J. Math. Soc. Japan
Vol. 35, No. 4, 1983

On the ring of Hilbert modular forms over Z

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(Received Jan. 5, 1982)
(Revised June 30, 1982)

Introduction.

In the theory of elliptic modular forms, it is known that every modular form whose Fourier coefficients lie in Z is represented as an isobaric polynomial in E_4, E_6, Δ with coefficients in Z, where E_k is the normalized Eisenstein series of weight k and $\Delta=2^{-6}\cdot 3^{-3}(E_4^3-E_6^2)$. On the other hand, in his paper [7], J. Igusa gave a minimal set of generators over Z of the graded ring of Siegel modular forms of degree two whose Fourier coefficients lie in Z. Also, some related topics and problems on the finite generation of an algebra of modular forms were discussed by W. L. Baily, Jr. in his recent paper [2].

In this paper, we give analogous results for symmetric Hilbert modular forms for the real quadratic fields $Q(\sqrt{2})$ and $Q(\sqrt{5})$. Let K be a real quadratic field and $A_Z(\Gamma_K)_k$ denote the Z-module of symmetric Hilbert modular forms of *even* weight k with rational integral Fourier coefficients and we put $A_Z(\Gamma_K)=\bigoplus A_Z(\Gamma_K)_k$. Denote by G_k the normalized Eisenstein series for the Hilbert modular group $\Gamma_K=SL(2, \mathfrak{o}_K)$. In the case of $K=Q(\sqrt{2})$, we put

$$H_4=2^{-6}\cdot 3^{-2}\cdot 11(G_2^2-G_4),$$

$$H_6=-2^{-8}\cdot 3^{-3}\cdot 13^{-1}\cdot 5\cdot 7^2 G_2^3+2^{-8}\cdot 3^{-2}\cdot 5^{-1}\cdot 13^{-1}\cdot 11\cdot 59 G_2G_4$$
$$-2^{-7}\cdot 3^{-3}\cdot 5^{-1}\cdot 13^{-1}\cdot 19^2 G_6.$$

Our first main result can be stated as follows:

THEOREM 1. *The modular forms G_2, H_4, H_6 all have integral Fourier coefficients, namely, $G_2\in A_Z(\Gamma_K)_2$, $H_k\in A_Z(\Gamma_K)_k$ $(k=4, 6)$. Furthermore, the elements G_2, H_4, H_6 form a minimal set of generators of $A_Z(\Gamma_K)$ over Z.*

In the case of $K=Q(\sqrt{5})$, we put

$$J_6=2^{-5}\cdot 3^{-3}\cdot 5^{-2}\cdot 67(G_2^3-G_6),$$

$$J_{10}=2^{-10}\cdot 3^{-5}\cdot 5^{-5}\cdot 7^{-1}(412751G_{10}-5\cdot 67\cdot 2293G_2^2G_6+2^2\cdot 3\cdot 7\cdot 4231G_2^5),$$

$$J_{12}=2^{-2}(J_6^2-G_2J_{10}).$$

The second main theorem is

THEOREM 2. *The four modular forms G_2, J_6, J_{10}, J_{12} all have rational integral Fourier coefficients, namely, $G_2 \in A_Z(\Gamma_K)_2$, $J_k \in A_Z(\Gamma_K)_k$ ($k=6$, 10, 12). Furthermore, the elements G_2, J_6, J_{10}, J_{12} form a minimal set of generators of $A_Z(\Gamma_K)$ over $\boldsymbol{Z}$.*

As we state in §4 and §5, the forms H_4 and J_{10} have the expressions as polynomials of theta series and their restrictions to the diagonal line vanish.

The author would like to thank Professor W. L. Baily, Jr. for his encouragement, and the referee for his valuable comments.

Notations.

We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers, the field of complex numbers, respectively. For any subring $\boldsymbol{B}$ of $\boldsymbol{C}$, we denote by $M_n(\boldsymbol{B})$ the ring of all matrices of size n with entries in $\boldsymbol{B}$. For any element A of $M_n(\boldsymbol{B})$, we denote the transpose of A by ${}^tA$. For a symmetric matrix Y in $M_n(\boldsymbol{R})$, we write $Y>0$ if Y is positive definite. Let $\boldsymbol{H}_n$ denote the Siegel upper-half space of degree n, namely the space of all complex symmetric matrices $Z=X+iY$ of degree n with imaginary parts $Y>0$. For an element α in a totally real algebraic number field $\boldsymbol{K}$, we write $\alpha \gg 0$ if α is totally positive. For a real number s, we denote by $[s]$ the largest integer $\leqq s$.

§1. Hilbert modular forms for real quadratic fields.

Let $\boldsymbol{K}$ be a real quadratic field and let $\mathfrak{o}_K$ denote the ring of integers in $\boldsymbol{K}$. We put $\boldsymbol{H}^2=\boldsymbol{H}_1\times\boldsymbol{H}_1$, as we stated in Notations, $\boldsymbol{H}_1$ is the upper-half plane. For any element α in $\boldsymbol{K}$, we write the conjugation of α by $\bar{\alpha}$. Then the Hilbert modular group $\Gamma_K=SL(2, \mathfrak{o}_K)$ acts on $\boldsymbol{H}^2$ by:

$$(z_1, z_2) \longmapsto \gamma\cdot(z_1, z_2)=\left(\frac{az_1+b}{cz_1+d}, \frac{\bar{a}z_2+\bar{b}}{\bar{c}z_2+\bar{d}}\right), \quad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_K .$$

For an element λ in $\boldsymbol{K}$ and for a point $\tau=(z_1, z_2)$ in $\boldsymbol{H}^2$, we denote

$$\lambda\tau=(\lambda z_1, \bar{\lambda}z_2), \quad N(\tau)=z_1z_2, \quad \mathrm{tr}(\tau)=z_1+z_2 . \tag{1.1}$$

DEFINITION 1.1. A holomorphic function $f(\tau)$ on $\boldsymbol{H}^2$ is called *a symmetric Hilbert modular form of weight k for $\boldsymbol{K}$* if it satisfies the following conditions:

(1) For any element $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Γ_K, $f(\tau)$ satisfies a functional equation of the form

$$f(\gamma\tau)=N(c\tau+d)^k f(\tau);$$

(2) $f((z_1, z_2))=f((z_2, z_1))$.

We denote by $A_C(\Gamma_K)_k$ the set of such functions. In this paper, we shall only consider the case of even weight k. We shall write $\mathfrak{d}_K$ the different of K. We define a subset Λ_K of K by

$$\Lambda_K=\{\lambda\in K \mid \lambda\in\mathfrak{d}_K{}^{-1},\ \lambda\gg 0 \text{ or } 0\}. \tag{1.2}$$

Then any element $f(\tau)$ in $A_C(\Gamma_K)_k$ admits a Fourier expansion of the form:

$$f(\tau)=\sum_{\nu\in\Lambda_K} a_f(\nu)\exp[2\pi i\mathrm{tr}(\nu\tau)], \qquad a_f(\nu)\in C. \tag{1.3}$$

For any subring R in C, we define an R-module $A_R(\Gamma_K)_k$ by:

$$A_R(\Gamma_K)_k=\{f\in A_C(\Gamma_K)_k \mid a_f(\nu)\in R \text{ for all } \nu\in\Lambda_K\}. \tag{1.4}$$

Then the sum $A_R(\Gamma_K)=\bigoplus_{k\geq 0} A_R(\Gamma_K)_k$ forms a graded ring over R. Similarly we denote by $A_C(SL(2, Z))_k$ a C-vector space of elliptic modular forms of weight k. It is well known that any element $f(z)$ in $A_C((SL(2, Z))_k$ has the following Fourier expansion:

$$f(z)=\sum_{n=0}^{\infty} a_f(n)\exp(2\pi inz).$$

By similar way, for any subring R, we can define $A_R(SL(2, Z))_k$ and $A_R(SL(2, Z))$.

Let $\sim$ denote an equivalence relation in $\mathfrak{o}_K\times\mathfrak{o}_K$ defined by:

$$(\alpha, \beta)\sim(\alpha', \beta') \text{ if } \alpha=\varepsilon'\alpha',\ \beta=\varepsilon'\beta' \text{ for some unit } \varepsilon' \text{ in } K.$$

For any even integer $k\geqq 2$, we define a series $G_k'(\tau)$ on H^2 as

$$G_k'(\tau)=\sum_{(\lambda,\mu)\in\mathfrak{o}_K\times\mathfrak{o}_K/\sim} N(\lambda\tau+\mu)^{-k},$$

where the summation runs through a set of representatives $(\lambda, \mu)\neq(0, 0)$ with respect to the above equivalence relation. It is known that the series is absolutely convergent and represents a symmetric Hilbert modular form of weight k for K. We normalize $G_k'(\tau)$ as:

$$G_k(\tau)=\zeta_K(k)^{-1}\cdot G_k'(\tau),$$

where $\zeta_K(s)$ is the Dedekind zeta function of K. The function $G_k(\tau)$ is called *the normalized Eisenstein series of weight k for Γ_K* and it has the following Fourier expansion:

$$G_k(\tau)=1+\sum_{\nu\in\Lambda_K-\{0\}} b_k(\nu)\exp[2\pi i\mathrm{tr}(\nu\tau)], \tag{1.5}$$

$$b_k(\nu)=\kappa_k\sum_{(\nu)\mathfrak{d}_K\subset\mathfrak{b}} |N(\mathfrak{b})|^{k-1},$$

$$\kappa_k=\zeta_K(k)^{-1}\cdot(2\pi)^{2k}\cdot[(k-1)!]^{-2}\cdot d_K{}^{1/2-k},$$

where d_K is the discriminant of $\boldsymbol{K}$. From Hecke's result it follows that

$$\zeta_K(k)=\pi^{2k}\cdot d_K{}^{1/2}\cdot(\text{rational number}),$$

therefore we have $G_k(\tau)\in \boldsymbol{A}_{\boldsymbol{Q}}(\Gamma_K)_k$.

