J. Math. Soc. Japan
Vol. 36, No. 3, 1984

Singular hyperbolic systems, V. Asymptotic expansions for Fuchsian hyperbolic partial differential equations

By Hidetoshi TAHARA

(Received July 28, 1983)

In this paper, we study the asymptotic behavior of solutions of Fuchsian hyperbolic partial differential equations (in Tahara [**9-III**]), and determine complete asymptotic expansions of solutions in $C^{\infty}((0, T)\times \boldsymbol{R}^{n})$ as $t\rightarrow+0$. Our result corresponds to the well-known result in the theory of ordinary differential equations with regular singularities.

Let $(t, x)\in[0, T)\times \boldsymbol{R}^{n}$ $(T>0)$ and let

$$P(t, x, \partial_{t}, \partial_{x})=t^{m}\partial_{t}^{m}+P_{1}(t, x, \partial_{x})t^{m-1}\partial_{t}^{m-1}+\cdots+P_{m}(t, x, \partial_{x})$$

be a linear partial differential operator of order m $(\geqq 1)$ with C^{∞} coefficients on $[0, T)\times \boldsymbol{R}^{n}$. Assume that P satisfies the following conditions:

(i) *order* $P_{j}(t, x, \partial_{x})\leqq j$ $(1\leqq j\leqq m)$,

(ii) *order* $P_{j}(0, x, \partial_{x})\leqq 0$ $(1\leqq j\leqq m)$.

Then, P is said to be a *Fuchsian type operator* with respect to t. Further, if P satisfies some hyperbolicity conditions, P is said to be a *Fuchsian hyperbolic operator* with respect to t. By (ii), $P_{j}(0, x, \partial_{x})$ $(1\leqq j\leqq m)$ are functions in x. We set $P_{j}(0, x, \partial_{x})=a_{j}(x)$ $(1\leqq j\leqq m)$. Then, the indicial polynomial $\mathcal{C}(\lambda, x)$ associated with P is defined by

$$\mathcal{C}(\lambda, x)=\lambda(\lambda-1)\cdots(\lambda-m+1)+a_{1}(x)\lambda(\lambda-1)\cdots(\lambda-m+2)+\cdots+a_{m}(x)$$

and the characteristic exponents $\rho_{1}(x), \cdots, \rho_{m}(x)$ of P are defined by the roots of the indicial equation $\mathcal{C}(\lambda, x)=0$ in λ.

In [**9-III**], we have solved the Cauchy problem in $C^{\infty}([0, T)\times \boldsymbol{R}^{n})$ for Fuchsian hyperbolic operators P under various assumptions of hyperbolicity. But, here, we want to consider the equation

$$\text{(S)} \qquad P(t, x, \partial_{t}, \partial_{x})u(t, x)=0$$

in $C^{\infty}((0, T)\times \boldsymbol{R}^{n})$ (not in $C^{\infty}([0, T)\times \boldsymbol{R}^{n})$) under the same assumptions as in

This research was partially supported by Grant-in-Aid for Scientific Research (No. 57740102), Ministry of Education.

[9-III]. Note that the difference between the study in $C^\infty([0, T)\times \boldsymbol{R}^n)$ and the study in $C^\infty((0, T)\times \boldsymbol{R}^n)$ lies in whether we admit solutions with some singularities at $t=0$ into our consideration or not. Especially, we are much interested in the following questions. *What kind of singularities appears at $t=0$ for solutions of* (S) *in $C^\infty((0, T)\times \boldsymbol{R}^n)$? By what quantities are the singularities characterized at $t=0$?* These are our first motivation of the study in this paper.

The main result of this paper is summarized as follows. Let $P(t, x, \partial_t, \partial_x)$ be a Fuchsian hyperbolic operator with C^∞ coefficients and with suitable hyperbolicity (that is, with (S-1)$\sim$(S-5) in § 1), and let $\rho_1(x), \cdots, \rho_m(x)$ be the characteristic exponents of P. Then, we have obtained the following result in this paper, which answers the above questions.

MAIN THEOREM. *Assume that $\rho_i(x)-\rho_j(x)\notin \boldsymbol{Z}$ holds for any $x\in \boldsymbol{R}^n$ and $1\leqq i\neq j\leqq m$. Then, we have the following results.*

(1) *Any solution $u(t, x)\in C^\infty((0, T)\times \boldsymbol{R}^n)$ of* (S) *can be expanded asymptotically into the form*

$$u(t, x)\sim \sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big) \qquad (*)$$

on $\boldsymbol{R}^n$ (as $t\to+0$) for some $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$. Further, such coefficients $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)$ are uniquely determined by $u(t, x)$.

(2) *Conversely, for any $\varphi_1(x), \cdots, \varphi_m(x)\in C^\infty(\boldsymbol{R}^n)$ we can find a solution $u(t, x)\in C^\infty((0, T)\times \boldsymbol{R}^n)$ of* (S) *and coefficients $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$ ($1\leqq i\leqq m$, $1\leqq k<\infty$ and $0\leqq h\leqq mk$) such that the asymptotic relation $(*)$ in* (1) *holds. Further, such a solution $u(t, x)$ and coefficients $\varphi_{k,h}^{(i)}(x)$ are uniquely determined by $\varphi_1(x), \cdots, \varphi_m(x)$.*

Here, the meaning of the asymptotic relation $(*)$ in (1) is as follows. Denote by $R_N(t, x)$ the N-th remainder term, that is,

$$R_N(t, x)=u(t, x)-\sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{N}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big).$$

Then, the asymptotic relation $(*)$ above is defined by the following: for any $s>0$ and any compact subset K of $\boldsymbol{R}^n$, there is an $N_0\in \boldsymbol{N}$ such that for any $N\geqq N_0$

$$\sup_{x\in K}|\partial_t^l\partial_x^\alpha R_N(t, x)|=o(t^{s-l})$$

(as $t\to+0$) holds for any l and α.

In the case of ordinary differential equations, the above result is well-known. For example, see Wasow [12]. In the case of analytic category, analogous characterizations are obtained in Froim [3], Tahara [7, 8], Chi Min-You [2] for general Fuchsian type partial differential equations. Note that the asymptotic expansion of the above form can be easily obtained from the development of the fundamental solutions constructed in Tahara [7] into formal series. See also Repin [5], Tersenov [11], Weinstein [13].

The basic idea of our discussion is as follows. By (ii), $P(t, x, \partial_t, \partial_x)$ is decomposed into the form

$$P(t, x, \partial_t, \partial_x)=\mathcal{C}(t\partial_t, x)-tR(t, x, t\partial_t, \partial_x)$$

for some linear differential operator $R(t, x, \partial_t, \partial_x)$ of order m with C^∞ coefficients. Therefore, we can construct formal asymptotic solutions of (S) by solving the ordinary differential equations $\mathcal{C}(t\partial_t, x)u_p=tR(t, x, t\partial_t, \partial_x)u_{p-1}$ ($p=0, 1, 2, \cdots$ and $u_{-1}\equiv 0$) inductively on p (see Proposition 7). Combining this with the C^∞ well posedness of the flat Cauchy problem in [**9-III**], we obtain (2). The proof of (1) is somewhat more complicated, but the essential idea is the same (see Proposition 6).

REMARK 1. When the hyperbolicity assumption is not satisfied, the situation becomes quite different from that in the above theorem. Let $P=t\partial_t+\sqrt{-1}t\partial_x-\rho$. Then, the general solution $u(t, x)\in C^\infty((0, T)\times \boldsymbol{R})$ of $Pu=0$ is given by $u(t, x)=t^\rho f(t+\sqrt{-1}x)$, where $f(z)$ is an arbitrary holomorphic function on $\{z\in \boldsymbol{C}; 0<\operatorname{Re} z<T\}$. In this case, it is impossible to characterize the singularities at $t=0$ of $u(t, x)$ by the notion of asymptotic expansions or asymptotic behaviors (as $t\to+0$).

REMARK 2. When $\rho_i(x)-\rho_j(x)\notin \boldsymbol{Z}$ ($x\in \boldsymbol{R}^n$ and $1\leqq i\neq j\leqq m$) does not hold, the asymptotic expansion of $u(t, x)$ (as $t\to+0$) will be much more complicated. For example, in the case $P=(t\partial_t-t\partial_x-t-x)t\partial_t$, the asymptotic expansion of solutions $u(t, x)\in C^\infty((0, T)\times \boldsymbol{R})$ of $Pu=0$ has the following form:

$$u(t, x)\sim\varphi(x)+\psi(x)Y_1(t, x)+(\partial_x\psi)(x)Y_2(t, x)+o(t^{x+1})$$

(as $t\to+0$), where $\varphi(x)$, $\psi(x)\in C^\infty(\boldsymbol{R})$ are arbitrary and $Y_1(t, x)$, $Y_2(t, x)\in C^\infty((0, T)\times \boldsymbol{R})$ are functions defined by

$$Y_1(t, x)=\begin{cases} \dfrac{t^x-1}{x}-\dfrac{1}{x+1}\Big(\dfrac{t^{x+1}-1}{x+1}-t^{x+1}\log t\Big), & \text{when } x\neq 0, -1, \\ 1-t+\log t+t\log t, & \text{when } x=0, \\ 1-\dfrac{1}{t}+\dfrac{1}{2}(\log t)^2, & \text{when } x=-1, \end{cases}$$

$$Y_2(t, x)=\begin{cases} \dfrac{t^{x+1}-1}{x+1}, & \text{when } x\neq -1, \\ \log t, & \text{when } x=-1. \end{cases}$$

The paper is organized as follows. In §1 we state our assumptions of hyperbolicity. In §2 we define our asymptotic expansions and give some elementary properties. From §3 to §5, we prove Main Theorem above: the part (1) is proved in §4 and the part (2) is proved in §5. §3 is a preparation for the discussion in § 4. In § 6 we give some typical examples which illustrate our assumptions in § 1. In § 7 we give some remarks and generalize our results to some extent.

§ 1. Assumptions of hyperbolicity.

First, we state our assumptions of hyperbolicity imposed on P in this paper. Let $(t, x)\in[0, T)\times \boldsymbol{R}^n$ $(T>0)$, let

$$P(t, x, \partial_t, \partial_x)=t^m\partial_t^m+P_1(t, x, \partial_x)t^{m-1}\partial_t^{m-1}+\cdots+P_m(t, x, \partial_x)$$

be a Fuchsian type partial differential operator of order m $(\geqq 1)$, and let $a_{j,\alpha}(t, x)\in B^\infty([0, T)\times \boldsymbol{R}^n)$ $(|\alpha|\leqq j)$ be the coefficients of $P_j(t, x, \partial_x)$, that is,

$$P_j(t, x, \partial_x)=\sum_{|\alpha|\leqq j} a_{j,\alpha}(t, x)\partial_x^\alpha \tag{1.1}$$

for $1\leqq j\leqq m$, where $B^\infty([0, T)\times \boldsymbol{R}^n)$ is the space of all such functions in $C^\infty([0, T)\times \boldsymbol{R}^n)$ that every derivative is bounded on $[0, T)\times \boldsymbol{R}^n$. Note that the condition (ii) (in the introduction) implies the following: when $|\alpha|>0$, $a_{j,\alpha}(0, x)=0$ holds on $\boldsymbol{R}^n$. On hyperbolicity, we assume the following five conditions (S-1)$\sim$(S-5) for P.

(S-1) (Factorizability). There is a positive number μ such that $a_{j,\alpha}(t, x)$ (when $|\alpha|=j$) has the form

$$a_{j,\alpha}(t, x)=t^{\mu j}b_{j,\alpha}(t, x)$$

for some $b_{j,\alpha}(t, x)\in B^0([0, T)\times \boldsymbol{R}^n)$ satisfying $(t\partial_t)^l\partial_x^\beta b_{j,\alpha}(t, x)\in B^0([0, T)\times \boldsymbol{R}^n)$ for any l and β, where $B^0([0, T)\times \boldsymbol{R}^n)$ is the space of all bounded continuous functions on $[0, T)\times \boldsymbol{R}^n$.

