J. Math. Soc. Japan
Vol. 36 No. 3, 1984

On spectral geometry of Kaehler submanifolds

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(Received Feb. 7, 1983)
(Revised July 7, 1983)

Introduction.

Let $x : M^n \to E^N$ be an isometric immersion of a compact Riemannian manifold into the Euclidean space. Let Δ be the Laplace-Beltrami operator of M acting on differentiable functions and $\mathrm{Spec}(M)=\{0<\lambda_1 \cdots \lambda_1<\lambda_2 \cdots \lambda_2< \cdots\}$ the spectrum of Δ, where each eigenvalue is repeated as many times as its multiplicity indicates. If $\phi : M \to E^N$ is a differentiable mapping we put $\phi=(\phi^1, \cdots, \phi^N)$, where ϕ^i is the i-th coordinate function of ϕ, and $\Delta\phi=(\Delta\phi^1, \cdots, \Delta\phi^N)$. We have the decomposition $x=\sum_k x_k$, $k \in \boldsymbol{N}$, where $x_k : M \to E^N$ is a differentiable mapping, $\Delta x_k=\lambda_k x_k$, and the addition is convergent, componentwise, for the L^2-topology on $C^\infty(M)$. Moreover x_0 is a constant mapping (it is the center of gravity of M) and $\{x_k\}_k$ are orthogonal mappings, that is

$$\int_M g(x_k, x_r)=0 \qquad \text{for all} \quad k, r, \quad k \neq r,$$

where g is the Euclidean metric on E^N. We have the relations

$$\Delta x=-nH=\sum_{k \geq 1} \lambda_k x_k,$$

$$\Delta^2 x=-n\Delta H=\sum_{k \geq 1} \lambda_k^2 x_k,$$

where H is the mean curvature vector of M in E^N. Let $k_1, k_2 \in \boldsymbol{N}$, $0 \leqq k_1 < k_2$. We say that *the immersion x is of order $\{k_1, k_2\}$* if

$$x_k=0 \qquad \text{for all} \quad k \in \boldsymbol{N}, \quad k \neq 0, k_1 \text{ or } k_2.$$

If $k_1=0$ we say simply that *the immersion is of order k_2*.

It is well-known that the complex projective space, $\boldsymbol{C}P^m$, with the Fubini-Study metric, admits an isometric imbedding of order 1 in the Euclidean space, which has parallel second fundamental form. In [**11**] S. Tai gives a simple version of this one. From this fact we can view any submanifold, M^n, of $\boldsymbol{C}P^m$ as a submanifold of the Euclidean space, $x : M^n \to E^N$. In [**10**] we have obtained some information about the spectral geometry of submanifolds in the complex projective space, studying Δx and the order k immersion. In this paper

we want to study spectral geometry of Kaehler submanifolds in $\boldsymbol{C}P^m$, corresponding to $\Delta^2 x$ and to immersion of order $\{k_1, k_2\}$. The main results are in section 5. We give a spectral inequality involving λ_1 and λ_2 in Spec(M), and we characterize some submanifolds in terms of these geometric invariants.

Manifolds are assumed to be connected and of real dimension $\geqq 2$, unless mentioned otherwise. For the necessary knowledge and notations of geometry of submanifolds see [3], for particular Kaehler submanifolds [9], and for spectral geometry see [2].

However we give the basic tools of submanifolds theory. Let M^n be an n-dimensional isometrically immersed submanifold of $\tilde{M}^m$. Let X, Y, Z (resp. ξ) be tangent (resp. normal) vector fields to M. Let $\tilde{\nabla}$ and ∇ be the Riemannian connections of $\tilde{M}$ and M respectively and $\nabla^\perp$ the normal connection of M in $\tilde{M}$. The second fundamental form σ and the Weingarten endomorphism A of the immersion are given by $\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$ and $A_\xi X=\nabla^\perp_X\xi-\tilde{\nabla}_X\xi$ respectively. Moreover the covariant derivative of σ is given by $(\nabla\sigma)_X(Y, Z)=\nabla^\perp_X\sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z)$, and the mean curvature vector of the immersion is $H=(1/n)\sum_i\sigma(E_i, E_i)$, where $\{E_i\}_{i=1,\cdots,n}$ is an orthonormal basis in the tangent space at any point of M.

1. The complex projective space.

For details in this section see [10].

Let $\mathrm{HM}(m)=\{A\in gl(m, \boldsymbol{C}) \mid \bar{A}=A^t\}$ be the space of $m\times m$-Hermitian matrices. We define on HM(m) the metric

$$g(A, B)=2\operatorname{tr}AB \quad \text{for all } A, B \text{ in } \mathrm{HM}(m). \tag{1.1}$$

We consider the submanifold $\boldsymbol{C}P^m=\{A\in\mathrm{HM}(m+1) \mid AA=A, \operatorname{tr}A=1\}$. It is known that $\boldsymbol{C}P^m$, with the metric induced by the one on HM($m+1$), is isometric to the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. The tangent space, with the usual identification, and the normal space at any point A of $\boldsymbol{C}P^m$ are given by

$$T_A(\boldsymbol{C}P^m)=\{X\in\mathrm{HM}(m+1) \mid XA+AX=X\}, \quad \text{and} \tag{1.2}$$

$$T^\perp_A(\boldsymbol{C}P^m)=\{Z\in\mathrm{HM}(m+1) \mid ZA=AZ\}, \tag{1.3}$$

respectively. The complex structure is given by

$$JX=\sqrt{-1}(I-2A)X \quad \text{for all } X \text{ in } T_A(\boldsymbol{C}P^m), \tag{1.4}$$

