)_*(a_{n-1})=s_{n-1}(Y_1)$ and $(h_1)_*(s_n(X))=s_n(Y_1)$. Iterating this construction, we obtain purely n-dimensional mod 2 Euler spaces $Y_1, Y_2, \cdots, Y_{n-1}$ and homotopy equivalences

$h_j: Y_{j-1} \to Y_j$, $2 \leqq j \leqq n-1$, such that

$$(h_j)_*(s_i(Y_{j-1})) = s_i(Y_j) \qquad i > n-j,$$

$$(h_j)_*((h_{j-1}h_{j-2}\cdots h_1)_*(a_{n-j})) = s_{n-j}(Y_j).$$

Put $Y = Y_{n-1}$ and $h = h_{n-1}h_{n-2}\cdots h_1$. Then the homotopy equivalence $h: X \to Y$ satisfies the required properties. This completes the proof of Theorem 1.

3. Proof of Theorem 2.

Let K be a locally finite k-dimensional simplicial complex. We say that K is a k-dimensional *pseudo-Euler complex* if, for any $(k-1)$-simplex σ in K, $\mathrm{lk}(\sigma, K)$ is nonvoid and consists of even vertices. Note that a classical pseudo-manifold is a complex K such that $\mathrm{lk}(\sigma, K)$ consists of two vertices for any $(k-1)$-simplex σ of K.

Let K be a k-dimensional pseudo-Euler complex. We define a set AK for K by $AK = \{(x, \sigma) \in |K| \times K^{(k)} \mid x \in \sigma\}$. A map $\mathrm{Asg}: AK \to \{-1, 0, 1\}$ is called an *attachment signal* of K, if for each k-simplex σ in K, there exist proper faces τ and μ of σ such that $\sigma = \tau * \mu$ satisfying the following conditions:

AS 1. $\quad \mathrm{Asg}(x, \sigma) = 0 \quad$ for x in $\mathrm{Int}\,\sigma$,

or x in $|\partial\,\mathrm{st}(\tau, \partial\sigma)| = |\partial\,\mathrm{st}(\mu, \partial\sigma)|$.

AS 2. $$\mathrm{Asg}(x, \sigma) = \begin{cases} \varepsilon & \text{for } x \text{ in } |\mathrm{st}(\tau, \partial\sigma)| - |\partial\,\mathrm{st}(\tau, \partial\sigma)|, \\ -\varepsilon & \text{for } x \text{ in } |\mathrm{st}(\mu, \partial\sigma)| - |\partial\,\mathrm{st}(\mu, \partial\sigma)|, \end{cases}$$

where $\varepsilon = \pm 1$.

Note that $|\mathrm{st}(\tau, \partial\sigma)|$ and $|\mathrm{st}(\mu, \partial\sigma)|$ are $(k-1)$-discs in $\partial\sigma$.

Let Asg be an attachment signal of a k-dimensional Euler complex K. For a subcomplex L of K and a point x in $|L|$, we write $\mathrm{Asg}(x, L) = \sum \mathrm{Asg}(x, \sigma)$, where σ runs over all k-simplexes in L such that $x \in \sigma$.

We have the following proposition, whose proof is given in Section 5 after preparations in Section 4.

PROPOSITION 3.1. *Let K be a k-dimensional pseudo-Euler complex. Then there exists an attachment signal* Asg *of the barycentric subdivision K' satisfying the relation*

$$\mathrm{Asg}(x, K')\ (= \sum_{x \in \sigma \in K'} \mathrm{Asg}(x, \sigma)) = 0,$$

for all x in $|K'|$.

In the rest of this section, we prove Theorem 2 by assuming Proposition 3.1. We need Proposition 3.1 when we prove Proposition 3.4.

Let X be a locally compact n-dimensional polyhedron. Let D and E be k-discs in X such that $D\cap E$ is a $(k-1)$-disc, and let $f:D\rightarrow E$ be a PL-homeomorphism such that $f|D\cap E$ is the identity. We give an equivalence relation in X by $f(x)\sim x$ and denote by X_f the quotient space. Let $\pi_f:X\rightarrow X_f$ be the projection. By subdividing sufficiently finely, we may assume that there exist triangulations T and T_f of X and X_f such that π_f is a simplicial map and that D, E, and $D\cap E$ are subpolyhedra in $|T|$. In the following, we also write D, E, and $D\cap E$ for the subcomplexes of T determining D, E, and $D\cap E$.

Obviously the following holds.

LEMMA 3.2. *The projection* $\pi_f:X\rightarrow X_f$ *is a homotopy equivalence.*

Denote by $[x]$ the point $\pi_f(x)$ in X_f. We have the following.

LEMMA 3.3. *Assume that* $\chi(X, X-x)=(-1)^n$ *for all* x *in* $E-D\cap E$. *If* $n-k$ *is even, then*

$$\chi(X_f, X_f-[x])-\chi(X, X-x)=\begin{cases}0 & \textit{for } x\in \operatorname{Int} D,\\ (-1)^k & \textit{for } x\in\partial D-D\cap E,\\ (-1)^{k-1} & \textit{for } x\in \operatorname{Int}(D\cap E),\\ 0 & \textit{for } x\in\partial(D\cap E).\end{cases}$$

PROOF. Let σ be a simplex in D such that $x\in\operatorname{Int}\sigma$ and put $i=\dim\sigma$. Let $[\sigma]$ denote the i-simplex $\pi_f(\sigma)$ in T_f. Since $\chi(X, X-x)=(-1)^i(1-\chi(\operatorname{lk}(\sigma, T)))$, for any polyhedron $X=|K|$ and i-simplex σ in X such that $x\in\operatorname{Int}\sigma$, by Lemma 1.1, we study $\operatorname{lk}(\sigma, T)$ and $\operatorname{lk}([\sigma], T_f)$. Firstly, assume that $\operatorname{Int}\sigma\subset D-D\cap E$. Then $\operatorname{lk}([\sigma], T_f)$ is equal to the union $\operatorname{lk}(\sigma, T)\cup\operatorname{lk}(f(\sigma), T)$ under the identification of $\operatorname{lk}(\sigma, D)$ with $\operatorname{lk}(\sigma, E)$. Noting that $\operatorname{lk}(\sigma, D)\cap\operatorname{lk}(\sigma, E)=\operatorname{lk}(\sigma, D\cap E)=\emptyset$, we obtain that

$$\chi(\operatorname{lk}[\sigma], T_f)-\chi(\operatorname{lk}(\sigma, T))=\chi(\operatorname{lk}(f(\sigma), T))-\chi(\operatorname{lk}(\sigma, D)).$$

