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Arithmetic Fuchsian groups with signature (1; e)

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§1. Introduction.

In the previous papers [17], [18] we determined all arithmetic triangle Fuchsian groups. The purpose of this paper is to determine all arithmetic Fuchsian groups with signature $(1; e)$. In §2, we prove that for arbitrary non-negative integers g and t there exist finitely many arithmetic Fuchsian groups with signature $(g; e_1, e_2, \cdots, e_t)$ up to $SL_2(\boldsymbol{R})$-conjugation (Theorem 2.1). In §3 we deal with arithmetic Fuchsian groups Γ with signature $(1; e)$ (i.e. $g=1$, $t=1$). We give a necessary and sufficient condition for such a group Γ to be arithmetic. More precisely, assume that Γ contains -1_2. Then Γ has the following presentation:

$$\Gamma=\langle \alpha, \beta, \gamma \mid \alpha\beta\alpha^{-1}\beta^{-1}\gamma=-1_2, \gamma^e=-1_2\rangle,$$

where α and β are hyperbolic elements of $SL_2(\boldsymbol{R})$ and γ is an elliptic (resp. a parabolic) element such that $\mathrm{tr}(\gamma)=2\cos(\pi/e)$. Among such triples (α, β, γ) of generators of Γ we can find a certain fundamental triple $(\alpha_0, \beta_0, \gamma_0)$. Let $x=\mathrm{tr}(\alpha_0)$, $y=\mathrm{tr}(\beta_0)$, $z=\mathrm{tr}(\alpha_0\beta_0)$. Then the condition for Γ to be arithmetic can be expressed in terms of x, y, z. We can also obtain an explicit expression of the quaternion algebra associated with Γ (Theorem 3.4). In §4 using Theorem 3.4 of §3 we determine all arithmetic Fuchsian groups with signature $(1; e)$ and list them up (Theorem 4.1). In Fricke-Klein [7] we can find some examples of arithmetic Fuchsian groups with signature $(1; e)$.

§2. Arithmetic Fuchsian groups.

We recall the definition of arithmetic Fuchsian groups. Let k be a totally real algebraic number field of degree n. Then we have n distinct $\boldsymbol{Q}$-embeddings φ_i $(1\leqq i\leqq n)$ of k into the real number field $\boldsymbol{R}$, where φ_1 is the identity. Let A be a quaternion algebra over k which is unramified at the place φ_1 and ramified at all other infinite places φ_i $(2\leqq i\leqq n)$. Then there exists an $\boldsymbol{R}$-isomorphism

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$$(2.1)\qquad \rho: A\otimes_{\boldsymbol{Q}}\boldsymbol{R} \longrightarrow M_2(\boldsymbol{R})+\boldsymbol{H}+\cdots+\boldsymbol{H},$$

where $\boldsymbol{H}$ is the Hamilton quaternion algebra over $\boldsymbol{R}$. Let ρ_1 (resp. ρ_i, $2\leqq i\leqq n$) be the composite of $\rho|_A$ with the projection to $M_2(\boldsymbol{R})$ (resp. $\boldsymbol{H}$). Then ρ_1 (resp. ρ_i) is a k-isomorphism of A into $M_2(\boldsymbol{R})$ (resp. $\boldsymbol{H}$). ρ_1 is uniquely determined up to $GL_2(\boldsymbol{R})$-conjugation. We may assume that $\rho_i|_k=\varphi_i$ $(2\leqq i\leqq n)$. Let O be an order of A. Put $U^{(1)}=\{\varepsilon\in O\mid n_A(\varepsilon)=1\}$, where $n_A(\ )$ is the reduced norm of A over k. Let $\Gamma^{(1)}(A, O)=\rho_1(U^{(1)})$. Then $\Gamma^{(1)}(A, O)$ is a Fuchsian group of the first kind (i.e. a discrete subgroup of $SL_2(\boldsymbol{R})$ acting discontinuously on the upper half plane $H=\{z\in\boldsymbol{C}\mid \mathrm{Im}(z)>0\}$ such that $\mathrm{vol}(H/\Gamma^{(1)}(A, O))<\infty$, where $\mathrm{vol}(\ )$ is the non-Euclidean volume on H.)

DEFINITION 1. Let Γ be a discrete subgroup of $SL_2(\boldsymbol{R})$ such that $\mathrm{vol}(H/\Gamma)<\infty$. If Γ is commensurable with some $\Gamma^{(1)}(A, O)$, then Γ is called *an arithmetic Fuchsian group*. We call *A the quaternion algebra associated with Γ.*

Let Γ be a Fuchsian group of the first kind with signature $(g; e_1, e_2, \cdots, e_t)$, where $2\leqq e_1\leqq e_2\leqq\cdots\leqq e_t\leqq\infty$. Then Γ is generated by $2g$ hyperbolic elements $\{\alpha_i, \beta_i\mid 1\leqq i\leqq g\}$ and t elliptic or parabolic elements $\{\gamma_j\mid 1\leqq j\leqq t\}$. The fundamental relations among them are given as follows:

$$(2.2)\qquad \begin{cases} \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1\cdots\gamma_t=\pm 1_2 \\ \gamma_j^{e_j}=\pm 1_2 \quad (1\leqq j\leqq t), \end{cases}$$

where we neglect the relation for $e_j=\infty$.

