

Lorenz attractors do not have the pseudo-orbit tracing property

By Motomasa KOMURO

(Received Feb. 21, 1984)

(Revised June 19, 1984)

§ 0. Introduction.

E. Lorenz studied the following system of differential equations in connection with problems in hydrodynamics [9];

$$(L) \quad \begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

where $(\sigma, r, b) = (10, 28, 8/3)$. J. Guckenheimer introduced in [4] a geometric description of a flow which seems to have the qualitative dynamics of the solutions of the Lorenz equation (L). This geometric Lorenz flow has a complicated attractor. R. F. Williams described in [14] this attractor, which is called the geometric Lorenz attractor, as the inverse limit of a semi-flow on a 2-dimensional branched manifold.

The aim of this paper is to study the condition under which the geometric Lorenz attractors have the pseudo-orbit tracing property (abbr. P.O.T.P.). P.O.T.P. plays important roles in several places in dynamics ([8], [12] and [13]). Suppose that ϕ is a continuous flow on a state space of a system. The pseudo-orbit of ϕ is viewed as the orbit realized in numerical calculation by computer, or in physical experiments. Then P.O.T.P. is interpreted as that, while a computer may not calculate the orbit which you hope for, what it does find is nonetheless an approximation to some true orbit of the system (P. 251 of [6]).

Let $(\tilde{K}_f, \tilde{\varphi}_f)$ be a geometric Lorenz attractor with the return map f from the interval $[0, 1]$ to itself (see § 2 for definition). Then our main theorem is stated as below.

THEOREM 1. *The geometric Lorenz attractor $(\tilde{K}_f, \tilde{\varphi}_f)$ has the finite P.O.T.P. iff the return map f satisfies that $f(0)=0$ and $f(1)=1$.*

This theorem implies that the geometric Lorenz attractors do not have P.O.T.P. except a special case of $f(0)=0$ and $f(1)=1$. If we suppose the geometric Lorenz

flows represent completely the dynamics of solutions of the Lorenz equation (L), the condition of $f(0)=0$ and $f(1)=1$ corresponds to the case of $r=r_A \approx 24.06$ under $(\sigma, b)=(10, 8/3)$ (P. 48 of [11]).

Combining above Theorem 1 and Guckenheimer-Williams-Robinson's theorem ([5], [15]), we can conclude the following;

THEOREM 3. *Let M be a 3-dimensional compact manifold and $\mathfrak{X}^2(M)$ the space of all vector fields on M with C^2 -topology. Then the set of vector fields with the strong P.O.T.P. is not dense in $\mathfrak{X}^2(M)$.*

A. Morimoto proposes in [10] a problem whether the set of diffeomorphisms with P.O.T.P. is residual in the space of all diffeomorphisms of a compact manifold. The answer of this problem is unknown yet. However, in the case of flows, Theorem 3 gives us a negative answer for this problem.

We give in §1 the definitions of the strong, weak and finite P.O.T.P. In §2 we study P.O.T.P. of the inverse limit system of a semi-flow on a compact metric space. The definition of the geometric Lorenz attractor is given in §3. Also we prepare in §3 a notation used in the proof of Theorem 1. We prove Theorem 1 in §4 and Theorems 2 and 3 in §5.

Throughout this paper the symbols \mathbf{R} , \mathbf{Z} and \mathbf{N} denote the set of all real numbers, the set of all integers and the set of all positive integers respectively.

§1. Definition of P.O.T.P.

Let (X, d) be a compact metric space with a distance function d . We denote

$$d(Y_1, Y_2) = \inf \{d(y_1, y_2) : y_i \in Y_i (i=1, 2)\}$$

for any subsets $Y_1, Y_2 \subset X$. A flow $\phi = \{\phi^t\}_{t \in \mathbf{R}}$ on X is a continuous map

$$\phi : X \times \mathbf{R} \longrightarrow X; \quad (x, t) \longrightarrow \phi(x, t) = \phi^t(x)$$

such that $\phi^{t+s} = \phi^t \circ \phi^s$ holds for every $s, t \in \mathbf{R}$ and ϕ^0 is the identity map (clearly ϕ^t is a homeomorphism on X for each $t \in \mathbf{R}$). This is often denoted by (X, ϕ) . For each $x \in X$ and $t \in \mathbf{R}$ we denote $x \cdot t = \phi^t(x)$. Also for any subsets $Y \subset X$ and $J \subset \mathbf{R}$ we denote

$$Y \cdot J = \{x \cdot t : x \in Y, t \in J\}.$$

A subset $Y \subset X$ is said to be ϕ -invariant if $Y = Y \cdot t$ for all $t \in \mathbf{R}$. The restriction of ϕ to a ϕ -invariant closed subset $Y \subset X$ is denoted by (Y, ϕ) . The nonwandering set of ϕ is a ϕ -invariant closed subset of X defined by

$$\Omega = \{x \in X : \text{for any open neighborhood } U \text{ of } x \text{ and any } T > 0, \\ U \cap (U \cdot [T, \infty)) \neq \emptyset\}.$$

DEFINITION 1. Given $\delta, T > 0$, a collection $\{x_a, x_{a+1}, \dots, x_b; t_a, t_{a+1}, \dots, t_b\}$ ($x_i \in X, t_i \geq 0, a \leq i \leq b; a = -\infty, b = \infty$ are permitted) is called a (δ, T) -chain of ϕ if

$$d(x_i \cdot t_i, x_{i+1}) \leq \delta \quad \text{and} \quad t_i \geq T \quad \text{for every } a \leq i \leq b-1.$$

A finite (δ, T) -chain $\{x_i; t_i\}_a^b$ ($-\infty < a \leq b < \infty$) is naturally extended to an infinite (δ, T) -chain $\{x_i; t_i\}_{i \in \mathbf{Z}}$. Put

$$S_i = \begin{cases} \sum_{j=0}^{i-1} t_j & (i > 0) \\ 0 & (i = 0) \\ -\sum_{j=i}^{-1} t_j & (i < 0). \end{cases}$$

Let $x_0 * t$ denote a point on a (δ, T) -chain t units time from x_0 . More precisely, for each $t \in \mathbf{R}$ we define

$$x_0 * t = x_i \cdot (t - S_i) \quad \text{if } S_i \leq t < S_{i+1}, \quad i \in \mathbf{Z}.$$

DEFINITION 2. Let $\varepsilon > 0$ be given. Define

$$\text{Rep} = \{g : \mathbf{R} \rightarrow \mathbf{R} : g \text{ is a monotone increasing homeomorphism with } g(0) = 0\}$$

and

$$\text{Rep}(\varepsilon) = \left\{ g \in \text{Rep} : \left| \frac{g(s) - g(t)}{s - t} - 1 \right| \leq \varepsilon \text{ for every } s, t \in \mathbf{R} \right\}.$$

Each element of Rep is called a *reparametrization*.

DEFINITION 3. A (δ, T) -chain $\{x_i; t_i\}_{i \in \mathbf{Z}}$ of ϕ is said to be weakly ε -traced (resp. strongly ε -traced) by a point $x \in X$ if there is a reparametrization $g \in \text{Rep}$ (resp. $g \in \text{Rep}(\varepsilon)$) such that

$$d(x_0 * t, x \cdot g(t)) \leq \varepsilon \quad \text{for every } t \in \mathbf{R}.$$

DEFINITION 4. A flow (X, ϕ) has the *weak P.O.T.P.* (resp. *the strong P.O.T.P.*) if for any $\varepsilon > 0$ there exist $\delta, T > 0$ such that every (δ, T) -chain of ϕ can be weakly ε -traced (resp. strongly ε -traced) by some point of X . A flow (X, ϕ) has the *finite P.O.T.P.* if for any $\varepsilon > 0$ there exist $\delta, T > 0$ such that every finite (δ, T) -chain $\{x_i; t_i\}_0^k$ ($0 \leq k < \infty$) of ϕ can be weakly ε -traced by some point x of X ; i.e. there is $g \in \text{Rep}$ with

$$d(x_0 * t, x \cdot g(t)) \leq \varepsilon \quad \text{for } 0 \leq t < S_{k+1}.$$

REMARK 1. The restriction of an Axiom A flow to its nonwandering set has the strong P.O.T.P. This fact can be proved from the Bowen's approximation theorem (Theorem 2.2 in [1]). The weak and finite P.O.T.P. have been

used in [3], [8] and [12] to study stability properties of flows. In general the following relation holds;

the strong P.O.T.P. \implies the weak P.O.T.P. \implies the finite P.O.T.P.

§ 2. Inverse limit systems.

In this section we study the finite P.O.T.P. of the inverse limit of a semi-flow.

DEFINITION 5. Let (X, d) be a compact metric space with a distance function d . A semi-flow $\varphi = \{\varphi^t\}_{t \geq 0}$ on X is a continuous map

$$\varphi : X \times [0, \infty) \longrightarrow X; \quad (x, t) \longrightarrow \varphi(x, t) = \varphi^t(x)$$

such that φ^0 is the identity map, $\varphi^t : X \rightarrow X$ is surjective and $\varphi^{t+s} = \varphi^t \circ \varphi^s$ holds for every $t, s \geq 0$. By (X, φ) we denote a semi-flow φ on X . Put $X_s = X$ for each $s \leq 0$. We define

$$\tilde{X} = \{\tilde{x} = (x^s)_{s \leq 0} \in \prod_{s \leq 0} X_s : x^t = \varphi^{t-s}(x^s), s \leq t \leq 0\}$$

and

$$\tilde{\varphi}^t(\tilde{x}) = \begin{cases} (\varphi^t x^s)_{s \leq 0} & \text{for } t \geq 0 \\ (x^{s+t})_{s \leq 0} & \text{for } t < 0. \end{cases}$$

The distance function on \tilde{X} is defined by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \int_0^\infty e^{-t} d(x^{-t}, y^{-t}) dt$$

for each $\tilde{x}, \tilde{y} \in \tilde{X}$. Then (\tilde{X}, \tilde{d}) is a compact metric space and $\tilde{\varphi} = \{\tilde{\varphi}^t\}_{t \in \mathbb{R}}$ is a flow on \tilde{X} . The flow $(\tilde{X}, \tilde{\varphi})$ is called the *inverse limit* of a semi-flow (X, φ) . We denote this by

$$(\tilde{X}, \tilde{\varphi}) = \varprojlim (X, \varphi).$$

DEFINITION 6. For a semi-flow (X, φ) , we denote $x \cdot t = \varphi^t(x)$ for every $x \in X$ and $t \geq 0$. Given $\delta, T > 0$, a collection $\{x_i; t_i\}_0^k$ ($x_i \in X, t_i \geq T, 0 \leq k < \infty$) is called a *finite* (δ, T) -chain of φ if

$$d(x_i \cdot t_i, x_{i+1}) \leq \delta \quad \text{and} \quad t_i \geq T \quad \text{for every } 0 \leq i \leq k-1.$$

For each $t \geq 0$ we define

$$x_0 \cdot t = x_i \cdot (t - S_i) \quad \text{if } S_i \leq t < S_{i+1} \quad \text{for some } 0 \leq i \leq k$$

where $S_0 = 0$ and $S_i = \sum_{j=0}^{i-1} t_j$ ($0 < i \leq k+1$).