where I is the $(m+1)\times(m+1)$-identity matrix. Let D be the Riemannian connection of HM($m+1$), $\tilde{\nabla}$ the induced connection in $\boldsymbol{C}P^m$, $\tilde{\sigma}$ the second fundamental form of the immersion, $\tilde{\nabla}^\perp$ and $\tilde{A}$ the normal connection and the Weingarten

endomorphism, and $\tilde{H}$ the mean curvature vector of $\boldsymbol{C}P^m$ in HM$(m+1)$. Then we have

$$\tilde{\sigma}(X, Y)=(XY+YX)(I-2A), \tag{1.5}$$

$$\tilde{A}_Z X=(XZ-ZX)(I-2A), \tag{1.6}$$

$$\tilde{H}=(1/2m)[I-(m+1)A], \tag{1.7}$$

$$\tilde{\sigma}(JX, JY)=\tilde{\sigma}(X, Y), \tag{1.8}$$

where X, Y are in $T_A(\boldsymbol{C}P^m)$, Z is in $T_A^{\perp}(\boldsymbol{C}P^m)$. Moreover

$$\tilde{\nabla}\tilde{\sigma}=0, \tag{1.9}$$

that is, the second fundamental form is parallel. The action of the unitary group $U(m+1)$ over $\boldsymbol{C}P^m$ is given by $(P, A)\mapsto PAP^{-1}$, where P is in $U(m+1)$ and A is in $\boldsymbol{C}P^m$. Hence the imbedding of $\boldsymbol{C}P^m$ in HM$(m+1)$ is $U(m+1)$-equivariant. The standard projection of $\boldsymbol{C}^{m+1}-\{0\}$ over $\boldsymbol{C}P^m$ is as follows

$$z\longmapsto (1/z\bar{z}^t)\bar{z}^t z, \tag{1.10}$$

where $z=(z^0, \cdots, z^m)$ is in $\boldsymbol{C}^{m+1}-\{0\}$. We put

$$B=\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Then B is in $\boldsymbol{C}P^m$. Moreover X is in $T_B(\boldsymbol{C}P^m)$ if and only if

$$X=\begin{pmatrix} 0 & b \\ \bar{b}^t & 0 \end{pmatrix}, \quad \text{where } b \text{ is in } \boldsymbol{C}^m. \tag{1.11}$$

From (1.1), (1.4), (1.5) and (1.11), and as the imbedding is equivariant we obtain

$$\begin{aligned} g(\tilde{\sigma}(X, Y), \tilde{\sigma}(V, W))&=(1/2)g(X, Y)g(V, W)+(1/4)\{g(X, W)g(Y, V) \\ &\quad +g(X, V)g(Y, W)+g(X, JW)g(Y, JV)+g(X, JV)g(Y, JW)\}, \\ &\quad X, Y, V, W\in T_A(\boldsymbol{C}P^m). \end{aligned} \tag{1.12}$$

Hence

$$\begin{aligned} \tilde{A}_{\tilde{\sigma}(X,Y)}V&=(1/2)g(X, Y)V+(1/4)\{g(Y, V)X+g(X, V)Y \\ &\quad +g(JY, V)JX+g(JX, V)JY\}, \qquad X, Y, V\in T_A(\boldsymbol{C}P^m). \end{aligned} \tag{1.13}$$

We also consider the relations

$$g(\tilde{\sigma}(X, Y), I)=0, \quad g(\tilde{\sigma}(X, Y), A)=-g(X, Y), \quad X, Y\in T_A(\boldsymbol{C}P^m). \tag{1.14}$$

2. An example: the complex quadric.

If we take in the $(n+1)$-dimensional complex projective space the homogeneous coordinate system determinated by the canonical projection (1.10), the (standard) complex quadric of complex dimension n, $\boldsymbol{Q}^n$, is given by the set of points $(z^0, \cdots, z^{n+1})$ verifying $\sum_i (z^i)^2=0$. Then $\boldsymbol{Q}^n$ is a Kaehler hypersurface of $\boldsymbol{C}P^{n+1}$ holomorphically isometric to the Hermitian symmetric space $SO(n+2)/SO(2)\times SO(n)$. Now we prove that $\boldsymbol{Q}^n$ is a submanifold of order $\{1, 2\}$ of $\mathrm{HM}(n+2)$. From (1.10) we have easily

$$\boldsymbol{Q}^n=\{A\in \boldsymbol{C}P^{n+1} \mid AA^t=0\}\ .$$

The action of $SO(n+2)$ over $\boldsymbol{Q}^n$ is given by $(P, A)\mapsto PAP^{-1}$, where P is in $SO(n+2)$ and A is in $\boldsymbol{Q}^n$. Hence the imbedding of $\boldsymbol{Q}^n$ in $\mathrm{HM}(n+2)$ is $SO(n+2)$-equivariant. We choose C in $\mathrm{HM}(n+2)$ as follows

$$C=\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{with} \quad c=\frac{1}{2}\begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}.$$

Then C is in $\boldsymbol{Q}^n$ and

$$T_C(\boldsymbol{Q}^n)=\{X\in T_C(\boldsymbol{C}P^{n+1}) \mid XC^t+CX^t=0\}\ .$$

Let $\{E_i, JE_i\}_{i=2,\cdots,n+1}$ be the orthonormal basis of $T_C(\boldsymbol{Q}^n)$ defined by

$$E_i=\frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & \begin{matrix} 0 & \cdots & \overset{(i)}{1} & \cdots & 0 \\ 0 & \cdots & -\sqrt{-1} & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 1 & \sqrt{-1} \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & 0 \end{pmatrix}_{(i)}.$$

