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Maximal surfaces with conelike singularities

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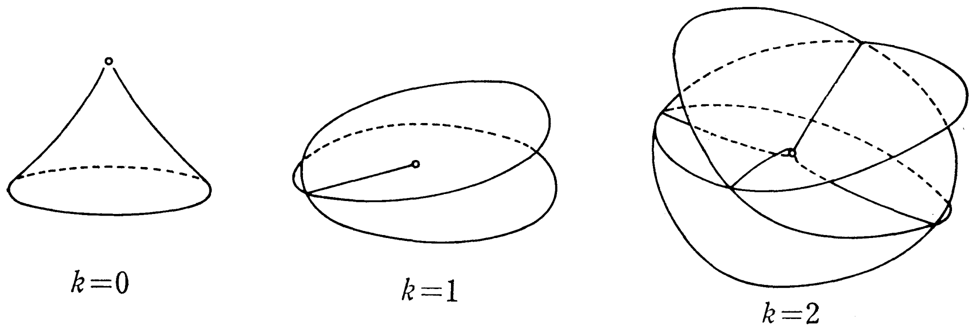
A spacelike surface in the 3-dimensional Minkowski space $L^3=(\boldsymbol{R}^3, dx^2+dy^2-dz^2)$ is said to be *maximal* if the mean curvature vanishes identically. Any spacelike surface in L^3 can be represented locally as the graph $\{z=u(x, y)\}$ of a smooth function u with $u_x^2+u_y^2<1$. Then the surface is maximal if u satisfies the equation:

$$(1-u_x^2)u_{yy}+2u_xu_yu_{xy}+(1-u_y^2)u_{xx}=0\,.$$

This equation is elliptic when $u_x^2+u_y^2<1$, but the ellipticity degenerates when $u_x^2+u_y^2$ tends to 1. Related to this fact, maximal surfaces often have singularities which are of different kinds from those appearing in minimal surfaces in the Euclidean space. For example, consider a surface S in L^3 defined by

$$\begin{cases} x(\rho, \theta)=\dfrac{1}{k+1}\sinh((k+1)\rho)\cos((k+1)\theta)+\dfrac{1}{k-1}\sinh((k-1)\rho)\cos((k-1)\theta)\,, \\ \qquad \left(=\dfrac{1}{2}\sinh 2\rho\cos 2\theta+\rho, \text{ if } k=1\right), \\ y(\rho, \theta)=\dfrac{1}{k+1}\sinh((k+1)\rho)\sin((k+1)\theta)-\dfrac{1}{k-1}\sinh((k-1)\rho)\sin((k-1)\theta)\,, \\ \qquad \left(=\dfrac{1}{2}\sinh 2\rho\sin 2\theta, \text{ if } k=1\right), \\ z(\rho, \theta)=-\dfrac{2}{k}\sinh(k\rho)\cos(k\theta)\,, \quad (=-2\rho, \text{ if } k=0)\,, \quad \rho>0\,, \quad 0\leqq\theta<2\pi\,, \end{cases}$$



$k=0$ $\qquad$ $k=1$ $\qquad$ $k=2$

Figure 1.

where k is a nonnegative integer. Then S is a maximal surface, and the origin $(x, y, z)=(0, 0, 0)$, which corresponds to the limit $\rho\to 0$, is an isolated singularity. It may be clear that this type of singularities never appear in minimal surfaces. Among such singularities of maximal surfaces, we shall consider in this paper especially those singularities that are similar to the above example for $k=0$, which will be called *conelike singularities*.

DEFINITION. Let S be a maximal surface in L^3, and $p=(x_0, y_0, z_0)$ be a point of the closure of S in L^3. Then p is called a *conelike singularity* of S if the following conditions are satisfied:

(i) In a neighbourhood of p, S is the graph of a smooth function u defined on $U\setminus(x_0, y_0)$, where U is a neighbourhood of (x_0, y_0) in the (x, y)-plane;

(ii) On $U\setminus(x_0, y_0)$, $u<z_0$ (or $u>z_0$). By setting $u(x_0, y_0)=z_0$, u is continuous on U;

(iii) $\lim_{(x,y)\to(x_0,y_0)}(u_x^2+u_y^2)=1$.