DEFINITION 7. A semi-flow (X, φ) has the *finite P.O.T.P.* if for every $\varepsilon > 0$

there exist $\delta, T > 0$ such that for every finite (δ, T) -chain $\{x_i; t_i\}_0^k$ of φ , there exist $x \in X$ and $g \in \text{Rep}$ with

$$d(x_0 * t, x \cdot g(t)) \leq \varepsilon \quad \text{for every } 0 \leq t < S_{k+1}.$$

DEFINITION 8. A semi-flow (X, φ) satisfies the condition (P) if for every $\varepsilon > 0$ there exist $N \geq 0, \delta > 0$ and $T > 0$ such that the following holds; for every finite (δ, T) -chain $\{x_i; t_i\}_0^k$ of φ with

$$d(x_i \cdot (t_i + s), x_{i+1} \cdot s) \leq \delta \quad \text{for every } 0 \leq s \leq N \text{ and } 0 \leq i \leq k,$$

there exist $x \in X$ and $g \in \text{Rep}$ such that

$$d((x_0 * t) \cdot N, x \cdot (g(t) + N)) \leq \varepsilon \quad \text{for every } 0 \leq t < S_{k+1}.$$

It is clear that the finite P.O.T.P. for semi-flows implies the condition (P). Indeed, we can take $N=0$ as the number $N \geq 0$ in the condition (P).

PROPOSITION 2.1. Let $(\tilde{X}, \tilde{\varphi})$ be the inverse limit of a semi-flow (X, φ) . Then $(\tilde{X}, \tilde{\varphi})$ has the finite P.O.T.P. iff (X, φ) satisfies the condition (P).

PROOF. At first we shall prove the "if" part. Suppose (X, φ) satisfies the condition (P). Let $\varepsilon > 0$ be given. There is $M > 0$ such that

$$D \int_M^\infty e^{-t} dt \leq \varepsilon/2$$

where $D = \text{diam}(X)$. Take $\varepsilon' > 0$ such that $d(x, y) \leq \varepsilon'$ ($x, y \in X$) implies $d(x \cdot t, y \cdot t) \leq \varepsilon/2$ for every $t \in [0, M]$. There are numbers $N > 0, \delta' > 0$ and $T > 0$ corresponding to ε' as in the condition (P). Take $\delta > 0$ such that $d(x, y) \leq e^{M+N+1}\delta$ implies $d(x \cdot t, y \cdot t) \leq \delta'$ for every $t \in [0, M+N+1]$. Let $\{\tilde{x}_i; t_i\}_0^k$ be a finite (δ, T) -chain of $(\tilde{X}, \tilde{\varphi})$. We express \tilde{x}_i by $(x_i^s)_{s \leq 0}$. Then we have

$$\begin{aligned} \delta &\geq \tilde{d}(\tilde{x}_i \cdot t_i, \tilde{x}_{i+1}) \geq \int_{M+N}^{M+N+1} e^{-t} d(x_i^{-t} \cdot t_i, x_{i+1}^{-t} \cdot t) dt \\ &\geq e^{-(M+N+1)} d(x_i^{-(M+N+\theta)} \cdot t_i, x_{i+1}^{-(M+N+\theta)} \cdot t) \end{aligned}$$

for some $\theta \in [0, 1]$. By the choice of δ we have

$$\begin{aligned} \delta' &\geq d(x_i^{-(M+N+\theta)} \cdot (t_i + t), x_{i+1}^{-(M+N+\theta)} \cdot t) \\ &= d(x_i^{t-(M+N+\theta)} \cdot t_i, x_{i+1}^{t-(M+N+\theta)}) \quad \text{for every } t \in [0, M+N+\theta], \end{aligned}$$

so that

$$d(x_i^{-t} \cdot t_i, x_{i+1}^{-t}) \leq \delta' \quad \text{for every } t \in [0, M+N].$$

Thus $\{x_i^{-(M+N)}; t_i\}_0^k$ is a finite (δ', T) -chain of φ satisfying

$$d(x_i^{-(M+N)} \cdot (t_i + s), x_{i+1}^{-(M+N)} \cdot s) \leq \delta' \quad \text{for every } s \in [0, N].$$

By the condition (P) there exist $y \in X$ and $g \in \text{Rep}$ such that

$$d((x_0^{-(M+N)*t}) \cdot N, y \cdot (g(t) + N)) \leq \varepsilon' \quad \text{for every } 0 \leq t < S_{k+1}$$

where $S_{k+1} = \sum_0^k t_i$. By the choice of ε' we have

$$\begin{aligned} d((x_0^{-M*t}) \cdot s, ((y \cdot N) \cdot g(t)) \cdot s) \\ = d((x_0^{-(M+N)*t}) \cdot (N+s), (y \cdot g(t)) \cdot (N+s)) \leq \varepsilon/2 \end{aligned}$$

for every $s \in [0, M]$ and $t \in [0, S_{k+1}]$. Take a point $\tilde{x} = (x^s)_{s \leq 0} \in \tilde{X}$ with $x^{-t} = y \cdot (M+N-t)$ for every $t \in [0, M]$. Let $0 \leq t < S_{k+1}$ be given and suppose that $S_i \leq t < S_{i+1}$ for some $0 \leq i \leq k$. Since

$$x_i^{-s} \cdot (t - S_i) = (x_i^{-M} \cdot (M-s)) \cdot (t - S_i) = (x_0^{-M*t}) \cdot (M-s)$$

for every $s \in [0, M]$, we have

$$\begin{aligned} \tilde{d}(\tilde{x} \cdot g(t), \tilde{x}_0 * t) &\leq \int_0^M e^{-s} d(x^{-s} \cdot g(t), x_i^{-s} \cdot (t - S_i)) ds + D \int_M^\infty e^{-s} ds \\ &\leq \int_0^M e^{-s} d((y \cdot (M+N-s)) \cdot g(t), (x_0^{-M*t}) \cdot (M-s)) ds + \varepsilon/2 \\ &= \int_0^M e^{-s} d(((y \cdot N) \cdot g(t)) \cdot (M-s), (x_0^{-M*t}) \cdot (M-s)) ds + \varepsilon/2 \\ &\leq (\varepsilon/2) \int_0^M e^{-s} ds + \varepsilon/2 \leq \varepsilon. \end{aligned}$$

This completes the proof of the "if" part.

Next we shall prove the "only if" part. Suppose $(\tilde{X}, \tilde{\varphi})$ has the finite P.O.T.P. Let $\varepsilon > 0$ be given. There is an $\varepsilon' > 0$ such that $d(x, y) \leq \varepsilon \varepsilon'$ ($x, y \in X$) implies $d(x \cdot t, y \cdot t) \leq \varepsilon$ for every $t \in [0, 1]$. By our assumption there are $\delta, T > 0$ such that every finite (δ, T) -chain of $(\tilde{X}, \tilde{\varphi})$ is weakly ε' -traced by some point in \tilde{X} . Take $N \geq 1$ with

$$D \int_N^\infty e^{-t} dt \leq \delta/2$$

where $D = \text{diam}(X)$. Let $\Gamma = \{x_i; t_i\}_0^k$ be a finite $(\delta/2, T)$ -chain of (X, φ) with

$$d(x_i \cdot (t_i + s), x_{i+1} \cdot s) \leq \delta/2 \quad \text{for every } s \in [0, N].$$

Take $\tilde{y}_i = (y_i^s)_{s \leq 0} \in \tilde{X}$ with

$$y_i^{-t} = x_i \cdot (N-t) \quad \text{for every } t \in [0, N].$$

Then we obtain that

$$\begin{aligned} & \tilde{d}(\tilde{y}_i \cdot t_i, \tilde{y}_{i+1}) \\ & \leq \int_0^N e^{-s} d((x_i \cdot t_i) \cdot (N-s), x_{i+1} \cdot (N-s)) ds + D \int_N^\infty e^{-s} ds \\ & \leq (\delta/2) \int_0^N e^{-s} ds + \delta/2 \leq \delta \end{aligned}$$

for every $0 \leq i \leq k-1$. Thus $\tilde{\Gamma} = \{\tilde{y}_i; t_i\}_0^k$ is a finite (δ, T) -chain of $(\tilde{X}, \tilde{\varphi})$. There exist $\tilde{y} = (y^s)_{s \leq 0} \in \tilde{X}$ and $g \in \text{Rep}$ by which $\tilde{\Gamma}$ is weakly ε' -traced. Let $0 \leq t < S_{k+1}$ be given and suppose that $S_i \leq t < S_{i+1}$ for some $0 \leq i \leq k$. Then we have

$$\begin{aligned} \varepsilon' & \geq \tilde{d}(\tilde{y} \cdot g(t), \tilde{y}_0 * t) \geq \int_0^1 e^{-s} d(y^{-s} \cdot g(t), y_i^{-s} \cdot (t - S_i)) ds \\ & = \int_0^1 e^{-s} d(y^{-N} \cdot (g(t) + N - s), x_i \cdot (t - S_i + N - s)) ds \\ & \geq e^{-1} d(y^{-N} \cdot (g(t) + N - \theta), x_i \cdot (t - S_i + N - \theta)) \end{aligned}$$

for some $\theta \in [0, 1]$. Therefore, by the choice of ε' we have

$$d(y^{-N} \cdot (g(t) + N), (x_0 * t) \cdot N) \leq \varepsilon \quad \text{for every } 0 \leq t < S_{k+1}.$$

This completes the proof of the “only if” part.

§ 3. Lorenz attractors.

In this section we give the definition of geometric Lorenz attractors and prepare a notation used in the proof of Theorem 1 of the next section.

DEFINITION 9. Let K be a 2-dimensional compact branched manifold (called a Lorenz branched manifold) illustrated as in Figure 1. The set of branch points of K is the line segment $\overline{b'c'}$. The point b (resp. c) is an intersection of the boundary of K and an extension of the line $\overline{c'b'}$ (resp. $\overline{b'c'}$). We permit the case of $b=b'$ or $c=c'$. The branched manifold K is embedded in \mathbf{R}^3 as a subset. We denote by d a distance function on K which is a usual distance function on \mathbf{R}^3 .

We suppose that a C^1 -semi-flow φ on K is defined as illustrated by some arrows in Figure 1. The point e is a unique singular point of φ . Near the point e , the linearized equation has the form

$$\begin{cases} \dot{x} = \mu_1 x \\ \dot{y} = -\mu_2 y \end{cases} \quad (0 < \mu_2 < \mu_1).$$

There is a unique point $a \in \overline{b'c'}$ such that $\varphi^t(a)$ does not return to the line segment \overline{bc} but converges to the point e as $t \rightarrow \infty$. The inverse limit of such a semi-flow (K, φ) ;

$$(\tilde{K}, \tilde{\varphi}) = \varprojlim (K, \varphi)$$

is called the *geometric Lorenz attractor*.