By the assumption $\chi(X, X-x)=(-1)^n$ for $x\in E-D\cap E$, we have $\chi(\operatorname{lk}(f(\sigma), T)$ $=1-(-1)^{n-i}$. Obviously we have

$$\chi(\operatorname{lk}(\sigma, D))=\begin{cases}1-(-1)^{k-i} & \text{if } \operatorname{Int}\sigma\in\operatorname{Int} D,\\ 1 & \text{if } \operatorname{Int}\sigma\in\partial D-D\cap E.\end{cases}$$

Applying Lemma 1.1, we obtain the first and the second equations of the lemma. Secondly, assume that $\sigma\in D\cap E$. Then $\operatorname{lk}([\sigma], T_f)$ is equal to the space made from $\operatorname{lk}(\sigma, T)$ under the identification of $\operatorname{lk}(\sigma, D)$ with $\operatorname{lk}(\sigma, E)$. Noting that $\operatorname{lk}(\sigma, D)\cap\operatorname{lk}(\sigma, E)=\operatorname{lk}(\sigma, D\cap E)$, we obtain

$$\chi(\operatorname{lk}([\sigma], T_f))-\chi(\operatorname{lk}(\sigma, T))=\chi(\operatorname{lk}(\sigma, D\cap E))-\chi(\operatorname{lk}(\sigma, D)).$$

Since $\sigma\in D\cap E\subset\partial D$, we have $\chi(\operatorname{lk}(\sigma, D))=1$ and

$$\chi(\mathrm{lk}(\sigma,\ D\cap E))=\begin{cases}1-(-1)^{k-i} & \text{if}\quad \mathrm{Int}\,\sigma\in\mathrm{Int}(D\cap E),\\ 1 & \text{if}\quad \mathrm{Int}\,\sigma\in\partial(D\cap E).\end{cases}$$

Applying Lemma 1.1, we obtain the third and the fourth equations of the lemma. The proof is complete.

Assuming Proposition 3.1, we have the following.

PROPOSITION 3.4. *Let X be a purely n-dimensional integral Euler space and let a be an element in $H_k(X;\boldsymbol{Z}_2)$. If $n-k$ is even, $k\neq 0$ and $k\neq n$, then there exist a purely n-dimensional integral Euler space Y and a homotopy equivalence $h:X\to Y$ such that $h_*(a)=s_k(Y)$ and $h_*(s_i(X))=s_i(Y)$ for $i>k$.*

PROOF. Let α be the element $s_k(X)-a$ in $H_k(X;\boldsymbol{Z}_2)$ and let $\sum_p \alpha_p$ be a mod 2 cycle representing α, where α_p are k-simplexes of a triangulation T of X. Let K be the simplicial complex consisting of all faces of all α_p. Since $\sum_p \alpha_p$ is a mod 2 cycle, K is a k-dimensional pseudo-Euler complex. By Proposition 3.1, there exists an attachment signal Asg of the barycentric subdivision K' of K such that $\mathrm{Asg}(x,K')=0$ for all x in $|K'|$. We construct an integral Euler space X_α according to the attachment signal Asg. To each k-simplex σ_j in K', choose a k-disc D_j in X satisfying the following conditions:

(1) $\sigma_j\cap D_j$ is a $(k-1)$-disc such that

$$\mathrm{Asg}(x,\sigma_j)=-1 \quad \text{for}\quad x\in\mathrm{Int}(\sigma_j\cap D_j),$$
$$\mathrm{Asg}(x,\sigma_j)=1 \quad \text{for}\quad x\in\sigma_j-(\sigma_j\cap D_j).$$

(2) $(\sigma_j-(\sigma_j\cap D_j))\cap(D_i-(\sigma_i\cap D_i))=\emptyset$ for any j, i.

(3) $(D_j-(\sigma_j\cap D_j))\cap(D_i-(\sigma_i\cap D_i))=\emptyset$ for $j\neq i$.

Then there exists a PL-homeomorphism $f_j:\sigma_j\to D_j$ for each j such that $f_j|\sigma_j\cap D_j$ is the identity. We give an equivalence relation in X by $x\sim f_j(x)$ for some j. Denote by X_α the quotient polyhedron. By Lemma 3.2, the projection $h:X\to X_\alpha$ is a homotopy equivalence. For any x in X, denote by $[x]$ the point $h(x)$ in X_α. From Lemma 3.3, we obtain that

$$\chi(X_\alpha,\ X_\alpha-[x])-\chi(X,\ X-x)=(-1)^k\sum\mathrm{Asg}(x,\ \sigma_j),$$

where σ_j runs over all k-simplexes in K' such that $x\in\sigma_j$. Then the equation $\mathrm{Asg}(x,K)=0$ implies that X_α is an integral Euler space. By Lemma 2.4, we have $h_*(s_i(X))=s_i(X_\alpha)$ for $i>k$ and $h_*(s_k(X)-\alpha)=s_k(X_\alpha)$. Since $\alpha=s_k(X)-a$, putting $Y=X_\alpha$, we get an integral Euler space Y and a homotopy equivalence $h:X\to Y$ satisfying the required properties. This completes the proof.

Using Proposition 3.4, we can prove Theorem 2 under the assumption of Proposition 3.1.