It is known that every *complete* maximal surface in L^3 is a plane, which is a great difference from minimal surfaces; there are numerous examples of complete minimal surfaces in the Euclidean 3-space. So, we want to modify the notion of completeness as follows: Let S be a maximal surface in L^3, and $\{p_k\}$ be the set of all conelike singularities of S. At each p_k, we round out S as shown below.

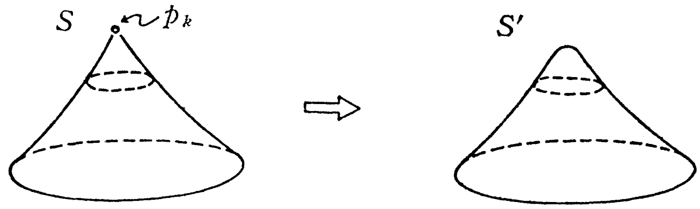


Figure 2.

Then we obtain a spacelike surface S'. The maximal surface S will be said to be *complete* if S' is complete in the usual sense (cf. the proof of Lemma 2.6).

The purpose of this paper is to show the following:

THEOREM. *Let S be a complete maximal surface in L^3 with at least one conelike singularity. Suppose that the Gauss map of S is* 1:1. *Then S is congruent to the surface defined by $\sqrt{x^2+y^2}+a\sinh(z/a)=0$, where a is a nonzero real constant.*

REMARK. Without the assumption on the Gauss map, the result is not true. An example will be given in § 1.

§1. Preliminaries.

Let H be a spacelike surface in L^3 defined by $x^2+y^2-z^2=-1$, $z<0$. Naturally, H is considered as the target space of the Gauss map for spacelike surfaces in L^3. H is conformal to the unit disk $\Delta=\{\zeta\in \boldsymbol{C} \mid |\zeta|<1\}$ in the complex plane. Indeed, the following gives a conformal isomorphism between them:

$$(1.1) \qquad \Delta \xrightarrow{\sim} H\,; \qquad \zeta \longmapsto \left(\frac{2\operatorname{Re}\zeta}{1-|\zeta|^2}, \frac{2\operatorname{Im}\zeta}{1-|\zeta|^2}, -\frac{1+|\zeta|^2}{1-|\zeta|^2}\right).$$

Hereafter, through this identification, we regard Δ as the target space of the Gauss map.

PROPOSITION 1.1 ([2]). *Let S be a maximal surface in L^3 and D $(\subset\Delta)$ the image of the Gauss map of S. Assume that the Gauss map is* 1:1. *Then there exists a holomorphic function f defined on D with no zeros in D such that S is represented as*

$$(1.2) \qquad \phi(\zeta)=\operatorname{Re}\int\left(\frac{1}{2}f(\zeta)(1+\zeta^2),\ \frac{\sqrt{-1}}{2}f(\zeta)(1-\zeta^2),\ -f(\zeta)\zeta\right)d\zeta\,, \qquad \zeta\in D\,.$$

(To simplify notation, we denote $\int^{\zeta}F(\omega)d\omega$ simply by $\int F(\zeta)d\zeta$.) Moreover the Gauss map is then the inverse of ϕ : $D\to S$.

REMARK. A surface given by (1.2) for a holomorphic function f is always a maximal surface, but the expression (1.2) sometimes represents a maximal surface whose Gauss map is not injective. For example, $f(\zeta)=1/(2\zeta^2-1)(\zeta^2-1)$ on $D=\Delta\setminus\{\pm 1/\sqrt{2}\}$ is the case (see Figure 3).