The integer g is the genus of the compact Riemann surface $(H/\Gamma)^*$ obtained by joining the finite number of cusps to H/Γ. We have the following formula:

$$(2.3)\qquad \mathrm{vol}(H/\Gamma)=(2\pi)^{-1}\int_{F(\Gamma)}\frac{dx\,dy}{y^2}=2g-2+\sum_{j=1}^{t}(1-1/e_j)>0,$$

where $F(\Gamma)$ denotes a fundamental domain of Γ.

Now we shall prove the following theorem.

THEOREM 2.1. *Let g and t be arbitrary non-negative integers. Then there exist only finitely many arithmetic Fuchsian groups with signature $(g; e_1, e_2, \cdots, e_t)$ up to $SL_2(\boldsymbol{R})$-conjugation.*

PROOF. In order to prove the above theorem we need several propositions and lemmas. Let Γ be an arithmetic Fuchsian group commensurable with $\Gamma^{(1)}(A, O)$. Then by the results of [**16**] we see that $k=\boldsymbol{Q}(\mathrm{tr}(\delta)\mid\delta\in\Gamma^{(2)})$, $\rho_1(A)=k[\Gamma^{(2)}]$, where $\Gamma^{(2)}$ is the subgroup of Γ generated by $\{\delta^2\mid\delta\in\Gamma\}$. Furthermore, $O_k[\Gamma^{(2)}]$ is an order of $\rho_1(A)$, where O_k is the ring of integers in k. Hence there exists a maximal order O_1 in A such that $\Gamma^{(2)}$ is a subgroup of finite index in $\Gamma^{(1)}(A, O_1)$.

PROPOSITION 2.2. *Let Γ be a Fuchsian group with signature $(g; e_1, e_2, \cdots, e_t)$. Then the following assertions hold:*

(i) *If* $t=0$, *then* $[\Gamma\cdot\{\pm 1_2\} : \Gamma^{(2)}\cdot\{\pm 1_2\}]=2^{2g}$.

(ii) *If* $t>0$, *then* $2^{2g}\leqq[\Gamma\cdot\{\pm 1_2\} : \Gamma^{(2)}\cdot\{\pm 1_2\}]\leqq 2^{2g+t-1}$.

PROOF OF PROPOSITION 2.2. Firstly consider the case (ii). Since $\Gamma\cdot\{\pm 1_2\}/\Gamma^{(2)}\cdot\{\pm 1_2\}$ is an elementary abelian group of type $(2, \cdots, 2)$ generated by $2g+t-1$ elements, we see that the second inequality holds. For an arbitrary element γ of Γ we have the expression $\gamma=\pm\delta_1^{m_1}\cdots\delta_r^{m_r}$, where $\delta_j\in\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_t\}$. We put

$$\nu_{\alpha_i}(\gamma)=\sum_{\delta_j=\alpha_i} m_j \pmod 2, \quad \nu_{\beta_i}(\gamma)=\sum_{\delta_j=\beta_i} m_j \pmod 2.$$

In view of (2.2), ν_{α_i}, ν_{β_i} $(1\leqq i\leqq g)$ are well-defined and they are homomorphisms of Γ onto $\boldsymbol{Z}/2\boldsymbol{Z}$. Let $\Gamma_{\alpha_i}=\mathrm{Ker}(\nu_{\alpha_i})$, $\Gamma_{\beta_i}=\mathrm{Ker}(\nu_{\beta_i})$. Then they are pair-wise distinct subgroups of index 2 in Γ. Since $\Gamma^{(2)}\cdot\{\pm 1_2\}$ is contained in $\bigcap_{1\leqq i\leqq g}(\Gamma_{\alpha_i}\cap\Gamma_{\beta_i})$, we obtain the first inequality. This proves the assertion (ii). By the same argument we can prove the assertion (i).

Let O_1 be a maximal order of A. Then by a formula of Shimizu [**14**] we have

$$\mathrm{vol}(H/\Gamma^{(1)}(A, O_1))=4(2\pi)^{-2n}d(k)^{3/2}\zeta_k(2)\prod_{\mathfrak{p}|D(A)}(n_{k/\boldsymbol{Q}}(\mathfrak{p})-1), \tag{2.4}$$

where $d(k)$ is the discriminant of k and $\zeta_k(2)$ is the value of the Dedekind zeta function of k at $s=2$ and $D(A)$ is the discriminant of A which is defined by the product of all finite places $\mathfrak{p}$ such that $A\otimes_k k_{\mathfrak{p}}$ is a division quaternion algebra.

Let Γ be an arithmetic Fuchsian group with signature $(g\,;e_1, \cdots, e_t)$ commensurable with $\Gamma^{(1)}(A, O_1)$. Then by (2.3) and (2.4) we have

$$4(2\pi)^{-2n}d(k)^{3/2}\zeta_k(2)\prod_{\mathfrak{p}|D(A)}(n_{k/\boldsymbol{Q}}(\mathfrak{p})-1)=d_1d_2^{-1}\{2g-2+\sum_{1\leq j\leq t}(1-1/e_j)\}, \tag{2.5}$$

where $d_1=[\Gamma\cdot\{\pm 1_2\} : \Gamma^{(2)}\cdot\{\pm 1_2\}]$, $d_2=[\Gamma^{(1)}(A, O_1) : \Gamma^{(2)}\cdot\{\pm 1_2\}]$. Since, $\zeta_k(2)>1$, $\prod_{\mathfrak{p}|D(A)}(n_{k/\boldsymbol{Q}}(\mathfrak{p})-1)\geqq 1$, by Proposition 2.2 we have

$$d(k)<(2\pi)^{4n/3}\cdot\{2^{2g+t-2}(2g+t-2)\}^{2/3}. \tag{2.6}$$

On the other hand the following result is proved by A. Odlyzko [**11**].