PROOF OF THEOREM 2. Since we have the relation $s_{i-1}(Y)=\beta s_i(Y)$ if $n-i$ is even, for n-dimensional integral Euler space Y, it is sufficient to construct a purely n-dimensional integral Euler space Y and a homotopy equivalence $h: X\to Y$ such that $h_*(a_i)=s_i(Y)$ for any $i>0$ such that $n-i$ is even. By Proposition 3.4, we have a purely n-dimensional integral Euler space Y_1 and a homotopy equivalence $h_1: X\to Y_1$ such that

$$(h_1)_*(s_i(X))=s_i(Y_1) \qquad \text{for} \quad n\geqq i>n-2,$$

$$(h_1)_*(a_{n-2})=s_{n-2}(Y).$$

Since $a_{n-3}=\beta a_{n-2}$, we have $(h_1)_*(a_{n-3})=s_{n-3}(Y)$. Iterating this procedure, we obtain purely n-dimensional Euler spaces $Y_2, Y_3, \cdots, Y_{[(n-1)/2]}$, and homotopy equivalences $h_j: Y_{j-1}\to Y_j$ $(2\leqq j\leqq[(n-1)/2])$, such that

$$(h_j)_*(s_i(Y_{j-1}))=s_i(Y_j) \qquad \text{for} \quad n\geqq i>n-2j,$$

$$(h_j)_*((h_{j-1}h_{j-2}\cdots h_1)_*(a_i))=s_i(Y_j) \quad \text{for} \quad i=n-2j,\ n-2j-1.$$

Put $Y=Y_{[(n-1)/2]}$ and $h=h_{[(n-1)/2]}h_{[(n-1)/2]-1}\cdots h_1$. Then the space Y and the homotopy equivalence $h: X\to Y$ satisfy the required properties.

4. Signal and checker signal.

In order to prove Proposition 3.1, we introduce a notion called signal. Let M be a triangulation of a k-dimensional PL-manifold with or without boundary. A map $\mathrm{sg}: M^{(k)}\to\{-1, 1\}$ is a *signal* on $M=M^k$ if $|\mathrm{sg}(\sigma)+\mathrm{sg}(\tau)+\mathrm{sg}(\mu)|=1$ for each σ, τ, μ in $M^{(k)}$ such that $\sigma\cap\tau$ and $\tau\cap\mu$ are $(k-1)$-simplexes in M.

Let W be a k-dimensional submanifold of M. Then there exist at most two k-simplexes in $\mathrm{st}(\sigma, W)$ for any $(k-1)$-simplex σ in W. For $\varepsilon=\pm 1$, denote by $\mathrm{NC}(W, \mathrm{sg}, \varepsilon)$ the set of all $(k-1)$-simplexes σ such that $\sharp\mathrm{st}(\sigma, W)^{(k)}=2$ and $\mathrm{sg}(\tau)=\varepsilon$ for any τ in $\mathrm{st}(\sigma, W)^{(k)}$. We denote by $\sharp\mathrm{NC}(W, \mathrm{sg}, \varepsilon)$ the number of $(k-1)$-simplexes in $\mathrm{NC}(W, \mathrm{sg}, \varepsilon)$. A signal $\mathrm{sg}: M^{(k)}\to\{-1, 1\}$ is called a *checker signal* if $\mathrm{NC}(M, \mathrm{sg}, \varepsilon)$ is empty for $\varepsilon=1$ and -1.

We have the following three lemmas. The proofs are easy and omitted.

LEMMA 4.1. *A checker signal is determined by the value on a k-simplex in M^k. Thus we have two checker signals on M if there exists one.*

LEMMA 4.2. *Let σ be a $(k+1)$-simplex. Then there exists a checker signal on the barycentric subdivision $K(\partial\sigma)'$ of $K(\partial\sigma)$.*

LEMMA 4.3. *Let Δ be a k-simplex and let* sg *be a checker signal of $K(\Delta)'$. Let b_Δ be the barycenter of Δ and let Δ_i be a $(k-1)$-face of Δ. Define a signal* sg_i *of $K(\Delta_i)'$ by $\mathrm{sg}_i(\sigma)=\mathrm{sg}(b_\Delta*\sigma)$ for each $(k-1)$-simplex σ in $K(\Delta_i)'$. Then* sg_i

is a checker signal.

Let τ be a $(k-3)$-simplex and let S be a triangulation of the circle S^1. Then the join $K(\tau)*S$ is a triangulation of a $(k-1)$-disc.

LEMMA 4.4. *Let* sg *be a signal of* $K(\tau)*S$ *such that* $\sum \mathrm{sg}(\sigma)=0$, *where* σ *runs over all* $(k-1)$*-simplexes in* $K(\tau)*S$. *Then* $\sharp\mathrm{NC}(K(\tau)*S, \mathrm{sg}, 1)=\sharp\mathrm{NC}(K(\tau)*S, \mathrm{sg}, -1)$.

PROOF. Define a signal sg′ of S by $\mathrm{sg}'(\mu)=\mathrm{sg}(\tau*\mu)$ for each 1-simplex μ of S. Then $\sum \mathrm{sg}'(\mu)=0$, where μ runs over all 1-simplexes in S. Obviously $\sharp\mathrm{NC}(S, \mathrm{sg}', \varepsilon)=\sharp\mathrm{NC}(K(\tau)*S, \mathrm{sg}, \varepsilon)$ for $\varepsilon=\pm 1$. Since S is a triangulation of S^1,

$$\sharp\{\mu \mid \mathrm{sg}'(\mu)=1\}-\sharp\mathrm{NC}(S, \mathrm{sg}', 1)=\sharp\{\mu \mid \mathrm{sg}'(\mu)=-1\}-\sharp\mathrm{NC}(S, \mathrm{sg}', -1).$$

Thus we have $\sharp\mathrm{NC}(S, \mathrm{sg}', 1)=\sharp\mathrm{NC}(S, \mathrm{sg}', -1)$ and $\sharp\mathrm{NC}(K(\tau)*S, \mathrm{sg}, 1)=\sharp\mathrm{NC}(K(\tau)*S, \mathrm{sg}, -1)$. This completes the proof.

LEMMA 4.5. *Let* Δ *be a* k*-simplex and let* sg *be a checker signal of* $K(\Delta)'$. *Suppose that* σ *is an* i*-simplex in* $K(\Delta)'$ *such that*

(1) $\sigma \in K(\Delta)'-K(\partial\Delta)'$, $\quad 0 \leqq i \leqq k-1$, *or*

(2) $\sigma \in K(\partial\Delta)'$, $\quad 0 \leqq i \leqq k-2$.

Then $\sum \mathrm{sg}(\tau)=0$, *where* τ *runs over all* k*-simplexes in* $\mathrm{st}(\sigma, K(\Delta)')$.

PROOF. Easy, e.g., by induction.