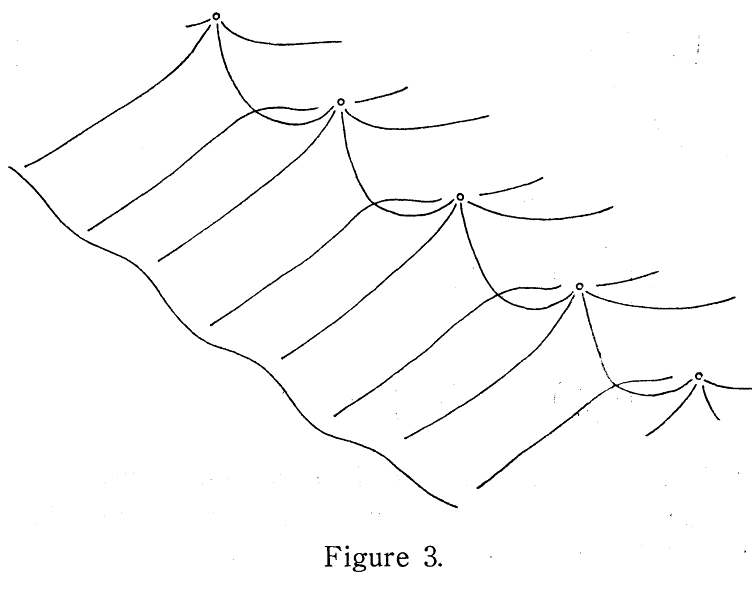


Figure 3.

Next, we consider how the formula (1.2) changes by a transformation of L^3. Let $\Phi: L^3 \to L^3$ be an isometry of L^3 with $\Phi(0)=0$. Such Φ's form a group denoted by $O(2, 1)$. $O(2, 1)$ is a Lie group with four connected components. Let $O_0(2, 1)$ be the identity component of $O(2, 1)$. Then, $\Phi \in O_0(2, 1)$ if and only if Φ is orientation preserving and $\Phi(H)=H$. Thus, using the identification (1.1), $\Phi \in O_0(2, 1)$ induces a linear fractional transformation Φ_Δ of Δ:

$$\Phi_\Delta(\zeta)=\varepsilon\frac{\zeta-\omega}{1-\bar{\omega}\zeta} \tag{1.3}$$

for some ε and ω with $|\varepsilon|=1$ and $|\omega|<1$. Conversely, given a linear transformation Φ_Δ defined by (1.3), we get the isometry Φ of L^3 by

$$\Phi=\begin{pmatrix}\cos(\alpha+\theta) & -\sin(\alpha+\theta) & 0\\ \sin(\alpha+\theta) & \cos(\alpha+\theta) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cosh\rho & 0 & \sinh\rho\\ 0 & 1 & 0\\ \sinh\rho & 0 & \cosh\rho\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{1.4}$$

where $\alpha=\arg\varepsilon$, $\theta=\arg\omega$ and $\rho=\log(1-|\omega|)-\log(1+|\omega|)$.

PROPOSITION 1.2. *Let S be a maximal surface given by* (1.2), *and $\Phi \in O_0(2, 1)$. Then, the maximal surface $\Phi(S)$ is represented as*

$$\tilde{\varphi}(\tilde{\zeta})=\mathrm{Re}\int\Big(\frac{1}{2}\tilde{f}(\tilde{\zeta})(1+\tilde{\zeta}^2),\ \frac{\sqrt{-1}}{2}\tilde{f}(\tilde{\zeta})(1-\tilde{\zeta}^2),\ -\tilde{f}(\tilde{\zeta})\tilde{\zeta}\Big)d\tilde{\zeta}, \qquad \tilde{\zeta}\in\tilde{D}, \tag{1.5}$$

where $\tilde{D}=\Phi_\Delta(D)$, and

$$\tilde{f}(\tilde{\zeta})=\frac{\varepsilon^2(1-|\omega|^2)^2}{(\varepsilon+\bar{\omega}\tilde{\zeta})^4}f\Big(\frac{\varepsilon\omega+\tilde{\zeta}}{\varepsilon+\bar{\omega}\tilde{\zeta}}\Big). \tag{1.6}$$