LEMMA 1.1 [3, 4]. (1) *In the case of* $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$, *we have*

$$\kappa_2=2^4\cdot 3,\quad \kappa_4=2^5\cdot 3\cdot 5\cdot 11^{-1},\quad \kappa_6=2^4\cdot 3^2\cdot 7\cdot 19^{-2}.$$

(2) *In the case of* $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$, *we have*

$$\kappa_2=2^3\cdot 3\cdot 5,\quad \kappa_4=2^4\cdot 3\cdot 5,\quad \kappa_6=2^3\cdot 3^2\cdot 5\cdot 7\cdot 67^{-1},$$

$$\kappa_{10}=2^3\cdot 3\cdot 5^2\cdot 11\cdot 412751^{-1}.$$

(Further numerical examples can be obtained by the method of Siegel [9]).

Let $E_k(z)$ be the Eisenstein series of weight k for $SL(2, \boldsymbol{Z})$ which is normalized as the constant term of the Fourier expansion is equal to unity.

The following Fourier expansions of E_k are well known:

$$E_4(z)=1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n,\qquad E_6(z)=1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n,$$

$$E_8(z)=1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n,\qquad \sigma_m(n)=\sum_{0<d\mid n}d^m,\qquad q=\exp(2\pi iz).$$

Now we define $\Delta(z)=2^{-6}\cdot 3^{-3}(E_4^3(z)-E_6^2(z))$. As is well known, $\Delta(z)$ is a cusp form of weight 12 and has the following expression:

$$\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

From this, we see easily that $\Delta(z)$ is an element of $\boldsymbol{A}_{\boldsymbol{Z}}(SL(2, \boldsymbol{Z}))_{12}$. The classical theory of elliptic modular forms tells us the following theorem.

THEOREM 1.1. *The modular forms* E_4, E_6, Δ *form a set of generators of* $\boldsymbol{A}_{\boldsymbol{Z}}(SL(2, \boldsymbol{Z}))$ *over* $\boldsymbol{Z}$, *i.e., every element* $f\in\boldsymbol{A}_{\boldsymbol{Z}}(SL(2, \boldsymbol{Z}))_k$ *can be written as an isobaric polynomial in* E_4, E_6, Δ *with integral coefficients. In particular, if* $k\equiv 0 \pmod 4$, *then every element* $f\in\boldsymbol{A}_{\boldsymbol{Z}}(SL(2, \boldsymbol{Z}))_k$ *can be expressed as an isobaric polynomial in* E_4, Δ *with integral coefficients.*

For any function $f((z_1, z_2))$ on $\boldsymbol{H}^2$, we define a function $\boldsymbol{D}(f)(z)$ on $\boldsymbol{H}_1$ by $\boldsymbol{D}(f)(z)=f((z, z))$. It is known that the map $\boldsymbol{D}$ induces an R-linear map $\boldsymbol{D}: \boldsymbol{A}_R(\Gamma_K)_k\rightarrow\boldsymbol{A}_R(SL(2, \boldsymbol{Z}))_{2k}$. In fact, for an element $f(\tau)=\sum a_f(\nu)\exp[2\pi i\operatorname{tr}(\nu\tau)]$ in $\boldsymbol{A}_R(\Gamma_K)_k$, the following formula holds:

$$\boldsymbol{D}(f)(z)=\sum_{n=0}^{\infty}c_f(n)\exp(2\pi inz),\qquad c_f(n)=\sum_{\operatorname{tr}\nu=n}a_f(\nu).$$

Direct calculation shows the following results.

PROPOSITION 1.1. (1) *If* $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$, *then we have*

$$\boldsymbol{D}(G_2)=E_4,\quad \boldsymbol{D}(G_4)=E_8=E_4^2,\quad \boldsymbol{D}(G_2^3-G_6)=2^7\cdot 3^3\cdot 5\cdot 13\cdot 19^{-2}\Delta .$$

(2) *If* $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$, *then we have*

$$\boldsymbol{D}(G_2)=E_4,\quad \boldsymbol{D}(G_4)=E_8,\quad \boldsymbol{D}(G_2^3-G_6)=2^6\cdot 3^3\cdot 5^2\cdot 67^{-1}\Delta .$$

§ 2. Theta constants.

In this section, we shall recall some properties of theta constants and we give some results which is required later. As we stated, let $\boldsymbol{H}_n$ denote the Siegel upper-half space of degree n and Z a point of $\boldsymbol{H}_n$; then the Siegel modular group $Sp(n, \boldsymbol{Z})$ acts discontinuously on $\boldsymbol{H}_n$. Let m', m'' denote elements of $\boldsymbol{Z}^n$ and put $\mathfrak{m}=(m'm'')$; then the theta constant $\theta_{\mathfrak{m}}(Z)$ with "characteristic" $\mathfrak{m}$ is a holomorphic function on $\boldsymbol{H}_n$ defined as

$$\theta_{\mathfrak{m}}(Z)=\sum_{p\in \boldsymbol{Z}^n}\exp[\pi i\{(p+m'/2)Z^t(p+m'/2)+(p+m'/2)^t m''\}]. \tag{2.1}$$

The function $\theta_{\mathfrak{m}}$ is different from the constant 0 if and only if $\mathfrak{m}$ is even in the sense that the integer $m'^t m''$ is even. If $\mathfrak{n}=(n'n'')$ is another element of $\boldsymbol{Z}^{2n}$, then we have $\theta_{\mathfrak{m}+2\mathfrak{n}}=(-1)^{m'^t n''}\theta_{\mathfrak{m}}$. Therefore we have only to consider theta constants with even characteristic in which entries are 0, 1. There are $2^{n-1}(2^n+1)$ such characteristics; they are (00), (01), (10) for $n=1$ and (0000), (0001), (0010), (0011), (0100), (0110), (1000), (1001), (1100), (1111) for $n=2$.

EXAMPLE 2.1. $(\theta_{00}\theta_{01}\theta_{10})^8=2^8\Delta$, where Δ is the elliptic cusp form of weight 12 introduced in § 1 (e.g. cf. [7]).

Now, in the case $n\leqq 2$ we give an expression of $\theta_{\mathfrak{m}}$ as Fourier series introduced by Igusa; cf. [7], pp. 155-156. First we put $r=\exp(\pi iz)$, $z\in\boldsymbol{H}_1$. Furthermore if we put

$$\boldsymbol{F}_0(r)=\sum_{p=1}^{\infty} r^{p^2},\qquad \boldsymbol{F}_1(r)=\sum_{p=1}^{\infty} r^{(p-1/2)^2}, \tag{2.2}$$

then we have

$$\theta_{00}=1+2\boldsymbol{F}_0(r),\quad \theta_{01}=1+2\boldsymbol{F}_0(-r),\quad \theta_{10}=2\boldsymbol{F}_1(r). \tag{2.3}$$

Next, for any element $Z=\begin{pmatrix} z_1 & z_{12}\\ z_{12} & z_2\end{pmatrix}$ in $\boldsymbol{H}_2$, we put $r_1=\exp(\pi iz_1)$, $r_2=\exp(\pi iz_2)$. $q_{12}=\exp(2\pi iz_{12})$. Furthermore we put

$$\boldsymbol{F}_0(r_1, r_2)=\boldsymbol{F}_0(r_1)+\boldsymbol{F}_0(r_2)+\sum_{p_1, p_2=1}^{\infty} A_{p_1,p_2} r_1^{p_1^2} r_2^{p_2^2}, \tag{2.4}$$

$$A_{p_1,p_2}=q_{12}^{p_1p_2}+q_{12}^{-p_1p_2},$$

$$\boldsymbol{F}_1(r_1, r_2)=\boldsymbol{F}_1(r_2)+\sum_{p_1, p_2=1}^{\infty} B_{p_1, p_2} r_1^{p_1^2} r_2^{(p_2-1/2)^2},$$

$$B_{p_1, p_2}=q_{12}^{p_1(p_2-1/2)}+q_{12}^{-p_1(p_2-1/2)},$$

$$\boldsymbol{F}_2(r_1, r_2)=\boldsymbol{F}_1(r_2, r_1),$$

$$\boldsymbol{F}_3(r_1, r_2)=\sum_{p_1, p_2=1}^{\infty} C_{p_1, p_2} r_1^{(p_1-1/2)^2} r_2^{(p_2-1/2)^2},$$

$$C_{p_1, p_2}=q_{12}^{(p_1-1/2)(p_2-1/2)}+q_{12}^{-(p_1-1/2)(p_2-1/2)},$$

$$\boldsymbol{F}_4(r_1, r_2)=\sum_{p_1, p_2=1}^{\infty}(-1)^{p_1+p_2-1} D_{p_1, p_2} r_1^{(p_1-1/2)^2} r_2^{(p_2-1/2)^2},$$

$$D_{p_1, p_2}=q_{12}^{(p_1-1/2)(p_2-1/2)}-q_{12}^{-(p_1-1/2)(p_2-1/2)}.$$

Then we have

(2.5)
$$\begin{aligned} &\theta_{0000}=1+2\boldsymbol{F}_0(r_1, r_2), && \theta_{0001}=1+2\boldsymbol{F}_0(r_1, -r_2), \\ &\theta_{0010}=1+2\boldsymbol{F}_0(-r_1, r_2), && \theta_{0011}=1+2\boldsymbol{F}_0(-r_1, -r_2), \\ &\theta_{0100}=2\boldsymbol{F}_1(r_1, r_2), && \theta_{0110}=2\boldsymbol{F}_1(-r_1, r_2), \\ &\theta_{1000}=2\boldsymbol{F}_2(r_1, r_2), && \theta_{1001}=2\boldsymbol{F}_2(r_1, -r_2), \\ &\theta_{1100}=2\boldsymbol{F}_3(r_1, r_2), && \theta_{1111}=2\boldsymbol{F}_4(r_1, r_2). \end{aligned}$$