LEMMA 4.6. *Let* sg *be a signal of* $K(\partial\Delta)'$ *such that the restriction of* sg *on* $K(\Delta_i)'$ *is a checker signal for each* $(k-1)$*-face* Δ_i *of a* k*-simplex* Δ. *Denote by* Δ_{ij} *the* $(k-2)$*-simplex* $\Delta_i \cap \Delta_j$. *Then the following holds:*

(1) *If* μ *is a* q*-simplex in* $K(\partial\Delta)'-\bigcup_{i\neq j} K(\Delta_{ij})'$ *such that* $q \leqq k-2$, *then* $\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, \varepsilon)$ *is empty for* $\varepsilon=\pm 1$.

(2) $\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, 1)=\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, -1)$.

(3) *If* μ *is a* q*-simplex in* $K(\Delta_{ij})'$ *for* $i\neq j$ *such that* $q \leqq k-3$, *then*

$$\sharp\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, 1)=\sharp\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, -1).$$

PROOF. (1) Since the restriction of sg on $K(\Delta_i)$ is a checker signal by Lemma 4.3, any simplex in $\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, \varepsilon)$, for $\varepsilon=\pm 1$, is contained in $K(\Delta_{ij})$ for some i, j. Thus $\mathrm{NC}(\mathrm{st}(\mu, \partial\Delta), \mathrm{sg}, \varepsilon)$ is empty.

(2) From Lemma 4.1, we deduce that $\sharp(\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, 1)\cap K(\Delta_{ij}))=\sharp(\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, -1)\cap K(\Delta_{ij}))$ for $i\neq j$. Since $\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, \varepsilon)=\sum_{i\neq j}\sharp(\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, \varepsilon)\cap K(\Delta_{ij}))$ for $\varepsilon=\pm 1$, we have $\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, 1)=\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, -1)$.

(3) First suppose that $q=\dim\mu=k-3$. Then $\mathrm{lk}(\mu, K(\partial\Delta)')$ is a triangulation of the circle S^1. From Lemma 4.4, it follows that $\sharp\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, 1)=\sharp\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, -1)$. Next assume that $q \leqq k-4$. By Lemma 4.3,

$$(\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, 1)\cup \mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, -1))\cap K(\Delta_{ij}),$$

for $i\neq j$, is equal to the set of all $(k-2)$-simplexes in $\mathrm{st}(\mu, K(\partial\Delta)')\cap K(\Delta_{ij})=\mathrm{st}(\mu, (K(\Delta_{ij}))')$ or empty. Since $q\leqq k-4$, by using Lemma 4.5, we obtain that

$$\sharp(\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, 1)\cap K(\Delta_{ij}))=\sharp(\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, -1)\cap K(\Delta_{ij})).$$

Consequently, it follows that

$$\sharp\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, 1)=\sharp\mathrm{NC}(\mathrm{st}(\mu, K(\partial\Delta)'), \mathrm{sg}, -1).$$

5. Proof of Proposition 3.1.

Let sg be a signal on the barycentric subdivision $K(\partial\Delta)'$ of the boundary $K(\partial\Delta)$ of a k-simplex Δ. By a q-ball, we mean a topological space homeomorphic to a q-simplex. We decompose $\partial\Delta$ as the union of balls as follows. An element in $B(\partial\Delta, \mathrm{sg}, \varepsilon)$, for $\varepsilon=\pm 1$, is one of the following:

(1) a $(k-1)$-simplex σ in $K(\partial\Delta)'$ such that $\mathrm{sg}(\sigma)=\varepsilon$ and $\mathrm{sg}(\tau)=-\varepsilon$ for any $\tau\in K(\partial\Delta)'^{(k-1)}$ such that $\sigma\cap\tau$ is a $(k-2)$-simplex,

(2) the union $\sigma\cup\tau$ in $\partial\Delta$, where $\sigma, \tau\in K(\partial\Delta)'^{(k-1)}$ such that $\mathrm{sg}(\sigma)=\mathrm{sg}(\tau)=\varepsilon$ and $\sigma\cap\tau$ is a $(k-2)$-simplex.

Then an element in $B(\partial\Delta, \mathrm{sg}, \varepsilon)$ is a $(k-1)$-ball. From the definition of the signal, we obtain

$$\partial\Delta=\bigcup_{\rho\in B(\partial\Delta, \mathrm{sg}, \pm 1)}\rho.$$

Let $c\rho$ denote the cone of the $(k-1)$-ball ρ in $B(\partial\Delta, \mathrm{sg}, \varepsilon)$. Then $c\rho$ is a k-ball. If we identify $c\rho$ with the join of ρ with the barycenter b_Δ, we have

$$\Delta=\bigcup_{\rho\in B(\partial\Delta, \mathrm{sg}, \pm 1)}\rho.$$

The cone $c\rho$ is either equal to a k-simplex $b_\Delta*\sigma$ or equal to the union of two k-simplexes $b_\Delta*\sigma$ and $b_\Delta*\tau$. The boundary $\partial\rho$ is homeomorphic to the $(k-2)$-sphere and the cone $c\partial\rho$ is a $(k-1)$-ball. The boundary $\partial(c\rho)$ is equal to the union $c\partial\rho\cup\rho$. We write $\mathrm{Int}(c\partial\rho)$ for the space $c\partial\rho-\partial\rho$.

We have the set $A(K(\Delta)')=\{(x, \sigma)\in\Delta\times K(\Delta)'^{(k-1)} \mid x\in\sigma\}$ for $K(\Delta)'$ as is defined in Section 3. We say that an attachment signal

$$\mathrm{Asg}: A(K(\Delta)')\longrightarrow\{1, 0, -1\}$$

is a *standard extension* of a signal $\mathrm{sg}: (\partial\Delta)'^{(k-1)}\to\{-1, 1\}$ if the following conditions are satisfied,

SE1. $\mathrm{Asg}(x, c\rho)=\varepsilon$ $\quad x\in\mathrm{Int}\,\rho$

SE2. $\mathrm{Asg}(x, c\rho)=-\varepsilon$ $\quad x\in\mathrm{Int}(c\partial\rho)$

SE3. $\mathrm{Asg}(x, c\rho)=0$ $\quad$ otherwise.

Here, as before, we write $\mathrm{Asg}(x, c\rho)=\sum_\mu \mathrm{Asg}(x, \mu)$, where μ ranges over all (although one or two) k-simplexes in $c\rho$.

The standard extension is not unique. But it is obvious that, for any signal sg on $K(\partial\Delta)'$, there exists a standard extension Asg of sg.