PROOF. Define a surface $\tilde{S}$ in L^3 by

$$\tilde{\varphi}(\zeta)=\mathrm{Re}\int\Big(\frac{1}{2}\frac{d\zeta}{d\tilde{\zeta}}f(\zeta)(1+\tilde{\zeta}^2),\ \frac{\sqrt{-1}}{2}\frac{d\zeta}{d\tilde{\zeta}}f(\zeta)(1-\tilde{\zeta}^2),\ -\frac{d\zeta}{d\tilde{\zeta}}f(\zeta)\tilde{\zeta}\Big)d\zeta, \quad \zeta\in D, \tag{1.7}$$

where $\tilde{\zeta}=\Phi_\Delta(\zeta)=\varepsilon\frac{\zeta-\omega}{1-\bar{\omega}\zeta}$. Then, if $\tilde{g}$ and $\tilde{h}$ denote the first and second fundamental forms of $\tilde{S}$ respectively, a direct calculation yields that

$$\tilde{g}=\frac{1}{4}\Big|\frac{d\zeta}{d\tilde{\zeta}}f\Big|^2|1-|\tilde{\zeta}|^2|^2|d\zeta|^2=\frac{1}{4}|f|^2|1-|\zeta|^2|^2=g$$

and

$$\tilde{h}=\mathrm{Re}\Big(\Big(\frac{d\zeta}{d\tilde{\zeta}}\Big)\cdot\Big(f\frac{d\tilde{\zeta}}{d\zeta}\Big)d\zeta^2\Big)=\mathrm{Re}(f\,d\zeta^2)=h,$$

where g and h are the first and second fundamental forms of S respectively. Hence, by the fundamental theorem of surfaces, S and $\tilde{S}$ are congruent. Using the variable $\tilde{\zeta}$, (1.7) is written as

$$(1.8)\qquad \tilde{\phi}(\tilde{\zeta})=\mathrm{Re}\int\left(\frac{1}{2}\left(\frac{d\zeta}{d\tilde{\zeta}}\right)^2 f(\zeta)(1+\tilde{\zeta}^2),\ \frac{\sqrt{-1}}{2}\left(\frac{d\zeta}{d\tilde{\zeta}}\right)^2 f(\zeta)(1-\tilde{\zeta}^2),\ -\left(\frac{d\zeta}{d\tilde{\zeta}}\right)^2 f(\zeta)\tilde{\zeta}\right)d\tilde{\zeta}\,,\quad \tilde{\zeta}\in\tilde{D}\,.$$

From this expression, we can regard $\tilde{\zeta}$ as the Gauss map of $\tilde{S}$. Since $\tilde{\zeta}=\Phi_{\Delta}(\zeta)$, we see easily that $\tilde{S}=\Phi(S)$. Now the assertion follows immediately. □

§ 2. Proof of Theorem.

Let S be a maximal surface satisfying the assumption of Theorem, and $p=(x_0, y_0, z_0)$ a conelike singularity of S. By the definition of the conelike singularity, we have a continuous function $u(x, y)$ defined on a neighbourhood U of (x_0, y_0) such that

(a) around p, S is the graph of $u|U\setminus(x_0, y_0)$;

(b) $u(x_0, y_0)=z_0$ and $u(x, y)<z_0$ for $(x, y)\in U\setminus(x_0, y_0)$;

(c) $\lim_{(x,y)\to(x_0,y_0)}(u_x^2+u_y^2)=1$.