(S-2) (Hyperbolicity). All the roots $\lambda_i(t, x, \xi)$ $(1\leqq i\leqq m)$ of the equation in λ

$$\lambda^m+\sum_{j=1}^m \sum_{|\alpha|=j} b_{j,\alpha}(t, x)\xi^\alpha\lambda^{m-j}=0$$

are real valued for any $(t, x, \xi)\in[0, T)\times \boldsymbol{R}^n\times \boldsymbol{R}^n$.

(S-3) (Distinctness). There are a positive constant c and a real quadratic form $Q(t, \xi)=\sum_{i,j=1}^m a_{i,j}(t)\xi_i\xi_j$ satisfying (i)$\sim$(iii) given below such that the estimate

$$|\lambda_i(t, x, \xi)-\lambda_j(t, x, \xi)|\geqq cQ(t, \xi)^{1/2}$$

holds for any $(t, x, \xi)\in[0, T)\times \boldsymbol{R}^n\times \boldsymbol{R}^n$ and $1\leqq i\neq j\leqq m$, where (i)$\sim$ (iii) are as follows: (i) $a_{i,j}(t)\in C^1([0, T))$, (ii) $Q(t, \xi)>0$ for $t>0$ and $\xi\in \boldsymbol{R}^n-\{0\}$, and (iii) $\max\{|\partial_t \log Q(t, \xi)|\,;\,|\xi|=1\}=O(1/t)$ (as $t\to+0$).

(S-4) (Estimates of principal part). For any β, there is a positive constant C_β such that the estimates

$$|\partial_x^\beta \sum_{|\alpha|=j} b_{j,\alpha}(t, x)\xi^\alpha| \leqq C_\beta Q(t, \xi)^{j/2},$$

$$|\partial_t \partial_x^\beta \sum_{|\alpha|=j} b_{j,\alpha}(t, x)\xi^\alpha| \leqq \frac{C_\beta Q(t, \xi)^{j/2}}{t}$$

hold for any $(t, x, \xi) \in (0, T) \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ and $1 \leqq j \leqq m$.

(S-5) (Estimates of lower order parts). For any β, there is a positive constant C_β such that the estimate

$$|\partial_x^\beta \sum_{|\alpha|<j} a_{j,\alpha}(t, x)(\sqrt{-1}\xi)^\alpha| \leqq C_\beta(1+t^{2\mu}Q(t, \xi))^{(j-1)/2}$$

holds for any $(t, x, \xi) \in [0, T) \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ and $1 \leqq j \leqq m$.

These (S-1)$\sim$(S-5) are our assumptions of hyperbolicity. The meanings of them will be illustrated by examples in § 6.

Note that P is nothing but a *Fuchsian hyperbolic operator of class* $(1, \mu)$ in the sense of Definition 1.1 in Tahara [**9-III**] and that the Cauchy problem for P is C^∞ well posed in the sense of Theorem 3.1 in [**9-III**]. In [**9-III**] we have dealt with much more general classes of Fuchsian hyperbolic operators, but here we will discuss only the case of class $(1, \mu)$, because it is the most fundamental and the generalization is not so difficult.

§ 2. Asymptotic series.

Secondly, we define our asymptotic series and present some elementary properties of them.

Let U be an open subset of $\boldsymbol{R}^n$. For $\phi(t, x) \in C^\infty((0, T) \times U)$ and $\rho(x) \in C^\infty(U)$, we denote by

$$\phi(t, x) = o(t^{\rho(x)}\,;\, \nabla^\infty) \qquad \text{on } U \text{ (as } t \to +0) \tag{2.1}$$

the following: for any l and α the function $(t\partial_t)^l \partial_x^\alpha(t^{-\rho(x)}\phi(t, x))$ converges to zero (as $t \to +0$), as functions in x, uniformly on any compact subset of U. For $u(t, x) \in C^\infty((0, T) \times U)$, $\rho_i(x) \in C^\infty(U)$ $(1 \leqq i \leqq m)$ and $\varphi_{k,h}^{(i)}(x) \in C^\infty(U)$ $(1 \leqq i \leqq m,$ $0 \leqq k < \infty$ and $0 \leqq h \leqq mk)$, we define the asymptotic relation

$$u(t, x) \sim \sum_{i=1}^{m} \sum_{k=0}^{\infty} \sum_{h=0}^{mk} \varphi_{k,h}^{(i)}(x) t^{\rho_i(x)+k} (\log t)^{mk-h} \qquad \text{on } U \text{ (as } t \to +0) \tag{2.2}$$

by the following: for any $s > 0$ and any compact subset K of U, there is an integer N_0 such that for any $N \geqq N_0$

$$u(t, x)-\sum_{i=1}^{m}\sum_{k=0}^{N}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}=o(t^s\,;\,\nabla^\infty) \quad \text{on } K \text{ (as } t\to+0) \tag{2.3}$$

holds. This is the meaning of our asymptotic expansions.

We now investigate some elementary properties. The following proposition is clear from the Taylor expansion in t.

PROPOSITION 1. *If $a(t, x)\in C^\infty([0, T)\times U)$, then we have*

$$a(t, x)\sim\sum_{k=0}^{\infty}\phi_k(x)t^k \tag{2.4}$$

on U (as $t\to+0$) for some $\phi_k(x)\in C^\infty(U)$ in the above sense.

Hence, we may assume that the coefficients $a_{j,\alpha}(t, x)$ of P are expanded asymptotically into the form (2.4) (as $t\to+0$). We will often use this fact in §§4 and 5.

PROPOSITION 2. *For any $\rho_i(x)\in C^\infty(U)$ $(1\leqq i\leqq m)$ and any $\varphi_{k,h}^{(i)}(x)\in C^\infty(U)$ $(1\leqq i\leqq m,\ 0\leqq k<\infty$ and $0\leqq h\leqq mk)$, there exists a function $u(t, x)\in C^\infty((0, T)\times U)$ such that the following asymptotic relation holds:*

$$u(t, x)\sim\sum_{i=1}^{m}\sum_{k=0}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}$$

on U (as $t\to+0$).

PROOF. To obtain this, we have only to construct a function $u_i(t, x)\in C^\infty((0, T)\times U)$ such that

$$u_i(t, x)\sim\sum_{k=0}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h} \tag{2.5}$$

on U (as $t\to+0$). Since (2.5) is equivalent to

$$w_i(t, x)\sim\sum_{k=0}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^k(\log t)^{mk-h} \tag{2.6}$$

under the relation $u_i(t, x)=t^{\rho_i(x)}w_i(t, x)$, we may construct $w_i(t, x)\in C^\infty((0, T)\times U)$ such that (2.6) holds on U (as $t\to+0$). Let $K_0\subset K_1\subset K_2\subset\cdots$ be a sequence of compact subsets of U such that $\bigcup_{k=0}^{\infty}K_k=U$ holds, and put

$$M_k=\sum_{|\alpha|\leqq k}\sum_{h=0}^{mk}\sup_{x\in K_k}|\partial_x^\alpha\varphi_{k,h}^{(i)}(x)|$$

for any k. Choose a sequence $\{a_k\,;\,0\leqq k<\infty\}$ of positive numbers such that the following conditions are satisfied: (i) $1\leqq a_0<a_1<a_2<\cdots$, (ii) $a_k\to+\infty$ (as $k\to+\infty$), (iii) $a_k>e^{mk}$ for any k, and (iv) the inequality

$$2(M_k)^{1/k}<\frac{\sqrt{a_k}}{|\log a_k|^m}$$

holds for any k sufficiently large. Let $\theta(t)\in C^{\infty}(\boldsymbol{R})$ such that $\theta(t)=1$ for $t\leqq 1/2$ and $\theta(t)=0$ for $t\geqq 1$. Then, $w_i(t, x)$ is given by

$$w_i(t, x)=\sum_{k=0}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)\theta(a_k t)t^k(\log t)^{mk-h}. \tag{2.7}$$

Since the right hand side of (2.7) is a locally finite sum on $(0, T)\times U$, $w_i(t, x)$ is well defined and satisfies $w_i(t, x)\in C^{\infty}((0, T)\times U)$. Further, by using (i)$\sim$(iv) we can easily see the following: $w_i(t, x)$ defined by (2.7) satisfies the asymptotic relation (2.6). Hence, we obtain Proposition 2. Q.E.D.

PROPOSITION 3. *Assume that $\rho_i(x)-\rho_j(x)\notin \boldsymbol{Z}$ holds for any $x\in U$ and $1\leqq i\neq j\leqq m$. Then, if $\varphi_{k,h}^{(i)}(x)\in C^{\infty}(U)$ $(1\leqq i\leqq m,\ 0\leqq k<\infty$ and $0\leqq h\leqq mk)$ satisfy*

$$0\sim\sum_{i=1}^{m}\sum_{k=0}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h} \tag{2.8}$$

on U (as $t\to+0$), we have $\varphi_{k,h}^{(i)}(x)=0$ on U for any i, k and h.

PROOF. Take any $x_0\in U$ and fix it. Put $\rho_i=\rho_i(x_0)$ and $a_{k,h}^{(i)}=\varphi_{k,h}^{(i)}(x_0)$. Then, $\rho_i\in \boldsymbol{C}$ $(1\leqq i\leqq m)$ and $a_{k,h}^{(i)}\in \boldsymbol{C}$ $(1\leqq i\leqq m,\ 0\leqq k<\infty$ and $0\leqq h\leqq mk)$ satisfy

$$0\sim\sum_{i=1}^{m}\sum_{k=0}^{\infty}\sum_{h=0}^{mk}a_{k,h}^{(i)}t^{\rho_i+k}(\log t)^{mk-h} \tag{2.9}$$

(as $t\to+0$) in the same sense as in (2.8). Therefore, to obtain Proposition 3 it is sufficient to prove that (2.9) implies $a_{k,h}^{(i)}=0$ for any i, k and h. We show this hereafter. Put $X(t)=t^{\rho_i+k}(\log t)^{mk-h}$ and $Y(t)=t^{\rho_j+p}(\log t)^{mp-q}$ for $(i, k, h)\neq (j, p, q)$. Then, $X(t)$ and $Y(t)$ satisfy the following: (i) if $\mathrm{Re}\rho_i+k\neq\mathrm{Re}\rho_j+p$ or if $mk-h\neq mp-q$, then either $(X/Y)(t)=o(1;\nabla^{\infty})$ (as $t\to+0$) or $(Y/X)(t)=o(1;\nabla^{\infty})$ (as $t\to+0$) holds, and (ii) if $\mathrm{Re}\rho_i+k=\mathrm{Re}\rho_j+p$, $mk-h=mp-q$ and $i\neq j$, then $(X/Y)(t)=t^{\sqrt{-1}\sigma}$ holds for some $\sigma\in\boldsymbol{R}-\{0\}$. In fact, (i) is clear and (ii) is verified by the condition $\rho_i-\rho_j\notin\boldsymbol{Z}$. Therefore, by placing $t^{\rho_i+k}(\log t)^{mk-h}$ $(1\leqq i\leqq m,\ 0\leqq k<\infty$ and $0\leqq h\leqq mk)$ in order of the degeneracy at $t=0$, we can obtain $X_{i,j}(t)\in C^{\infty}((0, T))$ $(0\leqq i<\infty$ and $j\in I_i$, where I_i $(\neq\emptyset)$ is a finite index set depending on i) such that the following conditions are satisfied: (iii) $\{X_{i,j}(t)\ ;\ i, j\}=\{t^{\rho_i+k}(\log t)^{mk-h}\ ;\ i, k, h\}$ holds, (iv) $(X_{i+1,j}/X_{i,k})(t)=o(1;\nabla^{\infty})$ (as $t\to+0$) holds for any i, $j\in I_{i+1}$ and $k\in I_i$, and (v) if $j, k\in I_i$ and $j\neq k$, then $(X_{i,j}/X_{i,k})(t)=t^{\sqrt{-1}\sigma}$ holds for some $\sigma\in\boldsymbol{R}-\{0\}$. Consequently, (2.9) is expressed in the form

$$0\sim\sum_{i=0}^{\infty}\sum_{j\in I_i}b_{i,j}X_{i,j}(t) \tag{2.10}$$

(as $t\to+0$) for some $b_{i,j}\in\boldsymbol{C}$ such that $\{b_{i,j}\ ;\ i, j\}=\{a_{k,h}^{(i)}\ ;\ i, k, h\}$ holds. Moreover, we may understand that (2.10) means the following: for any N and any $k\in I_N$

$$\left(\left(\sum_{i=0}^{N}\sum_{j\in I_i} b_{i,j}X_{i,j}\right)\Big/ X_{N,k}\right)(t)=o(1\,;\,\nabla^\infty)$$

(as $t\to+0$) holds. Therefore, by Lemma 1 given below and by the standard method we can easily obtain $b_{i,j}=0$ for any i and j. Hence, we also obtain $a_{k,h}^{(i)}=0$ for any i, k and h. Thus, to complete the proof we have only to show the following lemma.