Now we introduce some functions on $\boldsymbol{H}_2$ defined as polynomial with theta constants and study their properties. First, we put

(2.6)
$$\eta_{10}=2^{-12}\prod_{\mathfrak{m}}\theta_{\mathfrak{m}}^2,$$

where the product runs over ten even characteristics. On the other hand, it is known that, in the case $n=2$, there are fifteen syzygous quadruples (cf. [7], p. 158). We put

(2.7)
$$\eta_{12}=2^{-15}\sum(\theta_{\mathfrak{m}_1}\theta_{\mathfrak{m}_2}\theta_{\mathfrak{m}_3}\cdots\theta_{\mathfrak{m}_6})^4,$$

in which the summation is extended over the set of fifteen complements of syzygous quadruples. From transformation law of theta constants, it follows that η_{10} and η_{12} are Siegel modular forms of degree 2 and of respective weights 10 and 12. For later purposes, we prepare some results.

THEOREM 2.1 (J. Igusa [7]). *Under the above definitions, we have*

$$\eta_{10}\equiv\eta_{12}\equiv(\boldsymbol{F}_1\boldsymbol{F}_2\boldsymbol{F}_3)^4 \qquad (\operatorname{mod} 2^2),$$

where $\boldsymbol{F}_i$ is the power series defined in (2.4) *and the notation $\equiv$ means the Fourier coefficient-wise congruence as Fourier series in $r_1^{1/4}$, $r_2^{1/4}$, $q_{12}^{1/4}$.*

PROOF. For the proof, we refer to [7], Lemma 3, and a comment which was stated in [7], p. 159, line 22.

Next, we put

$$\xi_4 = 2^{-4}(\theta_{0000}\theta_{0011}\theta_{1100}\theta_{1111})^2, \tag{2.8}$$

$$\xi_6 = 2^{-5}\{(\theta_{0000}\theta_{0011}\theta_{1100})^4 + (\theta_{0000}\theta_{0011}\theta_{1111})^4 + (\theta_{0011}\theta_{1100}\theta_{1111})^4 + (\theta_{0000}\theta_{1100}\theta_{1111})^4\}.$$

LEMMA 2.1. *ξ_4 and ξ_6 have integral Fourier coefficients (as Fourier series in $r_1^{1/4}$, $r_2^{1/4}$, $q_{12}^{1/4}$). Furthermore, we have*

$$\xi_4 \equiv \xi_6 \quad (\mathrm{mod}\ 2^3). \tag{2.9}$$

PROOF. From (2.5), we have

$$(\theta_{0000}\theta_{0011}\theta_{1100}\theta_{1111})^2 = (1+2\boldsymbol{F}_0(r_1, r_2))^2(1+2\boldsymbol{F}_0(-r_1, -r_2))^2(2\boldsymbol{F}_3(r_1, r_2))^2(2\boldsymbol{F}_4(r_1, r_2))^2.$$

We recall that $\boldsymbol{F}_0(\pm r_1, \pm r_2)$, $\boldsymbol{F}_3(r_1, r_2)$, $\boldsymbol{F}_4(r_1, r_2)$ all have integral Fourier coefficients (as Fourier series in $r_1^{1/4}$, $r_2^{1/4}$, $q_{12}^{1/4}$). Since in any commutative ring $a \equiv b$ (mod 2) implies $a^2 \equiv b^2$ (mod 2^2) and since $\boldsymbol{F}_0(r_1, r_2) \equiv \boldsymbol{F}_0(-r_1, -r_2)$, $\boldsymbol{F}_3(r_1, r_2) \equiv \boldsymbol{F}_4(r_1, r_2)$ (mod 2), if we put $2\boldsymbol{B}(r_1, r_2) = \boldsymbol{F}_4(r_1, r_2) - \boldsymbol{F}_3(r_1, r_2)$; then we get

$$\begin{aligned} \xi_4 &\equiv (1+2\boldsymbol{F}_0(r_1, r_2))^4\{\boldsymbol{F}_3(r_1, r_2)(\boldsymbol{F}_3(r_1, r_2)+2\boldsymbol{B}(r_1, r_2))\}^2 \\ &\equiv \boldsymbol{F}_3(r_1, r_2)^4 + 4\boldsymbol{F}_3(r_1, r_2)^3\boldsymbol{B}(r_1, r_2) + 4\boldsymbol{F}_3(r_1, r_2)^2\boldsymbol{B}(r_1, r_2)^2 \quad (\mathrm{mod}\ 2^3). \end{aligned} \tag{2.10}$$

On the other hand, we get

$$\begin{aligned} 2^5\xi_6 &\equiv (1+2\boldsymbol{F}_0(r_1, r_2))^4(1+2\boldsymbol{F}_0(-r_1, -r_2))^4(2\boldsymbol{F}_3(r_1, r_2))^4 \\ &\quad + (1+2\boldsymbol{F}_0(r_1, r_2))^4(1+2\boldsymbol{F}_0(-r_1, -r_2))^4(2\boldsymbol{F}_4(r_1, r_2))^4 \\ &\equiv 2^5\{\boldsymbol{F}_3(r_1, r_2)^4 + 4\boldsymbol{F}_3(r_1, r_2)^3\boldsymbol{B}(r_1, r_2) + 12\boldsymbol{F}_3(r_1, r_2)^2\boldsymbol{B}(r_1, r_2)^2\} \quad (\mathrm{mod}\ 2^8). \end{aligned}$$

Therefore we have

$$\xi_6 \equiv \boldsymbol{F}_3(r_1, r_2)^4 + 4\boldsymbol{F}_3(r_1, r_2)^3\boldsymbol{B}(r_1, r_2) + 12\boldsymbol{F}_3(r_1, r_2)^2\boldsymbol{B}(r_1, r_2)^2 \quad (\mathrm{mod}\ 2^3). \tag{2.11}$$

Consequently, from (2.10) and (2.11), we obtain $\xi_4 \equiv \xi_6$ (mod 2^3). q.e.d.

§3. Modular imbeddings and modular forms.

In this section, we shall describe the modular imbeddings of $\boldsymbol{H}^2$. For the precise definition and the properties, we refer to [5]. First we consider the case of $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{2})$. If we put $\varepsilon_1 = 1+\sqrt{2}$, then ε_1 is the fundamental unit of $\boldsymbol{K} = \boldsymbol{Q}(\sqrt{2})$. We put

$$(3.1)\qquad A=\begin{pmatrix}\alpha & \bar{\alpha}\\ \bar{\alpha} & -\alpha\end{pmatrix},\quad \alpha=\sqrt{\varepsilon_1/2\sqrt{2}},\quad \bar{\alpha}=\sqrt{-\bar{\varepsilon}_1/2\sqrt{2}}\,.$$

Then we see that $A={}^tA=A^{-1}$. We now denote by Φ_1 the mapping $\Phi_1: H^2=H_1\times H_1\to H_2$ defined as

$$(3.2)\qquad \Phi_1(\tau)=\Phi_1((z_1,\ z_2))=A\begin{pmatrix}z_1 & 0\\ 0 & z_2\end{pmatrix}A$$

$$=\begin{pmatrix}\mathrm{tr}\,((\varepsilon_1/2\sqrt{2})\tau) & \mathrm{tr}\,((1/2\sqrt{2})\tau)\\ \mathrm{tr}\,((1/2\sqrt{2})\tau) & \mathrm{tr}\,((-\bar{\varepsilon}_1/2\sqrt{2})\tau)\end{pmatrix}.$$

Furthermore, we denote by Ψ_1 the mapping $\Gamma_K=SL(2,\ \mathfrak{o}_K)\to Sp(2,\ Z)$ defined as

$$(3.3)\qquad \Psi_1\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=\Psi_1\left(\begin{pmatrix}a_1+a_2\sqrt{2} & b_1+b_2\sqrt{2}\\ c_1+c_2\sqrt{2} & d_1+d_2\sqrt{2}\end{pmatrix}\right)$$

$$=\left(\begin{array}{cc|cc} a_1+a_2 & a_2 & b_1+b_2 & b_2\\ a_2 & a_1-a_2 & b_2 & b_1-b_2\\ \hline c_1+c_2 & c_2 & d_1+d_2 & d_2\\ c_2 & c_1-c_2 & d_2 & d_1-d_2\end{array}\right).$$

Then the pair $(\Phi_1,\ \Psi_1)$ defines a modular imbedding of $(H^2,\ \Gamma_K)$ into $(H_2, Sp(2,\ Z))$. Next, we consider the case of $K=Q(\sqrt{5})$. If we put $\varepsilon_2=(1+\sqrt{5})/2$, then ε_2 is the fundamental unit in K. We put

$$(3\ 4)\qquad B=\begin{pmatrix}\beta & \bar{\beta}\\ \bar{\beta} & -\beta\end{pmatrix},\quad \beta=\sqrt{\varepsilon_2/\sqrt{5}},\quad \bar{\beta}=\sqrt{-\bar{\varepsilon}_2/\sqrt{5}}\,.$$