From (a), the normal vector of S is given by

$$-\frac{1}{\sqrt{1-u_x^2-u_y^2}}(u_x, u_y, 1)\,.$$

Hence by (1, 1), $\zeta: S\to\Delta$ denoting the Gauss map, we get

$$(2.1)\qquad |\zeta|^2=\frac{1-\sqrt{1-u_x^2-u_y^2}}{1+\sqrt{1-u_x^2-u_y^2}}\,.$$

LEMMA 2.1. *The image D $(\subset\Delta)$ of the Gauss map of S contains $\{\zeta\in\Delta\,|\,|\zeta|>\delta\}$ for some $\delta<1$.*

PROOF. Take a small $r_0>0$ such that $\{(x-x_0)^2+(y-y_0)^2\leqq r_0^2\}\subset U$. For $r\leqq r_0$, set $C_r=\{(x, y, u(x, y))\,|\,(x-x_0)^2+(y-y_0)^2=r^2\}$ and $A_r=\{(x, y, u(x, y))\,|\,0<(x-x_0)^2+(y-y_0)^2<r^2\}$. Both C_r and A_r are contained in S. Let $\tilde{C}_r$ $(\subset\Delta)$ be the Gaussian image of C_r. Since the Gauss map of S is $1:1$, $\tilde{C}_r$ is a simple closed curve in Δ. $\tilde{A}_{r_0}$ denotes the outer part of $\tilde{C}_{r_0}$ in Δ. By (c) and (2.1), it is obvious that the Gaussian image of A_{r_0} is contained in $\tilde{A}_{r_0}$. Suppose that there is a point $\zeta_0\in\tilde{A}_{r_0}$ which is not in the Gaussian image of A_{r_0}. Then, ζ_0 lies in the outer part of $\tilde{C}_{r_0}$, and hence ζ_0 must lie in the outer part of $\tilde{C}_r$ for any r with $0<r\leqq r_0$. But, from (c) and (2.1), we have $\tilde{C}_\varepsilon\subset\{|\zeta|>|\zeta_0|\}$ for sufficiently small $\varepsilon>0$. This is a contradiction, and thus $\tilde{A}_{r_0}$ coincides with the Gaussian image of A_{r_0}, which shows the assertion. □

By Proposition 1.1, there is a holomorphic function f defined on D which has no zeros in D, and S is given by (1.2) using f. By the above argument, we see

$$(2.2)\qquad \lim_{|\zeta|\to1}\phi(\zeta)=p\,.$$

So, we need to observe the behavior of f near $\partial\Delta=\{|\zeta|=1\}$.

LEMMA 2.2. *f has a holomorphic extension to a neighbourhood of $\partial\Delta$ in $\boldsymbol{C}$, and then $f(\zeta)\zeta^2$ is real on $\partial\Delta$. Moreover, f has no zeros on $\partial\Delta$.*

PROOF. From (1.2) and (2.2), we have

$$\lim_{|\zeta|\to 1}\left(\mathrm{Re}\int f(\zeta)\zeta d\zeta\right)=-z_0\,. \tag{2.3}$$

Hence, by the reflection principle, $\mathrm{Re}\int f(\zeta)\zeta d\zeta$ has a harmonic extension to a neighbourhood of $\partial\Delta$. Therefore, f also has a holomorphic extension. Put $\zeta=e^{\rho+i\theta}$, where ρ and θ are real parameters, and we have

$$\begin{aligned}\mathrm{Re}\int f(\zeta)\zeta d\zeta&=\mathrm{Re}\int f(\zeta)\zeta^2 d(\log\zeta)\\&=\int(\mathrm{Re} f(\zeta)\zeta^2)d\rho-\int(\mathrm{Im} f(\zeta)\zeta^2)d\theta\,.\end{aligned}$$

This, together with (2.3), shows that $\mathrm{Im} f(\zeta)\zeta^2=0$ for $\zeta\in\partial\Delta$. That is, $f(\zeta)\zeta^2$ is real on $\partial\Delta$. By the property (b) of conelike singularity, $\mathrm{Re}\left(\int f(\zeta)\zeta d\zeta\right)>-z_0$ for any $\zeta\in D$ with $|\zeta|\doteqdot 1$. Hence, $f(\zeta)\zeta$, the derivative of $\int f(\zeta)\zeta d\zeta$, cannot vanish at each $\zeta\in\partial\Delta$. Therefore, f has no zeros on $\partial\Delta$. □

By the above lemma, it is easy to see that in a neighbourhood of $\partial\Delta$, $\zeta^2 f(\zeta)$ can be expanded as

$$\zeta^2 f(\zeta)=\cdots+a\zeta^{-1}+b+\bar{a}\zeta+\cdots, \tag{2.4}$$

where b is a nonzero real.