LEMMA 1. *Let $l\in \boldsymbol{N}$, $\sigma(i)\in \boldsymbol{R}$ $(1\leqq i\leqq l)$ and assume that $\sigma(i)\neq\sigma(j)$ holds for $1\leqq i\neq j\leqq l$. Then, if $c_i\in \boldsymbol{C}$ $(1\leqq i\leqq l)$ satisfy*

$$\sum_{i=1}^{l} c_i t^{\sqrt{-1}\,\sigma(i)}=o(1\,;\,\nabla^\infty) \tag{2.11}$$

(as $t\to+0$), we have $c_i=0$ for any i.

The proof of this lemma is as follows. Applying $(t\partial_t)^k$ to (2.11), we obtain

$$\sum_{i=1}^{l}(\sqrt{-1}\sigma(i))^k c_i t^{\sqrt{-1}\,\sigma(i)} \to 0 \tag{2.12}$$

(as $t\to+0$) for any k. Since $\sigma(i)\neq\sigma(j)$ holds for $1\leqq i\neq j\leqq l$, it follows from (2.12) that $c_i t^{\sqrt{-1}\,\sigma(i)}\to 0$ (as $t\to+0$) for any i. Hence, we obtain $c_i=0$ for any i.

Q.E.D.

By Proposition 3, we can conclude the following: when $\rho_i(x)-\rho_j(x)\notin \boldsymbol{Z}$ holds for any $x\in U$ and $1\leqq i\neq j\leqq m$, the coefficients $\varphi_{k,h}^{(i)}(x)$ of the asymptotic relation (2.2) are uniquely determined by $u(t, x)$.

§ 3. A priori estimates.

Thirdly, we give a priori estimates of the degree of singularities at $t=0$ for solutions of (S) in $C^\infty((0, T)\times \boldsymbol{R}^n)$.

PROPOSITION 4. *Assume that P satisfies* (S-1)$\sim$(S-5). *Then, there is a positive number s_1 such that any solution $u(t, x)\in C^\infty((0, T)\times \boldsymbol{R}^n)$ of* (S) *satisfies the estimate*

$$u(t, x)=o(t^{-s_1}\,;\,\nabla^\infty) \tag{3.1}$$

on $\boldsymbol{R}^n$ (as $t\to+0$).

We prove this by the same L^2-argument as in Proposition 5.1 of Tahara [**9-I**] or in Lemma 1 of [**9-IV**]. To do so, we note the following lemma.

LEMMA 2. *For any solution $u(t, x)\in C^\infty((0, T)\times \boldsymbol{R}^n)$ of* (S) *and any compact subset K of $\boldsymbol{R}^n$, there exists a solution $w(t, x)\in C^\infty((0, T), H^\infty(\boldsymbol{R}^n))$ of* (S) *such that $w(t, x)=u(t, x)$ holds on $(0, T)\times K$, where $H^\infty(\boldsymbol{R}^n)$ is the Sobolev space on $\boldsymbol{R}^n$.*

PROOF. Let L be a compact subset of $\boldsymbol{R}^n$ such that

$$\left\{x \in \boldsymbol{R}^n \ ; \ \min_{y \in K} |x-y| \leqq \frac{\lambda_{\max} T^{\mu}}{\mu}\right\} \subset L\,,$$

where μ is the positive number in (S-1), $\lambda_{\max}=\max\{|\lambda_i(t, x, \xi)| \ ; (t, x) \in [0, T) \times \boldsymbol{R}^n, |\xi|=1, 1 \leqq i \leqq m\}$, and $\lambda_i(t, x, \xi)$ $(1 \leqq i \leqq m)$ are the same as in (S-2). Let $\varphi(x)$ be a C^{∞} function with compact support such that $\varphi(x)=1$ in a neighborhood of L, and put $g(t, x)=P(t, x, \partial_t, \partial_x)(\varphi(x)u(t, x))$. Then, $g(t, x)$ satisfies $g(t, x)=0$ on $(0, T) \times L$ and $g(t, x) \in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^n))$. Therefore, we can find a solution $v(t, x) \in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^n))$ of the equation

$$\begin{cases} P(t, x, \partial_t, \partial_x)v(t, x)=g(t, x)\,, \\ \partial_t^i v(t, x)|_{t=T_0}=0 \quad (0 \leqq i \leqq m-1) \end{cases}$$

(where $0<T_0<T$) such that $v(t, x)=0$ on $(0, T) \times K$, because $P(t, x, \partial_t, \partial_x)$ is a regularly hyperbolic operator on $(\varepsilon, T) \times \boldsymbol{R}^n$ for any $\varepsilon>0$. Hence, by putting $w(t, x)=\varphi(x)u(t, x)-v(t, x)$ we obtain a desired solution in Lemma 2. Q.E.D.

In consequence of Lemma 2, we have only to show that (3.1) is valid for any solution $u(t, x)$ of (S) in $C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^n))$. Further, by using Sobolev's lemma (for example, Theorem 2.8 in Mizohata [**4**]) we can reduce the problem to the L^2-version (3.2) given below. Therefore, to obtain Proposition 4 it is sufficient to prove the following proposition.

PROPOSITION 5. *Assume that P satisfies* (S-1)$\sim$(S-5). *Then, there is a positive number s_2 such that any solution $u(t, x)(=u(t)) \in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^n))$ of* (S) *satisfies the estimate*

$$\|(t\partial_t)^l \partial_x^{\alpha} u(t)\|=o(t^{-s_2}) \tag{3.2}$$

(as $t \to +0$) for any l and α, where $\|\cdot\|$ is the L^2-norm on $\boldsymbol{R}^n$ and $\phi(t)=o(t^{-s})$ (as $t \to +0$) means that $t^s\phi(t)$ converges to zero (as $t \to +0$).

PROOF. First, we transform (S) into a symmetric first-order system of pseudo-differential equations. Let $u(t, x)(=u(t)) \in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^n))$ be a solution of (S), let $\rho(t) \in C^{\infty}(\boldsymbol{R})$ such that $\rho(t)=0$ for $t \leqq 1$, $0 \leqq \rho(t) \leqq 1$ for $1 \leqq t \leqq 2$ and $\rho(t)=1$ for $t \geqq 2$, let $\Theta(t)$ be the pseudo-differential operator defined by the symbol

$$\Theta(t, \xi)=\rho(4-Q(t, \xi)^{1/2})+Q(t, \xi)^{1/2}\rho(Q(t, \xi)^{1/2})$$

(where $Q(t, \xi)$ is the quadratic form in (S-3)) in Proposition 2.6 of Tahara [**9-II**], let $\vec{u}(t)={}^t(u_1(t), \cdots, u_m(t))$ be the m-column vector defined by

$$u_j(t)=(\sqrt{-1})^{m-j}(1+t^{\mu}\Theta(t))^{m-j}t^{j-1}\partial_t^{j-1}u(t)\,, \quad j=1, \cdots, m$$

(where μ is the positive number in (S-1)), and let Λ be the pseudo-differential operator defined by the symbol $\Lambda(\xi)=(1+|\xi|^2)^{1/2}$. Then, we have

LEMMA 3. *There are $m \times m$ matrices $N(t)$, $M(t)$, $R(t)$ and $2m \times 2m$ matrices $A(t)$, $B(t)$ of pseudo-differential operators such that the following conditions are* *satisfied:* (i) (S) *is transformed into*

$$t\partial_t U(t)+A(t)U(t)-t^{\mu}B(t)U(t)=0\,, \quad 0<t<T \tag{3.3}$$

under the relations $U(t)={}^t(U_1(t), U_2(t))$, $U_1(t)=N(t)\vec{u}(t)$, $U_2(t)=\Theta(t)^{-1}\vec{u}(t)$ *and* $\vec{u}(t)=M(t)U_1(t)+R(t)U_2(t)$, (ii) $N(t)$, $M(t)$, $R(t)$, $A(t)$ *and* $B(t)\Lambda^{-1}$ *are pseudo-differential operators of order* 0 *with a parameter* t $(0\leqq t<T)$ *in the sense of Tahara* [**9-II**, §1], *and* (iii) *there is a positive constant* b *such that* $|\mathrm{Re}(B(t)v, v)|\leqq b\|v\|^2$ *holds for any* $v\in H^{\infty}(\boldsymbol{R}^n)$ *and* $0\leqq t<T$, *where* $(\,,\,)$ *is the inner product in* $L^2(\boldsymbol{R}^n)$.

In fact, this is verified as follows. By the same calculation as in the proof of Theorem 6.1 (with $\sigma=1$) in Tahara [**9-II**] or in the proof of Proposition 2.1 (with $a=1$) in [**9-III**], we can transform (S) into a first-order system of the form

$$t\partial_t\vec{u}(t)+K(t)\vec{u}(t)-\sqrt{-1}\,t^{\mu}H(t)\Theta(t)\vec{u}(t)=0\,,$$

where $K(t)$ is a suitable $m\times m$ matrix of pseudo-differential operators of order 0, $H(t)$ is the matrix given by

$$H(t)=\begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -h_m(t) & -h_{m-1}(t) & \cdots & & -h_1(t) \end{bmatrix}$$

and $h_j(t)$ is the pseudo-differential operator defined by the symbol

$$h_j(t, x, \xi)=\sum_{|\alpha|=j} b_{j,\alpha}(t, x)\xi^{\alpha}\Theta(t, \xi)^{-j}$$

(where $b_{j,\alpha}(t, x)$ are the same as in (S-1)). Since the symbol $H(t, x, \xi)$ of $H(t)$ satisfies the conditions (I-1)$\sim$(I-4) in §3 of [**9-II**], by Theorem 3.1 in [**9-II**] we can find $m\times m$ matrices $N(t)$, $M(t)$, $D(t)$, $L(t)$, $T(t)$, $R(t)$ and $S(t)$ such that the following conditions are satisfied: (iv) $N(t)$, $tN_t'(t)$, $M(t)$, $tM_t'(t)$, $D(t)$, $tD_t'(t)$, $L(t)$, $T(t)$, $R(t)$ and $S(t)$ are pseudo-differential operators of order 0 with a parameter t $(0\leqq t<T)$ in the sense of [**9-II**], (v) $N(t)H(t)\Theta(t)=D(t)\Theta(t)N(t)+L(t)$, (vi) $D(t)\Theta(t)-(D(t)\Theta(t))^*=S(t)$ (where $(D(t)\Theta(t))^*$ is the formal adjoint operator of $D(t)\Theta(t)$), (vii) $M(t)N(t)=I-R(t)\Theta(t)^{-1}$, and (viii) $N(t)M(t)=I-T(t)\Theta(t)^{-1}$. Hence, by the same calculation as in the proof of Lemma 2 in Tahara [**9-IV**] we can easily obtain this lemma. Therefore, we may omit the details.