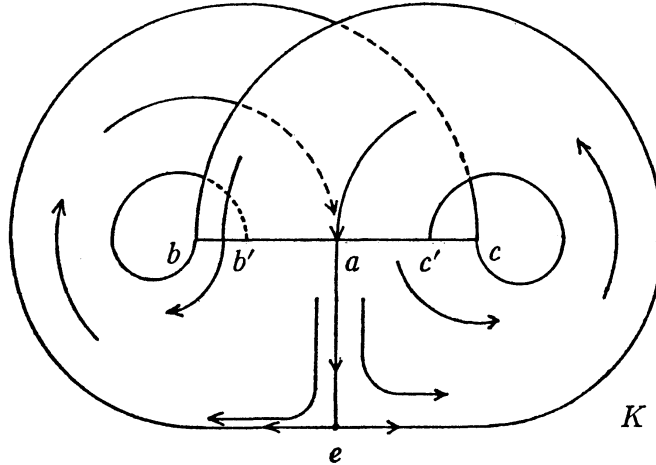


Figure 1.

We identify the line segment \overline{bc} with the interval $I = [0, 1]$ by $b=0$ and $c=1$. By the same symbol a we denote a point in I corresponding to the point $a \in \overline{bc}$. We denote

$$I_0^+ = (a, 1], \quad I_0^- = [0, a) \quad \text{and} \quad I_0 = I_0^+ \cup I_0^-.$$

Let $f : I_0 \rightarrow I$ be the return map of φ . More precisely, for each $x \in I_0$, $f(x)$ is defined by

$$f(x) = \varphi^T(x) \quad \text{where} \quad T = \inf \{s > 0 : x \cdot s \in I\}.$$

Under φ the point $a \in I = \overline{bc}$ does not return to $I = \overline{bc}$ by the definition of the point a . Thus the value $f(a)$ is not defined. However, to simplify we extend f to $f : I \rightarrow I$ by $f(a) = 0$. Then f is a piecewise C^1 -map with a unique discontinuity a . The return map f satisfies the following (see § 2 in [5]);

- (R. i) $f(a^+) = 0, \quad f(a^-) = 1;$
- (R. ii') $f' > \lambda \quad \text{for some } \lambda \geq 1;$
- (R. iii) $f(0) < a < f(1);$
- (R. iv) $f'(x) \rightarrow \infty \quad \text{as } x \rightarrow a^\pm.$

Here, as a simplifying assumption, we assume

$$(R. ii) \quad f' > \sqrt{2} \quad (\text{i. e. } \lambda = \sqrt{2} \text{ in (R. ii')}).$$

Under this assumption, f becomes *locally eventually onto*; i.e. for any open set $U \subset I$ there is an $n > 0$ such that $f^n(U) = I$ (Proposition 1 of [14]).

Conversely, if we have a piecewise C^1 -map $f: I \rightarrow I$ with a unique discontinuity a satisfying the above conditions (R. i)~(R. iv), then a Lorenz branched manifold K and a C^1 -semi-flow φ on K which has the return map f are uniquely determined up to the topological equivalence. Here a semi-flow (X_1, φ_1) is *topologically equivalent* to a semi-flow (X_2, φ_2) if there exist a homeomorphism $h: X_1 \rightarrow X_2$ and a continuous map $\sigma: X_1 \times [0, \infty) \rightarrow [0, \infty)$ such that

$\sigma(x, \cdot): [0, \infty) \rightarrow [0, \infty)$ is a surjective homeomorphism with $\sigma(x, 0) = 0$ and

$$\varphi_2(h(x), \sigma(x, t)) = h(\varphi_1(x, t)) \quad \text{for every } x \in X_1 \text{ and } t \in [0, \infty).$$

Therefore by (K_f, φ_f) we denote a semi-flow on a Lorenz branched manifold determined by the return map f with (R. i)~(R. iv). We call (K_f, φ_f) the geometric Lorenz semi-flow defined by the return map f . The inverse limit

$$(\tilde{K}_f, \tilde{\varphi}_f) = \varprojlim (K_f, \varphi_f)$$

is called the geometric Lorenz attractor defined by the return map f .

Let (K_f, φ_f) and $(\tilde{K}_f, \tilde{\varphi}_f)$ be a geometric Lorenz semi-flow and an attractor respectively defined by a return map $f: I \rightarrow I$ satisfying (R. i)~(R. iv). Without loss of generality we may assume that the length of \overline{ae} is equal to $1/2$ and the lengths of the arcs \widehat{eb} , \widehat{ec} , $\widehat{bb'}$ and $\widehat{cc'}$ are larger than one.

DEFINITION 10. We define

$$x \cdot t = \begin{cases} \varphi_f^t(x) & (t \geq 0) \\ \{y \in K_f : \varphi_f^{-t}(y) = x\} & (t < 0) \end{cases}$$

for each $x \in K_f$ and $t \in \mathbf{R}$. We denote

$$Y \cdot J = \{x \cdot t : x \in Y, t \in J\}$$

for subsets $Y \subset K_f$ and $J \subset \mathbf{R}$.

DEFINITION 11. Define a map $T: I_0 \rightarrow (0, \infty)$ by

$$T(x) = \inf \{s > 0 : x \cdot s \in I\}.$$

That is, $T(x)$ is the first return time of $x \in I_0$ to I under φ_f . Denote

$$C(x) = x \cdot [0, T(x)) \quad \text{for each } x \in I_0.$$

Define a map $L: I_0 \rightarrow (0, \infty)$ such that for each $x \in I_0$

$$L(x) \text{ is the length of } C(x).$$

As in Figure 2, we set

$$K_0^\sigma = \{x \cdot t : t \in [0, T(x)), x \in I_0^\sigma\} \quad (\sigma = +, -), \text{ and}$$

$$K_0 = K_0^+ \cup K_0^-.$$

DEFINITION 12. For each $u \in K_0$ there exists a unique point $x_u \in I_0$ with $u \in C(x_u)$. Thus a map $\pi : K_f \rightarrow I$ is well defined by

$$\pi(u) = \begin{cases} x_u & (u \in K_0) \\ a & (u \notin K_0). \end{cases}$$

Define a map $t : K_0 \rightarrow [0, \infty)$ such that for each $u \in K_0$

$$t(u) \text{ is the minimal time from } \pi(u) \text{ to } u \text{ under } \varphi_f.$$

Define a map $\chi : K_0 \rightarrow [0, \infty)$ such that for each $u \in K_0$

$$\chi(u) \text{ is the length of } \pi(u) \cdot [0, t(u)].$$

We set

$$L_*^\sigma = \{(x, \xi) \in \mathbf{R}^2 : \xi \in [0, L(x)), x \in I_0^\sigma\} \quad (\sigma = +, -),$$

$$T_*^\sigma = \{(x, t) \in \mathbf{R}^2 : t \in [0, T(x)), x \in I_0^\sigma\} \quad (\sigma = +, -),$$

$$L_* = L_*^+ \cup L_*^- \quad \text{and} \quad T_* = T_*^+ \cup T_*^-.$$

The distance function on L_* and T_* is defined by

$$d((x, s), (x', s')) = \max\{d(x, x'), |s - s'|\}.$$

For each $(x, \xi) \in L_*$ there exists a unique point $u_{x, \xi} \in K_0$ such that

$$\chi(u_{x, \xi}) = \xi \quad \text{and} \quad u_{x, \xi} \in C(x).$$

Define

$$\beta : L_* \longrightarrow K_0 \quad \text{by} \quad \beta(x, \xi) = u_{x, \xi} \quad \text{for all } (x, \xi) \in L_*$$

and

$$\theta : T_* \longrightarrow K_0 \quad \text{by} \quad \theta(x, t) = x \cdot t \quad \text{for all } (x, t) \in T_*.$$

DEFINITION 13. Let $q_0 = \inf\{T(x) : x \in I_0\} > 0$. Define

$$\zeta : I \cdot [-q_0/4, q_0/4] \longrightarrow I \quad \text{by} \quad \zeta(x \cdot t) = x$$

for each $x \in I$ and $t \in [-q_0/4, q_0/4]$. Given $\eta > 0$ and $\kappa_0 > 0$, we define the following (see Figure 2);

$$K(\kappa_0) = K_f - I \cdot (-\kappa_0, 0),$$

$$A(\eta) = \{(x, \xi) \in L_* : d(x, a) \leq \eta\},$$

$$L'_* = \{(x, \xi) \in L_* : 0 \leq \xi \leq 1/2\},$$

$$K'_0 = \beta(L'_*),$$

$$\begin{aligned}
 D(\eta) &= \text{Cl}\{\beta(x, \xi) : (x, \xi) \in A(\eta), |\xi - 1/2| \leq \eta\}, \\
 P(\eta) &= \text{Cl}\{\beta(x, \xi) : (x, \xi) \in A(\eta), 0 \leq \xi \leq 1/2 - \eta\}, \\
 Q^\sigma(\eta) &= \text{Cl}\{\beta(x, \xi) : (x, \xi) \in A(\eta) \cap L_*^\sigma, 1/2 + \eta \leq \xi < L(x)\} \quad (\sigma = +, -), \\
 Q(\eta) &= Q^+(\eta) \cup Q^-(\eta), \\
 Q^\sigma(\eta, \kappa_0) &= Q^\sigma(\eta) \cap K(\kappa_0) \quad (\sigma = +, -) \text{ and} \\
 Q(\eta, \kappa_0) &= Q^+(\eta, \kappa_0) \cup Q^-(\eta, \kappa_0)
 \end{aligned}$$

where $\text{Cl}(Y)$ denotes the closure of Y in K_f . For a subset $Y \subset K_f$ we denote the η -neighborhood of Y by

$$B(Y, \eta) = \{u \in K_f : d(u, Y) \leq \eta\}.$$

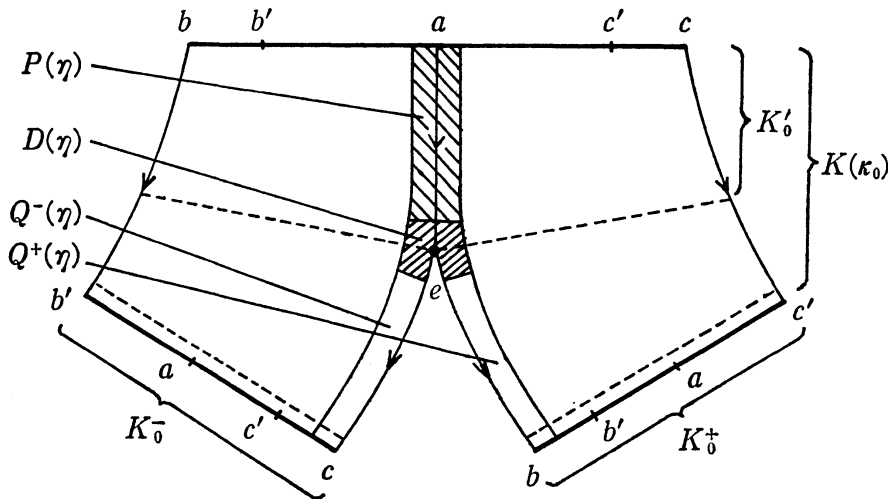


Figure 2.

LEMMA 3.1. Let $\eta > 0$ and $\kappa_0 > 0$ be given.