PROPOSITION 2.3 (A. Odlyzko). *Let k be a totally real algebraic number field of degree n and $d(k)$ be its discriminant. Then the following inequality holds:*

$$d(k)>a^n\exp(-b), \quad \text{where } a=29.099,\ b=8.3185. \tag{2.7}$$

REMARK. By using a computer he has made a table of the numerical values for a and b. We note that (2.7) is one of them.

If we fix the integers g and t, then by (2.6) and (2.7) we obtain an upper bound of the degree n of k and it is given by

$$n_0=(b+\log_e C(g, t))/\log_e(a/(2\pi)^{4/3}), \tag{2.8}$$

where $C(g, t)=2^{2g+t-2}(2g+t-2)^{2/3}$ and a and b are given in (2.7). We note that $\log_e(a/(2\pi)^{4/3})=0.920201\cdots$. Now we fix g, t and n. Then by (2.6) $d(k)$ is bounded. It is well-known that there exist only finitely many algebraic number fields k of given degree such that $d(k)$ is bounded up to $\boldsymbol{Q}$-isomorphisms.

Now we may fix the field k. By (2.5) $\prod_{\mathfrak{p}|D(A)}(n_{k/\boldsymbol{Q}}(\mathfrak{p})-1)$ is bounded. Therefore, if $\mathfrak{p}$ divides $D(A)$, then $n_{k/\boldsymbol{Q}}(\mathfrak{p})$ is bounded. Hence there exist only finitely many prime ideals $\mathfrak{p}$ dividing $D(A)$. Thus we have proved that $D(A)$ is of finite possibility. Since A satisfies (2.1), by the Hasse's principle in the theory of simple algebras we see that there exist only finitely many quaternion algebras over k associated with some arithmetic Fuchsian groups with given signature.

We may fix a quaternion algebra A. It is well-known that the type number of maximal orders in A (i.e. the number of conjugate classes of maximal orders under the invertible elements of A) is finite. Hence there exist only finitely many $\Gamma^{(1)}(A, O_1)$ up to $SL_2(\boldsymbol{R})$-conjugation. Now by (2.5) we see that d_2 is bounded. We need the following lemma.

LEMMA 2.4. *Let G be a finitely generated group. Then for an arbitrary positive integer d there exist only finitely many subgroups H of G such that $[G:H]\leqq d$.*

PROOF OF LEMMA 2.4. We see easily that we may assume that G is a free group. In this case this is a well-known fact (cf. Theorem 7.2.9 p 105 Hall [5]). Q.E.D.

By Lemma 2.4 we see that $\Gamma^{(2)}\cdot\{\pm 1_2\}$ is of finite possibility up to $SL_2(\boldsymbol{R})$-conjugation. Let $N(\Gamma^{(2)})$ be the normalizer of $\Gamma^{(2)}$ in $SL_2(\boldsymbol{R})$. Then we see that $\Gamma\cdot\{\pm 1_2\}\subset N(\Gamma^{(2)})$. We need the following

PROPOSITION 2.5. *Let Γ be a discrete subgroup of $SL_2(\boldsymbol{R})$ such that* $\mathrm{vol}(H/\Gamma)<\infty$. *Then the normalizer $N(\Gamma)$ of Γ in $SL_2(\boldsymbol{R})$ is also a discrete subgroup of $SL_2(\boldsymbol{R})$ such that* $\mathrm{vol}(H/N(\Gamma))<\infty$ *and* $[N(\Gamma):\Gamma]<\infty$.

The fact that $N(\Gamma)$ is discrete in $SL_2(\boldsymbol{R})$ is proved in [3] p. 5. Since we have $\mathrm{vol}(H/\Gamma)=[N(\Gamma):\Gamma\cdot\{\pm 1_2\}]\cdot\mathrm{vol}(H/N(\Gamma))$, we see that the assertion holds. Q.E.D.

By proposition 2.5 we see that there exist only finitely many $\Gamma\cdot\{\pm 1_2\}$ up to $SL_2(\boldsymbol{R})$-conjugation. This is valid for Γ. This proves Theorem 2.1.

§3. Arithmetic Fuchsian groups with signature (1; e).

From now on we treat Fuchsian groups Γ with signature $(1;e)$ (i.e. $g=1$, $t=1$). Since there is no essential difference between Γ and $\Gamma\cdot\{\pm 1_2\}$, we always assume that Γ contains -1_2. Then by Fricke-Klein [7] Γ has the following presentation:

(3.1) $$\Gamma=\langle\alpha,\ \beta,\ \gamma\mid\alpha\beta\alpha^{-1}\beta^{-1}\gamma=-1_2,\ \gamma^e=-1_2,\ \mathrm{tr}(\gamma)=2\cos(\pi/e)\rangle,$$

where α, β are hyperbolic elements.