We denote by Φ_2 the mapping $\Phi_2: H^2=H_1\times H_1\to H_2$ defined as

$$(3.5)\qquad \Phi_2(\tau)=\Phi_2((z_1,\ z_2))=B\begin{pmatrix}z_1 & 0\\ 0 & z_2\end{pmatrix}B$$

$$=\begin{pmatrix}\mathrm{tr}((\varepsilon_2/\sqrt{5})\tau) & \mathrm{tr}((1/\sqrt{5})\tau)\\ \mathrm{tr}((1/\sqrt{5})\tau) & \mathrm{tr}((-\bar{\varepsilon}_2/\sqrt{5})\tau)\end{pmatrix}.$$

Furthermore, we denote by Ψ_2 the mapping $\Gamma_K=SL(2,\ \mathfrak{o}_K)\to Sp(2,\ Z)$ defined as

$$(3.6)\qquad \Psi_2\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=\Psi_2\left(\begin{pmatrix}a_1+a_2\varepsilon_2 & b_1+b_2\varepsilon_2\\ c_1+c_2\varepsilon_2 & d_1+d_2\varepsilon_2\end{pmatrix}\right)$$

$$=\left(\begin{array}{cc|cc} a_1+a_2 & a_2 & b_1+b_2 & b_2\\ a_2 & a_1 & b_2 & b_1\\ \hline c_1+c_2 & c_2 & d_1+d_2 & d_2\\ c_2 & c_1 & d_2 & d_1\end{array}\right).$$

The pair $(\Phi_2,\ \Psi_2)$ defines a modular imbedding of $(H^2,\ \Gamma_K)$ into $(H_2,\ Sp(2,\ Z))$.

If f is a holomorphic function on $\boldsymbol{H}_2$, then by f^* we shall denote the restriction of f to the image of Φ_i $(i=1, 2)$, i.e., $f^*=f\circ\Phi_i$ $(i=1, 2)$. From this, if we put

$$(3.7) \qquad \chi_{10}=\eta_{10}{}^*=\eta_{10}\circ\Phi_2\,, \qquad \chi_{12}=\eta_{12}{}^*=\eta_{12}\circ\Phi_2\,,$$

(where η_k was defined in § 2), then χ_{10} and χ_{12} are symmetric Hilbert modular forms of respective weights 10 and 12 for $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$.

PROPOSITION 3.1. *We assume* $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$. *If we put* $\chi_k=\eta_k{}^*=\eta_k\circ\Phi_2$, *then* $\chi_k\in\boldsymbol{A}_C(\Gamma_{\boldsymbol{K}})_k$ $(k=10, 12)$.

In the following we consider the case $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$. First we define a subgroup $Sp(2, \boldsymbol{Z}, j)$ of $Sp(2, \boldsymbol{Z})$ as follows. Let $j=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and we put $J=\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$. We define

$$Sp(2, \boldsymbol{Z}, j)=\{M\in Sp(2, \boldsymbol{Z}) \mid {}^tJMJ\equiv M \pmod 2\}.$$

An integral row vector m with two components will be called mod 2 diagonal if $mj\equiv m \pmod 2$. Further, a characteristic $\mathfrak{m}=(m'm'')$ will be called mod 2 *diagonal* if the row vectors m' and m'' are mod 2 diagonal. For instance, the set of mod 2 diagonal characteristics for $n=2$ is given by

$$\mathfrak{S}=\{(0000), (0011), (1100), (1111)\}.$$

THEOREM 3.1 (Hammond [5]). (1) *If f is a holomorphic function on $\boldsymbol{H}_2$ which satisfies the functional equation for a modular form of weight k with respect to the operations on $\boldsymbol{H}_2$ of the elements of $Sp(2, \boldsymbol{Z}, j)$, then $f^*=f\circ\Phi_1$ is a symmetric Hilbert modular form of weight k for $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$.*

(2) *The set of* mod 2 *diagonal characteristics is stable under the operations of the group $Sp(2, \boldsymbol{Z}, j)$.*

Now we put

$$(3.8) \qquad \psi_4=\xi_4{}^*=\xi_4\circ\Phi_1\,, \qquad \psi_6=\xi_6{}^*=\xi_6\circ\Phi_1\,,$$

where ξ_4 and ξ_6 were defined in § 2. As an easy consequence of the above theorem, we get the following proposition.

PROPOSITION 3.2. *We assume* $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$. *If we put* $\psi_k=\xi_k{}^*=\xi_k\circ\Phi_1$, *then we have* $\psi_k\in\boldsymbol{A}_C(\Gamma_{\boldsymbol{K}})_k$ $(k=4, 6)$.

From the definition of modular imbedding and the form of Fourier expansions of symmetric Hilbert modular forms for $\boldsymbol{K}$, we obtain the following lemma.

LEMMA 3.1. *Let f and g be functions on $\boldsymbol{H}_2$ which are expressed as polynomials of theta constants of degree two. We assume that $f^*\in\boldsymbol{A}_C(\Gamma_{\boldsymbol{K}})_k$ and $g^*\in\boldsymbol{A}_C(\Gamma_{\boldsymbol{K}})_{k'}$. Then we have:*

(1) *If f has rational integral coefficients as Fourier series in $r_1{}^{1/4}$, $r_2{}^{1/4}$, $q_{12}{}^{1/4}$, then $f^*\in\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_k$.*

(2) *If $f\equiv g$* (mod n) *for an integer n, then $f^*\equiv g^*$* (mod n).

LEMMA 3.2. *Let $\boldsymbol{D}$ be the mapping defined in* § 1. *Then we get*

$$\boldsymbol{D}((\theta_{m_1m_2m_1'm_2'})^*)=\theta_{m_1m_1'}\cdot\theta_{m_2m_2'}\,.$$

PROOF. From the result of [7], p. 157 and the definitions of Φ_i, it follows that

$$\boldsymbol{D}((\theta_{m_1m_2m_1'm_2'})^*)=\theta_{m_1m_2m_1'm_2'}\left(\begin{pmatrix} z & 0 \\ 0 & z\end{pmatrix}\right)=\theta_{m_1m_1'}(z)\theta_{m_2m_2'}(z)\,. \qquad \text{q. e. d.}$$

THEOREM 3.2. *We assume that $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$. Let ψ_4 and ψ_6 be the functions defined in* (3.8). *Then we have the followings.*

(1) $\psi_4\in\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_4,\quad \psi_6\in\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_6$.

(2) $\psi_4\equiv\psi_6 \pmod{2^3}$.

(3) $\boldsymbol{D}(\psi_4)=0,\quad \boldsymbol{D}(\psi_6)=2^3\Delta$.

PROOF. (1) and (2) are consequences of Lemma 3.1 and Lemma 2.1. From Lemma 3.2, we know $\boldsymbol{D}(\theta^*_{1111})=0$. Therefore, we have

$$\boldsymbol{D}(\psi_4)=2^{-4}\boldsymbol{D}(\theta^*_{0000})^2\boldsymbol{D}(\theta^*_{0011})^2\boldsymbol{D}(\theta^*_{1100})^2\boldsymbol{D}(\theta^*_{1111})^2=0\,.$$

If we also note that $(\theta_{00}\theta_{01}\theta_{10})^8=2^8\Delta$ (see Example 2.1), then we get

$$\begin{aligned}\boldsymbol{D}(\psi_6)&=2^{-5}\boldsymbol{D}(\theta^*_{0000})^4\boldsymbol{D}(\theta^*_{0011})^4\boldsymbol{D}(\theta^*_{1100})^4=2^{-5}(\theta_{00}\theta_{01}\theta_{10})^8\\&=2^3\Delta\,. \qquad\qquad \text{q. e. d.}\end{aligned}$$

By similar way, we can obtain the following theorem.

THEOREM 3.3. *We assume that $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$. Let χ_{10} and χ_{12} be the functions defined in* (3.7). *Then we have:*

(1) $\chi_{10}\in\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_{10},\quad \chi_{12}\in\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_{\boldsymbol{K}})_{12}$.

(2) $\chi_{10}\equiv\chi_{12} \pmod{2^2}$.

(3) $\boldsymbol{D}(\chi_{10})=0,\quad \boldsymbol{D}(\chi_{12})=2^2\cdot 3\Delta^2$.