Next, we estimate the solution $U(t)$ of (3.3).

LEMMA 4. *Put* $a=\sup\{\|A(t)\|\ ;\ 0\leqq t<T\}$ (*where* $\|A(t)\|$ *is the operator norm of* $A(t)$ *in* $L^2(\boldsymbol{R}^n)$) *and define the sequence* $\{a_k\ ;\ 0\leqq k<\infty\}$ *by* $a_0=a$, $a_k=a_{k-1}+(1/2)^k$ $(k\geqq 1)$. *Then, any solution* $U(t)\in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^n))$ *of* (3.3) *satisfies the estimate*

$$\|\Lambda^k U(t)\| = O(t^{-a_k}) \tag{3.4}$$

(as $t \to +0$) *for any* $k \geqq 0$, *where* $\phi(t) = O(t^{-a})$ (as $t \to +0$) *means that* $|t^a \phi(t)| \leqq M$ *holds for some* $M > 0$.

This is verified by induction on k. For $t > 0$, we have

$$\begin{aligned}-t\frac{d}{dt}\|U(t)\|^2 &= 2\mathrm{Re}\Big(-t\frac{dU(t)}{dt},\, U(t)\Big)\\ &= 2\mathrm{Re}(A(t)U(t),\, U(t)) - 2t^\mu \mathrm{Re}(B(t)U(t),\, U(t))\,.\end{aligned}$$

Since $|\mathrm{Re}(AU, U)| \leqq a\|U\|^2$ and $|\mathrm{Re}(BU, U)| \leqq b\|U\|^2$, we have

$$-t\frac{d}{dt}\|U(t)\| - a\|U(t)\| - bt^\mu \|U(t)\| \leqq 0\,. \tag{3.5}$$

Therefore, multiplying (3.5) by $t^{a-1}\exp((b/\mu)t^\mu)$ and integrating from t to T_0 (where $0 < t < T_0 < T$) we have

$$t^a \|U(t)\| \exp\Big(\frac{b}{\mu}t^\mu\Big) \leqq T_0^a \|U(T_0)\| \exp\Big(\frac{b}{\mu}T_0^\mu\Big)\,.$$

Hence, we obtain $t^a\|U(t)\| = O(1)$ (as $t \to +0$), that is, $\|U(t)\| = O(t^{-a})$ (as $t \to +0$). This implies that (3.4) is valid for $k = 0$. Suppose that $l \geqq 1$ and that (3.4) is valid for $k = 0, 1, \cdots, l-1$. Then, (3.4) for $k = l$ is obtained in the following way. Note that $\Lambda^l U(t)$ satisfies

$$\begin{aligned}&t\frac{d}{dt}\Lambda^l U(t) + A(t)\Lambda^l U(t) - t^\mu (B(t) + lB_1(t))\Lambda^l U(t)\\ &= -\sum_{j=1}^{l}\binom{l}{j}A_j(t)\Lambda^{l-j}U(t) + t^\mu \sum_{j=2}^{l}\binom{l}{j}B_j(t)\Lambda^{l-j+1}U(t)\,,\end{aligned}$$

where $A_1 = [\Lambda, A]$, $A_j = [\Lambda, A_{j-1}]$ ($j \geqq 2$), $B_1 = [\Lambda, B\Lambda^{-1}]$ and $B_j = [\Lambda, B_{j-1}]$ ($j \geqq 2$) under the notation $[X, Y] = XY - YX$. Since $a_l > a$, by the same argument as in (3.5) we have

$$-t\frac{d}{dt}\|\Lambda^l U(t)\| - a_l\|\Lambda^l U(t)\| - b_l t^\mu \|\Lambda^l U(t)\| \leqq \sum_{k=0}^{l-1} c_k \|\Lambda^k U(t)\| \tag{3.6}$$

for some $b_l > 0$ and $c_k > 0$. Therefore, multiplying both sides of (3.6) by $t^{a_l - 1}\exp((b_l/\mu)t^\mu)$ and integrating from t to T_0 we obtain

$$\begin{aligned}t^{a_l}\|\Lambda^l U(t)\| \exp\Big(\frac{b_l}{\mu}t^\mu\Big) &\leqq T_0^{a_l}\|\Lambda^l U(T_0)\| \exp\Big(\frac{b_l}{\mu}T_0^\mu\Big)\\ &\quad + \sum_{k=0}^{l-1} c_k \int_t^{T_0} y^{a_l - 1}\|\Lambda^k U(y)\| \exp\Big(\frac{b_l}{\mu}y^\mu\Big)dy\,.\end{aligned} \tag{3.7}$$

Since $a_l > a_k$ for $k = 0, 1, \cdots, l-1$ and since (3.4) is valid for $k = 0, 1, \cdots, l-1$, the integral terms in the right hand side of (3.7) are convergent as $t\to+0$. Therefore, by (3.7) we obtain $t^{a_l}\|\Lambda^l U(t)\|=O(1)$ (as $t\to+0$), that is, $\|\Lambda^l U(t)\|=O(t^{-a_l})$ (as $t\to+0$). This implies that (3.4) is also valid for $k=l$. Thus, Lemma 4 is proved.

Now, let us prove the estimates in Proposition 5. Let $u(t, x)(=u(t))\in C^{\infty}((0, T), H^{\infty}(\boldsymbol{R}^n))$ be a solution of (S). Then, by Lemmas 3 and 4 we have the estimate $\|(t\partial_t)^l\partial_x^{\alpha}u(t)\|=o(t^{-a-1})$ (as $t\to+0$) for $l=0, 1, \cdots, m-1$ and α. Since $u(t, x)$ is a solution of (S), by operating $(t\partial_t)^k\partial_x^{\alpha}$ on (S) we have

$$(t\partial_t)^{m+k}\partial_x^{\alpha}u(t, x)=\sum_{\substack{j+|\beta|\leq m+k+|\alpha| \\ j<m+k}} c_{j,\beta}(t, x)(t\partial_t)^j\partial_x^{\beta}u(t, x)$$

for some $c_{j,\beta}(t, x)\in B^{\infty}([0, T)\times\boldsymbol{R}^n)$. Therefore, by induction on k and α we can easily see that $\|(t\partial_t)^{m+k}\partial_x^{\alpha}u(t)\|=o(t^{-a-1})$ (as $t\to+0$) holds for any $k\geqq 0$ and α. This immediately leads us to Proposition 5. Q.E.D.

Thus, Proposition 4 is also proved. In general, P is said to have the *tempered growth condition* (as $t\to+0$), if P satisfies the following: for any solution $u(t, x)\in C^{\infty}((0, T)\times\boldsymbol{R}^n)$ of $Pu=0$ and for any compact subset K of $\boldsymbol{R}^n$, there is a positive number s such that $u(t, x)=o(t^{-s}; \nabla^{\infty})$ on K (as $t\to+0$). Hence, by Proposition 4 we can conclude that our Fuchsian hyperbolic operator P has the tempered growth condition (as $t\to+0$).

§ 4. Asymptotic expansions.

Fourthly, we establish the following theorem.

THEOREM 1. *Assume that P satisfies* (S-1)$\sim$(S-5). *In addition, assume that $\rho_i(x)-\rho_j(x)\notin\boldsymbol{Z}$ holds for any $x\in\boldsymbol{R}^n$ and $1\leqq i\neq j\leqq m$. Then, any solution $u(t, x)\in C^{\infty}((0, T)\times\boldsymbol{R}^n)$ of* (S) *can be expanded asymptotically into the form*

$$u(t, x)\sim\sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big) \tag{4.1}$$

on $\boldsymbol{R}^n$ (as $t\to+0$) for some $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)\in C^{\infty}(\boldsymbol{R}^n)$. Further, such coefficients $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)$ are uniquely determined by $u(t, x)$. If $\varphi_i(x)=0$ on $\boldsymbol{R}^n$ for any i, then we have $\varphi_{k,h}^{(i)}(x)=0$ on $\boldsymbol{R}^n$ for any i, k and h.

Recall that the local uniqueness of the coefficients $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)$ of the asymptotic expansion (4.1) is already proved in Proposition 3. Therefore, to obtain the first half of Theorem 1 it is sufficient to show the following: for any $x_0\in\boldsymbol{R}^n$, there is an open neighborhood U of x_0 such that $u(t, x)$ is expanded asymptotically into the form (4.1) on U (as $t\to+0$) for some $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)\in C^{\infty}(U)$.

Take any $x_0\in\boldsymbol{R}^n$ and fix it. Let $\lambda(x)=\rho_i(x)+k$ and $\sigma(x)=\rho_j(x)+l$ for $(i, k)\neq(j, l)$. Then, one of the following three conditions holds: $\mathrm{Re}\lambda(x_0)>\mathrm{Re}\sigma(x_0)$, $\mathrm{Re}\lambda(x_0)=\mathrm{Re}\sigma(x_0)$ and $\mathrm{Re}\lambda(x_0)<\mathrm{Re}\sigma(x_0)$. Therefore, by placing $\rho_i(x)+k$ $(1\leqq i\leqq m$

and $0\leqq k<\infty$) in order of the value of $\mathrm{Re}\rho_i(x_0)+k$, we can obtain $\lambda_{j,l}(x)\in C^\infty(\boldsymbol{R}^n)$ ($1\leqq j<\infty$ and $l\in I_j$, where I_j ($\neq\emptyset$) is a finite index set depending on j) such that the following conditions are satisfied: (i) $\{\lambda_{j,l}(x)\ ;\ j,\ l\}=\{\rho_i(x)+k\ ;\ i,\ k\}$ holds, (ii) $\mathrm{Re}\lambda_{j,l}(x_0)=\mathrm{Re}\lambda_{j,k}(x_0)$ for any j and l, $k\in I_j$, and (iii) $\mathrm{Re}\lambda_{j,l}(x_0)<\mathrm{Re}\lambda_{j+1,k}(x_0)$ for any j, $l\in I_j$ and $k\in I_{j+1}$. Consequently, we also obtain the following: (iv) $\lambda_{j,l}(x)+1$ is expressed in the form $\lambda_{j,l}(x)+1=\lambda_{p,k}(x)$ for some p ($\geqq j+1$) and $k\in I_p$, and (v) there is a positive integer p_0 such that $\{\lambda_{j,l}(x)+1\ ;\ l\in I_j\}=\{\lambda_{j+p_0,k}(x)\ ;\ k\in I_{j+p_0}\}$ holds for any j sufficiently large. Hence, we can choose a common open neighborhood U of x_0 and a sequence $\{m_j\ ;\ 0\leqq j<\infty\}$ of real numbers so that the following inequalities hold:

$$\begin{cases} m_0<\inf\limits_{\substack{x\in U\\ l\in I_1}}(\mathrm{Re}\,\lambda_{1,l}(x))\,, & (4.2)\\ \sup\limits_{\substack{x\in U\\ l\in I_j}}(\mathrm{Re}\,\lambda_{j,l}(x))<m_j<\inf\limits_{\substack{x\in U\\ k\in I_{j+1}}}(\mathrm{Re}\,\lambda_{j+1,k}(x)) \qquad (j\geqq 1), & (4.3)\\ m_{j+1}<m_j+1 \qquad (j\geqq 0). & (4.4)\end{cases}$$

Hereafter, we fix $\{\lambda_{j,l}(x)\ ;\ 1\leqq j<\infty$ and $l\in I_j\}$, the open neighborhood U of x_0 and the sequence $\{m_j\ ;\ 0\leqq j<\infty\}$ above. For $\lambda_{j,l}(x)=\rho_i(x)+k$, we denote $K(\lambda_{j,l})=k$. Then, we can obtain the following proposition.