PROPOSITION 3.1. *Let Γ be a Fuchsian group with signature* $(1;e)$ $(2\leqq e\leqq\infty)$. *Let $\Gamma^{(2)}$ be the subgroup of Γ generated by $\{\delta^2\mid\delta\in\Gamma\}$. Then the signature of $\Gamma^{(2)}$ is $(1;e,e,e,e)$ and $[\Gamma:\Gamma^{(2)}\cdot\{\pm1_2\}]=4$. Furthermore, let (α,β,γ) be a triple of generators of Γ satisfying* (3.1). *Then $\Gamma^{(2)}\cdot\{\pm1_2\}$ is generated by $\{\alpha^2,\beta^2,\beta\gamma\beta^{-1},\beta\alpha\gamma\alpha^{-1}\beta^{-1},\gamma,\alpha\gamma\alpha^{-1}\}$ and the field $\boldsymbol{Q}(\mathrm{tr}(\delta)\mid\delta\in\Gamma^{(2)})$ is generated by $\{(\mathrm{tr}(\alpha))^2,(\mathrm{tr}(\beta))^2,\mathrm{tr}(\alpha)\mathrm{tr}(\beta)\mathrm{tr}(\alpha\beta)\}$ over $\boldsymbol{Q}$.*

PROOF. Let ν_α, ν_β be the same as defined in Proposition 2.2. Let $\Gamma_\alpha=\mathrm{Ker}(\nu_\alpha)$, $\Gamma_\beta=\mathrm{Ker}(\nu_\beta)$. Then we see that $\Gamma^{(2)}\cdot\{\pm1_2\}=\Gamma_\alpha\cap\Gamma_\beta$. It is easy to see that γ and $\alpha\gamma\alpha^{-1}$ represent all inequivalent conjugate classes of primitive elliptic (or parabolic if $e=\infty$) elements of Γ_α. Since Γ_α is of index 2 in Γ, we see that the signature of Γ_α is $(1;e,e)$. Moreover, we see that Γ_α is generated by $\{\alpha^2,\beta,\gamma,\alpha\gamma\alpha^{-1}\}$. To see this we denote by Γ' the subgroup of Γ generated by $\{\alpha^2,\beta,\gamma,\alpha\gamma\alpha^{-1}\}$. Then we see easily that Γ' is a normal subgroup of Γ such that $[\Gamma:\Gamma']\leqq2$. Since Γ' is contained in Γ_α, we see that $\Gamma_\alpha=\Gamma'$. Since $\{1_2,\beta\}$ is a complete set of representatives of $\Gamma_\alpha/\Gamma^{(2)}\cdot\{\pm1_2\}$, by the same argument as above we see that $\{\gamma,\alpha\gamma\alpha^{-1},\beta\gamma\beta^{-1},\beta\alpha\gamma\alpha^{-1}\beta^{-1}\}$ represent all inequivalent conjugate classes of primitive elliptic (or parabolic if $e=\infty$) elements of $\Gamma^{(2)}\cdot\{\pm1_2\}$ and that the signature of $\Gamma^{(2)}\cdot\{\pm1_2\}$ is $(1;e,e,e,e)$. Let Γ'' be the subgroup of Γ generated by $\{\alpha^2,\beta^2,\gamma,\alpha\gamma\alpha^{-1},\beta\gamma\beta^{-1},\beta\alpha\gamma\alpha^{-1}\beta^{-1}\}$. Then we see that $\Gamma''\subset\Gamma_\alpha\cap\Gamma_\beta=\Gamma^{(2)}\cdot\{\pm1_2\}$. Using the relations: $\beta\alpha^2\beta^{-1}=\gamma(\alpha\gamma\alpha^{-1})\alpha^2$, $\beta^{-1}\alpha^2\beta=\beta^{-2}(\beta\alpha^2\beta^{-1})\beta^2$, $\beta^{-1}(\alpha\gamma\alpha^{-1})\beta=\beta^{-2}(\beta\alpha\gamma\alpha^{-1}\beta^{-1})\beta^2$, we see that β normalizes Γ''. By the relation $\alpha\gamma\alpha^{-1}=\gamma^{-1}\beta\alpha^2\beta^{-1}\alpha^{-2}$ we see that Γ_α is generated by $\{\alpha^2,\beta,\gamma\}$. Therefore, Γ'' is a normal subgroup of Γ_α such that $[\Gamma_\alpha:\Gamma'']\leqq2$. Hence we see that $\Gamma''=\Gamma^{(2)}\cdot\{\pm1_2\}$. Let $k=\boldsymbol{Q}(\mathrm{tr}(\alpha^2),\mathrm{tr}(\beta^2),\mathrm{tr}(\alpha^2\beta^2))$. By the equations

(3.2) $$\begin{cases}\mathrm{tr}(\alpha^2)=\mathrm{tr}(\alpha)^2-2,\ \mathrm{tr}(\beta^2)=\mathrm{tr}(\beta)^2-2,\\ \mathrm{tr}(\alpha^2\beta^2)=\mathrm{tr}(\alpha)\mathrm{tr}(\beta)\mathrm{tr}(\alpha\beta)-\mathrm{tr}(\alpha)^2-\mathrm{tr}(\beta)^2+2,\end{cases}$$