PROPOSITION 5.1. *Let Δ be a k-simplex and let* sg *be a signal of $K(\Delta)'$ such that the restriction of* sg *on $K(\Delta_i)'$ is a checker signal for each $(k-1)$-face Δ_i of Δ. Then the standard extension* $\mathrm{Asg}: A(K(\Delta)')\to\{1, 0, -1\}$ *satisfies the following:*

(1) $\mathrm{Asg}(x, K(\Delta)')=\mathrm{sg}(\tau)$ *if* $x\in\mathrm{Int}\,\tau$, $\tau\in K(\partial\Delta)'^{(k-1)}$.

(2) $\mathrm{Asg}(x, K(\Delta)')=\varepsilon$ *if* $x\in\mathrm{Int}\,\mu$, $\mu\in\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, \varepsilon)^{(k-2)}$.

(3) $\mathrm{Asg}(x, K(\Delta)')=0$ *otherwise.*

PROOF. Let $\partial\Delta=\bigcup_{\rho\in B(\partial\Delta,\mathrm{sg},\pm1)}\rho$ be the ball decomposition. By SE3 of the standard extension, $\mathrm{Asg}(x, K(\Delta)')=0$ if x is neither contained in $\mathrm{Int}\,\rho$ nor contained $\mathrm{Int}(c\partial\rho)$ for any ball ρ in $\partial\Delta$. If x is contained in $\mathrm{Int}\,\rho$ for some $\rho\in B(\partial\Delta, \mathrm{sg}, \varepsilon)$, then $x\in\mathrm{Int}\,\tau$ for some $\tau\in K(\partial\Delta)'^{(k-1)}$ with $\mathrm{sg}(\tau)=\varepsilon$ or $x\in\mathrm{Int}\,\mu$ for $\mu\in\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, \varepsilon)^{(k-2)}$. In these cases, from SE1, we obtain that $\mathrm{Asg}(x, K(\Delta)')=\varepsilon$. This proves (1) and (2) of the proposition. Now assume that $x\in\mathrm{Int}(c\partial\rho)$ for some ρ. If $x=b_\Delta$, then by SE2,

$$\mathrm{Asg}(x, K(\Delta)')=\sharp B(K(\partial\Delta)', \mathrm{sg}, -1)-\sharp B(K(\partial\Delta)', \mathrm{sg}, 1).$$

From the definition of $B(K(\partial\Delta)', \mathrm{sg}, \varepsilon)$, we obtain that

$$\sharp B(K(\partial\Delta)', \mathrm{sg}, \varepsilon)=\sharp\{\sigma\in K(\partial\Delta)'^{(k-1)} \mid \mathrm{sg}(\sigma)=\varepsilon\}-\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, \varepsilon).$$

Thus we have

$$\mathrm{Asg}(x, K(\Delta)')=-\sum_{\sigma\in K(\Delta)'^{(k-1)}}\mathrm{sg}(\sigma)+\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, 1)-\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, -1).$$

The set of all $(k-1)$-simplexes in $K(\partial\Delta)'$ is equal to the set of all $(k-1)$-simplexes in $\bigcup_{i=0}^k \mathrm{st}(b_{\Delta_i}, K(\Delta_i)')$. By Lemma 4.5, $\sum\mathrm{sg}(\tau)=0$, where τ runs over all $(k-1)$-simplexes in $\mathrm{st}(b_{\Delta_i}, K(\Delta_i)')$. From (2) of Lemma 4.6, we have $\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, 1)=\sharp\mathrm{NC}(K(\partial\Delta)', \mathrm{sg}, -1)$. Consequently, we obtain that $\mathrm{Asg}(x, K(\Delta)')=0$. If $x\in\mathrm{Int}(c\partial\rho)$ and $x\neq b_\Delta$, then we can write $x\in\mathrm{Int}(b_\Delta*\mu)$ where $\mu\subset\partial\rho$, $\mu\in K(\partial\Delta)'$ and $\dim\mu\leqq k-2$. By SE2,

$$\mathrm{Asg}(x, K(\Delta)')=\sharp B(\mathrm{st}(\mu, K(\Delta)'), \mathrm{sg}, -1)-\sharp B(\mathrm{st}(\mu, K(\Delta)', \mathrm{sg}, 1).$$

If $\mu\in K(\Delta_{ij})'=K(\Delta_i\cap\Delta_j)'$ and $\dim\mu=k-2$, then $\sharp B(\mathrm{st}(\mu, K(\Delta)'), \mathrm{sg}, 1)=\sharp B(\mathrm{st}(\mu, K(\Delta)', \mathrm{sg}, -1)=1$. Hence we have $\mathrm{Asg}(x, K(\Delta)')=0$. The remaining case is when $\mu\in K(\Delta_{ij})'$ and $\dim\mu\leqq k-3$ or when $\mu\in K(\partial\Delta)'-\bigcup_{i\neq j}K(\Delta_{ij})'$ and $\dim\mu\leqq k-2$. In these cases, we can apply (1) and (3) of Lemma 4.6. Using Lemma 4.5 as before, we obtain that

$$\begin{aligned}\mathrm{Asg}(x,\ K(\varDelta)') &= -\sum_{\sigma\in \mathrm{st}(\mu,\ K(\varDelta)')} \mathrm{sg}(\sigma)+\sharp\mathrm{NC}(\mathrm{st}(\mu,\ K(\varDelta)'),\ \mathrm{sg},\ 1)\\ &\quad -\sharp\mathrm{NC}(\mathrm{st}(\mu,\ K(\varDelta)',\ \mathrm{sg},\ -1)\\ &=0.\end{aligned}$$

This completes the proof.

LEMMA 5.2. *Let K be a k-dimensional pseudo-Euler complex. Then there exist signals* $\mathrm{sg}^{\varDelta}$ *of* $K(\partial\varDelta)'$ *for all k-simplexes $\varDelta$ in K satisfying the following conditions:*

P1. *The restriction of* $\mathrm{sg}^{\varDelta}$ *to* $K(\varDelta_i)'$ *is a checker signal for each* $(k-1)$*-face* $\varDelta_i$ *of* $\varDelta$.

P2. *Let* K^{k-1} *denote the* $(k-1)$*-dimensional skeleton of K. Then for each* $(k-1)$*-simplex σ in* $(K^{k-1})'$, $\sum_{\varDelta}\mathrm{sg}^{\varDelta}(\sigma)=0$, *where $\varDelta$ ranges over all k-simplexes in K such that* $K(\varDelta)'\ni\sigma$.