PROOF. (1) and (2) are derived from Lemma 3.1 and Theorem 2.1. Since $\boldsymbol{D}(\theta^*_{1111})=0$, we see that $\boldsymbol{D}(\chi_{10})=\boldsymbol{D}(\prod(\theta^*_{\mathfrak{m}})^2)=\prod(\boldsymbol{D}(\theta^*_{\mathfrak{m}}))^2=0$. In order to show that $\boldsymbol{D}(\chi_{12})=2^2\cdot 3\Delta^2$, we observe that, if we denote ten characteristics in the order we have written in § 2 by 1, 2, 3, $\cdots$, 9, 0, then only the complements of (1490), (1680), (2390), (2670), (3580), (4570) have non zero contributions to $\boldsymbol{D}(\chi_{12})$, because the other complements contain the characteristic (1111). Thus we have

$$\begin{aligned}\boldsymbol{D}(\chi_{12})=2^{-15}\{&\boldsymbol{D}(\theta^*_{0001})^4\boldsymbol{D}(\theta^*_{0010})^4\boldsymbol{D}(\theta^*_{0100})^4\boldsymbol{D}(\theta^*_{0110})^4\boldsymbol{D}(\theta^*_{1000})^4\boldsymbol{D}(\theta^*_{1001})^4\\&+\boldsymbol{D}(\theta^*_{0001})^4\boldsymbol{D}(\theta^*_{0010})^4\boldsymbol{D}(\theta^*_{0011})^4\boldsymbol{D}(\theta^*_{0100})^4\boldsymbol{D}(\theta^*_{1000})^4\boldsymbol{D}(\theta^*_{1100})^4\\&+\boldsymbol{D}(\theta^*_{0000})^4\boldsymbol{D}(\theta^*_{0011})^4\boldsymbol{D}(\theta^*_{0100})^4\boldsymbol{D}(\theta^*_{0110})^4\boldsymbol{D}(\theta^*_{1000})^4\boldsymbol{D}(\theta^*_{1001})^4\\&+\boldsymbol{D}(\theta^*_{0000})^4\boldsymbol{D}(\theta^*_{0010})^4\boldsymbol{D}(\theta^*_{0011})^4\boldsymbol{D}(\theta^*_{0100})^4\boldsymbol{D}(\theta^*_{1001})^4\boldsymbol{D}(\theta^*_{1100})^4\end{aligned}$$

$$+\boldsymbol{D}(\theta^*_{0000})^4\boldsymbol{D}(\theta^*_{0001})^4\boldsymbol{D}(\theta^*_{0011})^4\boldsymbol{D}(\theta^*_{0110})^4\boldsymbol{D}(\theta^*_{1000})^4\boldsymbol{D}(\theta^*_{1100})^4$$

$$+\boldsymbol{D}(\theta^*_{0000})^4\boldsymbol{D}(\theta^*_{0001})^4\boldsymbol{D}(\theta^*_{0010})^4\boldsymbol{D}(\theta^*_{0110})^4\boldsymbol{D}(\theta^*_{1001})^4\boldsymbol{D}(\theta^*_{1100})^4\}$$

$$=2^{-16}\cdot 6(\theta_{00}\theta_{01}\theta_{10})^{16}, \qquad \text{by Lemma 3.2,}$$

$$=2^2\cdot 3\Delta^2, \qquad \text{by Example 1.1.} \qquad \text{q. e. d.}$$

§4. Integral Hibert modular forms for $Q(\sqrt{2})$.

In this section we shall construct a set of generators of $A_Z(\Gamma_K)$ over Z in the case $K=Q(\sqrt{2})$. Until the end of this section, we assume $K=Q(\sqrt{2})$. In this case, we have $d_K=8$, $\mathfrak{d}_K=(2\sqrt{2})$. Therefore, from § 1, the Fourier expansion of the Eisenstein series G_k is given by

$$G_k(\tau)=1+\kappa_k\sum_{\nu\in\Lambda_K-\{0\}} b'_k(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)], \tag{4.1}$$

$$b'_k(\nu)=\sum_{(\mu)|2\sqrt{2}\nu}|N(\mu)|^{k-1}.$$

From Lemma 1.1, (1), since $\kappa_2=2^4\cdot 3$, we see that $G_2(\tau)\in A_Z(\Gamma_K)_2$. Now we put

$$H_4=2^{-6}\cdot 3^{-2}\cdot 11(G_2{}^2-G_4). \tag{4.2}$$

LEMMA 4.1. $H_4\in A_Z(\Gamma_K)_4$.

PROOF. From Lemma 1.1, (1), we can write

$$G_2(\tau)=1+2^4\cdot 3\sum b'_2(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)],$$

$$G_4(\tau)=1+2^5\cdot 3\cdot 5\cdot 11^{-1}\sum b'_4(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)].$$

Hence we have

$$G_2{}^2(\tau)=1+2^5\cdot 3\sum b'_2(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)]$$

$$+2^8\cdot 3^2(\sum b'_2(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)])^2.$$

By comparing the terms of $G_2{}^2$ and G_4, it suffices to show that

$$2^5\cdot 3\cdot 11 b'_2(\nu)\equiv 2^5\cdot 3\cdot 5 b'_4(\nu)\ (\mathrm{mod}\,2^6\cdot 3^2) \qquad \text{for all } \nu\in\Lambda_K. \tag{4.3}$$

If we note that $11n\equiv 5n^3$ (mod $2\cdot 3$) for any integer n, then we have

$$2^5\cdot 3\cdot 11|N(\mu)|\equiv 2^5\cdot 3\cdot 5|N(\mu)|^3\ (\mathrm{mod}\ 2^6\cdot 3^2) \qquad \text{for all } \mu\in\mathfrak{o}_K.$$

Thus Lemma 4.1 is proved. q. e. d.

The direct calculation shows the following lemma.

LEMMA 4.2. *$H_4=\phi_4$, where ϕ_4 is the Hilbert modular form of weight 4 for $K=Q(\sqrt{2})$ defined in* (3.8).

We define a Hilbert modular form H_6 for $K=Q(\sqrt{2})$ as:

(4.4) $$H_6=2^{-3}(\psi_6-G_2\psi_4).$$

LEMMA 4.3. $H_6\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_6$.

PROOF. We first note that $\psi_4\equiv\psi_6 \pmod{2^3}$, (Theorem 3.2). Since $G_2\equiv 1 \pmod{2^3}$, we get

$$\psi_6\equiv\psi_4\equiv G_2\psi_4 \pmod{2^3}.$$

From this, we obtain $2^{-3}(\psi_6-G_2\psi_4)\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_6$. q. e. d.

By using Hammond's structure theorem for $\boldsymbol{A}_{\boldsymbol{C}}(\Gamma_K)$ (cf. [5]), we get the following polynomial expression of H_6 by Eisenstein series.

LEMMA 4.4.

$$H_6=-2^{-8}\cdot 3^{-3}\cdot 13^{-1}\cdot 5\cdot 7^2 G_2^3+2^{-8}\cdot 3^{-2}\cdot 5^{-1}\cdot 13^{-1}\cdot 11\cdot 59 G_2G_4$$
$$-2^{-7}\cdot 3^{-3}\cdot 5^{-1}\cdot 13^{-1}\cdot 19^2 G_6.$$

From these lemmas and Theorem 3.2, we have the following theorem.

THEOREM 4.1. *The above-defined modular forms G_2, H_4, H_6 have integral Fourier coefficients. Furthermore, we have*

$$\boldsymbol{D}(G_2)=E_4, \qquad \boldsymbol{D}(H_4)=0, \qquad \boldsymbol{D}(H_6)=\Delta.$$

In the following, we shall show that the modular forms G_2, H_4, H_6 form a minimal set of generators over $\boldsymbol{Z}$ of $\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)$. First, we recall the Fourier expansion of symmetric Hilbert modular form for $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$. For the set Λ_K defined in § 1, we shall define a linear order among the elements ν in Λ_K as follows: For any element ν in Λ_K, we write

$$\nu=(\alpha+\beta\sqrt{2})/2\sqrt{2}, \qquad \alpha,\ \beta\in\boldsymbol{Z}.$$

Then the conjugation $\bar{\nu}$ of ν is given as $\bar{\nu}=(-\alpha+\beta\sqrt{2})/2\sqrt{2}$ and $\mathrm{tr}(\nu)=\beta$.

(4.5) 1. We arrange ν in order of $\mathrm{tr}(\nu)$.
2. When the traces are equal, we arrange them in order of α in ν.

We write the numbers ν as $\nu_0, \nu_1, \nu_2, \nu_3, \cdots$ according to this order. We can make a list of the numbers ν_i for $\mathrm{tr}(\nu_i)\leqq 2$ as follows.

trace	$\nu\in\Lambda_K$
0	$\nu_0=0$
1	$\nu_1=(-1+\sqrt{2})/2\sqrt{2}$, $\nu_2=(0+\sqrt{2})/2\sqrt{2}$, $\nu_3=(1+\sqrt{2})/2\sqrt{2}$
2	$\nu_4=(-2+2\sqrt{2})/2\sqrt{2}$, $\nu_5=(-1+2\sqrt{2})/2\sqrt{2}$, $\nu_6=(0+2\sqrt{2})/2\sqrt{2}$
	$\nu_7=(1+2\sqrt{2})/2\sqrt{2}$, $\nu_8=(2+2\sqrt{2})/2\sqrt{2}$

If we write this order by $\prec$, then we see easily that $\nu \prec \nu'$ and $\mu \prec \mu'$ imply $\nu+\mu \prec \nu'+\mu'$.