(1) The following maps are uniformly continuous;

$$\begin{aligned}
 \pi|_{K(\kappa_0)} : K(\kappa_0) &\longrightarrow I, & \chi|_{K(\kappa_0)} : K(\kappa_0) &\longrightarrow [0, \infty), \\
 t|_{K(\kappa_0)} : K(\kappa_0) &\longrightarrow [0, \infty), & \zeta : I \cdot [-q_0/4, q_0/4] &\longrightarrow I \text{ and} \\
 L : I_0 &\longrightarrow (0, \infty).
 \end{aligned}$$

(2) The following maps are uniformly continuous injections:

$$\begin{aligned}
 \beta|_{L_*^\sigma} : L_*^\sigma &\longrightarrow K_0 \quad (\sigma = +, -), \\
 \beta|_{L'_*} : L'_* &\longrightarrow K_0 \quad \text{and} \quad \beta^{-1}|_{K(\kappa_0)} : K(\kappa_0) &\longrightarrow L_*.
 \end{aligned}$$

(3) The map T is continuous. The restriction of θ to $\{(x, t) \in T_* : t \neq 0\}$ is

a homeomorphism onto its image.

$$(4) \quad P(\eta) \cap Q(\eta, \kappa_0) = \emptyset \quad \text{and} \quad Q^+(\eta) \cap Q^-(\eta) = \emptyset.$$

(5) For any $u \in D(\eta)$ and $s > 0$, if

$$u \cdot s \in D(\eta) \quad \text{and} \quad u \cdot [0, s] \cap I = \emptyset,$$

then $u \cdot [0, s] \subset D(\eta)$.

PROOF. These statements are easily verified from each definition.

§4. Proof of Theorem 1.

Let f be a piecewise C^1 -map of $I = [0, 1]$ with a unique discontinuity $a \in (0, 1)$ satisfying (R. i)~(R. iv) in Definition 9. Let (K_f, φ_f) be the geometric Lorenz semi-flow defined by f . We recall $I_0^+ = (a, 1]$, $I_0^- = [0, a)$ and $I_0 = I_0^+ \cup I_0^-$. For $x \in I_0$ we denote

$$I(x) = \begin{cases} I_0^- & (0 \leq x < a) \\ I_0^+ & (a < x \leq 1). \end{cases}$$

DEFINITION 14. A sequence $\{x_i\}_0^k$ ($0 \leq k < \infty$) of points in I is a *finite δ -pseudo-orbit* of f in I_0 if $d(fx_i, x_{i+1}) \leq \delta$ and $x_i \in I_0$ for each $0 \leq i \leq k-1$. (I, f) has the finite P.O.T.P. if for every $\varepsilon > 0$ there is a $\delta > 0$ with the following property; for every finite δ -pseudo-orbit $\{x_i\}_0^k$ of f in I_0 , there exists $x \in I_0$ such that

$$d(f^i x, x_i) \leq \varepsilon \quad \text{and} \quad f^i x \in I(x_i) \quad \text{for every} \quad 0 \leq i \leq k.$$

Now our main theorem is stated as below;

THEOREM 1. *The following are equivalent to each other.*

- (A) *The geometric Lorenz attractor $(\tilde{K}_f, \tilde{\varphi}_f)$ has the finite P.O.T.P.*
- (B) *The geometric Lorenz semi-flow (K_f, φ_f) satisfies the condition (P).*
- (C) *The geometric Lorenz semi-flow (K_f, φ_f) has the finite P.O.T.P.*
- (D) *The return map (I, f) has the finite P.O.T.P.*
- (E) *The return map (I, f) satisfies $f(0) = 0$ and $f(1) = 1$.*

In Proposition 2.1 we have proved that (A) and (B) are equivalent. By the remark below Definition 8, (C) implies (B). We shall prove the remaining statements by a series of propositions.

PROPOSITION 4.1. *If $f(0) = 0$ and $f(1) = 1$, then (I, f) has the finite P.O.T.P.*

PROOF. Assume that $f(0) = 0$ and $f(1) = 1$. Define

$$C = \bigcup_{j=0}^{\infty} f^{-j}(a) \cup \{0, 1\} \quad \text{and} \quad J = I - C.$$

Since C is countable, J is dense in I . By the assumption, we have

$$f(J \cap I_0) = J \quad (\sigma = +, -).$$

Put $\lambda = \inf \{f'(x) : x \in I\} \geq \sqrt{2}$. Let $0 < \varepsilon < \min\{a, 1-a\}$ be given. For any $x, y \in J$, if $d(fx, y) < \varepsilon$, then there is $y' \in I(x) \cap J$ with

$$d(x, y') < \varepsilon/\lambda \quad \text{and} \quad f(y') = y.$$

Take $\delta > 0$ with $\delta/(1-\lambda^{-1}) < \varepsilon$. Let a finite δ -pseudo-orbit $\{x_i\}_0^k$ of f in I_0 be given. Since J is dense in I_0 , without loss of generality we may assume that $x_i \in J$ ($0 \leq i \leq k$). Put $y_k = x_k$. Since $d(fx_{k-1}, y_k) \leq \delta$, there is $y_{k-1} \in I(x_{k-1}) \cap J$ such that

$$d(x_{k-1}, y_{k-1}) \leq \delta\lambda^{-1} \quad \text{and} \quad f(y_{k-1}) = y_k.$$

Since

$$d(fx_{k-2}, y_{k-1}) \leq d(fx_{k-2}, x_{k-1}) + d(x_{k-1}, y_{k-1}) \leq \delta(1+\lambda^{-1}) < \varepsilon,$$

there is $y_{k-2} \in I(x_{k-2}) \cap J$ such that

$$d(x_{k-2}, y_{k-2}) \leq \delta(\lambda^{-1} + \lambda^{-2}) \quad \text{and} \quad f(y_{k-2}) = y_{k-1}.$$

Repeating this process we get y_i ($0 \leq i \leq k$) such that

$$y_i \in I(x_i) \cap J, \quad d(x_i, y_i) \leq \delta(\lambda^{-1} + \lambda^{-2} + \dots + \lambda^{i-k}) < \varepsilon$$

and

$$f(y_i) = y_{i+1}.$$

Put $x = y_0$. Since $f^i x = y_i$, we have

$$d(f^i x, x_i) \leq \varepsilon \quad \text{and} \quad f^i x \in I(x_i) \quad \text{for every } 0 \leq i \leq k.$$

This completes the proof of Proposition 4.1.

PROPOSITION 4.2. *If (K_f, φ_f) satisfies the condition (P), then $f(0) = 0$ and $f(1) = 1$.*

PROOF. Assume that (K_f, φ_f) satisfies the condition (P) and $f(0) \neq 0$. We shall derive a contradiction. Notice that we identified the line segment \overline{bc} in K_f and the unit interval $I = [0, 1]$ in \mathbf{R} by $b = 0$ and $c = 1$. We denote

Arc(e, b) = the arc in the boundary of K_f from e to b ,

Arc(e, c) = the arc in the boundary of K_f from e to c ,

$[a, e) = a \cdot [0, \infty)$ (i.e. the positive half orbit of a) and

$$C = \bigcup_{j=0}^{\infty} f^{-j}(a).$$

Since f is locally eventually onto, for any open set $U \subset I$ there is an $n > 0$ such that $f^n(U) = I$. From this, it follows that C is dense in I . Let $\varepsilon > 0$ be given. Without loss of generality we may assume that $\varepsilon < f(0)/2$ and $B(b, \varepsilon) \cap b \cdot [T(b)/2, T(b)] = \emptyset$. There are numbers $N > 0$, $\delta > 0$ and $T > 0$ as in the condition (P).

First we consider the case of $b \in C$. Take $x_0 \in \text{Arc}(e, b)$ and $t_0 \geq \max\{T, N\}$ such that

$$x_0 \cdot N = b, \quad x_0 \cdot t_0 \in [a, e) \quad \text{and}$$

$$x_0 \cdot (t_0 + s) \in B(e, \delta/2) \quad \text{for every } 0 \leq s \leq N$$

where $B(e, \delta/2) = \{u \in K_f : d(e, u) \leq \delta/2\}$. Take $x_1 \in \text{Arc}(e, c)$ and $t_1 \geq \max\{T, N\}$ such that

$$x_1 \cdot (t_1 + N) = c \quad \text{and}$$

$$x_1 \cdot s \in B(e, \delta/2) \quad \text{for every } 0 \leq s \leq N.$$

Then $\{x_i; t_i\}_0^1$ is a finite (δ, T) -chain of φ_f with

$$d(x_0 \cdot (t_0 + s), x_1 \cdot s) \leq \delta \quad \text{for every } 0 \leq s \leq N.$$

There are $x \in K_f$ and $g \in \text{Rep}$ such that

$$d(x_0 \cdot (s + N), x \cdot (g(s) + N)) \leq \varepsilon \quad \text{for every } 0 \leq s \leq t_0$$

and

$$(*) \quad d(x_1 \cdot (s + N), x \cdot (g(t_0 + s) + N)) \leq \varepsilon \quad \text{for every } 0 \leq s \leq t_1.$$

Since $d(b, x \cdot N) = d(x_0 \cdot N, x \cdot N) \leq \varepsilon$, the point $x \cdot N$ must be ε -near and in the left hand side of the point b with respect to the positive direction of the semi-flow φ_f (see Figure 1.) Similarly the point $(x \cdot N) \cdot g(s)$ must be ε -near and in the left hand side of the point $b \cdot s = x_0 \cdot (s + N)$ for every $s \in [0, t_0]$, thus we conclude that

$$x \cdot (g(t_0) + N) \in \text{Cl}(K_0^+).$$

Since $d(x_1 \cdot N, x \cdot (g(t_0) + N)) \leq \varepsilon$, we have

$$x \cdot (g(t_0) + N) \in B(x_1 \cdot N, \varepsilon) \cap \text{Cl}(K_0^+).$$

Since $x_1 \cdot N \in \text{Cl}(K_0^-)$ and $x_1 \cdot (t_1 + N) = c$, there is $t \in [0, t_1]$ such that

$$x \cdot (g(t_0 + t) + N) \in \text{Cl}(K_0^+) - B(\text{Cl}(K_0^-), \varepsilon) \quad \text{or}$$

$$x_1 \cdot (t + N) \in \text{Cl}(K_0^-) - B(\text{Cl}(K_0^+), \varepsilon).$$

This contradicts (*).