PROOF. Since K is a pseudo-Euler complex, to each $(k-1)$-simplex τ in K and to all k-simplexes μ_j such that $\mu_j>\tau$, we can give checker signals $\mathrm{sg}_{\tau}{}^{\mu_j}$ of $K(\tau)'$ such that $\sum_{\mu_j>\tau}\mathrm{sg}_{\tau}{}^{\mu_j}(\sigma)=0$ for any $(k-1)$-simplex σ in $K(\tau)'$. Let $\varDelta$ be a k-simplex in K and let σ be a $(k-1)$-simplex in $K(\partial\varDelta)'$. Then there exists a $(k-1)$-simplex τ in K such that $\sigma\in K(\tau)'$. Define a checker signal $\mathrm{sg}^{\varDelta}$ of $K(\partial\varDelta)'$ by $\mathrm{sg}^{\varDelta}(\sigma)=\mathrm{sg}_{\tau}^{\varDelta}(\sigma)$. Then the collection $\{\mathrm{sg}^{\varDelta}\}$ satisfies the conditions (1) and (2). This completes the proof

Now we are in a position to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1. Let $\mathrm{sg}^{\varDelta}$ be the set of signals of $K(\partial\varDelta)'$ for all k-simplexes $\varDelta$ in K satisfying P1 and P2 of Lemma 5.2. Let $\mathrm{Asg}^{\varDelta}$ be the standard extension of $\mathrm{sg}^{\varDelta}$. We define an attachment signal Asg of K'

$$\mathrm{Asg}\ :\ AK=\{(x,\ \sigma)\in|K'|\times K'^{(k)}\,|\,x\in\sigma\}\longrightarrow\{-1,\ 0,\ 1\}$$

by $\mathrm{Asg}(x,\ \sigma)=\mathrm{Asg}^{\varDelta}(x,\ \sigma)$ where $\sigma\in K(\varDelta)'$. We now show that $\mathrm{Asg}(x,\ K(\varDelta)')=0$ for any x in $|K|$. By Proposition 5.1, it is sufficient to prove the following cases:

(1) $x\in\mathrm{Int}\,\tau$ for some $(k-1)$-simplex τ in $K(\partial\varDelta)'$, $\varDelta\in K^{(k)}$.

(2) $x\in\mathrm{Int}\,\mu$, $\mu\in\mathrm{NC}(K(\partial\varDelta)',\ \mathrm{sg},\ \varepsilon)$, $\varDelta\in K^{(k)}$.

In the case (1), we have $\mathrm{Asg}(x,\ K(\varDelta)')=0$ by P2 of Lemma 5.2. Now we consider the case (2). By P1, $\mu\in K(\varDelta_{ij})'$ and so $\mu\in K'^{(k-2)}$. Let σ be a k-simplex in K such that $K(\sigma)'\ni\mu$. We have two $(k-1)$-simplexes τ_+ and τ_- in $K(\partial\sigma)'$ such that $\tau_+>\mu$ and $\tau_->\mu$. If $\mathrm{sg}(\tau_+)\mathrm{sg}(\tau_-)=1$, then $\mu\in\mathrm{NC}(K(\partial\sigma)',\ \mathrm{sg},\ \varepsilon)$ and

$$\mathrm{Asg}(x,\ K(\sigma)')=\varepsilon=\frac{1}{2}(\mathrm{sg}^{\sigma}(\tau_+)+\mathrm{sg}^{\sigma}(\tau_-)).$$

If $\mathrm{sg}(\tau_+)\mathrm{sg}(\tau_-)=-1$, then $\mathrm{Asg}(x,\ K(\sigma)')=0=(1/2)(\mathrm{sg}^{\sigma}(\tau_+)+\mathrm{sg}^{\sigma}(\tau_-))$. Consequently, we obtain that, if $x\in\operatorname{Int}\mu$ for some $(k-2)$-simplex μ in K', then $\operatorname{Asg}(x, K')=\sum(1/2)(\operatorname{sg}^{\sigma}(\tau_{+})+\operatorname{sg}^{\sigma}(\tau_{-}))$, where σ runs over all k-simplexes in K such that $K(\partial\sigma)'\ni\mu$. Thus

$$\operatorname{Asg}(x, K')=\frac{1}{2}\sum_{\sigma\in K^{(k)}}\sum_{\tau\in K(\partial\sigma)'^{(k-1)},\tau>\mu}\operatorname{sg}^{\sigma}(\tau)$$

$$=\frac{1}{2}\sum_{\tau\in K'^{(k-1)},\tau>\mu}\sum_{\sigma\in K^{(k)},K(\partial\sigma)'\ni\tau}\operatorname{sg}^{\sigma}(\tau)\,.$$

By P2 of Lemma 5.2,

$$\sum_{\sigma\in K^{(k)},K(\partial\sigma)'\ni\tau}\operatorname{sg}^{\sigma}(\tau)=0\,.$$

Hence we obtain that $\operatorname{Asg}(x, K')=0$ in the case (2). The proof is complete.

6. Counterexamples to Halperin conjecture.

In order to explain Halperin's conjecture, we first give the definition of the normally nonsingular map according to [**4**]. For that, as in [**4**], we introduce the bivariant language.

We consider a simple situation. Let X and Y be compact polyhedra and let $f:X\to Y$ be a continuous map. Since X is embeddable as a closed subspace of $\boldsymbol{R}^{n}$ for some n, there is a mapping $\phi:X\to\boldsymbol{R}^{n}$ such that $(f,\phi):X\to Y\times\boldsymbol{R}^{n}$ is a closed embedding. Write X_{ϕ} for the image of X in $Y\times\boldsymbol{R}^{n}$. Define $H^{i}(X\xrightarrow{f}Y)$ by

$$H^{i}(X\xrightarrow{f}Y)=H^{i+n}(Y\times\boldsymbol{R}^{n},\ Y\times\boldsymbol{R}^{n}-X_{\phi}\,;\boldsymbol{Z}_{2})\,.$$

This definition is independent of the choice of ϕ. If f is the identity, we have the natural isomorphism

$$H^{i}(X\xrightarrow{\mathrm{id}}X)=H^{i}(X\,;\boldsymbol{Z}_{2})\,,$$

and if Y is the point, we have

$$H^{-i}(X\to\mathrm{pt.})=H_{i}(X\,;\boldsymbol{Z}_{2})\,.$$

If X is a subpolyhedron of Y, let (L, K) be a triangulation of (Y, X). Let N be the second derived neighborhood of K in L and let ∂N be its boundary. Then $H^{i}(X\xrightarrow{\iota}Y)=H^{i}(N,\partial N\,;\boldsymbol{Z}_{2})$, where $\iota:X\to Y$ is the inclusion.