Let H be the mean curvature vector of $\boldsymbol{Q}^n$ in $\mathrm{HM}(n+2)$. As $\boldsymbol{Q}^n$ is minimal in $\boldsymbol{C}P^{n+1}$, using (1.8) we have

$$H_C=\frac{1}{n}\sum_{i=2,\cdots,n+1}\tilde{\sigma}(E_i, E_i)\ .$$

A direct calculation proves that

$$H_C=\frac{1}{2n}\begin{pmatrix} -nc & \cdots\ 0\ \cdots \\ \begin{matrix} \vdots \\ 0 \\ \vdots \end{matrix} & I_{n\times n} \end{pmatrix}=\frac{1}{2n}[I-(n+1)C-C^t]\ ,$$

and the imbedding being equivariant we obtain

$$H=(1/2n)[I-(n+1)A-A^t], \qquad \text{for all } A \text{ in } \boldsymbol{Q}^n.$$

Hence

$$\Delta\frac{1}{2}(A-A^t)=n\frac{1}{2}(A-A^t),$$

$$\Delta\Big[\frac{1}{2}(A+A^t)-\frac{1}{n+2}I\Big]=(n+2)\Big[\frac{1}{2}(A+A^t)-\frac{1}{n+2}I\Big],$$

where Δ is the Laplace-Beltrami operator of $\boldsymbol{Q}^n$. Moreover

$$A=\frac{1}{n+2}I+\frac{1}{2}(A-A^t)+\Big[\frac{1}{2}(A+A^t)-\frac{1}{n+2}I\Big].$$

So from table 1 the imbedding of $\boldsymbol{Q}^n$ in HM$(n+2)$ is of order $\{1, 2\}$.

3. Kaehler submanifolds.

Let M^n be a Kaehler submanifold, of complex dimension n, immersed in the $(n+p)$-dimensional complex projective space, and let $A : M^n \to \boldsymbol{C}P^{n+p}$ be the immersion. Let $E_1, \cdots, E_n, E_{1^*}=JE_1, \cdots, E_{n^*}=JE_n, \xi_1, \cdots, \xi_p, \xi_{1^*}=J\xi_1, \cdots, \xi_{p^*}=J\xi_p$ be a local field of orthonormal frames of $\boldsymbol{C}P^{n+p}$, such that, restricted to M, $E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}$ are tangent to M.[†] Let ∇, σ, $\nabla^\perp$ and A be the Riemannian connection, the second fundamental form, the normal connection and the Weingarten endomorphism of M^n in $\boldsymbol{C}P^{n+p}$ respectively, and H the mean curvature vector of M^n in HM$(n+p+1)$. We write A_λ for A_{ξ_λ} and we put $\sigma(E_i, E_j)=\sum_\lambda h_{ij}^\lambda \xi_\lambda$. We denote by Δ the Laplace-Beltrami operator of M acting on functions in $C^\infty(M)$. The natural extension of this operator to the space of differentiable mappings of M^n in HM$(n+p+1)$ is also denoted by Δ.

LEMMA 3.1. *We have the following relations:*

$$H=\frac{1}{2n}\sum_i \tilde{\sigma}(E_i, E_i), \tag{3.1}$$

$$\Delta H=(n+1)H+\frac{1}{n}\sum_{ij}\tilde{\sigma}(A_{\sigma(E_i,E_j)}E_i, E_j) \tag{3.2}$$

$$-\frac{1}{n}\sum_{ij}\tilde{\sigma}(\sigma(E_i, E_j), \sigma(E_i, E_j)).$$

[†] For the range of indices we use the following convention throughout sections 3 and 4:

$a, b=1, \cdots, n$
$i, j, k, r, s=1, \cdots, n, 1^*, \cdots, n^*$
$\alpha, \beta=1, \cdots, p$
$\lambda, \mu, x, y=1, \cdots, p, 1^*, \cdots, p^*$

$$i^*=\begin{cases} a^* & \text{if } i=a \\ a & \text{if } i=a^* \end{cases}$$

$$\lambda^*=\begin{cases} \alpha^* & \text{if } \lambda=\alpha \\ \alpha & \text{if } \lambda=\alpha^*. \end{cases}$$

PROOF. Since Kaehler submanifolds are minimal submanifolds we have (3.1). We compute the differential of the mapping $H : M^n \to \mathrm{HM}(n+p+1)$.

$$dH(E_j)=D_{E_j}H=\frac{1}{2n}\sum_i[\tilde{\nabla}^{\perp}_{E_j}\tilde{\sigma}(E_i, E_i)-\tilde{A}_{\tilde{\sigma}(E_i,E_i)}E_j]$$

$$=\frac{1}{n}\sum_i\tilde{\sigma}(\tilde{\nabla}_{E_j}E_i, E_i)-\frac{1}{2n}\sum_i\tilde{A}_{\tilde{\sigma}(E_i,E_i)}E_j$$

$$=\frac{1}{n}\sum_i\tilde{\sigma}(\nabla_{E_j}E_i+\sigma(E_i, E_j), E_i)-\frac{n+1}{2n}E_j,$$

where we have used (1.9) and (1.13). Now we prove that $\sum_i\tilde{\sigma}(\nabla_{E_j}E_i, E_i)=0$. Effectively, from (1.12) we obtain

$$g\Big(\sum_i\tilde{\sigma}(\nabla_{E_j}E_i, E_i), \tilde{\sigma}(E_r, E_s)\Big)=0 \qquad \text{for all} \quad j, r, s.$$

But $\sum_i\tilde{\sigma}(\nabla_{E_j}E_i, E_i)\in\mathrm{Span}\{\tilde{\sigma}(E_r, E_s)\}_{r,s}$. So

$$dH(E_j)=\frac{1}{n}\sum_i\tilde{\sigma}(\sigma(E_i, E_j), E_i)-\frac{n+1}{2n}E_j.$$