we see that $k=\boldsymbol{Q}(\mathrm{tr}(\alpha)^2,\mathrm{tr}(\beta)^2,\mathrm{tr}(\alpha)\mathrm{tr}(\beta)\mathrm{tr}(\alpha\beta))$. Let A be the vector space spanned by $\{1_2,\alpha^2,\beta^2,\alpha^2\beta^2\}$ over k in $M_2(\boldsymbol{R})$. By the equations $\beta^2\alpha^2=\mathrm{tr}(\alpha^2\beta^2)1_2-\alpha^{-2}\beta^{-2}$, $\alpha^{-2}=\mathrm{tr}(\alpha^2)1_2-\alpha^2$, $\beta^{-2}=\mathrm{tr}(\beta^2)1_2-\beta^2$, we see that A is an algebra over k. Using the equation $\delta=\mathrm{tr}(\delta)^{-1}(\delta^2+1_2)$ for $\delta\in SL_2(\boldsymbol{R})$ such that $\mathrm{tr}(\delta)\neq0$, we see that $\gamma=-\beta\alpha\beta^{-1}\alpha^{-1}=\mathrm{tr}(\alpha)^{-2}\mathrm{tr}(\beta)^{-2}(\beta^2+1_2)(\alpha^2+1_2)(\beta^{-2}+1_2)(\alpha^{-2}+1_2)\in A$. In the same way we see that $\alpha\gamma\alpha^{-1}$, $\beta\gamma\beta^{-1}$ and $\alpha\beta\gamma\beta^{-1}\alpha^{-1}$ are also contained in A. It follows that $A=k[\Gamma^{(2)}]$ and $k=\boldsymbol{Q}(\mathrm{tr}(\delta)\mid\delta\in\Gamma^{(2)})$. Q.E.D.

Let Γ be a Fuchsian group with signature $(1;e)$. Let $\{\alpha,\beta,\gamma\}$ be a triple of generators of Γ satisfying (3.1). Let $x=\mathrm{tr}(\alpha)$, $y=\mathrm{tr}(\beta)$, $z=\mathrm{tr}(\alpha\beta)$. Then by the equation $\mathrm{tr}(\delta\varepsilon)+\mathrm{tr}(\delta\varepsilon^{-1})=\mathrm{tr}(\delta)\mathrm{tr}(\varepsilon)$ for $\delta,\varepsilon\in SL_2(\boldsymbol{R})$ and by (3.1) we have the following equation (cf. Fricke-Klein [7] p. 306)

$$x^2+y^2+z^2-xyz=2-2\cos(\pi/e)\,. \tag{3.3}$$

Now we consider the transformations:

(i) $\alpha_1=-\alpha,\ \ \beta_1=-\beta,\ \ \gamma_1=\gamma,$

(ii) $\alpha_2=-\alpha,\ \ \beta_2=\beta,\ \ \gamma_2=\gamma,$

(iii) $\alpha_3=\alpha,\ \ \beta_3=-\beta,\ \ \gamma_3=\gamma,$

(iv) $\alpha_4=\beta,\ \ \beta_4=\alpha,\ \ \gamma_4=\gamma^{-1},$

(v) $\alpha_5=\alpha\beta,\ \ \beta_5=\alpha^{-1},\ \ \gamma_5=\gamma,$

(vi) $\alpha_6=\alpha^{-1},\ \ \beta_6=\alpha\beta\alpha^{-1},\ \ \gamma_6=\gamma^{-1}.$

Then each $(\alpha_i, \beta_i, \gamma_i)$ $(1\leqq i\leqq 6)$ is also a triple of generators of Γ satisfying (3.1). Let $x_i=\mathrm{tr}(\alpha_i)$, $y_i=\mathrm{tr}(\beta_i)$, $z_i=\mathrm{tr}(\alpha_i\beta_i)$. Then (x_i, y_i, z_i) is given by

(i)′ $(x_1, y_1, z_1)=(-x, -y, z),$

(ii)′ $(x_2, y_2, z_2)=(-x, y, -z),$

(iii)′ $(x_3, y_3, z_3)=(x, -y, -z),$

(iv)′ $(x_4, y_4, z_4)=(y, x, z),$

(v)′ $(x_5, y_5, z_5)=(z, x, y),$

(vi)′ $(x_6, y_6, z_6)=(x, y, xy-z).$

We note that each (x_i, y_i, z_i) $(1\leqq i\leqq 6)$ also satisfies (3.3).

DEFINITION 2. Let notations be the same as above. Each transformation $(\alpha, \beta, \gamma)\to(\alpha_i, \beta_i, \gamma_i)$ $(1\leqq i\leqq 6)$ is called *an elementary operation for* (α, β, γ).

These operations are introduced in Fricke-Klein [**7**].

DEFINITION 3. Let (α, β, γ) be a triple of generators of Γ satisfying (3.1). We denote *the height* of (α, β, γ) by

$$h(\alpha, \beta, \gamma)=\mathrm{tr}(\alpha)^2+\mathrm{tr}(\beta)^2+\mathrm{tr}(\alpha\beta)^2\,. \tag{3.4}$$

This notion is a modified one given in Mordell [**10**] p. 107. We note here that each permutation of (x, y, z) can be realized by a finite number of the elementary operations. The height $h(\alpha, \beta, \gamma)$ is unchanged under the operations (i), (ii), (iii), (iv), (v) and by the operation (vi) we have

$$h(\alpha_6, \beta_6, \gamma_6)=h(\alpha, \beta, \gamma)+x^2y^2-2xyz\,. \tag{3.5}$$

DEFINITION 4. Let (α, β, γ) and $(\alpha', \beta', \gamma')$ be arbitrary triples of generators of Γ satisfying (3.1). If the one can be obtained from the other under a finite number of the elementary operations, we say that they are *equivalent to each other* and we denote $(\alpha, \beta, \gamma)\sim(\alpha', \beta', \gamma')$.