PROPOSITION 6. *Let $\{\lambda_{j,l}(x)\ ;\ 1\leqq j<\infty$ and $l\in I_j\}$ and U be as above. Then, any solution $u(t,x)\in C^\infty((0,T)\times\boldsymbol{R}^n)$ of* (S) *can be expanded asymptotically into the form*

$$\begin{aligned} u(t,x)\sim &\sum_{j=1}^{\infty}\sum_{\substack{l\in I_j\\ K(\lambda_{j,l})=0}}\varphi_{j,l}(x)t^{\lambda_{j,l}(x)}\\ &+\sum_{j=1}^{\infty}\sum_{\substack{l\in I_j\\ K(\lambda_{j,l})\neq 0}}\sum_{h=0}^{mK(\lambda_{j,l})}\varphi_{j,l}^h(x)t^{\lambda_{j,l}(x)}(\log t)^{mK(\lambda_{j,l})-h} \end{aligned} \tag{4.5}$$

on U (as $t\to+0$) for some $\varphi_{j,l}(x)$, $\varphi_{j,l}^h(x)\in C^\infty(U)$. Further, if $\varphi_{j,l}(x)=0$ on U for any j and l, then we have $\varphi_{j,l}^h(x)=0$ on U for any j, l and h.

This immediately leads us to Theorem 1. So, from now on we confine ourselves to proving Proposition 6. Here, we prepare some lemmas which are necessary to the proof of Proposition 6.

LEMMA 5. *Let $\rho(x)\in C^\infty(U)$, A, $B\in\boldsymbol{R}$ and $f(t,x)\in C^\infty((0,T)\times U)$. Assume that $f(t,x)=o(t^A\,;\,\nabla^\infty)$ on U (as $t\to+0$). In addition, assume that $B<A<\mathrm{Re}\rho(x)$ holds for any $x\in U$, or that $\mathrm{Re}\rho(x)<B<A$ holds for any $x\in U$. Then, there exists a solution $\tilde{u}(t,x)\in C^\infty((0,T)\times U)$ of the equation $(t\partial_t-\rho(x))\tilde{u}(t,x)=f(t,x)$ which satisfies $\tilde{u}(t,x)=o(t^B\,;\,\nabla^\infty)$ on U (as $t\to+0$).*

PROOF. When $B<A<\mathrm{Re}\rho(x)$ holds for any $x\in U$, $\tilde{u}(t, x)$ is given by

$$\tilde{u}(t, x)=-t^{\rho(x)}\int_t^{T_0}\tau^{-\rho(x)-1}f(\tau, x)d\tau$$

(where $0<T_0<T$). When $\mathrm{Re}\rho(x)<B<A$ holds for any $x\in U$, $\tilde{u}(t, x)$ is given by

$$\tilde{u}(t, x)=t^{\rho(x)}\int_0^t\tau^{-\rho(x)-1}f(\tau, x)d\tau .$$

In any case, details are verified by direct calculations. Q. E. D.

LEMMA 6. *Let $A, B\in\boldsymbol{R}$ and $f(t, x)\in C^\infty((0, T)\times U)$, let $\rho_1(x), \cdots, \rho_m(x)$ be the characteristic exponents of P, and let $\mathcal{C}(\lambda, x)$ be the indicial polynomial associated with P. Assume that $f(t, x)=o(t^A; \nabla^\infty)$ on U (as $t\to+0$). In addition, for any i $(1\leqq i\leqq m)$ we assume that $B<A<\mathrm{Re}\rho_i(x)$ holds for any $x\in U$, or that $\mathrm{Re}\rho_i(x)<B<A$ holds for any $x\in U$. Then, we have the following results.* (1) *There exists a solution $\tilde{u}(t, x)\in C^\infty((0, T)\times U)$ of the equation $\mathcal{C}(t\partial_t, x)\tilde{u}(t, x)=f(t, x)$ which satisfies $\tilde{u}(t, x)=o(t^B; \nabla^\infty)$ on U (as $t\to+0$).* (2) *Any solution $u(t, x)\in C^\infty((0, T)\times U)$ of the equation $\mathcal{C}(t\partial_t, x)u(t, x)=f(t, x)$ is expressed in the form*

$$u(t, x)=\sum_{j=1}^{\infty}\sum_{\substack{l\in I_j\\ K(\lambda_{j,l})=0}}\varphi_{j,l}(x)t^{\lambda_{j,l}(x)}+o(t^B; \nabla^\infty) \tag{4.6}$$

on U (as $t\to+0$) for some $\varphi_{j,l}(x)\in C^\infty(U)$.

PROOF. Note that $\mathcal{C}(t\partial_t, x)$ is decomposed into

$$\mathcal{C}(t\partial_t, x)=(t\partial_t-\rho_1(x))(t\partial_t-\rho_2(x))\cdots(t\partial_t-\rho_m(x)) .$$

Therefore, applying Lemma 5 m-times we obtain (1). The proof of (2) is as follows. Let $\tilde{u}(t, x)$ be the solution obtained in (1) and $v(t, x)=u(t, x)-\tilde{u}(t, x)$. Then, $v(t, x)$ satisfies $\mathcal{C}(t\partial_t, x)v(t, x)=0$. Since $\rho_i(x)\neq\rho_j(x)$ holds for any $x\in U$ and $1\leqq i\neq j\leqq m$, we have

$$v(t, x)=\sum_{i=1}^{m}\varphi_i(x)t^{\rho_i(x)}$$

for some $\varphi_i(x)\in C^\infty(U)$. This immediately leads us to (4.6), because $u(t, x)=v(t, x)+\tilde{u}(t, x)$ and $\{\lambda_{j,l}(x)\ ;\ 1\leqq j<\infty,\ l\in I_j \text{ and } K(\lambda_{j,l})=0\}=\{\rho_i(x)\ ;\ 1\leqq i\leqq m\}$ hold. Q. E. D.

LEMMA 7. *Let $u(t, x)$, $f(t, x)\in C^\infty((0, T)\times U)$ and let $\{m_j;\ 0\leqq j<\infty\}$ be the sequence chosen in (4.2)$\sim$(4.4). Assume that $u(t, x)$ and $f(t, x)$ satisfy the equation $\mathcal{C}(t\partial_t, x)u(t, x)=f(t, x)$. Then, we have the following results.* (1) *If $f(t, x)=o(t^{m_0-a}; \nabla^\infty)$ on U (as $t\to+0$) for some $a>0$, then we have $u(t, x)=o(t^{m_0-b}; \nabla^\infty)$ on U (as $t\to+0$) for any $b>a$.* (2) *If $f(t, x)=o(t^{m_1+\varepsilon}; \nabla^\infty)$ on U (as $t\to+0$) for* *some* $\varepsilon>0$, *then we have*

$$u(t, x)=\sum_{l\in I_1}\varphi_{1,l}(x)t^{\lambda_{1,l}(x)}+o(t^{m_1}; \nabla^\infty)$$

on U *(as* $t\to+0$*) for some* $\varphi_{1,l}(x)\in C^\infty(U)$. (3) *If* $u(t, x)=o(t^{m_{p-1}}; \nabla^\infty)$ *on* U *(as* $t\to+0$*) for some* p $(\geqq 2)$ *and if* $f(t, x)$ *has the form*

$$f(t, x)=\sum_{\substack{l\in I_p \\ K(\lambda_{p,l})\neq 0}}\sum_{h=0}^{mK(\lambda_{p,l})}\psi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h}+o(t^{m_p+\varepsilon}; \nabla^\infty) \tag{4.7}$$

on U *(as* $t\to+0$*) for some* $\psi_{p,l}^h(x)\in C^\infty(U)$ *and some* $\varepsilon>0$, *then we have*

$$\begin{aligned} u(t, x)=&\sum_{\substack{l\in I_p \\ K(\lambda_{p,l})=0}}\varphi_{p,l}(x)t^{\lambda_{p,l}(x)} \\ &+\sum_{\substack{l\in I_p \\ K(\lambda_{p,l})\neq 0}}\sum_{h=0}^{mK(\lambda_{p,l})}\varphi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h}+o(t^{m_p}; \nabla^\infty) \end{aligned} \tag{4.8}$$

on U *(as* $t\to+0$*) for some* $\varphi_{p,l}(x)$, $\varphi_{p,l}^h(x)\in C^\infty(U)$. (4) *In* (3), *the coefficients* $\{\varphi_{p,l}^h(x);\ l\in I_p,\ K(\lambda_{p,l})\neq 0$ *and* $0\leqq h\leqq mK(\lambda_{p,l})\}$ *are uniquely determined by* $\{\psi_{p,l}^h(x);\ l\in I_p,\ K(\lambda_{p,l})\neq 0$ *and* $0\leqq h\leqq mK(\lambda_{p,l})\}$ *and are characterized as the unique solution of the equation*

$$\begin{aligned} &C(t\partial_t, x)\Big(\sum_{h=0}^{mK(\lambda_{p,l})}\varphi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h}\Big) \\ &\quad=\sum_{h=0}^{mK(\lambda_{p,l})}\psi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h} \end{aligned} \tag{4.9}$$

for any $l\in I_p$ *such that* $K(\lambda_{p,l})\neq 0$.

PROOF. The proof of (1) is as follows. By (2) of Lemma 6, $u(t, x)$ is expressed in the form

$$u(t, x)=\sum_{j=1}^{\infty}\sum_{\substack{l\in I_j \\ K(\lambda_{j,l})=0}}\varphi_{j,l}(x)t^{\lambda_{j,l}(x)}+o(t^{m_0-b}; \nabla^\infty)$$

on U (as $t\to+0$) for some $\varphi_{j,l}(x)\in C^\infty(U)$. Hence, we obtain $u(t, x)=o(t^{m_0-b}; \nabla^\infty)$ on U (as $t\to+0$), because by (4.2) we have $\varphi(x)t^{\lambda_{j,l}(x)}=o(t^{m_0-b}; \nabla^\infty)$ on U (as $t\to+0$) for any $j\geqq 1$, $l\in I_j$ and $\varphi(x)\in C^\infty(U)$. Thus, (1) is proved. Since by (4.3) we also have $\varphi(x)t^{\lambda_{j,l}(x)}=o(t^{m_1}; \nabla^\infty)$ on U (as $t\to+0$) for any $j\geqq 2$, $l\in I_j$ and $\varphi(x)\in C^\infty(U)$, (2) may be proved in the same way as (1). The proof of (3) is as follows. Divide $f(t, x)$ (in (4.7)) into the following:

$$f(t, x)=\sum_{\substack{l\in I_p \\ K(\lambda_{p,l})\neq 0}}g_l(t, x)+h(t, x),$$

$$g_l(t, x)=\sum_{h=0}^{mK(\lambda_{p,l})} \psi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h},$$

where $\psi_{p,l}^h(x)\in C^\infty(U)$ are the same as in (4.7) and $h(t, x)=o(t^{m_p+\varepsilon}; \nabla^\infty)$ on U (as $t\to+0$). Note that $C(\lambda_{p,l}(x), x)\neq 0$ holds on U for any $\lambda_{p,l}(x)$ such that $K(\lambda_{p,l})\neq 0$, because $\rho_i(x)-\rho_j(x)\notin \boldsymbol{Z}$ holds for any $x\in U$ and $1\leqq i\neq j\leqq m$. Therefore, by an easy calculation we can determine the coefficients $\varphi_{p,l}^h(x)\in C^\infty(U)$ $(0\leqq h\leqq mK(\lambda_{p,l}))$ of

$$w_l(t, x)=\sum_{h=0}^{mK(\lambda_{p,l})} \varphi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h}$$