For continuous maps $f:X\to Y$ and $g:Y\to Z$ of compact polyhedra, the cup product defines the bilinear map

$$H^{i}(X\xrightarrow{f}Y)\times H^{j}(Y\xrightarrow{g}Z)\ \xrightarrow{\cdot}\ H^{i+j}(X\xrightarrow{gf}Z)\,.$$

A map $f:X\to Y$ is called *homologically normally nonsingular* if there is an element θ in $H^{d}(X\xrightarrow{f}Y)$, for some $d\in\boldsymbol{Z}$, such that, for any compact polyhedron

W and a continuous map $g: W \to X$, the homomorphism

$$H^i(W \xrightarrow{g} X) \xrightarrow{\cdot\theta} H^{i+d}(W \xrightarrow{fg} Y)$$

is an isomorphism. We say that θ is a strong orientation with codimension d. In particular, if f has a normal bundle, the Thom class is a strong orientation. Remark that Fulton-MacPherson used the sheaf cohomology $R^i(\mathrm{Hom}(Rf_! \boldsymbol{Z}_{2X}, \boldsymbol{Z}_{2Y})$. But the definitions agree since we work on the category of compact polyhedra [**4**, p. 86]. Note that θ is a strong orientation if the homomorphism $\cdot\theta$ is an isomorphism for any compact subpolyhedron W in X and the inclusion g [**4**, p. 85].

If $f: X \to Y$ is a homologically normally nonsingular map, we have the Gysin map

$$f^! : H_j(Y, \boldsymbol{Z}_2) \longrightarrow H_{j-d}(X, \boldsymbol{Z}_2)$$

defined by $f^!(a) = \theta \cdot a$ for $a \in H_j(Y; \boldsymbol{Z}_2) = H^{-j}(Y \to \mathrm{pt.})$. The i-dimensional Stiefel-Whitney cohomology class $w^i(N_f)$ in $H^i(X; \boldsymbol{Z}_2)$ of the normal space of f ($i \geqq 0$) is defined by

$$w^i(N_f) = (\cdot\theta)^{-1} \mathrm{Sq}^i(\theta),$$

where Sq^i is the i-th squaring operation of Steenrod on $H^i(X \xrightarrow{f} Y)$. Put $w(N_f) = \sum_{i \geq 0} w^i(N_f)$. Since $w^0(N_f) = 1$, we have the inverse $w(N_f)^{-1}$ in $H^*(X; \boldsymbol{Z}_2)$.

Let X and Y be compact mod 2 Euler spaces and let $f: X \to Y$ be a homologically normally nonsingular map. Then Halperin's conjecture is the following equation:

$$\text{(H)} \qquad s_*(X) = w(N_f)^{-1} \cap f^! s_*(Y).$$

Now we give some simple examples where the relation (H) does not hold.

We define a PL-manifold Z which is PL-homeomorphic to $D^2 \times S^1$ as follows. Put $Q = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid |x_1| + |x_2| \leqq 1\}$. Then Z is the quotient space of $Q \times [-1, 1] \subset \boldsymbol{R}^3$ under the identification $x \times \{-1\} \sim x \times \{1\}$ for $x \in Q$. Let $\sigma^\pm$ be two 2-discs in $Q \times \{0\}$ defined by

$$\sigma^+ = \{(x_1, x_2, x_3) \in Q \times \{0\} \mid x_1 \geqq 0\}$$

$$\sigma^- = \{(x_1, x_2, x_3) \in Q \times \{0\} \mid x_1 \leqq 0\}.$$

Obviously $\sigma^+ \cup \sigma^- = Q \times \{0\}$ and $\sigma^+ \cap \sigma^-$ is a 1-disc. Further we have two 2-discs $\tau^\pm$ in Z defined by

$$\tau^+ = \{(x_1, x_2, x_3) \mid x_1 = 0,\ 0 \leqq x_2 \leqq 1,\ 0 \leqq x_3 \leqq (1/10)x_2\}$$

$$\tau^- = \{(x_1, x_2, x_3) \mid x_1 = 0,\ 0 \geqq x_2 \geqq -1,\ 0 \geqq x_3 \geqq (1/10)x_2\}.$$

The intersection of $\sigma^\pm$ or $\tau^\pm$ with ∂Z is a 1-disc such that

$$(\sigma^+ \cap \partial Z) \cap (\tau^+ \cap \partial Z) = \mathrm{pt.}, \qquad (\sigma^- \cap \partial Z) \cap (\tau^- \cap \partial Z) = \mathrm{pt.}$$

We have two PL-homeomorphisms $h^{\pm}$ from $\sigma^{\pm}$ onto $\tau^{\pm}$ such that

$$h^{\pm}|\sigma^{\pm}\cap\tau^{\pm}=\text{identity},\qquad h^{\pm}(\sigma^{\pm}\cap\partial Z)=\tau^{\pm}\cap\partial Z.$$

Let Z_h and ∂Z_h be the quotient polyhedron of Z and ∂Z under the identification $x\sim h^{\pm}(x)$. Let $\pi: Z\to Z_h$ be the projection. Then π is a homotopy equivalence of pairs; $\pi:(Z, \partial Z)\to(Z_h, \partial Z_h)$. Let Y be the double of $Z_h: Y=Z_h\bigcup_{\partial Z_h}Z_h$. By the arguments in Section 2, Y is a mod 2 Euler space homotopy equivalent to $S^2\times S^1$. The circle $\{0\}\times S^1=\{(x_1, x_2, x_3)\,|\,x_1=x_2=0\}/\sim$ in Z is mapped by π identically. We write X for $\{0\}\times S^1$ and also for the image $\pi(\{0\}\times S^1)\subset Z_h\subset Y$.