Next we consider the case of $b \notin C$. Take $x_0 \in \text{Arc}(e, b)$ and $t_0 \geq \max\{T, N\}$ such that

$$x_0 \cdot N = b \quad \text{and} \quad x_0 \cdot t_0 \in I.$$

Since C is dense in I , there are $x_1 \in C$ and $t_1 \geq \max\{T, N\}$ such that

$$x_1 < x_0 \cdot t_0 \quad (\text{with respect to the usual order of } I \subset \mathbf{R}),$$

$$d(x_0 \cdot (t_0 + s), x_1 \cdot s) \leq \delta \quad \text{for every } 0 \leq s \leq N,$$

$$x_1 \cdot t_1 \in [a, e) \quad \text{and}$$

$$x_1 \cdot (t_1 + s) \in B(e, \delta/2) \quad \text{for every } 0 \leq s \leq N.$$

Take $x_2 \in \text{Arc}(e, c)$ and $t_2 \geq \max\{T, N\}$ such that

$$x_2 \cdot (t_2 + N) = c \quad \text{and} \quad x_2 \cdot s \in B(e, \delta/2) \quad \text{for every } 0 \leq s \leq N.$$

Then $\{x_i; t_i\}_0^2$ is a finite (δ, T) -chain of φ_f with

$$d(x_i \cdot (t_i + s), x_{i+1} \cdot s) \leq \delta \quad \text{for every } 0 \leq s \leq N \text{ and } i=0, 1.$$

There are $x \in K_f$ and $g \in \text{Rep}$ such that

$$(**) \quad d((x_0 * s) \cdot N, x \cdot (g(s) + N)) \leq \varepsilon \quad \text{for every } 0 \leq s \leq t_0 + t_1 + t_2.$$

Since $d(b, x \cdot N) = d(x_0 \cdot N, x \cdot N) \leq \varepsilon < f(0)/2$, the point $x \cdot N$ must be ε -near and in the left hand side of the point b with respect to the positive direction of the semi-flow φ_f (see Figure 1). And, since $x_1 < x_0 \cdot t_0$, the point $x \cdot (g(t_0) + N)$ must be also in the left hand side of x_1 . Moreover the point $x \cdot (g(s) + N)$ must be ε -near and in the left hand side of the point $(x_0 * s) \cdot N$ for every $s \in [0, t_0 + t_1]$, thus we conclude that

$$x \cdot (g(t_0 + t_1) + N) \in \text{Cl}(K_0^+).$$

Since $d(x_2 \cdot N, x \cdot (g(t_0 + t_1) + N)) = d((x_0 * (t_0 + t_1)) \cdot N, x \cdot (g(t_0 + t_1) + N)) \leq \varepsilon$, we have

$$x \cdot (g(t_0 + t_1) + N) \in B(x_2 \cdot N, \varepsilon) \cap \text{Cl}(K_0^+).$$

Since $x_2 \cdot N \in \text{Cl}(K_0^-)$ and $x_2 \cdot (t_2 + N) = c$, there is $t \in [0, t_2]$ such that

$$\begin{aligned} x \cdot (g(t_0 + t_1 + t) + N) &\in \text{Cl}(K_0^+) - B(\text{Cl}(K_0^-), \varepsilon) \quad \text{or} \\ x_2 \cdot (t + N) &= (x_0 * (t_0 + t_1 + t)) \cdot N \in \text{Cl}(K_0^-) - B(\text{Cl}(K_0^+), \varepsilon). \end{aligned}$$

This contradicts (**).

Similarly, assuming the condition (P) and $f(1) \neq 1$, we can derive a contradiction. This completes the proof of Proposition 4.2.

The following Proposition completes the proof of Theorem 1.

PROPOSITION 4.3. *If (I, f) has the finite P.O.T.P., then (K_f, φ_f) has the finite P.O.T.P.*

PROOF. To simplify the notation, we denote $(K, \varphi) = (K_f, \varphi_f)$. Assume that (I, f) has the finite P.O.T.P. We shall prove that (K, φ) has the finite P.O.T.P.

Let $\varepsilon > 0$ be given. By Lemma 3.1 (2), there is an $\varepsilon_0 > 0$ with $\varepsilon_0 < \varepsilon/2$ such that

$$d((x, \xi), (x', \xi')) \leq 2\varepsilon_0 \quad \text{implies} \quad d(\beta(x, \xi), \beta(x', \xi')) \leq \varepsilon/4$$

for every $(x, \xi), (x', \xi') \in L_*^\sigma$ ($\sigma = +, -$) or L'_* . Since $L : I_0 \rightarrow (0, \infty)$ is uniformly continuous (Lemma 3.1 (1)), we can take an $\varepsilon_1 > 0$ with $\varepsilon_1 < \varepsilon_0/4$ such that

$$d(x, y) \leq \varepsilon_1 \quad \text{implies} \quad |L(x) - L(y)| \leq \varepsilon_0/4$$

for every $x, y \in I_0$. By our assumption, there is a $\delta_1 > 0$ with $\delta_1 < \varepsilon_1$ as in the definition of the finite P.O.T.P. of f corresponding to ε_1 . Take a $\kappa_0 > 0$ with $\kappa_0 < q_0/12$ such that $d(u, u \cdot t) \leq \varepsilon/3$ holds for every $t \in [-9\kappa_0, 9\kappa_0]$ and $u \in K$. There is a $\delta_0 > 0$ with $\delta_0 < \delta_1/3$ such that $d(u, v) \leq \delta_0$ implies

$$d(\pi(u), \pi(v)) \leq \delta_1/3 \quad \text{and} \quad d(\beta^{-1}(u), \beta^{-1}(v)) \leq \varepsilon_0/12$$

for every $u, v \in K(\kappa_0)$. We choose $\delta > 0$ and $\eta > 0$ with $\eta < \varepsilon_0/12$ which satisfy the following;

$$(C. 1) \quad B(D(\eta), \delta) \subset D(2\eta), \quad \text{diam} D(2\eta) < \delta_0,$$

$$D(2\eta) \cap I \cdot [-3\kappa_0, 3\kappa_0] = \emptyset;$$

$$(C. 2) \quad B(P(\eta), \delta) \cap B(Q(\eta, \kappa_0), \delta) = \emptyset, \quad B(Q^+(\eta, \kappa_0), \delta) \cap K_0^- = \emptyset,$$

$$B(Q^+(\eta, \kappa_0), \delta) \cap B(Q^-(\eta, \kappa_0), \delta) = \emptyset,$$

$$B(P(\eta), \delta) \cap K(\kappa_0) \subset K'_0;$$

$$(C. 3) \quad B(I \cdot [-j\kappa_0, j\kappa_0], \delta) \subset I \cdot [-(j+1)\kappa_0, (j+1)\kappa_0] \quad (j=1, 2);$$

$$(C. 4) \quad d(u, v) \leq \delta \quad \text{implies} \quad d(\zeta(u), \zeta(v)) \leq \delta_0$$

$$\text{for every } u, v \in I \cdot [-3\kappa_0, 3\kappa_0].$$

Take $T_0 > 0$ such that

$$(C. 5) \quad \text{if } u \in K - (D(\eta) \cup P(\eta)), \quad \text{then } (u \cdot (0, T_0]) \cap I \neq \emptyset,$$

$$(C. 6) \quad \text{if } u \in K - (D(\eta) \cup Q(\eta)), \quad \text{then } (u \cdot (-T_0, 0]) \cap I \neq \emptyset.$$

Put $T = T_0 + 6\kappa_0$.

Let a finite (δ, T) -chain $\{u_i; t_i\}_0^k$ of φ be given. We define $\hat{u}_i = u_i \cdot t_i$ for $0 \leq i \leq k$. Without loss of generality we may assume that $u_i \in K_0$ for each $0 \leq i \leq k$, because K_0 is dense in K . We must show that $\{u_i; t_i\}_0^k$ is weakly ε -traced by some point in K . This process is divided into three steps.

Step 1 (Normalization of chain). We construct a new chain $\Gamma = \{v_i; s_i\}_0^k$ from $\{u_i; t_i\}_0^k$. Put $v_0 = u_0$ and $t'_0 = t_0$. Recursively define $v_i \in K_0$ and $s_i \geq 0$ as below. If

$$u_i \in I \cdot [-2\kappa_0, 2\kappa_0] \quad (\text{notice then } \hat{u}_{i-1} \in I \cdot [-3\kappa_0, 3\kappa_0]),$$

or

$$u_i \in I \cdot ([-3\kappa_0, 3\kappa_0] - [-2\kappa_0, 2\kappa_0]) \quad \text{and} \quad \hat{u}_{i-1} \in I \cdot [-2\kappa_0, 2\kappa_0],$$

then define

$$v_i = \zeta(u_i),$$

$$t'_i = t_i - \tau'_i \quad \text{where } \tau'_i \in [-3\kappa_0, 3\kappa_0] \text{ with } \zeta(u_i) = u_i \cdot \tau'_i,$$

$$s_{i-1} = t'_{i-1} + \hat{\tau}_{i-1} \quad \text{where } \hat{\tau}_{i-1} \in [-3\kappa_0, 3\kappa_0] \text{ with } \zeta(\hat{u}_{i-1}) = \hat{u}_{i-1} \cdot \hat{\tau}_{i-1}.$$

Otherwise, define

$$v_i = u_i, \quad t'_i = t_i \quad \text{and} \quad s_{i-1} = t'_{i-1}.$$

Finally define $s_k = t'_k$.

Using (C.3) and (C.4), we can verify that the above construction of $\{v_i; s_i\}_0^k$ implies

$$d(v_i \cdot s_i, v_{i+1}) \leq \delta_0 \quad \text{and} \quad s_i \geq T - 6\kappa_0 = T_0$$

for every $0 \leq i \leq k$. Thus $\{v_i; s_i\}_0^k$ is a (δ_0, T_0) -chain of φ with $v_i \in K_0$ ($0 \leq i \leq k$). This is called *the normalized chain* of $\{u_i; t_i\}_0^k$.

LEMMA 4.4. *Let $\{v_i; s_i\}_0^k$ be a normalized chain and denote $\hat{v}_i = v_i \cdot s_i$. Then the following holds.*

- (i) $v_i, \hat{v}_i \in I \cup (K - I \cdot (-2\kappa_0, 2\kappa_0)) \subset K(\kappa_0)$;
- (ii) $v_i \in I$ iff $\hat{v}_{i-1} \in I$;
- (iii) $v_i \notin I$ implies $\hat{v}_{i-1} \notin I$ and $d(\hat{v}_{i-1}, v_i) \leq \delta$;
- (iv) $\hat{v}_i \notin I$ implies $v_{i+1} \notin I$ and $d(\hat{v}_i, v_{i+1}) \leq \delta$.

PROOF. These are immediately obtained from the above construction of normalized chain.

LEMMA 4.5. *If the normalized chain $\{v_i; s_i\}_0^k$ is weakly $\varepsilon/2$ -traced by some point $v \in K$, then the original chain $\{u_i; t_i\}_0^k$ is weakly ε -traced by the same point v .*

PROOF. Put $S_0 = R_0 = 0$, $S_i = \sum_0^{i-1} s_j$ and $R_i = \sum_0^{i-1} t_j$ for $0 < i \leq k+1$. We define $h(S_i) = R_i$ ($0 \leq i \leq k+1$) and extend h linearly between these points. Then h belongs to Rep. Let $g \in \text{Rep}$ be a reparametrization under which $\{v_i; s_i\}_0^k$ is weakly $\varepsilon/2$ -traced by v . Put $g' = g \circ h$. Then g' belongs to Rep.

For each $0 \leq i \leq k+1$, take $\sigma_i \in [-3\kappa_0, 3\kappa_0]$ with $v_i = u_i \cdot \sigma_i$. For every $t \in [R_i, R_{i+1}]$, since $h(t) - S_i = (t - R_i)s_i/t_i$, we have

$$|\sigma_i + t - R_i - h(t) + S_i| \leq |\sigma_i| + |t - s_i| \leq 9\kappa_0.$$

By the choice of κ_0 , it follows that

$$\begin{aligned} d(u_i \cdot t, v \cdot g'(t)) &\leq d(u_i \cdot (t - R_i), u_i \cdot (\sigma_i + h(t) - S_i)) \\ &\quad + d(v_i \cdot (h(t) - S_i), v \cdot g(h(t))) \leq \varepsilon \end{aligned}$$

for every $t \in [0, S_{k+1}]$. This completes the proof of Lemma 4.5.