Let x be an arbitrary point of M. We may assume without loss of generality that $\nabla_{E_j}E_i=0$ at x. We compute ΔH at x.

$$\Delta H(x)=-\sum_j D_{E_j}(dH(E_j))_x=-\frac{1}{n}\sum_{ij}D_{E_j}\tilde{\sigma}(\sigma(E_i, E_j), E_i)+\frac{n+1}{2n}\sum_j D_{E_j}E_j$$

$$=-\frac{1}{n}\sum_{ij}[\tilde{\nabla}^{\perp}_{E_j}\tilde{\sigma}(\sigma(E_i, E_j), E_i)$$

$$-\tilde{A}_{\tilde{\sigma}(\sigma(E_i,E_j),E_i)}E_j]+\frac{n+1}{2n}\sum_j[\sigma(E_j, E_j)+\tilde{\sigma}(E_j, E_j)].$$

From (1.9) we have

$$\tilde{\nabla}^{\perp}_{E_j}\tilde{\sigma}(\sigma(E_i, E_j), E_i)=\tilde{\sigma}(\tilde{\nabla}_{E_j}\sigma(E_i, E_j), E_i)+\tilde{\sigma}(\sigma(E_i, E_j), \tilde{\nabla}_{E_j}E_i)$$

$$=\tilde{\sigma}(\nabla^{\perp}_{E_j}\sigma(E_i, E_j)-A_{\sigma(E_i,E_j)}E_j, E_i)+\tilde{\sigma}(\sigma(E_i, E_j), \sigma(E_i, E_j)).$$

From (1.13) we obtain

$$\sum_{ij}\tilde{A}_{\tilde{\sigma}(\sigma(E_i,E_j),E_i)}E_j=\sum_j\sigma(E_j, E_j)=0.$$

Finally

$$\frac{n+1}{2n}\sum_j[\sigma(E_j, E_j)+\tilde{\sigma}(E_j, E_j)]=(n+1)H.$$

So

$$\Delta H(x)=-\frac{1}{n}\sum_{ij}\tilde{\sigma}((\nabla\sigma)_{E_j}(E_i, E_j), E_i)+\frac{1}{n}\sum_{ij}\tilde{\sigma}(A_{\sigma(E_i,E_j)}E_j, E_i)$$

$$-\frac{1}{n}\sum_{ij}\tilde{\sigma}(\sigma(E_i, E_j), \sigma(E_i, E_j))+(n+1)H.$$

Now, it is enough to prove that the first term is zero. From Codazzi equation

$$(\nabla\sigma)_{E_j}(E_i, E_j)=(\nabla\sigma)_{E_i}(E_j, E_j)=-(\nabla\sigma)_{E_i}(E_{j^*}, E_{j^*}).$$

Then we have (3.2). Q.E.D.

REMARK. H and ΔH are in $T^{\perp}_{A(x)}(\boldsymbol{C}P^{n+p})$ for all $x\in M$. If we put $\sigma(E_i, E_j)=\sum_\lambda h^\lambda_{ij}\xi_\lambda$, we have

$$\Delta H=(n+1)H+\frac{1}{n}\sum_{ijk\lambda}h^\lambda_{ij}h^\lambda_{ik}\tilde{\sigma}(E_k, E_j)-\frac{1}{n}\sum_{ij\lambda\mu}h^\lambda_{ij}h^\mu_{ij}\tilde{\sigma}(\xi_\lambda, \xi_\mu). \tag{3.3}$$

LEMMA 3.2. *We have the following relations:*

$$g(A, A)=2, \tag{3.4}$$

$$g(A, H)=-1, \tag{3.5}$$

$$g(A, \Delta H)=-(n+1), \tag{3.6}$$

$$g(H, H)=\frac{n+1}{2n}, \tag{3.7}$$

$$g(H, \Delta H)=\frac{(n+1)^2}{2n}+\frac{1}{2n^2}\|\sigma\|^2, \tag{3.8}$$

$$g(\Delta H, \Delta H)=\frac{(n+1)^3}{2n}+\left(\frac{n+1}{n^2}\right)\|\sigma\|^2+\frac{1}{n^2}\sum_{\lambda\mu}(\mathrm{tr}A_\lambda A_\mu)^2+\frac{1}{n^2}\mathrm{tr}\left(\sum_\lambda A^2_\lambda\right)^2. \tag{3.9}$$

The proof can be obtained from lemma 3.1 and a systematic use of (1.12) and (1.14).

The normal space of M^n in $\boldsymbol{C}P^{n+p}$ at x is denoted by $T^{\perp}_x(M^n)$. We define the tensor $T : T^{\perp}_x(M)\times T^{\perp}_x(M)\to\boldsymbol{R}$ by

$$T(\xi, \eta)=\mathrm{tr}(A_\xi A_\eta), \quad \text{for all } \xi, \eta\in T^{\perp}_x(M).$$

LEMMA 3.3.

$$\frac{1}{2p}\|\sigma\|^4\leqq\|T\|^2\leqq\frac{1}{2}\|\sigma\|^4. \tag{3.10}$$

The first equality holds if and only if $T=kg$, where k is a real number and g is the metric restricted to $T^{\perp}_x(M)$.

PROOF. We have $\|T\|^2=\sum_{\lambda\mu}(\mathrm{tr}A_\lambda A_\mu)^2$, $\|\sigma\|^2=\sum_\lambda\mathrm{tr}A^2_\lambda$. There exists an orthonormal basis in $T^{\perp}_x(M)$, we suppose that this basis is $\xi_1, \cdots, \xi_p, \xi_{1^*}, \cdots, \xi_{p^*}$, such that the $2p\times 2p$ matrix $(\mathrm{tr}A_\lambda A_\mu)_{\lambda\mu}$ is a diagonal matrix, see [9]. We take $v=(1, \cdots, 1)$ and $w=(\mathrm{tr}A^2_\lambda)_\lambda$ in $\boldsymbol{R}^{2p}$. Then Schwartz inequality, $(vw^t)^2\leqq(vv^t)(ww^t)$, is just the first inequality in (3.10). The equality holds if and only

if $w=kv$, $k\in\boldsymbol{R}$, that is $T=kg$. The second inequality is proved in [9].