(where $l\in I_p$ and $K(\lambda_{p,l})\neq 0$) so that $w_l(t, x)$ becomes a solution of (4.9), that is, $C(t\partial_t, x)w_l(t, x)=g_l(t, x)$. On the other hand, since $h(t, x)=o(t^{m_p+\varepsilon}; \nabla^\infty)$ on U (as $t\to+0$), by (1) of Lemma 6 we can find a function $\tilde{u}(t, x)\in C^\infty((0, T)\times U)$ which satisfies $C(t\partial_t, x)\tilde{u}(t, x)=h(t, x)$ and $\tilde{u}(t, x)=o(t^{m_p}; \nabla^\infty)$ on U (as $t\to+0$). Consequently, $u(t, x)$ is expressed in the form

$$u(t, x)=\sum_{j=1}^{\infty} \sum_{\substack{l\in I_j \\ K(\lambda_{j,l})=0}} \varphi_{j,l}(x)t^{\lambda_{j,l}(x)}+\sum_{\substack{l\in I_p \\ K(\lambda_{p,l})\neq 0}} w_l(t, x)+o(t^{m_p}; \nabla^\infty) \tag{4.10}$$

on U (as $t\to+0$) for some $\varphi_{j,l}(x)\in C^\infty(U)$. Since $u(t, x)=o(t^{m_{p-1}}; \nabla^\infty)$ on U (as $t\to+0$), we have $\varphi_{j,l}(x)=0$ on U for any $j\leqq p-1$ and $l\in I_j$. Therefore, (4.10) immediately leads us to (4.8), because $\varphi(x)t^{\lambda_{j,l}(x)}=o(t^{m_p}; \nabla^\infty)$ on U (as $t\to+0$) for any $j\geqq p+1$, $l\in I_j$ and $\varphi(x)\in C^\infty(U)$. Thus, we obtain (3). Moreover, by the same argument as in the proof of Proposition 3 we can prove that the coefficients $\varphi_{p,l}(x)$, $\varphi_{p,l}^h(x)$ of (4.8) are uniquely determined by $u(t, x)$. Therefore, (4) is clear from the discussion of (3). Q.E.D.

LEMMA 8. *Let $u(t, x)\in C^\infty((0, T)\times U)$ and let $R(t, x, \partial_t, \partial_x)$ be a linear differential operator of order m with coefficients in $C^\infty([0, T)\times U)$. Then, we have the following results.* (1) *If $u(t, x)=o(t^{m_p}; \nabla^\infty)$ on U (as $t\to+0$), then we have $tR(t, x, t\partial_t, \partial_x)u(t, x)=o(t^{m_{p+1}+\varepsilon}; \nabla^\infty)$ on U (as $t\to+0$) for some $\varepsilon>0$.* (2) *If $u(t, x)$ has the form*

$$u(t, x)=\sum_{q\in I_p} \sum_{h=0}^{mK(\lambda_{p,q})} \varphi_{p,q}^h(x)t^{\lambda_{p,q}(x)}(\log t)^{mK(\lambda_{p,q})-h}$$

for some p $(\geqq 1)$ and $\varphi_{p,q}^h(x)\in C^\infty(U)$, then we have

$$tR(t, x, t\partial_t, \partial_x)u(t, x)\sim\sum_{j=p+1}^{\infty} \sum_{\substack{l\in I_j \\ K(\lambda_{j,l})\neq 0}} \sum_{h=0}^{mK(\lambda_{j,l})} \psi_{j,l}^h(x)t^{\lambda_{j,l}(x)}(\log t)^{mK(\lambda_{j,l})-h}$$

on U (as $t\to+0$) for some $\psi_{j,l}^h(x)\in C^\infty(U)$.

PROOF. (1) is clear from (4.4). (2) is verified by Proposition 1 and the following facts: (i) $\lambda_{p,q}(x)+k$ (where $k\in\boldsymbol{N}$) is expressed in the form $\lambda_{p,q}(x)+k$

$=\lambda_{j,l}(x)$ for some j ($\geqq p+k$) and $l\in I_j$, and (ii) when $\lambda_{p,q}(x)+k=\lambda_{j,l}(x)$, we have $K(\lambda_{p,q})+k=K(\lambda_{j,l})$ and therefore $mK(\lambda_{p,q})+m\leqq mK(\lambda_{j,l})$. Q. E. D.

Thus, preparations are completed. Now, let us return to Proposition 6. Note that by the condition (ii) in the introduction we can decompose P into the following two parts

$$P(t, x, \partial_t, \partial_x)=\mathcal{C}(t\partial_t, x)-tR(t, x, t\partial_t, \partial_x), \tag{4.11}$$

where $\mathcal{C}(\lambda, x)$ is the indicial polynomial associated with P and $R(t, x, \partial_t, \partial_x)$ is a suitable linear differential operator of order m with coefficients in $C^\infty([0, T)\times \boldsymbol{R}^n)$. Consequently, the equation (S) is expressed in the form

$$\mathcal{C}(t\partial_t, x)u(t, x)=tR(t, x, t\partial_t, \partial_x)u(t, x).$$

For simplicity, we denote $tR(t, x, t\partial_t, \partial_x)u(t, x)$ by $t\mathcal{R}[u](t, x)$. Then, by using the results in Lemmas 7 and 8 we can give a proof of Proposition 6 as follows.

Proof of Proposition 6. Let $u(t, x)\in C^\infty((0, T)\times \boldsymbol{R}^n)$ be a solution of (S) and let us consider the equation

$$\mathcal{C}(t\partial_t, x)u(t, x)=t\mathcal{R}[u](t, x) \tag{4.12}$$

on $(0, T)\times U$. Note that by Proposition 4 we have $u(t, x)=o(t^{m_0-N}; \nabla^\infty)$ on U (as $t\to+0$) for some $N\in \boldsymbol{N}$ sufficiently large, and that this implies that $t\mathcal{R}[u](t, x)=o(t^{m_0-a_1}; \nabla^\infty)$ on U (as $t\to+0$) for any $a_1>N-1$. Therefore, applying (1) of Lemma 7 to (4.12) we obtain $u(t, x)=o(t^{m_0-b_1}; \nabla^\infty)$ on U (as $t\to+0$) for any $b_1>N-1$. This also implies that $t\mathcal{R}[u](t, x)=o(t^{m_0-a_2}; \nabla^\infty)$ on U (as $t\to+0$) for any $a_2>N-2$. Again, applying (1) of Lemma 7 to (4.12) we obtain $u(t, x)=o(t^{m_0-b_2}; \nabla^\infty)$ on U (as $t\to+0$) for any $b_2>N-2$. Repeating the same argument N-times, we can obtain $u(t, x)=o(t^{m_0-b}; \nabla^\infty)$ on U (as $t\to+0$) for any $b>0$, and hence we can also obtain $t\mathcal{R}[u](t, x)=o(t^{m_1+\varepsilon}; \nabla^\infty)$ on U (as $t\to+0$) for some $\varepsilon>0$. Therefore, by (2) of Lemma 7, $u(t, x)$ is expressed in the form

$$u(t, x)=\sum_{l\in I_1}\varphi_{1,l}(x)t^{\lambda_{1,l}(x)}+o(t^{m_1}; \nabla^\infty) \tag{4.13}$$

on U (as $t\to+0$) for some $\varphi_{1,l}(x)\in C^\infty(U)$. Thus, we have obtained the first terms of the asymptotic series in (4.5).

Now, let us determine all the terms of the asymptotic series in (4.5). To do so, it is sufficient to find functions $w_j(t, x)\in C^\infty((0, T)\times U)$ ($j\geqq 1$) of the form

$$\begin{aligned} w_j(t, x)= &\sum_{\substack{l\in I_j\\ K(\lambda_{j,l})=0}} \varphi_{j,l}(x)t^{\lambda_{j,l}(x)} \\ &+\sum_{\substack{l\in I_j\\ K(\lambda_{j,l})\neq 0}}\sum_{h=0}^{mK(\lambda_{j,l})} \varphi^h_{j,l}(x)t^{\lambda_{j,l}(x)}(\log t)^{mK(\lambda_{j,l})-h} \end{aligned} \tag{4.14}$$

(where $\varphi_{j,l}(x)$, $\varphi_{j,l}^h(x)\in C^\infty(U)$) such that the following conditions are satisfied:

$$u(t, x)-\sum_{j=1}^{k} w_j(t, x)=o(t^{m_k}; \nabla^\infty), \tag{4.15}$$

$$C(t\partial_t, x)\Big(\sum_{j=1}^{k} w_j(t, x)\Big)=t\mathcal{R}\Big[\sum_{j=1}^{k-1} w_j\Big](t, x)+o(t^{m_k}; \nabla^\infty) \tag{4.16}$$

on U (as $t\to+0$) for any $k\geqq 1$. Put

$$w_1(t, x)=\sum_{l\in I_1}\varphi_{1,l}(x)t^{\lambda_{1,l}(x)},$$

where $\varphi_{1,l}(x)\in C^\infty(U)$ are the same as in (4.13). Then, (4.15) and (4.16) are valid for $k=1$. Suppose that $p\geqq 2$ and that we have already obtained $w_1(t, x)$, $\cdots$, $w_{p-1}(t, x)\in C^\infty((0, T)\times U)$ of the form (4.14) such that (4.15) and (4.16) are valid for $k=1, \cdots, p-1$. Then, we can find $w_p(t, x)$ in the following way. Put $u_p(t, x)=u(t, x)-\sum_{j=1}^{p-1}w_j(t, x)$. Then, $u_p(t, x)$ satisfies $u_p(t, x)=o(t^{m_{p-1}}; \nabla^\infty)$ on U (as $t\to+0$). Therefore, by (1) of Lemma 8 we obtain $t\mathcal{R}[u_p](t, x)=o(t^{m_p+\varepsilon}; \nabla^\infty)$ on U (as $t\to+0$) for some $\varepsilon>0$. On the other hand, by (4.12) we have

$$C(t\partial_t, x)u_p(t, x)=-C(t\partial_t, x)\Big(\sum_{j=1}^{p-1} w_j(t, x)\Big)+t\mathcal{R}\Big[\sum_{j=1}^{p-2} w_j\Big](t, x)$$
$$+t\mathcal{R}[w_{p-1}](t, x)+t\mathcal{R}[u_p](t, x).$$

Hence, by (4.15) for $k=p-1$ and by (2) of Lemma 8 we can obtain the following equation

$$C(t\partial_t, x)u_p(t, x)=\sum_{\substack{l\in I_p\\ K(\lambda_{p,l})\neq 0}}\sum_{h=0}^{mK(\lambda_{p,l})}\phi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h}+o(t^{m_p+\varepsilon}; \nabla^\infty), \tag{4.17}$$

where $\phi_{p,l}^h(x)\in C^\infty(U)$ are known functions determined by $w_1(t, x), \cdots, w_{p-1}(t, x)$. Consequently, by (3) of Lemma 7, $u_p(t, x)$ is expressed in the form $u_p(t, x)=w_p(t, x)+o(t^{m_p}; \nabla^\infty)$ on U (as $t\to+0$) for some

$$w_p(t, x)=\sum_{\substack{l\in I_p\\ K(\lambda_{p,l})=0}}\varphi_{p,l}(x)t^{\lambda_{p,l}(x)}+\sum_{\substack{l\in I_p\\ K(\lambda_{p,l})\neq 0}}\sum_{h=0}^{mK(\lambda_{p,l})}\varphi_{p,l}^h(x)t^{\lambda_{p,l}(x)}(\log t)^{mK(\lambda_{p,l})-h}, \tag{4.18}$$

where $\varphi_{p,l}(x)$, $\varphi_{p,l}^h(x)\in C^\infty(U)$. Thus, we have obtained $w_p(t, x)\in C^\infty((0, T)\times U)$ of the form (4.14) such that (4.15) and (4.16) are also valid for $k=p$. Repeating the same argument as above, we can obtain all the terms $w_j(t, x)\in C^\infty((0, T)\times U)$ $(j\geqq 1)$ of the form (4.14) such that

$$u(t, x) \sim \sum_{j=1}^{\infty} w_j(t, x)$$

on U (as $t \to +0$). This immediately leads us to (4.5). Thus, we have proved the first half of Proposition 6. Further, by (4) of Lemma 7 we have the following: if $w_j(t, x)=0$ on $(0, T) \times U$ for $j=1, \cdots, p-1$, then in (4.17) we have $\phi^h_{p,l}(x)=0$ on U for any l and h, and therefore in (4.18) we also have $\varphi^h_{p,l}(x)=0$ on U for any l and h. Therefore, the latter half of Proposition 6 is clear. Q. E. D.