PROPOSITION 6.1. *The inclusion $f: X\to Y$ is a homologically normally non-singular map.*

PROOF. We give a homotopy inverse $\omega:(Z_h, \partial Z_h)\to(Z, \partial Z)$ as follows. Let A be the subspace of Z defined by

$$A=\{(x_1, x_2, x_3)\ |\ |x_1|+|x_2|\leqq 1,\ |x_3|\geqq 2/10\}/\sim.$$

Then A is a 3-disc and A is mapped by π identically. Let B be the subspace of ∂Z defined by

$$B=\{(x_1, x_2, x_3)\ |\ |x_1|+|x_2|=1,\ |x_3|\leqq 2/10\}.$$

Then B is a PL-manifold homeomorphic to $S^1\times D^1$ and let ∂B be the boundary. Put $B_h=\pi(B)$ and $\partial B_h=\pi(\partial B)$. Since $\partial B\subset A$, we may identify ∂B with ∂B_h. By the construction, $\pi|B:(B, \partial B)\to(B_h, \partial B_h)$ is a homotopy equivalence. We have a map $\omega^B: B_h\to B$ such that $\omega^B|\partial B_h=$identity and $\omega^B:(B_h, \partial B_h)\to(B, \partial B)$ is a homotopy inverse of $\pi|B:(B, \partial B)\to(B_h, \partial B)$. Put

$$E=\{(x_1, x_2, x_3)\ |\ |x_1|+|x_2|\leqq 1,\ |x_3|=2/10\}.$$

Then $E\subset A$ and E is homeomorphic to $D^2\times S^0$. The union $E\cup B$ in Z is homeomorphic to S^2 and the union $E\cup B_h$ in Z_h is homotopy equivalent to S^2. Let $C(E\cup B)$ and $C(E\cup B_h)$ denote their cones. Then we have

$$Z=A\cup_E C(E\cup B),\qquad Z_h=A\cup_E C(E\cup B_h).$$

We define a map $\omega: Z_h\to Z$ by $\omega|A=$identity, $\omega|B_h=\omega^B$ and by the cone extension of (identity$\cup\omega^B$) on $C(E\cup B_h)$. Then $\omega|X$ is the identity. Since the inclusion $s: X\to Z$ is equal to the zero section of trivial D^2-bundle, we have the strong orientation θ_0 in $H^2(Z, Z-X; \boldsymbol{Z}_2)=H^2(X\overset{s}{\to}Z)$. Put

$$\theta=\omega^*\theta_0\in H^2(Z_h, Z_h-X; \boldsymbol{Z}_2)=H^2(X\overset{f}{\to}Y).$$

To show that θ is a strong orientation, it is sufficient to take W to be an interval or the point in X containing $O=(0, 0, 0)$. Notice that we can triangulate $C(E\cup B_h)$ by the cone extension of a triangulation of $E\cup B_h$. Consequently, we have the natural isomorphism $H^i(O\overset{fg}{\to}Y)=H^i(C(E\cup B_h), E\cup B_h ; \boldsymbol{Z}_2)$, where g: $O\to X$ is the inclusion. Since ω^* maps $H^i(C(E\cup B), E\cup B ; \boldsymbol{Z}_2)$ isomorphically onto $H^i(C(E\cup B_h), E\cup B_h ; \boldsymbol{Z}_2)$, we obtain that

$$H^i(W\overset{g}{\to}X)\xrightarrow{\cdot\theta}H^i(W\overset{fg}{\to}Y)$$

is an isomorphism for any i, if $W=O$. The proof is similar when W is an interval containing O. This completes the proof.

By the arguments in Section 2, we have $0\neq s_2(Y)\in H_2(Y ; \boldsymbol{Z}_2)\cong H_2(S^2\times S^1 ; \boldsymbol{Z}_2)\cong\boldsymbol{Z}_2$. The Gysin homomorphism $f^!$ maps $H_2(Y ; \boldsymbol{Z}_2)$ onto $H_0(Y ; \boldsymbol{Z}_2)$ isomorphically. Since $s_0(X)=s_0(S^1)=0$ and $w(N_f)^{-1}=1$, we obtain the following:

PROPOSITION 6.2. *$s_0(X)$ is not equal to $w(N_f)^{-1}\cap f^! s_2(Y)$.*

This shows that Halperin's conjecture is not true in our case.

Our construction of Z_h can naturally be extended, for example, to constructions of mod 2 Euler spaces $Z^{p,q}$ homotopy equivalent to $S^p\times S^q$ if $p\geqq 2$ and $q\geqq 1$. We have the inclusion of S^q in $Z^{p,q}$ which is homologically normally nonsingular, but Halperin's equation (H) does not hold.

References

[1] E. Akin, Stiefel-Whitney homology classes and bordism, Trans. Amer. Math. Soc., **205** (1975), 341-359.

[2] J. Blanton and C. McCrory, An axiomatic proof of Stiefel's conjecture, Proc. Amer. Math. Soc., **77** (1979), 409-414.

[3] J. Cheeger, A combinatorial formula for Stiefel-Whitney classes, Topology of Manifolds, Markham Publ., Chicago, 1971, 470-471.

[4] W. Fulton and R. MacPherson, Categorical framework for the study of singular spaces, Mem. Amer. Math. Soc., **243** (1981).

[5] R. Goldstein, A Wu formula for Euler mod 2 spaces, Compositio Math., **32** (1976), 33-39.

[6] S. Halperin and D. Toledo, Stiefel-Whitney homology classes, Ann. of Math., **96** (1972), 511-525.

[7] S. Halperin and D. Toledo, The product formula for Stiefel-Whitney homology classes, Proc. Amer. Math. Soc., **48** (1975), 239-244.

[8] A. Matsui, Stiefel-Whitney homology classes of $\boldsymbol{Z}_2$-Poincaré-Euler spaces, Tôhoku Math. J., **35** (1983), 321-339.

[9] D. Sullivan, Combinatorial invariants of analytic spaces, Proc. Liverpool Singularities I, Lecture Notes in Math., **192**, Springer, 1971, 165-168.

[10] L. Taylor, Stiefel-Whitney homology classes, Quart. J. Oxford, **28** (1977), 381-387.

[11] D. Veljan, Axioms for Stiefel-Whitney homology classes of some singular spaces, Trans. Amer. Math. Soc., **277** (1983), 285-305.

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