LEMMA 2.3. *In* (2.4), $b^2>|a|^2$.

PROOF. First, recall the formulas:

$$a=\frac{1}{2\pi i}\int_{|\zeta|=1}\zeta^2 f(\zeta)d\zeta \quad\text{and}\quad b=\frac{1}{2\pi i}\int_{|\zeta|=1}\zeta f(\zeta)d\zeta\,.$$

By Lemma 2.2, $\zeta^2 f(\zeta)$ is real on $\partial\Delta$, and may be assumed to be positive on $\partial\Delta$. Hence, putting $F(\theta)=(e^{2i\theta}f(e^{i\theta})/2\pi)^{1/2}$, we have

$$a=\int_0^{2\pi}e^{i\theta}F(\theta)^2 d\theta \quad\text{and}\quad b=\int_0^{2\pi}F(\theta)^2 d\theta\,.$$

Therefore, by the Schwarz inequality,

$$\begin{aligned}|a|^2&=\left|\int_0^{2\pi}(e^{i\theta}F(\theta))F(\theta)d\theta\right|^2\leqq\int_0^{2\pi}|e^{i\theta}F(\theta)|^2 d\theta\cdot\int_0^{2\pi}F(\theta)^2 d\theta\\&=\left(\int_0^{2\pi}F(\theta)^2 d\theta\right)^2=b^2\,.\end{aligned}$$

Since $F(\theta)$ is real valued and positive, the strict inequality holds. $\square$

Next, we make use of an isometry of L^3 to simplify the function f. Remark that Lemmas 2.1, 2.2 and hence 2.3 are valid for $\tilde{D}$, $\tilde{f}$ in Proposition 1.2.

LEMMA 2.4. *There is a* $\Phi\in O_0(2, 1)$ *such that* $\int_{|\tilde{\zeta}|=1}\tilde{f}(\zeta)d\tilde{\zeta}=0$, *where* $\tilde{f}$ *is the function defined by* (1.6).

PROOF. If $a=0$ in (2.4), then obviously $\int_{|\zeta|=1}fd\zeta=0$, hence $\Phi=$identity is the desired isometry. So we assume $a\neq 0$. Consider a quadratic equation $\lambda^2-2b\lambda+|a|^2=0$. By Lemma 2.3, the discriminant $b^2-|a|^2$ is positive. Hence, we have two distinct real roots λ_+ and λ_-. Since $\lambda_+\cdot\lambda_-=|a|^2$ and $\lambda_+\neq\lambda_-$, λ_+ or λ_-, say λ_-, satisfies the inequality $|\lambda_-|<|a|$. We put $\omega=\lambda_-/\bar{a}$. Namely, ω satisfies

$$a\bar{\omega}^2-2b\bar{\omega}+\bar{a}=0 \tag{2.5}$$

and

$$|\omega|<1 . \tag{2.6}$$

It follows from (2.4) that the left hand side of (2.5) is exactly the coefficients of ζ^{-1} in the Laurent expansion of $(1-\bar{\omega}\zeta)^2f(\zeta)$. Therefore

$$\int_{|\zeta|=1}(1-\bar{\omega}\zeta)^2f(\zeta)d\zeta=0 . \tag{2.7}$$

Now, let $\Phi_\Delta(\zeta)=(\zeta-\omega)/(1-\bar{\omega}\zeta)$, and Φ be as defined by (1.4). Then, from (1.6), we have