Thus, the proof of Theorem 1 is completed. Note that the hyperbolicity is not used explicitly in the discussion of this section. Therefore, Theorem 1 is also valid for any operator P which satisfies the following two conditions: (i) P is decomposed into the form (4.11), and (ii) P has the tempered growth condition (as $t \to +0$). However, it seems to the author that such operators are closely related to Fuchsian hyperbolic operators in a suitable sense. For example, we can easily see the following: if $P=t\partial_t+\alpha t^m \partial_x+\beta$ (where $m \in \boldsymbol{Z}$ and $\alpha, \beta \in \boldsymbol{C}$) has the tempered growth condition (as $t \to +0$), then we have $m \geqq 1$ and $\alpha \in \boldsymbol{R}$.

§5. Completion of the proof of Main Theorem.

Fifthly, we complete the proof of Main Theorem in the introduction. Since the part (1) of Main Theorem is already proved in Theorem 1, to obtain Main Theorem it is sufficient to show the following result.

THEOREM 2. *Assume that P satisfies* (S-1)$\sim$(S-5). *In addition, assume that $\rho_i(x)-\rho_j(x) \notin \boldsymbol{Z}$ holds for any $x \in \boldsymbol{R}^n$ and $1 \leqq i \neq j \leqq m$. Then, for any $\varphi_1(x), \cdots, \varphi_m(x) \in C^\infty(\boldsymbol{R}^n)$ we can find a solution $u(t, x) \in C^\infty((0, T) \times \boldsymbol{R}^n)$ of* (S) *and coefficients $\varphi^{(i)}_{k,h}(x) \in C^\infty(\boldsymbol{R}^n)$ $(1 \leqq i \leqq m,\ 1 \leqq k < \infty$ and $0 \leqq h \leqq mk)$ such that the following asymptotic relation holds:*

$$u(t, x) \sim \sum_{i=1}^{m} \Big(\varphi_i(x) t^{\rho_i(x)} + \sum_{k=1}^{\infty} \sum_{h=0}^{mk} \varphi^{(i)}_{k,h}(x) t^{\rho_i(x)+k} (\log t)^{mk-h} \Big) \tag{5.1}$$

on $\boldsymbol{R}^n$ (as $t \to +0$). Further, such a solution $u(t, x)$ and coefficients $\varphi^{(i)}_{k,h}(x)$ are uniquely determined by $\varphi_1(x), \cdots, \varphi_m(x)$.

Before the proof of Theorem 2, we prepare two propositions: the one asserts the existence of formal solutions of (S) and the other asserts the C^∞ well posedness of the flat Cauchy problem for P.

PROPOSITION 7. *Assume that $\rho_i(x)-\rho_j(x) \notin \boldsymbol{Z}$ holds for any $x \in \boldsymbol{R}^n$ and $1 \leqq i \neq j \leqq m$. Then, for any $\varphi_1(x), \cdots, \varphi_m(x) \in C^\infty(\boldsymbol{R}^n)$ we can determine the coefficients $\varphi^{(i)}_{k,h}(x) \in C^\infty(\boldsymbol{R}^n)$ $(1 \leqq i \leqq m,\ 1 \leqq k < \infty$ and $0 \leqq h \leqq mk)$ of the formal sum*

$$\hat{u}(t, x) = \sum_{i=1}^{m} \Big(\varphi_i(x) t^{\rho_i(x)} + \sum_{k=1}^{\infty} \sum_{h=0}^{mk} \varphi^{(i)}_{k,h}(x) t^{\rho_i(x)+k} (\log t)^{mk-h} \Big)$$

so that $\hat{u}(t, x)$ becomes a formal solution of (S) *in the following sense: for any $s>0$ and any compact subset K of $\boldsymbol{R}^n$, there is an integer N_0 such that for any $N \geqq N_0$*

$$P(t, x, \partial_t, \partial_x)\Big(\sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{N}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big)\Big) \\ =o(t^s; \nabla^\infty) \quad \text{on } K \text{ (as } t\to+0). \tag{5.2}$$

PROOF. To obtain this, we have only to discuss a formal solution $\hat{u}_i(t, x)$ of the form

$$\hat{u}_i(t, x)=\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h} \tag{5.3}$$

corresponding to $\rho_i(x)$ and $\varphi_i(x)$ only. Recall that P is decomposed in (4.11) into the form

$$P(t, x, \partial_t, \partial_x)=\mathcal{C}(t\partial_t, x)-tR(t, x, t\partial_t, \partial_x)$$

for some linear differential operator $R(t, x, \partial_t, \partial_x)$ of order m with coefficients in $C^\infty([0, T)\times\boldsymbol{R}^n)$. Therefore, to obtain a formal solution $\hat{u}_i(t, x)$ of the form (5.3), it is sufficient to show that we can determine the coefficients $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$ $(0\leqq h\leqq mk)$ of

$$w_k(t, x)=\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}$$

(where $k\geqq 0$ and $\varphi_{0,0}^{(i)}(x)=\varphi_i(x)$) so that the following conditions are satisfied:

$$\mathcal{C}(t\partial_t, x)\Big(\sum_{j=0}^{k}w_j(t, x)\Big)=tR(t, x, t\partial_t, \partial_x)\Big(\sum_{j=0}^{k-1}w_j(t, x)\Big)+o(t^{\rho_i(x)+k}; \nabla^\infty) \tag{5.4}$$

on $\boldsymbol{R}^n$ (as $t\to+0$) for any $k\geqq 0$. Note that $\mathcal{C}(\rho_i(x), x)=0$ holds on $\boldsymbol{R}^n$ and that

$$\mathcal{C}(t\partial_t, x)(\varphi_i(x)t^{\rho_i(x)})=\mathcal{C}(\rho_i(x), x)\varphi_i(x)t^{\rho_i(x)}=0$$

holds on $(0, T)\times\boldsymbol{R}^n$ for any $\varphi_i(x)\in C^\infty(\boldsymbol{R}^n)$. Therefore, we can take the coefficient $\varphi_{0,0}^{(i)}(x)(=\varphi_i(x))\in C^\infty(\boldsymbol{R}^n)$ of $w_0(t, x)$ arbitrarily. Suppose that $p\geqq 1$ and that the coefficients $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$ $(0\leqq k\leqq p-1$ and $0\leqq h\leqq mk)$ of $w_0(t, x), \cdots, w_{p-1}(t, x)$ are already determined so that (5.4) is valid for $k=0, \cdots, p-1$. Then, by an easy calculation we can transform (5.4) for $k=p$ into the form

$$\mathcal{C}(t\partial_t, x)w_p(t, x)=\sum_{h=0}^{mp}\psi_{p,h}^{(i)}(x)t^{\rho_i(x)+p}(\log t)^{mp-h}+o(t^{\rho_i(x)+p}; \nabla^\infty) \tag{5.5}$$

on $\boldsymbol{R}^n$ (as $t\to+0$) for some known functions $\psi_{p,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$. Since $\mathcal{C}(\rho_i(x)+p, x)\neq 0$ holds on $\boldsymbol{R}^n$, by (5.5) we can uniquely determine the coefficients $\varphi_{p,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$ $(0\leqq h\leqq mp)$ of $w_p(t, x)$. Thus, by induction on k we can obtain all the coefficients $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$ $(0\leqq k<\infty$ and $0\leqq h\leqq mk)$ of $w_k(t, x)$ $(k\geqq 0)$ such that (5.4) is valid for any $k\geqq 0$. Hence, we obtain a formal solution $\hat{u}_i(t, x)$ of the form

(5.3) by

$$\hat{u}_i(t, x) = \sum_{k=0}^{\infty} w_k(t, x)$$

for any $\varphi_{0,0}^{(i)}(x)(=\varphi_i(x)) \in C^\infty(\boldsymbol{R}^n)$. Q. E. D.

PROPOSITION 8. *Assume that P satisfies* (S-1)$\sim$(S-5). *Then, for any $g(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$ such that $g(t, x)\sim 0$ on $\boldsymbol{R}^n$ (as $t\to +0$), there exists a unique solution $v(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$ of $P(t, x, \partial_t, \partial_x)v(t, x)=g(t, x)$ which satisfies $v(t, x)\sim 0$ on $\boldsymbol{R}^n$ (as $t\to +0$).*

PROOF. Note that the following two conditions on $u(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$ are equivalent to each other: (i) $u(t, x)\sim 0$ on $\boldsymbol{R}^n$ (as $t\to +0$), and (ii) $u(t, x) \in C^\infty([0, T)\times \boldsymbol{R}^n)$ and $(\partial_t^l u)(0, x)=0$ on $\boldsymbol{R}^n$ for any $l\geqq 0$. Therefore, Proposition 8 is an easy consequence of Theorem 3.1 in Tahara [**9-III**] and its proof.

Q. E. D.

Now, let us give a proof of Theorem 2 by using the above propositions.

PROOF OF THEOREM 2. The proof of the first half is as follows. Let $\varphi_1(x), \cdots, \varphi_m(x) \in C^\infty(\boldsymbol{R}^n)$ and let $\hat{u}(t, x)$ be the formal solution of (S) constructed in Proposition 7 corresponding to $\varphi_1(x), \cdots, \varphi_m(x)$. Then, by Proposition 2 we can find $w(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$ such that $w(t, x)\sim \hat{u}(t, x)$ on $\boldsymbol{R}^n$ (as $t\to +0$). Put $g(t, x)=P(t, x, \partial_t, \partial_x)w(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$. Then, we have $g(t, x)\sim 0$ on $\boldsymbol{R}^n$ (as $t\to +0$), because $P\hat{u}(t, x)\sim 0$ on $\boldsymbol{R}^n$ (as $t\to +0$) in the sense of (5.2). Therefore, by Proposition 8 we obtain $v(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$ such that $P(t, x, \partial_t, \partial_x)v(t, x)=g(t, x)$ and $v(t, x)\sim 0$ on $\boldsymbol{R}^n$ (as $t\to +0$). Hence, by putting $u(t, x)=w(t, x)-v(t, x)$ we obtain a solution $u(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$ of (S) which satisfies $u(t, x)\sim \hat{u}(t, x)$ on $\boldsymbol{R}^n$ (as $t\to +0$). This implies (5.1). Thus, the first half of Theorem 2 is proved. The proof of the latter half of Theorem 2 is as follows. Assume that a solution $u(t, x) \in C^\infty((0, T)\times \boldsymbol{R}^n)$ of (S) and functions $\varphi_{k,h}^{(i)}(x) \in C^\infty(\boldsymbol{R}^n)$ ($1\leqq i\leqq m$, $1\leqq k<\infty$ and $0\leqq h\leqq mk$) satisfy the following asymptotic relation

$$u(t, x)\sim \sum_{i=1}^{m} \sum_{k=1}^{\infty} \sum_{h=0}^{mk} \varphi_{k,h}^{(i)}(x) t^{\rho_i(x)+k} (\log t)^{mk-h} \tag{5.6}$$

on $\boldsymbol{R}^n$ (as $t\to +0$). Then, by the last part of Theorem 1 we have $\varphi_{k,h}^{(i)}(x)=0$ on $\boldsymbol{R}^n$ for any i, k and h. This implies that $u(t, x)\sim 0$ on $\boldsymbol{R}^n$ (as $t\to +0$). Therefore, by the uniqueness part of Proposition 8 we also have $u(t, x)=0$ on $(0, T)\times \boldsymbol{R}^n$. Thus, we have proved the following: if $u(t, x)$ is a solution of (S), then (5.6) implies $u(t, x)=0$ on $(0, T)\times \boldsymbol{R}^n$ and $\varphi_{k,h}^{(i)}(x)=0$ on $\boldsymbol{R}^n$ for any i, k and h. This immediately leads us to the latter half of Theorem 2. Q.E.D.