The Fourier expansion of $f(\tau)$ in $A_C(\Gamma_K)_k$ can be rewritten as:

$$f(\tau)=\sum_{j=0}^{\infty} a_f(\nu_j)\exp[2\pi i\,\mathrm{tr}(\nu_j\tau)].$$

EXAMPLE 4.1. From § 1, we get the following formulas.

$$a_{G_k}(\nu_1)=\kappa_k, \qquad a_{G_k}(\nu_2)=\kappa_k(1+2^{k-1}), \qquad a_{G_k}(\nu_3)=\kappa_k,$$
$$a_{G_k}(\nu_4)=\kappa_k(1+2^{k-1}+4^{k-1}), \qquad a_{G_k}(\nu_5)=\kappa_k(1+7^{k-1}),$$
$$a_{G_k}(\nu_6)=\kappa_k(1+2^{k-1}+4^{k-1}+8^{k-1}), \qquad a_{G_k}(\nu_7)=\kappa_k(1+7^{k-1}),$$
$$a_{G_k}(\nu_8)=\kappa_k(1+2^{k-1}+4^{k-1}).$$

EXAMPLE 4.2. If we write

$$\boldsymbol{q}=\exp[\pi i(z_1+z_2)], \qquad \boldsymbol{x}=\exp[\pi i(z_1-z_2)/\sqrt{2}], \quad z_1, z_2\in \boldsymbol{H}_1, \tag{4.6}$$

then, for any element $\nu=(\alpha+\beta\sqrt{2})/2\sqrt{2}$ in Λ_K, we obtain

$$\exp[2\pi i\,\mathrm{tr}(\nu\tau)]=\boldsymbol{x}^{\alpha}\boldsymbol{q}^{\beta}. \tag{4.7}$$

For example, the Fourier expansion of f can be expressed as:

$$f=a_f(\nu_0)+a_f(\nu_1)\boldsymbol{x}^{-1}\boldsymbol{q}+a_f(\nu_2)\boldsymbol{q}+a_f(\nu_3)\boldsymbol{x}\boldsymbol{q}+a_f(\nu_4)\boldsymbol{x}^{-2}\boldsymbol{q}^2$$
$$+a_f(\nu_5)\boldsymbol{x}^{-1}\boldsymbol{q}^2+a_f(\nu_6)\boldsymbol{q}^2+a_f(\nu_7)\boldsymbol{x}\boldsymbol{q}^2+a_f(\nu_8)\boldsymbol{x}^2\boldsymbol{q}^2+\cdots.$$

This way of writing Fourier series of Hilbert modular forms is convenient if one wants to see the effect of interchanging z_1 and z_2 (corresponding to $\boldsymbol{x}\to\boldsymbol{x}^{-1}$) or of restricting to the diagonal line (corresponding to $\boldsymbol{x}=1$).

By using the formulas in Example 4.1, we get the following numerical examples:

$$G_2=1+2^4\cdot 3\{(\boldsymbol{x}^{-1}+3+\boldsymbol{x})\boldsymbol{q}+(7\boldsymbol{x}^{-2}+8\boldsymbol{x}^{-1}+15+8\boldsymbol{x}+7\boldsymbol{x}^2)\boldsymbol{q}^2+\cdots\}.$$

$$G_4=1+2^5\cdot 3\cdot 5\cdot 11^{-1}\{(\boldsymbol{x}^{-1}+9+\boldsymbol{x})\boldsymbol{q}+(73\boldsymbol{x}^{-2}+344\boldsymbol{x}^{-1}+585+344\boldsymbol{x}+73\boldsymbol{x}^2)\boldsymbol{q}^2+\cdots\}.$$

$$G_6=1+2^4\cdot 3^2\cdot 7\cdot 19^{-2}\{(\boldsymbol{x}^{-1}+33+\boldsymbol{x})\boldsymbol{q}+(1057\boldsymbol{x}^{-2}+16808\boldsymbol{x}^{-1}+33825$$
$$+16808\boldsymbol{x}+1057\boldsymbol{x}^2)\boldsymbol{q}^2+\cdots\}.$$

$$H_4=2^{-6}\cdot 3^{-2}\cdot 11(G_2{}^2-G_4)=(\boldsymbol{x}^{-1}-2+\boldsymbol{x})\boldsymbol{q}+(-4\boldsymbol{x}^{-2}-8\boldsymbol{x}^{-1}+24-8\boldsymbol{x}-4\boldsymbol{x}^2)\boldsymbol{q}^2$$
$$+(-2\boldsymbol{x}^{-4}+26\boldsymbol{x}^{-3}+16\boldsymbol{x}^{-2}-14\boldsymbol{x}^{-1}-52-14\boldsymbol{x}+16\boldsymbol{x}^2$$
$$+26\boldsymbol{x}^3-2\boldsymbol{x}^4)\boldsymbol{q}^3+\cdots.$$

$$H_6=-2^{-8}\cdot 3^{-3}\cdot 13^{-1}\cdot 5\cdot 7^2 G_2^3+2^{-8}\cdot 3^{-2}\cdot 5^{-1}\cdot 13^{-1}\cdot 11\cdot 59 G_2G_4$$
$$-2^{-7}\cdot 3^{-3}\cdot 5^{-1}\cdot 13^{-1}\cdot 19^2 G_6$$
$$=\boldsymbol{q}+(-2\boldsymbol{x}^{-2}-16\boldsymbol{x}^{-1}+12-16\boldsymbol{x}-2\boldsymbol{x}^2)\boldsymbol{q}^2+\cdots.$$
$$\boldsymbol{D}(H_6)=q-24q^2+252q^3-\cdots, \qquad q=\exp(2\pi iz).$$

Now we shall prove some lemmas which will be used later.

LEMMA 4.5. *Let R be a subring of $\boldsymbol{C}$. Suppose $f\in \boldsymbol{A}_R(\Gamma_K)_k$, $g\in \boldsymbol{A}_R(\Gamma_K)_{k'}$ $(k\geqq k')$. Furthermore, we assume that the first non zero coefficient of g is invertible in R. If $f=gh$, $h\in \boldsymbol{A}_C(\Gamma_K)_{k-k'}$, then we get $h\in \boldsymbol{A}_R(\Gamma_K)_{k-k'}$.*

PROOF. Let $g(\tau)=\sum_{m=n}^{\infty} a_g(\nu_m)\exp[2\pi i\,\mathrm{tr}(\nu_m\tau)]$, $(a_g(\nu_n)\neq 0)$, and $h(\tau)=\sum_{j=l}^{\infty} a_h(\nu_j)$ $\cdot\exp[2\pi i\,\mathrm{tr}(\nu_j\tau)]$, $(a_h(\nu_l)\neq 0)$. By assumption, $a_g(\nu_n)$ is invertible in R. Now we suppose that $h\notin \boldsymbol{A}_R(\Gamma_K)_{k-k'}$. We assume $a_h(\nu_i)$ is the first coefficient which does not belong to R. Then the coefficient of $\exp[2\pi i\,\mathrm{tr}(\nu_n+\nu_i)\tau]$ in the expansion of $f(\tau)=g(\tau)h(\tau)$ is $a_g(\nu_n)a_h(\nu_i)+\sum a_g(\nu_s)a_h(\nu_t)$, where the sum runs over numbers ν_s and ν_t ($s>n$ and $i>t$) such that $\nu_s+\nu_t=\nu_n+\nu_i$. By assumption, the second sum of above expression must be contained in R. Hence we get $a_g(\nu_n)a_h(\nu_i)\in R$. Since $a_g(\nu_n)$ is invertible in R, we have $a_h(\nu_i)\in R$, which is a contradiction. q.e.d.

LEMMA 4.6. *Let R be a subring of $\boldsymbol{C}$. If $f\in \boldsymbol{A}_R(\Gamma_K)_k$ satisfies $\boldsymbol{D}(f)=0$, then f is divisible by H_4 in $\boldsymbol{A}_R(\Gamma_K)$, i.e., $f=H_4f'$ with f' in $\boldsymbol{A}_R(\Gamma_K)_{k-4}$.*

PROOF. From the result of Hammond [5], p. 514, we have known that the ideal of symmetric Hilbert modular forms for $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{2})$ of even weight which vanish on the diagonal line is the principal ideal generated by H_4, and $f/H_4\in \boldsymbol{A}_C(\Gamma_K)_{k-4}$. We can actually show that $f'=f/H_4\in \boldsymbol{A}_R(\Gamma_K)_{k-4}$. Since the first non zero Fourier coefficient of H_4 is one, we can take $g=H_4$ in the previous lemma. So, from Lemma 4.5, we get $f'=f/H_4\in \boldsymbol{A}_R(\Gamma_K)_{k-4}$. q.e.d.

By using this lemma, we can prove that the graded $\boldsymbol{Z}$-algebra $\boldsymbol{A}_Z(\Gamma_K)=\bigoplus_{k\geq 0}\boldsymbol{A}_Z(\Gamma_K)_k$ is generated over $\boldsymbol{Z}$ by G_2, H_4, H_6.

THEOREM 4.2. *The elements G_2, H_4, H_6 form a minimal set of generators of $\boldsymbol{A}_Z(\Gamma_K)$ over $\boldsymbol{Z}$.*

PROOF. If $f\in \boldsymbol{A}_Z(\Gamma_K)_k$, then we have

$$\boldsymbol{D}(f)\in \boldsymbol{A}_Z(SL(2, \boldsymbol{Z}))_{2k}, \qquad 2k\equiv 0 \pmod 4. \tag{4.8}$$

From Theorem 1.1, we can write

$$\boldsymbol{D}(f)=\sum_{4a+12b=2k}\gamma_{ab}E_4^a\Delta^b, \qquad \gamma_{ab}\in \boldsymbol{Z}. \tag{4.9}$$

On the other hand, since $\boldsymbol{D}(G_2)=E_4$ and $\boldsymbol{D}(H_6)=\Delta$, the element $f'=f-\sum\gamma_{ab}G_2^aH_6^b$ in $\boldsymbol{A}_Z(\Gamma_K)_k$ satisfies the condition $\boldsymbol{D}(f')=0$. Therefore, we can apply Lemma 4.6 in the case $R=\boldsymbol{Z}$, so we can express as $f'=H_4 f''$ for some $f''\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{k-4}$. Consequently, we can express as:

$$f=\boldsymbol{P}_0(G_2, H_6)+H_4 f'', \qquad f''\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{k-4}.$$

Then, by induction, we have the following expression:

$$f=\boldsymbol{P}_0(G_2, H_6)+\boldsymbol{P}_1(G_2, H_6)H_4+\cdots+\boldsymbol{P}_j(G_2, H_6)H_4{}^j, \quad \boldsymbol{P}_i(X_1, X_2)\in \boldsymbol{Z}[X_1, X_2].$$

The minimality is derived from Hammond's structure theorem for $\boldsymbol{A}_{\boldsymbol{C}}(\Gamma_K)$. This completes the proof of Theorem 4.2.