It remains only to prove that the normalized chain $\{v_i; s_i\}_0^k$ is weakly $\varepsilon/2$ -traced by some $v \in K$ and $g \in \text{Rep}$.

Step 2 (Signs). For the normalized (δ_0, T_0) -chain $\Gamma = \{v_i; s_i\}_0^k$, we denote $\gamma_i = v_i \cdot [0, s_i]$. Define

$$\operatorname{sgn}(\gamma_i) = \begin{cases} 1 & \text{if } (v_i \cdot (0, s_i]) \cap I \neq \emptyset \\ -1 & \text{if } (v_i \cdot (0, s_i]) \cap I = \emptyset \text{ and } v_i, \hat{v}_i \in B(D(\gamma), \delta) \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality we may assume that $\operatorname{sgn}(\gamma_0) = \operatorname{sgn}(\gamma_k) = 1$ and $v_0, \hat{v}_k \in I_0$ (prolong γ_0 for negative direction and γ_k for positive direction if necessary).

LEMMA 4.6. *For each $0 \leq i \leq k$ the following holds.*

(i) *If $\operatorname{sgn}(\gamma_i) \neq 1$ and $\operatorname{sgn}(\gamma_{i+1}) \neq 1$, then $\hat{v}_i, v_{i+1} \in B(D(\gamma), \delta)$.*

(ii) *If $\operatorname{sgn}(\gamma_{i-1}) = -1$, $\operatorname{sgn}(\gamma_i) = 0$ and $\operatorname{sgn}(\gamma_{i+1}) = 1$, then*

$$\hat{v}_i, v_{i+1} \in K_0^+ \quad \text{or} \quad \hat{v}_i, v_{i+1} \in K_0^-.$$

(iii) *If $\operatorname{sgn}(\gamma_{i-1}) = 1$, $\operatorname{sgn}(\gamma_i) = 0$ and $\operatorname{sgn}(\gamma_{i+1}) = -1$, then $\hat{v}_{i-1}, v_i \in K'_0$.*

PROOF. Put $D = D(\gamma)$, $P = P(\gamma)$ and $Q = Q(\gamma, \kappa_0)$. At first we shall prove

(i). Since $(v_{i+1} \cdot (0, T_0]) \cap I \subset (v_{i+1} \cdot (0, s_{i+1}]) \cap I = \emptyset$, we have $v_{i+1} \in D \cup P$ by (C.5). Since $(\hat{v}_i \cdot (-T_0, 0]) \cap I \subset (v_i \cdot (0, s_i]) \cap I = \emptyset$, we have

$$\hat{v}_i \in (D \cup Q(\gamma)) \cap (K - I \cdot (-2\kappa_0, 2\kappa_0)) \subset D \cup Q$$

by (C.6) and Lemma 4.4 (i). Also Lemma 4.4 (iv) implies that

$$d(\hat{v}_i, v_{i+1}) \leq \delta.$$

Using the first condition of (C.2), we obtain that

$$\hat{v}_i \in (D \cup Q) \cap (B(D, \delta) \cup B(P, \delta)) = D \cup (Q \cap B(D, \delta)) \subset B(D, \delta).$$

Similarly we can prove that $v_{i+1} \in B(D, \delta)$.

Next we shall prove (ii). Since $\operatorname{sgn}(\gamma_{i-1}) \neq 1$ and $\operatorname{sgn}(\gamma_i) \neq 1$, the above (i) implies that $v_i \in B(D, \delta)$. Since $\operatorname{sgn}(\gamma_i) \neq -1$, $\hat{v}_i \notin B(D, \delta)$. From this and the fact that $(\hat{v}_i \cdot (-T_0, 0]) \cap I \subset (v_i \cdot (0, s_i]) \cap I = \emptyset$, using (C.6), we have

$$\hat{v}_i \in (D \cup Q) - B(D, \delta) \subset Q.$$

Since $d(\hat{v}_i, v_{i+1}) \leq \delta$, it follows that

$$\hat{v}_i, v_{i+1} \in B(Q^+(\gamma, \kappa_0), \delta) \quad \text{or} \quad \hat{v}_i, v_{i+1} \in B(Q^-(\gamma, \kappa_0), \delta).$$

By the second condition of (C.2), we obtain (ii).

Finally we shall prove (iii). Since $\operatorname{sgn}(\gamma_i) \neq 1$ and $\operatorname{sgn}(\gamma_{i+1}) \neq 1$, the above (i) implies that $\hat{v}_i \in B(D, \delta)$. Since $\operatorname{sgn}(\gamma_i) \neq -1$, we have $v_i \notin B(D, \delta)$. From this and the fact that $(v_i \cdot (0, T_0]) \cap I \subset (v_i \cdot (0, s_i]) \cap I = \emptyset$, using (C.5), we have

$$v_i \in P - B(D, \delta).$$

Since $d(\hat{v}_{i-1}, v_i) \leq \delta$, $\hat{v}_{i-1} \in B(P, \delta) - D$. From the last condition of (C. 2) and the fact that $\hat{v}_{i-1}, v_i \in K(\kappa_0)$, we obtain (iii). The proof of Lemma 4.6 is completed.

LEMMA 4.7. Take γ_i with $\text{sgn}(\gamma_i) = 1$ and put $C = \{\text{sgn}(\gamma_a)\}_{a=i}^{i+m}$ for $m \geq 1$. If $C = \{1, \sigma, -1, -1, \dots, -1, \sigma', 1\}$ ($\sigma, \sigma' = 0$ or -1), then

$$d(\pi(\hat{v}_i), \pi(v_{i+m})) \leq \delta_1.$$

PROOF. Since $d(\hat{v}_i, v_{i+1}) \leq \delta_0$ and $d(\hat{v}_{i+m-1}, v_{i+m}) \leq \delta_0$, we have

$$d(\pi(\hat{v}_i), \pi(v_{i+1})) \leq \delta_1/3 \quad \text{and} \quad d(\pi(\hat{v}_{i+m-1}), \pi(v_{i+m})) \leq \delta_1/3.$$

By Lemma 4.6 (i) we obtain that

$$\hat{v}_{i+1}, v_{i+m-1} \in B(D(\eta), \delta).$$

Using (C. 1), we have $d(\hat{v}_{i+1}, v_{i+m-1}) \leq \delta_0$, so that

$$d(\pi(\hat{v}_{i+1}), \pi(v_{i+m-1})) \leq \delta_1/3.$$

Since $\pi(v_{i+a}) = \pi(\hat{v}_{i+a})$ for $a = 1$ and $m - 1$, we get the conclusion by the triangle inequality. The proof of Lemma 4.7 is completed.

To construct a δ_1 -pseudo-orbit of f , we set $x_i^0 = \pi(v_i)$ for each γ_i which satisfies $\text{sgn}(\gamma_i) = 1$. Take an orbit of x_i^0 under f ;

$$\{x_i^0, x_i^1, \dots, x_i^{k(i)}\}$$

where

$$x_i^{j+1} = f(x_i^j) = x_i^j \cdot T(x_i^j) \quad \text{for } 0 \leq j \leq k(i) - 1$$

and

$$\sum_{j=0}^{k(i)} T(x_i^j) \leq s_i + t(v_i) < \sum_{j=0}^{k(i)+1} T(x_i^j).$$

For $\{i_1, \dots, i_m\} = \{i : 0 \leq i \leq k, \text{sgn}(\gamma_i) = 1\}$, we define

$$G = \{x_{i_1}^0, x_{i_1}^1, \dots, x_{i_1}^{k_1-1}, x_{i_2}^0, \dots, x_{i_m}^{k_m}\}$$

where $k_1 = k(i_1), \dots, k_m = k(i_m)$. By Lemma 4.7, we have

$$d(fx_{i_j}^{k_j-1}, x_{i_{j+1}}^0) = d(\pi(\hat{v}_{i_j}), \pi(v_{i_{j+1}})) \leq \delta \quad \text{for } 0 \leq j \leq m - 1.$$

Thus G is a finite δ_1 -pseudo-orbit of f in I_0 . For simplicity we denote $G = \{y_0, y_1, \dots, y_n\}$. By the choice of δ_1 , there is $y \in I_0$ such that

$$d(f^i y, y_i) \leq \epsilon_1 \quad \text{and} \quad f^i(y) \in I(y_i) \quad \text{for every } 0 \leq i \leq n.$$

Put $S_{k+1} = \sum_0^k s_i$. We separate $[0, S_{k+1}]$ into

$$0 = r_0 < r_1 < \dots < r_n = S_{k+1}$$

such that $v_0^*(r_i - 0) = f y_{i-1}$ where $v_0^*(r_i - 0) = \lim_{t \uparrow r_i} v_0^* t$.

Step 3 (Reparametrization).

LEMMA 4.8. For each $0 \leq i \leq n-1$, there exists a continuous map $g_i: [0, r_{i+1} - r_i] \rightarrow \mathbf{R}$ such that

- (i) $g_i(0) = 0$ and $g_i(s) \leq g_i(t)$ for every $s \leq t$,
- (ii) $(f^i y) \cdot g_i(r_{i+1} - r_i) = f^{i+1} y$,
- (iii) $d(v_0^*(r_i + t), (f^i y) \cdot g_i(t)) \leq \varepsilon/2$ for every $t \in [0, r_{i+1} - r_i]$.

If Lemma 4.8 holds, then the proof of Proposition 4.3 is completed as follows. Define a continuous map $\bar{g}: \mathbf{R} \rightarrow \mathbf{R}$ by

$$\bar{g}(t) = \sum_0^{i-1} g_j(r_{j+1} - r_j) + g_i(t - r_i) \quad \text{for } t \in [r_i, r_{i+1}], \quad 0 \leq i \leq n-1,$$

and $g: \mathbf{R} \rightarrow \mathbf{R}$ by

$$g(t) = \bar{g}(t) + \lambda(t) \quad (t \in \mathbf{R})$$

where $\lambda > 0$ small enough. Then we have $g \in \text{Rep}$ such that $\{v_i; s_i\}_0^k$ is weakly $\varepsilon/2$ -traced by y under g . Thus it remains only to prove Lemma 4.8.

PROOF. Let $0 \leq i \leq n$ be given. For $v_0^*[r_i - 0, r_{i+1}]$, there are $0 \leq j \leq k$ and $m > 0$ such that

$$v_0^*(r_i - 0) \in \gamma_j, \quad v_0^* r_{i+1} \in \gamma_{j+m} \quad \text{and} \quad v_0^*[r_i - 0, r_{i+1}] \subset \bigcup_{a=0}^m \gamma_{j+a}.$$

Put $C = \{\text{sgn}(\gamma_a)\}_{a=j}^{j+m}$. The following five cases are possible;

- (a) $C = \{1\}$,
- (b) $C = \{1, 1\}$,
- (c) $C = \{1, \sigma, 1\}$,
- (d) $C = \{1, \sigma, \sigma', 1\}$.
- (e) $C = \{1, \sigma, -1, -1, \dots, -1, \sigma', 1\}$

where $\sigma, \sigma' = 0$ or -1 .