This is obviously an equivalence relation.

DEFINITION 5. Let $(\alpha_0, \beta_0, \gamma_0)$ be a triple of generators of Γ satisfying (3.1). We call $(\alpha_0, \beta_0, \gamma_0)$ *a fundamental triple of generators* if it satisfies the following conditions:

$$2<\mathrm{tr}(\alpha_0)\leqq\mathrm{tr}(\beta_0)\leqq\mathrm{tr}(\alpha_0\beta_0)\,, \tag{3.6}$$

$$h(\alpha_0, \beta_0, \gamma_0)=\mathrm{Min}\{h(\alpha, \beta, \gamma)\,|\,(\alpha, \beta, \gamma)\sim(\alpha_0, \beta_0, \gamma_0)\}\,. \tag{3.7}$$

This definition is motivated by the notion given in Mordell [**10**] p. 107.

PROPOSITION 3.2. *Let* (α, β, γ) *be a triple of generators of* Γ *satisfying* (3.1). *Then by a finite number of the elementary operations* (α, β, γ) *can be transformed to a fundamental triple of generators of* Γ.

PROOF. Let $h=h(\alpha, \beta, \gamma)$. Let C_h be the set of all triples $(\alpha', \beta', \gamma')$ such that $(\alpha', \beta', \gamma')\sim(\alpha, \beta, \gamma)$ and $h(\alpha', \beta', \gamma')\leqq h$. Then we have $|\mathrm{tr}(\alpha')|\leqq h^{1/2}$, $|\mathrm{tr}(\beta')|\leqq h^{1/2}$. By a result of [**3**] p. 88 (and Takeuchi [**15**]) the set $\mathrm{tr}(\Gamma)$ has no limit point in $\boldsymbol{R}$. Hence C_h is a finite set. Therefore, we can find a triple $(\alpha_0, \beta_0, \gamma_0)$ equivalent to (α, β, γ) satisfying (3.7). Now we need the following

LEMMA 3.3. *Let* Γ *be a Fuchsian group with signature* $(1; e)$. *Let* (α, β, γ) *be a triple of generators of* Γ *satisfying* (3.1). *Then* $\alpha\beta$ *is a hyperbolic element and* $\mathrm{tr}(\alpha)\mathrm{tr}(\beta)\mathrm{tr}(\alpha\beta)\geqq 10$.

PROOF. Assume that $\alpha\beta$ is non-hyperbolic. Then we have the expression $\alpha\beta=\pm\delta^{-1}\gamma^m\delta$ for $\delta\in\Gamma$. Since $\nu_\alpha(\alpha\beta)=1$ and $\nu_\alpha(\delta^{-1}\gamma^m\delta)=\nu_\alpha(\gamma^m)=0$, we have a contradiction. This shows $\alpha\beta$ is a hyperbolic element. Since $|\mathrm{tr}(\alpha)|>2$, $|\mathrm{tr}(\beta)|>2$, $|\mathrm{tr}(\alpha\beta)|>2$, by (3.3) we have $\mathrm{tr}(\alpha)\mathrm{tr}(\beta)\mathrm{tr}(\alpha\beta)>10+2\cos(\pi/e)\geqq 10$. This proves Lemma 3.3.

By Lemma 3.3 under a finite number of operations (i)-(v) we obtain a triple of generators of Γ satisfying (3.6) and (3.7). This proves Proposition 3.2. Q.E.D.

In order to determine all arithmetic Fuchsian groups with signature $(1; e)$ we shall prove the following theorem.

THEOREM 3.4. *Let* Γ *be an arithmetic Fuchsian groups with signature* $(1; e)$. *Let* A *be the quaternion algebra over* k *associated with* Γ. *Assume that* Γ *contains* -1_2. *Let* (α, β, γ) *be a fundamental triple of generators of* Γ *satisfying* (3.1). *Put*

$$x=\mathrm{tr}(\alpha), \quad y=\mathrm{tr}(\beta), \quad z=\mathrm{tr}(\alpha\beta). \tag{3.8}$$

Then the following assertions hold:

(i) $k=\boldsymbol{Q}(x^2, y^2, z^2, xyz)$ and k *contains* $\cos(\pi/e)$.

(ii) x, y *and* z *are algebraic integers satisfying* (3.9), (3.10), (3.11):

$$x^2+y^2+z^2-xyz=c_e, \text{ where } c_e=2-2\cos(\pi/e) \ (c_e=0 \text{ if } e=\infty). \tag{3.9}$$

$$\begin{cases} 2<x<3 \ (2<x\leqq 3 \text{ if } e=\infty), \\ 4(x^2-c_e)/(x^2-4)\leqq y^2\leqq(x^2-c_e)/(x-2), \\ x\leqq y\leqq z=(xy-\sqrt{x^2y^2-4x^2-4y^2+4c_e})/2. \end{cases} \tag{3.10}$$

$$\begin{cases} 0<\varphi_i(y^2)\leqq\varphi_i(4(x^2-c_e)/(x^2-4))<4, \\ 0<\varphi_i(z^2)\leqq\varphi_i(4(y^2-c_e)/(y^2-4))<4, \\ 0<\varphi_i(x^2)\leqq\varphi_i(4(z^2-c_e)/(z^2-4))<4 \ (2\leqq i\leqq n). \end{cases} \tag{3.11}$$