§ 5. Integral Hilbert modular forms for $\boldsymbol{Q}(\sqrt{5})$.

In this section we construct a minimal set of generators of $\boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)$ over $\boldsymbol{Z}$ for $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$. From now on, we assume $\boldsymbol{K}=\boldsymbol{Q}(\sqrt{5})$. Then we have $d_K=5$, $\mathfrak{d}_K=(\sqrt{5})$. The Fourier expansion of the Eisenstein series G_k is given as follows:

$$G_k(\tau)=1+\kappa_k \sum_{\nu\in\Lambda_K-\{0\}} b_k'(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)], \tag{5.1}$$

$$b_k'(\nu)=\sum_{(\mu)|\sqrt{5}\nu}|N(\mu)|^{k-1}.$$

From Lemma 1.1, (2), since $\kappa_2=2^3\cdot 3\cdot 5$, we see that $G_2(\tau)\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_2$. We put

$$J_6=2^{-5}\cdot 3^{-3}\cdot 5^{-2}\cdot 67(G_2{}^3-G_6). \tag{5.2}$$

LEMMA 5.1. $J_6\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_6$.

PROOF. From Lemma 1.1, (2), we can write

$$G_2(\tau)=1+2^3\cdot 3\cdot 5\sum b_2'(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)],$$

$$G_6(\tau)=1+2^3\cdot 3^2\cdot 5\cdot 7\cdot 67^{-1}\sum b_6'(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)].$$

So, we can obtain

$$\begin{aligned} G_2{}^3(\tau)=1&+2^3\cdot 3^2\cdot 5\sum b_2'(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)]\\ &+2^6\cdot 3^3\cdot 5^2(\sum b_2'(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)])^2\\ &+2^9\cdot 3^3\cdot 5^3(\sum b_2'(\nu)\exp[2\pi i\,\mathrm{tr}(\nu\tau)])^3. \end{aligned}$$

By comparing the terms of $G_2{}^3$ and G_6, it suffices to show that

$$2^3\cdot 3^2\cdot 5\cdot 67 b_2'(\nu)\equiv 2^3\cdot 3^2\cdot 5\cdot 7 b_6'(\nu) \pmod{2^5\cdot 3^3\cdot 5^2}, \qquad \text{for all } \nu\in\Lambda_K. \tag{5.3}$$

Since $67n\equiv 7n^5 \pmod{3\cdot 5}$ for any integer n, we see

$$67|N(\mu)|\equiv 7|N(\mu)|^5 \pmod{3\cdot 5} \qquad \text{for } \mu\in\mathfrak{o}_K. \tag{5.4}$$

On the other hand, from the fact that $|N(\mu)|\not\equiv 2 \pmod{2^2}$ for all $\mu\in\mathfrak{o}_K$, we see that

$$(5.5)\qquad 67|N(\mu)|\equiv 7|N(\mu)|^5 \pmod{2^2}.$$

The congruence (5.3) is an immediate consequence of (5.4) and (5.5). q. e. d.

For convenience of writing, we put

$$(5.6)\qquad J_{10}=\chi_{10},\qquad J_{12}=2^{-2}(J_6{}^2-G_2\chi_{10}),$$

where χ_{10} is the modular form defined in (3.7).

By calculating the Fourier coefficients of χ_{10} and χ_{12}, we get the following lemma (cf. Gundlach [**3**]).

LEMMA 5.2. *Let χ_{10} and χ_{12} be the modular forms defined in* (3.7). *Then we have the following expressions.*

$$\chi_{10}=2^{-10}\cdot 3^{-5}\cdot 5^{-5}\cdot 7^{-1}(412751G_{10}-5\cdot 67\cdot 2293G_2{}^2G_6+2^2\cdot 3\cdot 7\cdot 4231G_2{}^5),$$

$$\chi_{12}=3J_6{}^2-2G_2\chi_{10}.$$

LEMMA 5.3. $J_{10}\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{10}$, $J_{12}\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{12}$.

PROOF. The result $J_{10}\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{10}$ has been proved in Theorem 3.3, (2). Since $\chi_{10}\equiv\chi_{12} \pmod{2^2}$ from Theorem 3.3, (2) and since $G_2\equiv 1 \pmod{2^3}$, we see that

$$(5.7)\qquad G_2\chi_{10}\equiv\chi_{10}\equiv\chi_{12} \pmod{2^2}.$$

First we show that $4^{-1}(3J_6{}^2+G_2\chi_{10})\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{12}$. From Lemma 5.2, we get

$$(5.8)\qquad 3J_6{}^2+G_2\chi_{10}=(\chi_{12}-G_2\chi_{10})+4G_2\chi_{10}.$$

Since $\chi_{12}-G_2\chi_{10}\equiv 0 \pmod{2^2}$ from (5.7) and since $4G_2\chi_{10}\equiv 0 \pmod{2^2}$, from (5.8), we get $4^{-1}(3J_6{}^2+G_2\chi_{10})\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{12}$. If we note that $4^{-1}(J_6{}^2-G_2\chi_{10})=J_6{}^2-4^{-1}(3J_6{}^2+G_2\chi_{10})$, then we obtain $J_{12}=4^{-1}(J_6{}^2-G_2\chi_{10})\in \boldsymbol{A}_{\boldsymbol{Z}}(\Gamma_K)_{12}$. This completes the proof of Lemma 5.3. q. e. d.

From Proposition 1.1, (2) and the definition of J_6, we see that $\boldsymbol{D}(J_6)=2\Delta$. On the other hand, from Theorem 3.3, we get $\boldsymbol{D}(J_{10})=\boldsymbol{D}(\chi_{10})=0$ and $\boldsymbol{D}(J_{12})=4^{-1}\boldsymbol{D}(J_6{}^2-G_2\chi_{10})=\Delta^2$. Summing up these results, we obtain the following:

THEOREM 5.1. *The modular forms G_2, J_6, J_{10}, J_{12} have rational integral Fourier coefficients. Furthermore we have*

$$\boldsymbol{D}(G_2)=E_4,\quad \boldsymbol{D}(J_6)=2\Delta,\quad \boldsymbol{D}(J_{10})=0,\quad \boldsymbol{D}(J_{12})=\Delta^2.$$

Now, following the same argument in § 4, (4.5), we determine a linear order among the elements in Λ_K for $K=\boldsymbol{Q}(\sqrt{5})$. It should be noted that any element ν in Λ_K can be written as $\nu=(\alpha+\beta\sqrt{5})/2\sqrt{5}$, $\alpha\equiv\beta \pmod 2$ and $\mathrm{tr}(\nu)=\beta$.

trace	$\nu\in\Lambda_K$
0	$\nu_0=0$
1	$\nu_1=(-1+\sqrt{5})/2\sqrt{5}$, $\nu_2=(1+\sqrt{5})/2\sqrt{5}$
2	$\nu_3=(-4+2\sqrt{5})/2\sqrt{5}$, $\nu_4=(-2+2\sqrt{5})/2\sqrt{5}$, $\nu_5=(0+2\sqrt{5})/2\sqrt{5}$
	$\nu_6=(2+2\sqrt{5})/2\sqrt{5}$, $\nu_7=(4+2\sqrt{5})/2\sqrt{5}$

For instance, a few examples of Fourier coefficients of Eisenstein series $G_k(\tau)$ are given as follows.

EXAMPLE 5.1.

$$a_{G_k}(\nu_1)=a_{G_k}(\nu_2)=a_{G_k}(\nu_3)=a_{G_k}(\nu_7)=\kappa_k,$$

$$a_{G_k}(\nu_5)=\kappa_4(1+5^{k-1}),\quad a_{G_k}(\nu_4)=a_{G_k}(\nu_6)=\kappa_k(1+4^{k-1}).$$

EXAMPLE 5.2. In the same manner of Example 4.2, we write

(5.9) $$\boldsymbol{q}=\exp[\pi i(z_1+z_2)],\quad \boldsymbol{x}=\exp[\pi i(z_1-z_2)/\sqrt{5}],\quad z_1, z_2\in \boldsymbol{H}_1.$$

Then, for any element $\nu=(\alpha+\beta\sqrt{5})/2\sqrt{5}$ in Λ_K, we get

(5.10) $$\exp[2\pi i\,\mathrm{tr}(\nu\tau)]=\boldsymbol{x}^{\alpha}\boldsymbol{q}^{\beta},\quad \tau=(z_1, z_2)\in \boldsymbol{H}^2.$$

Then the Fourier expansion of f in $\boldsymbol{A}_C(\Gamma_K)_k$ can be rewritten as:

(5.11) $$f=a_f(\nu_0)+a_f(\nu_1)\boldsymbol{x}^{-1}\boldsymbol{q}+a_f(\nu_2)\boldsymbol{x}\boldsymbol{q}+a_f(\nu_3)\boldsymbol{x}^{-4}\boldsymbol{q}^2$$
$$+a_f(\nu_4)\boldsymbol{x}^{-2}\boldsymbol{q}^2+a_f(\nu_5)\boldsymbol{q}^2+a_f(\nu_6)\boldsymbol{x}^2\boldsymbol{q}^2+a_f(\nu_7)\boldsymbol{x}^4\boldsymbol{q}^2+\cdots.$$

The direct calculation shows the following numerical examples.