Q.E.D.

Now we give some other well-known results for Kaehler submanifolds in $\boldsymbol{C}P^{n+p}$. Let S be the Ricci tensor of M. Then we have

$$S(X, Y)=\left(\frac{n+1}{2}\right)g(X, Y)-\sum_{\lambda}g(A_{\lambda}^{2}X, Y)\,. \tag{3.11}$$

Let r be the scalar curvature of M. Then we have

$$r=n(n+1)-\|\sigma\|^{2}\,. \tag{3.12}$$

Let R be the curvature tensor of M. Then

$$\|S\|^{2}=\frac{1}{2}n(n+1)^{2}-(n+1)\|\sigma\|^{2}+\mathrm{tr}\left(\sum_{\lambda}A_{\lambda}^{2}\right)^{2}, \tag{3.13}$$

$$\|R\|^{2}=2n(n+1)-4\|\sigma\|^{2}+2\sum_{\lambda\mu}(\mathrm{tr}\,A_{\lambda}A_{\mu})^{2}\,, \tag{3.14}$$

$$-\frac{1}{2}\Delta\|\sigma\|^{2}=\|\nabla\sigma\|^{2}+\frac{1}{2}(n+2)\|\sigma\|^{2}-2\mathrm{tr}\left(\sum_{\lambda}A_{\lambda}^{2}\right)^{2}-\sum_{\lambda\mu}(\mathrm{tr}\,A_{\lambda}A_{\mu})^{2}\,. \tag{3.15}$$

Moreover, with the same notations, for any Kaehler manifold we have

$$\frac{1}{2}(n+1)n\|R\|^{2}\geqq 2n\|S\|^{2}\geqq r^{2}\,. \tag{3.16}$$

The first equality holds if and only if M has constant holomorphic sectional curvature, and the second equality holds if and only if M is Einstein. From (3.12), (3.13) and (3.16) we obtain

$$\mathrm{tr}\left(\sum_{\lambda}A_{\lambda}^{2}\right)^{2}\geqq\frac{1}{2n}\|\sigma\|^{4}\,. \tag{3.17}$$

The equality holds if and only if M is Einstein.

4. Immersions of order $\{k_1, k_2\}$.

Let $x : M^{n}\to E^{N}$ be an isometric immersion of a compact Riemannian manifold into the Euclidean space. If x is of order $\{k_1, k_2\}$ we have $x=x_0+x_{k_1}+x_{k_2}$, $-nH=\lambda_{k_1}x_{k_1}+\lambda_{k_2}x_{k_2}$ and $-n\Delta H=\lambda_{k_1}^{2}x_{k_1}+\lambda_{k_2}^{2}x_{k_2}$. Hence $\Delta H=(\lambda_{k_1}+\lambda_{k_2})H+(1/n)\lambda_{k_1}\lambda_{k_2}(x-x_0)$. Conversely, if $\Delta H=aH+b(x-x_0)$, for some real constants a, b, we have $\sum_{k>0}\lambda_{k}^{2}x_{k}=a\sum_{k>0}\lambda_{k}x_{k}-nb\sum_{k>0}x_{k}$, and so $\sum_{k>0}(\lambda_{k}^{2}-a\lambda_{k}+nb)x_{k}=0$. Then $x_{k}=0$ except, possibly, for two different values of k. So x is of order $\{k_1, k_2\}$, for some k_1, k_2, if and only if $\Delta H=aH+b(x-x_0)$, for some real constants a, b.

THEOREM 4.1. *Let M^{n} be a compact Kaehler submanifold, of complex dimension n in $\boldsymbol{C}P^{n+p}$ such that the immersion $A : M^{n}\to\boldsymbol{C}P^{n+p}$ is full. Then*

M is a submanifold of order $\{k_1, k_2\}$ in $\mathrm{HM}(n+p+1)$ *for some natural numbers k_1, k_2, if and only if M is an Einstein submanifold with $T=kg$.*

PROOF. We suppose that $\Delta H=aH+b(A-Q)$, where Q is the center of gravity of M. If $b=0$ then the immersion is of order k, for some k. From [**10**] we have that M is totally geodesic in $\boldsymbol{C}P^{n+p}$. Hence we consider $b\neq 0$. As A, H and ΔH are normal to $\boldsymbol{C}P^{n+p}$ we have that $Q\in T^{\perp}_{A(x)}(\boldsymbol{C}P^{n+p})$, for all x in M. So, from (1.3), $A(M)$ is contained in the linear subspace L of $\mathrm{HM}(n+p+1)$ defined by the equation $AQ=QA$.