$$\tilde{f}(\zeta)=\left(\frac{d\zeta}{d\tilde{\zeta}}\right)^2f(\zeta)=\frac{d\zeta}{d\tilde{\zeta}}\frac{(1-\bar{\omega}\zeta)^2}{1-|\omega|^2}f(\zeta) , \tag{2.8}$$

where $\tilde{\zeta}=\Phi_\Delta(\zeta)$. Hence by (2.7) and (2.8), we get

$$\begin{aligned}\int_{|\tilde{\zeta}|=1}\tilde{f}(\zeta)d\tilde{\zeta}&=\int_{|\zeta|=1}\frac{(1-\bar{\omega}\zeta)^2}{1-|\omega|^2}f(\zeta)d\zeta\\&=\frac{1}{1-|\omega|^2}\int_{|\zeta|=1}(1-\bar{\omega}\zeta)^2f(\zeta)d\zeta=0 .\end{aligned}$$ $\square$

By the above lemma and Proposition 1.2, we may assume without loss of generality that

$$\int_{|\zeta|=1}f(\zeta)d\zeta=0 . \tag{2.9}$$

From (2.2), S has only one conelike singularity. Hence, by the completeness of S, it is easy to see that S must be represented globally as the graph of a function $u(x, y)$ defined on $\boldsymbol{R}^2\setminus(x_0, y_0)$. That is, S is homeomorphic to $S^1\times\boldsymbol{R}$, hence, so is D. Thus, (2.9) implies that the following is a well-defined holomorphic function on $D\cup\partial\Delta$:

$$F : D\cup\partial\Delta \longrightarrow \boldsymbol{C} \ ; \qquad \zeta \longmapsto \int_1^{\zeta} f(\omega)d\omega . \tag{2.10}$$

LEMMA 2.5. *The image γ of $\partial\Delta$ by F is a regular simple closed curve in $\boldsymbol{C}$.*

PROOF. Put $\gamma(\theta)=F(e^{i\theta})$, and we have $\frac{d}{d\theta}\gamma(\theta)=if(e^{i\theta})e^{i\theta}$. By Lemma 2.2, $f(e^{i\theta})e^{2i\theta}$ is real, hence, the Frenet frame of γ is given by $e_1(\theta)=ie^{-i\theta}$, $e_2(\theta)=-e^{-i\theta}$. Then the curvature κ of γ is easily computed and $\kappa(\theta)=-1/|f(e^{i\theta})|<0$. On the other hand, the rotation index of γ is $(1/2\pi)\int_0^{2\pi}\kappa(\theta)\left|\frac{d}{d\theta}\gamma(\theta)\right|d\theta=-1$. Therefore, γ is a regular simple closed curve. □

LEMMA 2.6. *Let $c : [0, 1]\rightarrow \boldsymbol{C}$ be a smooth curve such that $c(0)\in\gamma$ and $c(t)$ lies in the outer region of γ in $\boldsymbol{C}$ for $t>0$. Then, along c, the inverse function of F can be defined.*

PROOF. From the proof of Lemma 2.5, F is invertible in a neighbourhood of $\partial\Delta$, and an inward normal vector of $\partial\Delta$ is carried to an outward normal vector of γ by dF. Hence, for a sufficiently small $t>0$, F^{-1} can be defined along $c|[0, t)$. If F^{-1} is defined on $c|[0, t_0]$, then $\zeta_0=F^{-1}(c(t_0))\in D$ and $f(\zeta_0)\neq 0$. Therefore, by the inverse function theorem, F^{-1} can be extended to $c|[0, t_0+\varepsilon)$ for a sufficiently small $\varepsilon>0$. Hence, $T=\{t\in[0, 1] \mid F^{-1}$ can be defined along $c|[0, t]\}$ is a nonempty open set in $[0, 1]$.