(5.12) $$G_2=1+2^3\cdot 3\cdot 5\{(\boldsymbol{x}^{-1}+\boldsymbol{x})\boldsymbol{q}+(\boldsymbol{x}^{-4}+5\boldsymbol{x}^{-2}+6+5\boldsymbol{x}^2+\boldsymbol{x}^4)\boldsymbol{q}^2+\cdots\}.$$
$$J_6=(\boldsymbol{x}^{-1}+\boldsymbol{x})\boldsymbol{q}+(\boldsymbol{x}^{-4}+20\boldsymbol{x}^{-2}-90+20\boldsymbol{x}^2+\boldsymbol{x}^4)\boldsymbol{q}^2+\cdots.$$
$$\boldsymbol{D}(J_6)=2\Delta=2q-48q^2+504q^3-\cdots.$$
$$J_{10}=(\boldsymbol{x}^{-1}-\boldsymbol{x})^2\boldsymbol{q}^2-2(\boldsymbol{x}^{-1}-\boldsymbol{x})(\boldsymbol{x}^{-4}+10\boldsymbol{x}^{-2}-10\boldsymbol{x}^2-\boldsymbol{x}^4)\boldsymbol{q}^3+\cdots.$$
$$J_{12}=\boldsymbol{q}^2+(\boldsymbol{x}^{-5}-15\boldsymbol{x}^{-3}-10\boldsymbol{x}^{-1}-10\boldsymbol{x}-15\boldsymbol{x}^3+\boldsymbol{x}^5)\boldsymbol{q}^3+\cdots.$$
$$\boldsymbol{D}(J_{12})=\Delta^2=q^2-48q^3+\cdots.$$

THEOREM 5.2. *The elements G_2, J_6, J_{10}, J_{12} form a minimal set of generators of $\boldsymbol{A}_Z(\Gamma_K)$ over $\boldsymbol{Z}$.*

In order to carry out the proof of this theorem, we shall prepare some results.

LEMMA 5.4. *Let R be a subring of $\boldsymbol{C}$. If $f\in \boldsymbol{A}_R(\Gamma_K)_k$ satisfies $\boldsymbol{D}(f)=0$,* *then f is divisible by J_{10} in $\boldsymbol{A}_R(\Gamma_K)$, i.e., $f=J_{10}f'$ with f' in $\boldsymbol{A}_R(\Gamma_K)_{k-10}$.*

PROOF. From Gundlach's result [3], we can write $f=J_{10}f'$ with $f'\in\boldsymbol{A}_C(\Gamma_K)_{k-10}$. Since we can take $g=J_{10}$ in Lemma 4.5, we see that $f'\in\boldsymbol{A}_R(\Gamma_K)_{k-10}$. q.e.d.

LEMMA 5.5. *Let $f=\sum_{m=0}^{\infty} b_m(\boldsymbol{x}^{-1}, \boldsymbol{x})\boldsymbol{q}^m$ be the Fourier expansion of $f\in\boldsymbol{A}_Z(\Gamma_K)_k$ with $b_m(\boldsymbol{x}^{-1}, \boldsymbol{x})\in\boldsymbol{Z}[\boldsymbol{x}^{-1}, \boldsymbol{x}]$. Let $b_n(\boldsymbol{x}^{-1}, \boldsymbol{x})$ be the first term of $b_m(\boldsymbol{x}^{-1}, \boldsymbol{x})$ such that $b_n(\boldsymbol{x}^{-1}, \boldsymbol{x})|_{x=1}\neq 0$. If n is odd, then*

$$\boldsymbol{D}(f)=2cE_4{}^{k/2-3n}\Delta^n+(\text{the higher order terms of } \Delta)\in\boldsymbol{Z}[E_4, \Delta] \tag{5.13}$$

with $c\in\boldsymbol{Z}-\{0\}$.

PROOF. Since n is odd, we can write

$$b_n(\boldsymbol{x}^{-1}, \boldsymbol{x})=a_1(\boldsymbol{x}^{-1}+\boldsymbol{x})+a_3(\boldsymbol{x}^{-3}+\boldsymbol{x}^3)+\cdots+a_{2j+1}(\boldsymbol{x}^{-(2j+1)}+\boldsymbol{x}^{2j+1})$$

with $a_i\in\boldsymbol{Z}$. Therefore, $b_n|_{x=1}$ is a non zero even integer. We put $b_n|_{x=1}=2c$, $c\in\boldsymbol{Z}-\{0\}$. Since $\boldsymbol{D}(f)=2cq^n+\cdots$, we get (5.13). q.e.d.

LEMMA 5.6. *For an element f in $\boldsymbol{A}_Z(\Gamma_K)_k$, we have*

$$\boldsymbol{D}(f)=c_0E_4{}^{k/2}+c_1E_4{}^{k/2-3}\Delta+\cdots+c_iE_4{}^{k/2-3i}\Delta^i+\cdots\in\boldsymbol{Z}[E_4, \Delta]$$

with $c_1\equiv c_3\equiv c_5\equiv c_7\equiv\cdots\equiv 0 \pmod 2$.

PROOF. We put $\boldsymbol{D}(f)=\boldsymbol{P}(E_4, \Delta)$ with $\boldsymbol{P}(X_1, X_2)\in\boldsymbol{Z}[X_1, X_2]$. Then we see that $\deg_\Delta\boldsymbol{P}(E_4, \Delta)\leqq[k/6]$. Let n be the first integer such that $c_n\neq 0$. We shall prove the assertion by induction on $[k/6]-n$. If $[k/6]-n=0$, the assertion is immediate from Lemma 5.5. Let $[k/6]-n\geqq 1$ and

$$\boldsymbol{D}(f)=c_nE_4{}^{k/2-3n}\Delta^n+c_{n+1}E_4{}^{k/2-3(n+1)}\Delta^{n+1}+\cdots+c_iE_4{}^{k/2-3i}\Delta^i+\cdots.$$

If n is even, then

$$\boldsymbol{D}(f-c_nG_2{}^{k/2-3n}J_{12}{}^{n/2})=c_{n+1}E_4{}^{k/2-3(n+1)}\Delta^{n+1}+\cdots+c_iE_4{}^{k/2-3i}\Delta^i+\cdots$$

satisfies the induction hypothesis. Hence we have $c_j\equiv 0 \pmod 2$ for any odd j. If n is odd, then we see by Lemma 5.5 that c_n is even.

$$\boldsymbol{D}(f-(c_n/2)G_2{}^{k/2-3n}J_6J_{12}{}^{(n-1)/2})=c_{n+1}E_4{}^{k/2-3(n+1)}\Delta^{n+1}+\cdots$$

also satisfies the induction hypothesis. Therefore $c_j\equiv 0 \pmod 2$ for any odd j. q.e.d.

The following is a consequence of Theorem 5.1 and Lemma 5.6.

COROLLARY 5.1. $\boldsymbol{D}(\boldsymbol{A}_Z(\Gamma_K))=\boldsymbol{D}(\boldsymbol{Z}[G_2, J_6, J_{12}])$.

PROOF OF THEOREM 5.2. We take an element f in $A_Z(\Gamma_K)_k$. From Corollary 5.1, we can write

$$D(f)=\sum \gamma_{abc} D(G_2)^a D(J_6)^b D(J_{12})^c$$

with $\gamma_{abc} \in \boldsymbol{Z}$. Then, by Lemma 5.4, $f-\sum \gamma_{abc} G_2{}^a J_6{}^b J_{12}{}^c$ is divisible by J_{10} in $A_Z(\Gamma_K)$. Therefore, by similar argument in the proof of Theorem 4.2, we can write

$$f=\sum \delta_{abcd} G_2{}^a J_6{}^b J_{10}{}^c J_{12}{}^d$$

with $\delta_{abcd} \in \boldsymbol{Z}$. Consequently, we obtain $A_Z(\Gamma_K)=\boldsymbol{Z}[G_2, J_6, J_{10}, J_{12}]$. The minimality of the generators G_2, J_6, J_{10}, J_{12} is derived from Gundlach's structure theorem of $A_C(\Gamma_K)$ (Gundlach [3]) and the way of construction of the generators. q. e. d.

REMARK 1. In [1], W. L. Baily, Jr. proved that under certain conditions, the graded ring of integral automorphic forms, with respect to an arithmetic group operating on a tube domain, is generated as a graded algebra over $\boldsymbol{C}$ by a finite number of automorphic forms having rational integral Fourier coefficients. In our cases, from the structure theorems [3], [5], we see that

$$A_C(\Gamma_K)=\boldsymbol{C}[G_2, H_4, H_6], \qquad \text{for } K=\boldsymbol{Q}(\sqrt{2})$$

$$A_C(\Gamma_K)=\boldsymbol{C}[G_2, J_6, J_{10}], \qquad \text{for } K=\boldsymbol{Q}(\sqrt{5}).$$

REMARK 2. As Hammond [5] has observed, modular imbeddings exist for a given quadratic field if and only if the discriminant of the field is the sum of two squares. Namely, the first few discriminants for which modular imbeddings exist are 5, 8, 13 and 17. In connection with this problem, the author also studied the structures of $A_Z(\Gamma_K)$ for $K=\boldsymbol{Q}(\sqrt{13})$, $\boldsymbol{Q}(\sqrt{17})$.

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