We can suppose that Q is a diagonal matrix (otherwise we can use an isometry of $\mathrm{HM}(n+p+1)$ of the type $A\mapsto PAP^{-1}$, where $P\in U(n+p+1)$). We put

$$Q=\begin{pmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_r \\ & & & & & \ddots \\ & & & & & & a_r \end{pmatrix}, \quad \begin{matrix} m_1 \\ \\ \\ \\ m_r \\ \\ \end{matrix}, \qquad a_i\neq a_j\ (i\neq j).$$

Then $\boldsymbol{C}P^{n+p}\cap L$ is a disjoint union of the linear subvarieties in $\boldsymbol{C}P^{n+p}$, $S_1, \cdots, S_r$, with $\dim S_i=m_i$. Since M is connected and the immersion is full we conclude that Q is a scalar matrix. But $\mathrm{tr}A=1$ for all A in $\boldsymbol{C}P^{n+p}$, then $Q=1/(n+p+1)I$. Hence

$$\Delta H=aH+b(A-1/(n+p+1)I). \tag{4.1}$$

Take r, s, such that $r\neq s, s^*$. Applying $g(\tilde{\sigma}(E_r, E_s), -)$ to both sides of (4.1), from (1.12), (1.14) and (3.3) we obtain $\sum_{i\lambda}h^{\lambda}_{ir}h^{\lambda}_{is}=0$. On the other hand $\sum_{i\lambda}h^{\lambda}_{ir}h^{\lambda}_{ir^*}=0$. That is $\sum_{\lambda}A^2_{\lambda}$ is a diagonal matrix. Applying $g(\tilde{\sigma}(E_r, E_r), -)$ to (4.1) we obtain

$$\frac{(n+1)^2}{2n}+\frac{1}{n}\sum_{i\lambda}h^{\lambda}_{ir}h^{\lambda}_{ir}=\frac{n+1}{2n}a-b.$$

Hence

$$\sum_{\lambda}A^2_{\lambda}=\left[\frac{n+1}{2}a-nb-\frac{(n+1)^2}{2}\right]I_{2n\times 2n}. \tag{4.2}$$

From (3.11) and (4.2) we see that M is Einstein. We choose x, y such that $x\neq y, y^*$. Applying $g(\tilde{\sigma}(\xi_x, \xi_y), -)$ to both sides of (4.1) we obtain $\sum_{ij}h^x_{ij}h^y_{ij}=0$. On the other hand $\sum_{ij}h^x_{ij}h^{x^*}_{ij}=0$. Hence the matrix $(\mathrm{tr}A_{\lambda}A_{\mu})_{\lambda\mu}$ is a diagonal matrix. Applying $g(\tilde{\sigma}(\xi_x, \xi_x), -)$ to (4.1) we obtain

$$\frac{n+1}{2}-\frac{1}{n}\sum_{ij}h_{ij}^{x}h_{ij}^{x}=\frac{1}{2}a-b\,.$$

So

$$\operatorname{tr} A_xA_x=\frac{1}{2}n(n+1-a+2b)\,.$$

Then we have finally

$$T=\frac{1}{2}n(n+1-a+2b)g\,. \tag{4.3}$$

Conversely, suppose that M^n is an Einstein submanifold such that $T=kg$. Then, if $\sum_\lambda A_\lambda^2=hI_{2n\times 2n}$, we have

$$\begin{aligned}\Delta H&=(n+1)H+\frac{1}{n}\sum_{ijk\lambda}h_{ij}^{\lambda}h_{ik}^{\lambda}\tilde{\sigma}(E_k,\,E_j)-\frac{1}{n}\sum_{ijk\mu}h_{ij}^{\lambda}h_{ij}^{\mu}\tilde{\sigma}(\xi_\lambda,\,\xi_\mu)\\&=(n+1)H+\frac{h}{n}\sum_j\tilde{\sigma}(E_j,\,E_j)-\frac{k}{n}\sum_\lambda\tilde{\sigma}(\xi_\lambda,\,\xi_\lambda)\\&=(n+1)H+\frac{h+k}{n}\sum_j\tilde{\sigma}(E_j,\,E_j)-\frac{k}{n}\Big[\sum_j\tilde{\sigma}(E_j,\,E_j)+\sum_\lambda\tilde{\sigma}(\xi_\lambda,\,\xi_\lambda)\Big]\\&=[n+1+2(h+k)]H-\frac{2k(n+p)}{n}\widetilde{H}\,,\end{aligned}$$

where $\widetilde{H}$ is the mean curvature vector of $\boldsymbol{C}P^{n+p}$ in $\mathrm{HM}(n+p+1)$. Now from (1.7) we have that the immersion of M into $\mathrm{HM}(n+p+1)$ is of order $\{k_1,\,k_2\}$.

Q.E.D.

COROLLARY 4.2. *Let M^n be a compact Kaehler submanifold, of complex dimension n, in $\boldsymbol{C}P^{n+p}$ such that the immersion is full. Suppose that M is Einstein and $T=kg$. Then*

$$\frac{1}{2}\Big[n+1+\frac{p+n}{pn}\|\sigma\|^2\pm\sqrt{\Big(n+1-\frac{n+p}{pn}\|\sigma\|^2\Big)^2+\frac{4}{n}\|\sigma\|^2}\Big] \tag{4.4}$$

are in $\mathrm{Spec}(M)$.

PROOF. We suppose that M is Einstein with $T=kg$. Then we have

$$\Delta H=aH+b\Big(A-\frac{1}{n+p+1}I\Big)\qquad\text{for some } a,\,b \text{ in } \boldsymbol{R}. \tag{4.5}$$

Applying $g(A,\,-)$ and $g(H,\,-)$ to both sides of (4.5) we obtain

$$\begin{cases}a=n+1+\dfrac{p+n}{pn}\|\sigma\|^2,\\[2ex] b=\dfrac{p+n+1}{2pn}\|\sigma\|^2.\end{cases} \tag{4.6}$$

From theorem 4.1 we have

$$(4.7)\qquad \begin{cases} A-\dfrac{1}{n+p+1}I=A_{k_1}+A_{k_2}, \\ -2nH=\lambda_{k_1}A_{k_1}+\lambda_{k_2}A_{k_2}, \\ -2n\Delta H=\lambda_{k_1}^2A_{k_1}+\lambda_{k_2}^2A_{k_2}. \end{cases}$$

By (4.5) and (4.7) we obtain

$$(4.8)\qquad \begin{cases} a=\lambda_{k_1}+\lambda_{k_2}, \\ b=\dfrac{1}{2n}\lambda_{k_1}\lambda_{k_2}. \end{cases}$$

From (4.6) and (4.8) we have (4.4). Q.E.D.