Here we prove only the case of $C = \{1, 0, -1, -1, \dots, -1, 0, 1\}$ (the other cases are similarly proved). Put $z = f^i y$. For every $x \in I_0$ we define an increasing homeomorphism $\alpha_x: [0, L(x)] \rightarrow [0, T(x)]$ by $\alpha_x(0) = 0$, $\alpha_x(L(x)) = T(x)$ and

$$\beta(x, \xi) = \theta(x, \alpha_x(\xi)) \quad \text{for every } \xi \in [0, L(x)].$$

This homeomorphism α_x is well-defined, because β is one-to-one by Lemma 3.1 (2), and θ is a homeomorphism from $\{(x, t) \in T_* : t \neq 0\}$ to its image by Lemma 3.1 (3). Put $s'_{j+1} = \inf \{s \in [0, s_{j+1}] : v_{j+1} \cdot s \in D(\eta)\}$. We set the following notation.

- (1') $w_1 = f y_{i-1}, \quad \hat{w}_1 = \hat{v}_j; \quad w_2 = v_{j+1}, \quad \hat{w}_2 = v_{j+1} \cdot s'_{j+1};$
 $w_3 = \hat{w}_3 = \hat{w}_2; \quad w_4 = v_{j+m-1}, \quad \hat{w}_4 = \hat{v}_{j+m-1};$
 $w_5 = v_{j+m}, \quad \hat{w}_5 = f y_i.$
- (2') $z_i = \pi(w_i) \quad \text{for each } 1 \leq i \leq 5.$

$$(3') \quad S_0=0; \quad S_1=t(\hat{w}_1); \quad S_2=S_1+s'_{j+1};$$

$$R_a=S_1+(s_{j+1}+\dots+s_{j+a}) \quad (1 \leq a \leq m-1);$$

$$S_3=R_{m-2}; \quad S_4=R_{m-1}; \quad S_5=R_{m-1}+T(y_i)-t(w_5).$$

$$(4') \quad \xi_i=\chi(w_i) \quad (1 \leq i \leq 5); \quad \hat{\xi}_i=\chi(\hat{w}_i) \quad (1 \leq i \leq 4) \quad \text{and} \quad \hat{\xi}_5=L(z_5).$$

It is clear that $\hat{\xi}_2=\hat{\xi}_3=\hat{\xi}_5$, $z_1=fy_{i-1}$ and $z_5=y_i$. Put $L_0=0$ and $L_i=\sum_{j=1}^i(\hat{\xi}_j-\xi_j)$ ($1 \leq i \leq 5$). Define $H(0)=0$, $H(L_2)=1/2$ and $H(L_5)=L(z)$, and extend H linearly between these points. Then we have an increasing homeomorphism $H: [0, L_5] \rightarrow [0, L(z)]$. We claim the following; for each $1 \leq i \leq 4$,

- (i') $d(\hat{w}_i, w_{i+1}) \leq \delta_0$, (ii') $|\xi_{i+1}-\hat{\xi}_i| \leq \epsilon_0/12$,
- (iii') $|\hat{\xi}_2-1/2| \leq \epsilon_0/12$, (iv') $|\xi_{i+1}-L_i| \leq \epsilon_0/4$,
- (v') $|H(\xi)-\xi| \leq 2\epsilon_0/3$ for every $\xi \in [0, L_5]$,
- (vi') $d(z, z_i) \leq \epsilon_0/2$,
- (vii') $d(\hat{w}_2, v_{j+1} \cdot t) \leq \delta_0$ for every $t \in [s'_{j+1}, s_{j+1}]$,
- (viii') $d(\hat{w}_2, v_{j+a} \cdot t) \leq \delta_0$ for every $t \in [0, s_{j+a}]$, $2 \leq a \leq m-2$.

Indeed, (i') is clear for $i=1, 2$ or 4 , because $\{v_i: s_i\}_0^k$ is a (δ_0, T_0) -chain. If $i=3$, we have $\hat{w}_3, w_4 \in B(D(\eta), \delta)$ by Lemma 4.6 (i). Since $\text{diam} B(D(\eta), \delta) \leq \delta_0$, we get (i') for $i=3$. By the choice of δ_0 , we have

$$|\xi_{i+1}-\hat{\xi}_i| \leq d((z_{i+1}, \xi_{i+1}), (z_i, \hat{\xi}_i)) = d(\beta^{-1}(w_{i+1}), \beta^{-1}(\hat{w}_i))$$

$$\leq \epsilon_0/12,$$

so that (ii') holds. Using $\hat{w}_2 = \beta(z_2, \hat{\xi}_2) \in D(\eta) \cap K'_0$ and $\eta \leq \epsilon_0/12$, we get (iii'). Since $|\xi_3-\hat{\xi}_2|=0$, it follows that

$$|\xi_{i+1}-L_i| \leq \sum_1^4 |\xi_{i+1}-\hat{\xi}_i| \leq \epsilon_0/4,$$

so that (iv') holds. To see (v'), let $\xi \in [0, L_5]$ be given. If $\xi \in [0, L_2]$, then

$$|H(\xi)-\xi| \leq |1/2-L_2| \leq \epsilon_0/6$$

by (ii') and (iii'). If $\xi \in [L_2, L_5]$, then it follows that

$$|H(\xi)-\xi| \leq |1/2-L_2| + |L(z)-1/2-L_5+L_2| \leq (\epsilon_0/3) + |L(z)-L(z_5)| \leq 2\epsilon_0/3,$$

because $d(z, z_5) \leq \epsilon_1$ holds. This proves (v'). To see (vi') we notice that

$$d(z_i, z_{i+1}) \leq \delta_1/3 \leq \epsilon_0/12 \quad \text{for each } 1 \leq i \leq 4 \quad (\text{by (i')}).$$

From $d(z, z_5) \leq \epsilon_1$ and $d(z_1, z_5) \leq \delta_1$, we obtain (vi). Using Lemma 4.6 (i) and

Lemma 3.1 (5), we have

$$v_{j+1} \cdot [s'_{j+1}, s_{j+1}] \subset D(2\eta).$$

Since $\text{diam} D(2\eta) \leq \delta_0$, (vii') is proved. Similarly we get (viii').

Now we define $g_i(t)$ for $t \in [0, S_5]$ by

$$g_i(t) = \begin{cases} \alpha_z(H(\alpha_{z_a}^{-1}(t - S_{a-1} + t(w_a)) + L_{a-1} - \xi_a)) & \text{for } t \in [S_{a-1}, S_a], \quad a=1, 2, 4, 5, \\ \alpha_z(H(\alpha_{z_2}^{-1}(S_2 - S_1 + t(w_2)) + L_1 - \xi_2)) & \text{for } t \in [S_2, S_3]. \end{cases}$$

By the definitions of α_z and H , the above g_i satisfies (i). We shall show that g_i satisfies (ii) and (iii). Let $t \in [0, S_5]$ be given.

If $t \in [S_{a-1}, S_a]$ ($a=1, 2$), put $\xi = \alpha_{z_a}^{-1}(t - S_{a-1} + t(w_a)) + L_{a-1} - \xi_a$. By Lemma 4.6 (iii), we have $z_a \cdot (t - S_{a-1} + t(w_a)) = w_a \cdot t \in K'_0$. Since $\xi \in [0, L_2]$, we have $H(\xi) \in [0, 1/2]$, so that $z \cdot \alpha_z(H(\xi)) = \beta(z, H(\xi)) \in K'_0$. Since

$$\begin{aligned} d((z_a, \xi - L_{a-1} + \xi_a), (z, H(\xi))) &\leq \max\{d(z_a, z), |H(\xi) - \xi| + |L_{a-1} - \xi_a|\} \\ &\leq \varepsilon_0, \end{aligned}$$

we get

$$\begin{aligned} d(v_0^*(r_i + t), z \cdot g_i(t)) &= d(z_a \cdot (t - S_{a-1} + t(w_a)), z \cdot \alpha_z(H(\xi))) \\ &= d(\beta(z_a, \xi - L_{a-1} + \xi_a), \beta(z, H(\xi))) \leq \varepsilon/4 \end{aligned}$$

by the uniform continuity of β .

If $t \in [S_2, S_3]$, then $v_0^*(r_i + t) \in B(D(\eta), \delta)$. Since $\hat{w}_2 \in D(\eta)$, we have $d(v_0^*(r_i + t), \hat{w}_2) \leq \delta_0 \leq \varepsilon_0/4$. Since $g_i(t) = g_i(S_2)$, we get

$$\begin{aligned} d(v_0^*(r_i + t), z \cdot g_i(t)) &\leq d(v_0^*(r_i + t), \hat{w}_2) + d(\hat{w}_2, z \cdot g_i(S_2)) \\ &\leq \varepsilon/2. \end{aligned}$$

If $t \in [S_{a-1}, S_a]$ ($a=4, 5$), put $\xi = \alpha_{z_a}^{-1}(t - S_{a-1} + t(w_a)) + L_{a-1} - \xi_a$. Using Lemma 4.6 (ii), we obtain $z_4 \in I(z_5)$. By the definition of the finite P.O.T.P. of f , we have $z \in I(z_5)$. Thus two points $(z_a, \xi - L_{a-1} + \xi_a) = \beta^{-1}(z_a \cdot (t - S_{a-1} + t(w_a)))$ and $(z, H(\xi))$ belong to either L_*^+ or L_*^- . Since

$$\begin{aligned} d((z_a, \xi - L_{a-1} + \xi_a), (z, H(\xi))) &\leq \max\{d(z_a, z), |H(\xi) - \xi| + |L_{a-1} - \xi_a|\} \\ &\leq \varepsilon_0, \end{aligned}$$

we have

$$d(v_0^*(r_i + t), z \cdot g_i(t)) = d(\beta(z_a, \xi - L_{a-1} + \xi_a), \beta(z, H(\xi))) \leq \varepsilon/4.$$

If $t = S_5$, then

$$\begin{aligned} g_i(S_5) &= \alpha_z(H(\alpha_{z_5}^{-1}(T(z_5)) + L_4 - \xi_5)) = \alpha_z(H(L(z_5) + L_4 - \xi_5)) \\ &= \alpha_z(H(L_5)) = \alpha_z(L(z)) = T(z). \end{aligned}$$

Thus it follows that

$$z \cdot g_i(S_5) = z \cdot T(z) = f(z) \quad \text{and}$$

$$d(v_0^*(r_i + S_5), z \cdot g_i(S_5)) \leq d(fy_i, y_{i+1}) + d(y_{i+1}, fz) \leq \epsilon/4.$$

It has been proved that g_i satisfies (ii) and (iii). This completes the proof of Lemma 4.8. Therefore the proof of Proposition 4.3 is completed.

§ 5. Non-denseness of vector fields with the strong P.O.T.P.

In this section we study the strong P.O.T.P. of flows.