(iii) $A\cong\left(\frac{a, b}{k}\right)$, *where* $a=x^2(x^2-4)$, $b=-(2+2\cos(\pi/e))x^2y^2$. *We denote by* $\left(\frac{a, b}{k}\right)$ *a quaternion algebra over* k *defined as follows:*

$$\left(\frac{a, b}{k}\right)=k1_2+k\omega+k\Omega+k\omega\Omega,\ \omega^2=a,\ \Omega^2=b,\ \omega\Omega+\Omega\omega=0.$$

Conversely, let x, y *and* z *be algebraic integers satisfying* (i), (ii). *Let* α, β *be two elements of* $SL_2(\boldsymbol{R})$ *determined by* (3.8). *Then the subgroup of* $SL_2(\boldsymbol{R})$ *generated by* $\{\alpha, \beta\}$ *is an arithmetic Fuchsian group with signature* $(1; e)$.

REMARK. By (3.11) in particular we have

$$0<\varphi_i(x^2),\quad \varphi_i(y^2),\quad \varphi_i(z^2)<\varphi_i(c_e)\quad (2\leqq i\leqq n). \tag{3.12}$$

In case $e=\infty$, this means that $n=1$. Hence $k=\boldsymbol{Q}$. In fact $A\cong M_2(\boldsymbol{Q})$.

PROOF OF THEOREM 3.4. Let Γ be commensurable with $\Gamma^{(1)}(A, O)$. Then there exists a maximal order O_1 of A such that $\Gamma^{(2)}$ is a subgroup of index finite in $\Gamma^{(1)}(A, O_1)$. $k=\boldsymbol{Q}(\mathrm{tr}(\delta)\,|\,\delta\in\Gamma^{(2)})$ and $\mathrm{tr}(\Gamma^{(2)})$ is contained in the ring O_k of integers in k (cf. [16]). Since $\rho_i(A)$ $(2\leqq i\leqq n)$ is contained in $\boldsymbol{H}$, we have $\varphi_i(\mathrm{tr}(\alpha^2))=\mathrm{tr}_{\boldsymbol{H}/\boldsymbol{R}}(\rho_i(\alpha^2))$ is contained in the interval $(-2, 2)$. By the equation $x^2=\mathrm{tr}(\alpha^2)+2$ we see that x^2 is an algebraic integer in k such that $4<x^2$, $0<\varphi_i(x^2)<4$ $(2\leqq i\leqq n)$. Hence x is totally real. In the same way we see that y and z are also totally real algebraic integers.

Since (α, β, γ) is a fundamental triple of generators of Γ, we have $h(\alpha, \beta, \gamma)\leqq h(\alpha_6, \beta_6, \gamma_6)$. By (3.5) we see that $x\leqq y\leqq z\leqq xy/2$. Hence by (3.3) $x^2y^2-4x^2-4y^2+4c_e\geqq 0$ and $z=(xy-\sqrt{x^2y^2-4x^2-4y^2+4c_e})/2$. Let $f(t)=t^2-xyt$ $(y\leqq t\leqq xy/2)$. Then we see easily that $y^2(1-x)\geqq f(t)\geqq -x^2y^2/4$. Hence by (3.3) we have the second and third inequality of (3.10). Now we shall prove the first inequality of (3.10). By the inequality $3z^2-xyz\geqq x^2+y^2+z^2-xyz=c_e>0$ in case $e<\infty$, we have $xy/3<z\leqq xy/2$. Hence $-xy/6<z-xy/2\leqq 0$. By (3.3) $x^2+y^2+(z-xy/2)^2-x^2y^2/4=c_e$. Thus we have $2y^2(9-x^2)/9\geqq x^2+y^2-2x^2y^2/9>c_e\geqq 0$. Hence we have $2<x<3$ in case $e<\infty$. In case $e=\infty$ by the slight modification of the above argument we have $2<x\leqq 3$ (cf. Mordell [10] p. 91). Since z is totally real, by (3.3) we have $\varphi_i(x^2y^2-4x^2-4y^2+4c_e)\geqq 0$. By the same argument we can prove all inequalities of (3.11).

We shall prove the assertion (iii). By Proposition 3.1 and its proof we see that $k=\boldsymbol{Q}(\mathrm{tr}(\delta)\,|\,\delta\in\Gamma^{(2)})=\boldsymbol{Q}(x^2, y^2, xyz)$ and $k\ni c_e$. Let $A_0=k[\Gamma^{(2)}]$ be the vector space spanned by $\Gamma^{(2)}$ over k in $M_2(\boldsymbol{R})$. Then $A_0=k1_2+k\alpha^2+k\beta^2+k\alpha^2\beta^2=\rho_1(A)$. Let $\xi=y_01_2+y_1\alpha^2+y_2\beta^2+y_3\alpha^2\beta^2$ be an arbitrary element of A_0 $(y_i\in k)$. Let $c_1=\mathrm{tr}(\alpha^2)$, $c_2=\mathrm{tr}(\beta^2)$, $c_3=\mathrm{tr}(\alpha^2\beta^2)$, $c_4=\mathrm{tr}(\alpha^2\beta^{-2})$. Then the reduced norm $n_{A_0}(\xi)$ of ξ is given by

$$n_{A_0}(\xi)=(y_0, y_1, y_2, y_3)\,D_0\,{}^t(y_0, y_1, y_2, y_3),$$

where

$$D_0=\begin{pmatrix} 1, & c_1/2, & c_2/2, & c_3/2 \\ c_1/2, & 1, & c_4/2, & c_2/2 \\ c_2/2, & c_4/2, & 1, & c_1/2 \\ c_3/2, & c_2/2, & c_1/2, & 1 \end{pmatrix}.$$