LEMMA 4.3. *Let M^n be a compact Kaehler submanifold, of complex dimension n, in $\boldsymbol{C}P^{n+p}$ such that the immersion is full.*

i) *If M is Einstein and $T=kg$, then we have*

$$(4.9)\qquad \|\sigma\|^2\geqq\frac{np(n+2)}{2p+n}.$$

The equality holds if and only if $\nabla\sigma=0$, that is the second fundamental form of M in $\boldsymbol{C}P^{n+p}$ is parallel.

ii) *If $\|\sigma\|^2=\dfrac{pn(n+2)}{2p+n}$ then, $\nabla\sigma=0$ if and only if M is Einstein with $T=kg$.*

PROOF. i) From (3.10), (3.15) and (3.17) we obtain

$$\|\nabla\sigma\|^2=\Big(\frac{2p+n}{2pn}\|\sigma\|^2-\frac{n+2}{2}\Big)\|\sigma\|^2.$$

So we conclude (4.9). In the same way we prove ii). Q.E.D.

REMARK. In the case $\nabla\sigma=0$, $\sigma\neq0$, the eigenvalues in (4.4) are $n(n+p+1)/(2p+n)$ and $n+2$. Let $\boldsymbol{C}P^n(1/2)$ be the complex projective space with constant holomorphic sectional curvature 1/2. There exists an isometric imbedding of $\boldsymbol{C}P^n(1/2)$ in $\boldsymbol{C}P^{n+(1/2)(n+1)n}$ such that $\nabla\sigma=0$. We call this imbedding the Veronese imbedding. From lemma 4.3 we have easily that the Veronese imbedding is of order $\{1, 2\}$ in $\mathrm{HM}(n+(1/2)(n+1)n+1)$.

LEMMA 4.4. *Let M^n be an n-dimensional compact Kaehler submanifold of $\boldsymbol{C}P^{n+p}$ such that the immersion is full. Suppose that M is Einstein and $T=kg$. Then*

$$(4.10)\qquad \frac{1}{2}(n+1)n\geqq p.$$

The equality holds if and only if M is the Veronese submanifold.

PROOF. From (3.10), (3.12), (3.14) and (3.16) we obtain (4.10). The equality holds if and only if M has constant sectional holomorphic curvature. But the

only possibility, for this codimension, is the Veronese imbedding [9]. Q.E.D.

5. Spectral inequalities.

Let M^n be a compact manifold, of real dimension n, and $x : M^n \to E^N$ an isometric immersion in the Euclidean space. We have, following the introduction section,

$$x = \sum_{k \geq 0} x_k ,$$

$$\Delta x = -nH = \sum_{k \geq 1} \lambda_k x_k ,$$

$$\Delta^2 x = -n\Delta H = \sum_{k \geq 1} \lambda_k^2 x_k .$$

Moreover

$$\int_M g(x_k, x_r) = 0 \qquad \text{if} \quad k \neq r .$$

We put

$$\int_M g(x_k, x_k) = a_k , \qquad \text{for all } k \text{ in } \boldsymbol{N}.$$

Then

$$-n\int_M g(x, H) = \sum_{k \geq 1} \lambda_k a_k ,$$

$$n^2 \int_M g(H, H) = \sum_{k \geq 1} \lambda_k^2 a_k ,$$

$$n^2 \int_M g(H, \Delta H) = \sum_{k \geq 1} \lambda_k^3 a_k .$$

We put

$$\Xi = n^2 \int_M g(H, H) + n\lambda_1 \int_M g(x, H) , \qquad \text{and}$$

$$\Omega = n^2 \int_M g(H, \Delta H) - n^2 \lambda_1 \int_M g(H, H) .$$

Then from the above relations we obtain

$$\Xi = \sum_{k \geq 2} \lambda_k (\lambda_k - \lambda_1) a_k \geqq 0 , \tag{5.1}$$

$$\Omega = \sum_{k \geq 2} \lambda_k^2 (\lambda_k - \lambda_1) a_k \geqq 0 , \tag{5.2}$$

$$\Omega - \lambda_2 \Xi = \sum_{k \geq 3} \lambda_k (\lambda_k - \lambda_1)(\lambda_k - \lambda_2) a_k \geqq 0 . \tag{5.3}$$

The equality in (5.1) holds if and only if the immersion is of order 1. For the equality in (5.2) we have the same condition. The equality in (5.3) holds if and only if the immersion is of order $\{1, 2\}$.

Now for Kaehler submanifolds in the complex projective space we have

THEOREM 5.1 [**4, 10**]. *Let M^n be an immersed compact Kaehler submanifold of complex dimension n in CP^m. Let λ_1 be the first eigenvalue of the Laplace-Beltrami operator of M acting on functions. Then*

$$n+1\geqq\lambda_1 . \tag{5.4}$$

The equality holds if and only if M is totally geodesic.

PROOF. From lemma 3.2 we obtain $\Xi=2n(n+1-\lambda_1)\mathrm{vol}(M)$ and

$$\Omega=2n(n+1)(n+1-\lambda_1)\mathrm{vol}(M)+2\int_M\|\sigma\|^2,$$

where $\mathrm{vol}(M)$ denotes the volume of M. By (5.1) we have (5.4). If $\Xi=0$ then $\Omega=0$, so $\|\sigma\|=0$. The converse is well-known. Q.E.D.