By Theorem 1 (in §4) and Theorem 2 (in this section), we obtain Main Theorem in the introduction. Thus, the proof of Main Theorem is completed at last.

§ 6. Examples.

Sixthly, we give some typical examples to which we can apply our results.

EXAMPLE 1. Let P_1 be the Euler-Poisson-Darboux operator of the form

$$P_1=\partial_t^2-\Delta+\frac{\alpha}{t}\partial_t ,$$

where $(t, x)\in[0, T)\times\boldsymbol{R}^n$, $\Delta=\partial_{x_1}^2+\cdots+\partial_{x_n}^2$ and $\alpha\in\boldsymbol{C}$. Then, t^2P_1 satisfies our conditions with $\mu=1$ and $Q=\xi_1^2+\cdots+\xi_n^2$, and the characteristic exponents are given by $\lambda=0, 1-\alpha$. Note that by the change of variable $t^2\to t$ we can transform t^2P_1 into the form $\tilde{P}_1=4t^2\partial_t^2-t\Delta+2(1+\alpha)t\partial_t$, and that $\tilde{P}_1$ also satisfies our conditions with $\mu=1/2$ and $Q=\xi_1^2+\cdots+\xi_n^2$. Therefore, we can obtain the following: if $\alpha\neq\pm1, \pm3, \pm5, \cdots$, then any solution $u(t, x)\in C^\infty((0, T)\times\boldsymbol{R}^n)$ of $P_1u=0$ is characterized by the asymptotic expansion (as $t\to+0$) of the form

$$u(t, x)\sim\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{1+\alpha}{2}\right)\Delta^k\varphi_1(x)}{2^{2k}\Gamma(k+1)\Gamma\left(k+\frac{1+\alpha}{2}\right)}t^{2k}$$

$$+\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{3-\alpha}{2}\right)\Delta^k\varphi_2(x)}{2^{2k}\Gamma(k+1)\Gamma\left(k+\frac{3-\alpha}{2}\right)}t^{2k+1-\alpha}.$$

EXAMPLE 2. Let P_2 be of the form

$$P_2=t^2\partial_t^2-t^k\partial_x^2+t^la(t, x)\partial_x+b(t, x)t\partial_t+c(t, x) ,$$

where $(t, x)\in[0, T)\times\boldsymbol{R}$, k and l are positive integers such that $l\geqq k/2$, and $a(t, x)$, $b(t, x)$, $c(t, x)\in B^\infty([0, T)\times\boldsymbol{R})$. Then, P_2 satisfies our conditions with $\mu=k/2$ and $Q=\xi^2$, and the characteristic exponents $\rho_\pm(x)$ are given by

$$\rho_\pm(x)=\frac{b(0, x)-1}{2}\pm\frac{\sqrt{(b(0, x)-1)^2-4c(0, x)}}{2}.$$

Therefore, if $\rho_+(x)-\rho_-(x)\notin\boldsymbol{Z}$ holds for any $x\in\boldsymbol{R}$, then any solution $u(t, x)\in C^\infty((0, T)\times\boldsymbol{R})$ of $P_2u=0$ is characterized by the asymptotic expansion (as $t\to+0$) of the form

$$u(t, x)\sim\sum_\pm(\varphi_\pm(x)t^{\rho_\pm(x)}+\cdots).$$

EXAMPLE 3. Let P_3 be of the form

$$P_3=R(t, x, t\partial_t, t^\kappa\partial_x) ,$$

$$t^\kappa\partial_x=(t^{\kappa_1}\partial_{x_1}, \cdots, t^{\kappa_n}\partial_{x_n}) ,$$

where $(t, x)\in[0, T)\times\boldsymbol{R}^n$, κ_i $(1\leqq i\leqq n)$ are positive integers, and $R(t, x, \partial_t, \partial_x)$ is a regularly hyperbolic operator (for example, see Mizohata [4]) with coefficients in $B^\infty([0, T)\times\boldsymbol{R}^n)$. Then, P_8 satisfies our conditions with $\mu=\min\{\kappa_i;\ 1\leqq i\leqq n\}$ and $Q=\xi_1^{2(\kappa_1-\mu)}+\cdots+\xi_n^{2(\kappa_n-\mu)}$, and the indicial polynomial is given by $\mathcal{C}(\lambda, x)=R(0, x, \lambda, 0)$. Therefore, we can apply our results to the equation $P_8u=0$ in $C^\infty((0, T)\times\boldsymbol{R}^n)$. See Tahara [10].

§ 7. Supplementary remarks.

Lastly, we give some remarks. Throughout our whole discussions before, we have assumed that the coefficients of P belong to $B^\infty([0, T)\times\boldsymbol{R}^n)$ or $C^\infty([0, T)\times\boldsymbol{R}^n)$. However, this condition is too restricted from the view point of asymptotic analysis. Here, we give a wider condition and generalize our results to some extent.

Let $P(t, x, \partial_t, \partial_x)$ be a Fuchsian type partial differential operator of order m and let $a_{j,\alpha}(t, x)$ $(1\leqq j\leqq m$ and $|\alpha|\leqq j)$ be the coefficients of P in (1.1). Instead of the condition $a_{j,\alpha}(t, x)\in B^\infty([0, T)\times\boldsymbol{R}^n)$, we assume the following two conditions on $a_{j,\alpha}(t, x)$ for any j and α:

(T-1) $a_{j,\alpha}(t, x)\in B^0([0, T)\times\boldsymbol{R}^n)\cap C^\infty((0, T)\times\boldsymbol{R}^n)$ and $(t\partial_t)^l\partial_x^\beta a_{j,\alpha}(t, x)\in B^0([0, T)\times\boldsymbol{R}^n)$ for any l and β.

(T-2) There is a positive integer N (independent of j and α) such that $a_{j,\alpha}(t, x)$ is expanded asymptotically into the form

$$a_{j,\alpha}(t, x)\sim\sum_{k=0}^{\infty}a_{j,\alpha}^{(k)}(x)t^{k/N}$$

on $\boldsymbol{R}^n$ (as $t\to+0$) for some $a_{j,\alpha}^{(k)}(x)\in C^\infty(\boldsymbol{R}^n)$.

Then, under (S-1)$\sim$(S-5) we can obtain Propositions 4 and 8 also in this case. In fact, to obtain these results we do not necessarily need the condition $a_{j,\alpha}(t, x)\in B^\infty([0, T)\times\boldsymbol{R}^n)$ but we need only the condition (T-1). Therefore, carrying out the change of variables $t^{1/N}\to t$ and $x\to x$, and applying the same argument as developed in the previous sections, we obtain the following theorem.

THEOREM 3. *Assume that P satisfies* (T-1), (T-2) *and* (S-1)$\sim$(S-5). *In addition, assume that $\rho_i(x)-\rho_j(x)\notin\boldsymbol{Z}/N=\{k/N;\ k\in\boldsymbol{Z}\}$ holds for any $x\in\boldsymbol{R}^n$ and $1\leqq i\neq j\leqq m$. Then, we have the following results.*

(1) *Any solution $u(t, x)\in C^\infty((0, T)\times\boldsymbol{R}^n)$ of* (S) *can be expanded asymptotically into the form*

$$u(t, x)\sim\sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k/N}(\log t)^{mk-h}\Big)\qquad(**)$$

on $\boldsymbol{R}^n$ (as $t\to+0$) for some $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)\in C^\infty(\boldsymbol{R}^n)$. Further, such coefficients $\varphi_i(x)$, $\varphi_{k,h}^{(i)}(x)$ are uniquely determined by $u(t, x)$.

(2) *Conversely, for any* $\varphi_1(x), \cdots, \varphi_m(x) \in C^\infty(\boldsymbol{R}^n)$ *we can find a solution* $u(t, x) \in C^\infty((0, T) \times \boldsymbol{R}^n)$ *of* (S) *and coefficients* $\varphi_{k,h}^{(i)}(x) \in C^\infty(\boldsymbol{R}^n)$ $(1 \leqq i \leqq m,\ 1 \leqq k < \infty$ *and* $0 \leqq h \leqq mk)$ *such that the asymptotic relation* (**) *in* (1) *holds. Further, such a solution* $u(t, x)$ *and coefficients* $\varphi_{k,h}^{(i)}(x)$ *are uniquely determined by* $\varphi_1(x), \cdots, \varphi_m(x)$.

Note that the functions t^q, $t^q a(t, x)$, $a(t^q, x), \cdots$ satisfy (T-1) and (T-2), if q is a positive rational number and $a(t, x) \in B^\infty([0, T) \times \boldsymbol{R}^n)$. Therefore, we can apply Theorem 3 to the operator P_2 (in Example 2) when k and l are positive rational numbers, and also to the operator P_3 (in Example 3) when κ_i $(1 \leqq i \leqq n)$ are positive rational numbers.

References

[1] M. S. Baouendi and C. Goulaouic, Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math., **26** (1973), 455-475.

[2] Chi Min-You, On Fuchsian type partial differential equations, Sci. Rep. Wuhan Univ., **4** (1977), 1-6 (Chinese).

[3] V. Kh. Froim, Linear scalar partial differential equations with regular singularities on a hypersurface, Differencial'nye Uravnenija, **9**-3 (1973), 533-541 (Russian).

[4] S. Mizohata, The theory of partial differential equations, Cambridge Univ. Press, 1973.

[5] O. A. Repin, Determination of the general solution and solution of the Cauchy problem for an equation of hyperbolic type, Differencial'nye Uravnenija (Ryazan), **6** (1975), 155-161 (Russian).

[6] R. Sakamoto, Cauchy problem for degenerate hyperbolic equations, Comm. Pure Appl. Math., **33** (1980), 785-816.

[7] H. Tahara, Fuchsian type equations and Fuchsian hyperbolic equations, Japan. J. Math. New Ser., **5** (1979), 245-347.

[8] H. Tahara, Fundamental systems of analytic solutions of Fuchsian type partial differential equations, Funkcialaj Ekvacioj, **24** (1981), 135-140.

[9] H. Tahara, Singular hyperbolic systems.
I. Existence, uniqueness and differentiability, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **26** (1979), 213-238.
II. Pseudo-differential operators with a parameter and their applications to singular hyperbolic systems, ibid., **26** (1979), 391-412.
III. On the Cauchy problem for Fuchsian hyperbolic partial differential equations, ibid., **27** (1980), 465-507.
IV. Remarks on the Cauchy problem for singular hyperbolic partial differential equations, Japan. J. Math. New Ser., **8** (1982), 297-308.

[10] H. Tahara, Asymptotic expansions of solutions of Fuchsian hyperbolic partial differential equations, Proc. Japan Acad., **59** (1983), 120-121.

[11] S. A. Tersenov, On a singular Cauchy problem, Dokl. Akad. Nauk SSSR, **196** (1971), 1032-1035 (Russian).

[12] W. Wasow, Asymptotic expansions for ordinary differential equations, Interscience, 1965.

[13] A. Weinstein, The singular solutions and the Cauchy problem for generalized Tricomi equations, Comm. Pure Appl. Math., **7** (1954), 105-116.

Hidetoshi TAHARA
Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku
Tokyo 102, Japan