Next, suppose that F^{-1} is defined on $c|[0, t_1)$. From (1.2), the first fundamental form of S is $\frac{1}{4}|f|^2|1-|\zeta|^2|^2|d\zeta|^2$. That is, S is isometric to $\left(D, \frac{1}{4}|f|^2 |1-|\zeta|^2|^2|d\zeta|^2\right)$. Let d be the distance function defined by the Riemannian metric. Then, for $s_1, s_2\in[0, t_1)$,

$$\begin{aligned} d(\zeta_1, \zeta_2) &\leqq \int_{\zeta_1}^{\zeta_2}\frac{1}{2}|f|\,|1-|\zeta|^2|\,|d\zeta| \\ &\leqq \frac{1}{2}\int_{\zeta_1}^{\zeta_2}|fd\zeta| = \frac{1}{2}L(c|[s_1, s_2]) , \end{aligned}$$

where $\zeta_i=F^{-1}(c(s_i))$, $i=1, 2$, and $L(c|[s_1, s_2])$ is the Euclidean length of $c|[s_1, s_2]$ in $\boldsymbol{C}$. Hence, for any sequence $\{s_n\}\subset[0, t_1)$ with $\lim s_n=t_1$, $\{F^{-1}(c(s_n))\}$ is a Cauchy sequence of (D, d). On the other hand, there is a $\delta>0$ such that $|F^{-1}(c(s))|<1-\delta$ for any s sufficiently close to t_1 because $c(t_1)\notin\gamma$ and $F(\partial\Delta)=\gamma$. That is, $F^{-1}(c(s_n))$ keeps away from the conelike singularity for large n. Hence, by the completeness of S, $\{F^{-1}(c(s_n))\}$ converges to some ζ_1 in D, which shows that F^{-1} can be extended to $c|[0, t_1]$. Thus T is closed, hence $T=[0, 1]$. □

LEMMA 2.7. *There is a $\zeta_0\in\Delta$ such that $D=\Delta\setminus\{\zeta_0\}$, and f is meromorphic* *on $\varDelta$.*

PROOF. Let D_γ be the outer part of γ in $\boldsymbol{C}$. From Lemmas 2.5 and 2.6, F^{-1} is a well-defined holomorphic function on D_γ. Since $F^{-1}(D_\gamma)\subset D\subset\varDelta$, ∞ is a removable singularity of F^{-1}. Set $\zeta_0=F^{-1}(\infty)\in\varDelta$. Then it is easy to see that $D_\gamma\cup\{\infty\}$ and $\varDelta$ are biholomorphic. Hence, F has a meromorphic extension to $\varDelta$ with a pole at ζ_0. Consequently, $D=\varDelta\setminus\{\zeta_0\}$, and f is meromorphic on $\varDelta$. $\square$

It follows from Lemmas 2.2, 2.7 and the argument principle that f has a pole of order 2 at ζ_0. Moreover by the reflection principle, f has a meromorphic extension to $\boldsymbol{C}\cup\{\infty\}$ and satisfies $f(\zeta)=\zeta^{-4}\overline{f(1/\bar{\zeta})}$. Hence, f has a zero of order 4 at ∞ and poles of order 2 at ζ_0 and $1/\bar{\zeta}_0$. Therefore, f must be of the form;

(2.11) $f(\zeta)=a/(1-\zeta/\zeta_0)^2(\bar{\zeta}_0\zeta-1)^2$ for some nonzero constant a, if $\zeta_0\neq 0$; or

(2.12) $f(\zeta)=a\zeta^{-2}$ for some nonzero real constant a, if $\zeta_0=0$.

However, in the case of (2.11), a direct calculation shows $\int_{|\zeta|=1}f(\zeta)d\zeta\neq 0$, which contradicts (2.9). Hence, $\zeta_0=0$, and $f(\zeta)=a\zeta^{-2}$, $a\in\boldsymbol{R}\setminus\{0\}$. Then by Proposition 1.1, it is immediately seen that our maximal surface S is just the maximal surface defined by $\sqrt{x^2+y^2}+a\sinh(z/a)=0$, which completes the proof of Theorem.

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