REMARK. This proof of theorem 5.1 differs from that in [**10**]. It is due essentially to N. Ejiri [**4**].

THEOREM 5.2. *Let M^n be an immersed compact Kaehler submanifold of complex dimension n in CP^m. Then*

$$n[n+1+(n+1-\lambda_1)(n+1-\lambda_2)]\mathrm{vol}(M)\geqq\int_M r , \tag{5.5}$$

being λ_1 and λ_2 the first and the second eigenvalues of the Laplace-Beltrami operator of M, $\mathrm{vol}(M)$ the volume of M and r the scalar curvature of M. If the equality holds then M is Einstein and (if the immersion is full) $T=kg$.

PROOF. From lemma 3.2, (3.12) and (5.3) we obtain (5.5). If the equality holds, let CP^{n+p} be the smallest linear subvariety of CP^m which contain $A(M)$, being A the immersion. From theorem 4.1 we conclude the proof. Q.E.D.

For complete intersection we have the next result

COROLLARY 5.3. *Let M^n be an n-dimensional compact Kaehler submanifold imbedded in CP^{n+p}. If M is a complete intersection of p non-singular hypersurfaces of degree $a_1, \cdots, a_p$ in CP^{n+p}, then*

$$(n+1-\lambda_1)(n+1-\lambda_2)\geqq p-\sum a_\alpha . \tag{5.6}$$

The equality holds if and only if $M=CP^n$ is a linear subvariety of CP^{n+p} or $M=Q^n$ is the complex quadric in some linear subvariety CP^{n+1} of CP^{n+p}.

PROOF. For complete intersection K. Ogiue has proved the formula [**9**]

$$\int_M r=n(n+p+1-\sum a_\alpha)\mathrm{vol}(M) . \tag{5.7}$$

From (5.5) and (5.7) we obtain (5.6). If the equality holds, then M is Einstein. Then by a result of J. Hano [**5**], M is a linear subvariety or the complex quadric in some $(n+1)$-linear subvariety of CP^{n+p}. The converse follows from table 1. Q.E.D.

COROLLARY 5.4. *Let M^n be an n-dimensional compact Kaehler submanifold immersed in CP^m. Let λ_1 and λ_2 be the first and the second eigenvalues of the Laplace-Beltrami operator of M. If $\lambda_1=\left(\int_M r\right)/(n\mathrm{vol}(M))$, and M is not totally geodesic, then*

$$n+2\geqq\lambda_2\,. \tag{5.8}$$

If the equality holds then M is an Einstein submanifold and the second fundamental form of the immersion is parallel.

PROOF. By (5.5) we have $n+1+(n+1-\lambda_1)(n+1-\lambda_2)\geqq\left(\int_M r\right)/(n\mathrm{vol}(M))=\lambda_1$. So $(n+1-\lambda_1)(n+2-\lambda_2)\geqq 0$. If M is not totally geodesic, from theorem 5.1 we have $n+1-\lambda_1>0$. Hence we obtain (5.8).

If the equality holds, the equality holds in (5.5). From (4.6) and (4.8) we obtain the equality in (4.9). Q.E.D.

In [7] H. Nakagawa and R. Takagi, see also M. Takeuchi [12], give a classification of Kaehler submanifolds in the complex projective space with parallel second fundamental form. From their result we have that every compact Einstein Kaehler submanifold in the complex projective space with parallel second fundamental form is a linear subvariety, or an imbedded submanifold congruent to the standard imbedding of some M_i in table 1.

In the following table n is the complex dimension, p the full complex codimension, r the scalar curvature, and λ_1, λ_2, the first and the second eigenvalues of the Laplace-Beltrami operator of M_i. For every homogeneous compact Einstein Kaehler manifold, with $r>0$, we know from M. Obata [8] that $\lambda_1=r/n$. On the other hand eigenvalues for classical symmetric spaces are computed by T. Nagano [6]. The author does not know λ_2 for M_6.

Table 1.

Submanifold	n	p	r	λ_1	λ_2
$M_1=\boldsymbol{C}P^n(1/2)$	n	$(1/2)n(n+1)$	$(1/2)n(n+1)$	$(1/2)(n+1)$	$n+2$
$M_2=\boldsymbol{Q}^n$	n	1	n^2	n	$n+2$
$M_3=\boldsymbol{C}P^n\times\boldsymbol{C}P^n$	$2n$	n^2	$2n(n+1)$	$n+1$	$2n+2$
$M_4=U(s+2)/U(s)\times U(2)$ $s\geqq 3$	$2s$	$(1/2)s(s-1)$	$2s(s+2)$	$s+2$	$2s+2$
$M_5=SO(10)/U(5)$	10	5	80	8	12
$M_6=E_6/Spin(10)\times T$	16	10	192	12	

Because $\dim(M)$, $\mathrm{vol}(M)$ and $\int_M r$ are spectral invariants, from corollary 5.4 and table 1, we have the following inverse result.

COROLLARY 5.5. *Let M^n be an n-dimensional compact Kaehler submanifold immersed in $\boldsymbol{C}P^m$. If $\mathrm{Spec}(M)=\mathrm{Spec}(M_i)$ for some $i=1, \cdots, 5$ then M is congruent to the standard imbedding of M_i.*

REMARK. If $i=2$, that is, for the complex quadric, corollary 5.5 is obtained in [1].

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Added in proof. Recently B. Y. Chen has studied compact submanifolds in Euclidean space for which the decomposition $x=\sum_k x_k$ has only a finite number of non zero terms. Some of his results are similar to general arguments exposed in the beginning of sections 4 and 5 ("On the total curvature of immersed manifolds VI: Submanifolds of finite type and their applications", Bull. Math. Acad. Sinica, **11** (1983), 309-328).