DEFINITION 15. Let $\phi = \{\phi^t\}_{t \in \mathbf{R}}$ be a flow on a compact metric space (X, d) . (X, ϕ) is *topologically transitive* if for every non-empty open subsets U, V of X and every $T \geq 0$, there is $t \geq T$ such that $U \cdot t \cap V \neq \emptyset$. A ϕ -invariant subset A of the nonwandering set Ω of ϕ is said to be a *basic set* if A is closed and open in Ω and (A, ϕ) is topologically transitive.

DEFINITION 16. Let M be a compact C^1 -manifold and $\mathfrak{X}^1(M)$ the space of all C^1 -vector fields on M endowed with the C^1 -topology. A vector field $\xi \in \mathfrak{X}^1(M)$ has the *strong P.O.T.P.* if the flow ϕ_ξ on M defined by ξ has the strong P.O.T.P.

DEFINITION 17. Two flows $(X_1, \phi_1), (X_2, \phi_2)$ are isomorphic to each other if there is a homeomorphism $h : X_1 \rightarrow X_2$ such that

$$h\phi_1^t(x) = \phi_2^t(h(x)) \quad \text{for every } x \in X_1 \text{ and } t \in \mathbf{R}.$$

THEOREM 2. Let $\phi = \{\phi^t\}_{t \in \mathbf{R}}$ be a flow on a compact manifold M . If ϕ has a basic set $A \subset M$ such that (A, ϕ) is isomorphic to a geometric Lorenz attractor $(\tilde{K}_f, \tilde{\phi}_f)$ with $f(0) \neq 0$ or $f(1) \neq 1$, then (M, ϕ) does not have the strong P.O.T.P.

PROOF. By Theorem 1, $(\tilde{K}_f, \tilde{\phi}_f)$ with $f(0) \neq 0$ or $f(1) \neq 1$ does not have the strong P.O.T.P., so that (A, ϕ) does not have the strong P.O.T.P. Thus Theorem 2 is immediately obtained from the next Proposition 5.1.

PROPOSITION 5.1. Let $\phi = \{\phi^t\}_{t \in \mathbf{R}}$ be a flow on a compact metric space (X, d) and A a basic set of ϕ . If (X, ϕ) has the strong P.O.T.P., then so has (A, ϕ) .

To prove Proposition 5.1, we prepare two lemmas.

LEMMA 5.2. Let $\epsilon > 0$ and a sequence $\{g_n\}_{n \in \mathbf{N}}$ of elements of $\text{Rep}(\epsilon)$ be given. Then there exists an element g of $\text{Rep}(\epsilon)$ such that for every $N > 0$ there is a subsequence $\{m\} \subset \mathbf{N}$ such that

$$g_m|[-N, N] \rightarrow g|[-N, N] \quad \text{uniformly as } m \rightarrow \infty.$$

PROOF. The definition of $\text{Rep}(\epsilon)$ implies that $\{g_n|[-N, N]\}_{n \in \mathbf{N}}$ is uniformly

bounded and equi-continuous for every $N > 0$. Using the Ascoli-Arzelà's theorem, we obtain Lemma 5.2.

LEMMA 5.3. *The following are equivalent.*

- (1) (X, ϕ) has the strong P.O.T.P.
- (2) For every $\varepsilon > 0$ there exist $\delta, T > 0$ such that, for every finite (δ, T) -chain $\{x_i; t_i\}_0^k$ ($0 \leq k < \infty$) of ϕ , there are $x \in X$ and $g \in \text{Rep}(\varepsilon)$ with

$$d(x_0 * t, x \cdot g(t)) \leq \varepsilon \quad \text{for every } t \in [0, S_{k+1}].$$

PROOF. Clearly (1) implies (2). We prove that (2) implies (1). Given $\varepsilon > 0$, let $\delta, T > 0$ be as in (2). Let $\Gamma = \{x_i; t_i\}_{i \in \mathbf{Z}}$ be a (δ, T) -chain of ϕ . For each $n \in \mathbf{N}$ there are $y_n \in X$ and $g_n \in \text{Rep}(\varepsilon)$ such that

$$d(x_0 * t, y_n \cdot g_n(t)) \leq \varepsilon \quad \text{for every } t \in [S_{-n}, S_n].$$

Let $\{y_n\}$ converge to a point $y \in X$. By Lemma 5.2, a reparametrization $g \in \text{Rep}(\varepsilon)$ is determined from $\{g_n\}$. Then we can easily verify that Γ is strongly ε -traced by y and g .

PROOF OF PROPOSITION 5.1. Since A is closed and open in Ω , the distance between A and $\Omega - A$ is positive; i.e.

$$\delta_0 = d(A, \Omega - A) > 0.$$

Let $\delta_0/2 > \varepsilon > 0$ be given. There are $\delta, T > 0$ as in Lemma 5.3 (2). Without loss of generality we may assume that $\delta < \delta_0$. Let a finite (δ, T) -chain $\Gamma = \{x_i; t_i\}_0^k$ of (A, ϕ) be given. By topological transitivity of (A, ϕ) , there are $x_{k+1} \in A$ and $t_{k+1} \geq T$ such that

$$d(x_k \cdot t_k, x_{k+1}) \leq \delta \quad \text{and} \quad d(x_{k+1} \cdot t_{k+1}, x_0) \leq \delta.$$

Define

$$z_i = x_{i-n(k+1)} \quad \text{and} \quad s_i = t_{i-n(k+1)}$$

for every $n(k+1) \leq i < (n+1)(k+1)$, $n \in \mathbf{Z}$. Then $\{z_i; s_i\}_{i \in \mathbf{Z}}$ is a (δ, T) -chain of (A, ϕ) . Thus there are $z \in X$ and $h \in \text{Rep}(\varepsilon)$ such that $d(z_0 * t, z \cdot h(t)) \leq \varepsilon$ for every $t \in \mathbf{R}$. Put $p = \sum_0^{k+1} t_i$ and suppose a sequence $\{z \cdot h(np)\}_{n \geq 0}$ of points in X .

If $\{z \cdot h(np)\}_{n \geq 0}$ is finite, then z is a periodic point, so that $z \in \Omega$. Since $d(z, x_0) \leq \varepsilon < \delta_0$, we have $z \in A$. Thus Γ is strongly ε -traced by z under h .

Suppose $\{z \cdot h(np)\}_{n \geq 0}$ is infinite, then a subsequence converges to some point $y \in X$. It is easy to see that y belongs to A . Define $g_n \in \text{Rep}(\varepsilon)$ by

$$g_n(t) = h(np+t) - h(np) \quad \text{for } t \in \mathbf{R}.$$

By Lemma 5.2 there are $g \in \text{Rep}(\varepsilon)$ and a subsequence $\{m\} \subset \mathbf{N}$ such that

$$z \cdot h(mp) \rightarrow y \quad \text{and}$$

$$g_m|_{[0, p]} \rightarrow g|_{[0, p]} \quad \text{uniformly as } m \rightarrow \infty.$$

Then, for every $t \in [0, \sum_0^k t_i] \subset [0, p]$, we have

$$\begin{aligned} d(x_0 * t, y \cdot g(t)) &= \lim_{m \rightarrow \infty} d(x_0 * (mp+t), (z \cdot h(mp)) \cdot g_m(t)) \\ &= \lim_{m \rightarrow \infty} d(x_0 * (mp+t), z \cdot h(mp+t)) \leq \varepsilon. \end{aligned}$$

Thus Γ is strongly ε -traced by y under g .

In any case Γ is strongly ε -traced by some point in Λ . By Lemma 5.3, (Λ, ϕ) has the strong P.O.T.P. This completes the proof of Proposition 5.1.

THEOREM ([5], [15]). *There is an open set U in the space of all C^2 -vector fields on \mathbf{R}^3 such that each $\xi \in U$ has a basic set isomorphic to a geometric Lorenz attractor $(\tilde{K}_f, \tilde{\varphi}_f)$ with $f(0) \neq 0$ or $f(1) \neq 1$.*

Let M be a compact 3-manifold and suppose $\mathbf{R}^3 \subset M$. Then the above theorem implies that

there is an open set $U \subset \mathfrak{X}^2(M)$ such that each $\xi \in U$ has a basic set isomorphic to a geometric Lorenz attractor $(\tilde{K}_f, \tilde{\varphi}_f)$ with $f(0) \neq 0$ or $f(1) \neq 1$.

Combining this fact and Theorem 2, we conclude the following.

THEOREM 3. *Let M be a compact 3-manifold and $\mathfrak{X}^2(M)$ the space of all C^2 -vector fields on M endowed with the C^2 -topology. Then there exists an open set $U \subset \mathfrak{X}^2(M)$ such that each $\xi \in U$ does not have the strong P.O.T.P.*

REMARK. A. Morimoto proposed in [10] the next problem. "Is the set of diffeomorphisms with P.O.T.P. residual in the set of all diffeomorphisms of a compact manifold?" Theorem 3 implies that the answer of this problem for flow case is negative.

References

- [1] R. Bowen, Periodic orbits for hyperbolic flows, *Amer. J. Math.*, **94** (1972), 1-30.
- [2] R. Bowen and P. Walters, Expansive one-parameter flows, *J. Differential Equations*, **12** (1972), 180-193.
- [3] J. E. Franke and J. F. Selgrade, Hyperbolicity and chain recurrence, *J. Differential Equations*, **26** (1977), 27-36.
- [4] J. Guckenheimer, A strange, strange attractor, in *The Hopf Bifurcation Theorem and its Applications*, ed. by J. E. Marsden and M. McCracken, Springer-Verlag, 1976, 368-381.
- [5] J. Guckenheimer and R. F. Williams, Structural stability of Lorenz attractors, *IHES Publ. Math.*, **50** (1979), 59-72.
- [6] J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields, *Applied Mathematical Sciences*, **42**, Springer,

- Berlin - Heidelberg - New York, 1983.
- [7] K. Kato, Pseudo-orbit and stabilities of flows, Mem. Fac. Sci. Kochi Univ. (Math.), 5 (1984), 45-62.
 - [8] K. Kato and A. Morimoto, Topological Ω -stability of Axiom A flows with no Ω -explosions, J. Differential Equations, 34 (1979), 464-481.
 - [9] E. Lorenz, Deterministic nonperiodic flow, J. Atmospheric Sciences, 20 (1963), 130-141.
 - [10] A. Morimoto, The method of pseudo-orbit tracing and stability of dynamical systems, Seminar Note, 39, Dept. Math. Tokyo Univ., 1979, (in Japanese).
 - [11] C. Sparrow, The Lorenz Equations, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
 - [12] R. F. Thomas, Stability properties of one-parameter flows, Proc. London Math. Soc., 45 (1982), 479-505.
 - [13] P. Walters, On the pseudo-orbit tracing property and its relationship to stability, Lecture Notes in Math., 668 (1978), Springer, 231-244.
 - [14] R. F. Williams, Structure of Lorenz attractors, IHES Publ. Math., 50 (1979), 73-99.
 - [15] R.C. Robinson, Differentiability of the stable foliation for the model Lorenz equations, Lecture Notes in Math., 898 (1981), Springer, 302-315.

Motomasa KOMURO
Department of Mathematics
Tokyo Metropolitan University
Setagayaku, Tokyo 158
Japan