By the following linear transformation:

$$\begin{cases} Y_0=y_0+(c_1y_1+c_2y_2+c_3y_3)/2\,, \\ Y_1=y_1/2-((c_1c_2-2c_3)y_2+(c_1c_3-2c_2)y_3)/(2(4-c_1^2))\,, \\ Y_2=y_3/2\,, \\ Y_3=(y_2+c_1y_3/2)/(4-c_1^2)\,, \end{cases}$$

we have

$$n_{A_0}(\xi)=Y_0^2+(4-c_1^2)Y_1^2-(c_1^2+c_2^2+c_3^2-c_1c_2c_3-4)Y_2^2$$
$$-(4-c_1^2)(c_1^2+c_2^2+c_3^2-c_1c_2c_3-4)Y_3^2.$$

Since $c_1=x^2-2$, $c_2=y^2-2$, $c_3=-x^2-y^2+xyz+2$, by an easy calculation we see that A_0 is isomorphic to $\left(\frac{a,\,b}{k}\right)$, where a, b are as given in (iii).

Conversely, let x, y, z be algebraic integers satisfying (i), (ii). Let α, β be two elements of $SL_2(\boldsymbol{R})$ determined by (3.8). Then α, β are uniquely determined up to $GL_2(\boldsymbol{R})$-conjugation. We can define γ so that (α, β, γ) satisfies (3.1). Now we need the following proposition proved in Fricke-Klein [7] pp. 335-353 and Purzitsky-Rosenberger [**13**].

PROPOSITION 3.5. *Let α, β be two elements of $SL_2(\boldsymbol{R})$ such that* $2<\mathrm{tr}(\alpha)$, $2<\mathrm{tr}(\beta)$, $\mathrm{tr}(\alpha\beta\alpha^{-1}\beta^{-1})=-2\cos(\pi/e)$ $(=-2$ *if* $e=\infty)$. *Then the subgroup of $SL_2(\boldsymbol{R})$ generated by $\{\alpha, \beta\}$ is a Fuchsian group of the first kind with signature* $(1\,;e)$.

By Proposition 3.5 the subgroup Γ of $SL_2(\boldsymbol{R})$ generated by $\{\alpha, \beta\}$ is a Fuchsian group with signature $(1\,;e)$. Let $k=\boldsymbol{Q}(\mathrm{tr}(\delta)\,|\,\delta\in\Gamma^{(2)})$ and $A_0=k[\Gamma^{(2)}]$. Then by the same argument as before we see that $k=\boldsymbol{Q}(x^2, y^2, xyz)$ and k contains $\cos(\pi/e)$ and $A_0=\left(\frac{a,\,b}{k}\right)$. By (3.12) we see that A_0 is unramified at φ_1 and ramified at all other φ_i $(2\leqq i\leqq n)$. Since Γ is generated by $\{\alpha, \beta\}$, by Lemma 2 in [**17**] p. 95 we see that $\mathrm{tr}(\Gamma)$ is contained in the ring of integers in $\boldsymbol{Q}(x, y, z)$. Let $O=O_k[\Gamma^{(2)}]$ be the O_k-module generated by $\Gamma^{(2)}$ in $M_2(\boldsymbol{R})$. Then O is an order of A_0 and $\Gamma^{(2)}$ is a subgroup of $\Gamma^{(1)}(A_0, 0)$ of finite index. This shows that Γ is arithmetic. This completes the proof of Theorem 3.4. Q.E.D.

The following theorem is useful to determine all arithmetic Fuchsian groups with signature $(1\,;e)$.

THEOREM 3.6. *Let k be a totally real algebraic number field of degree n such that k contains* $\cos(\pi/e)$ $(2\leqq e<\infty)$. *Let* $c_e=2-2\cos(\pi/e)$. *If there exists an algebraic integer X in k satisfying the inequalities:*

(3.13) $$4<X<9,\quad 0<\varphi_i(X)<\varphi_i(c_e)\quad (2\leqq i\leqq n),$$

then (e, n) *is one of pairs listed below:*

$$(e, n)=(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2), (4, 4), (4, 6), (4, 8), (5, 2), (5, 4), (6, 2), (6, 4), (6, 6), (7, 3), (7, 6), (8, 4), (8, 8), (9, 3), (9, 6), (10, 4), (11, 5), (12, 4), (12, 8), (13, 6), (14, 6), (15, 4), (15, 8), (16, 8), (17, 8), (18, 6), (19, 9), (20, 8), (21, 6), (24, 8), (25, 10), (27, 9), (30, 8), (33, 10).$$

PROOF. By (3.13) we have

$$0<X(X-c_e)<9(9-c_e), \quad 0<\varphi_i(X(c_e-X))\leqq\varphi_i(c_e^2)/4 \quad (2\leqq i\leqq n).$$

Since X is an algebraic integer in k, we have

$$1\leqq|n_{k/Q}(X(c_e-X))|<(9/c_e)(9/c_e-1)n_{k/Q}(c_e^2)/4^{n-1}. \tag{3.14}$$

Hence we have

$$4^{n-1}<(9/c_e)(9/c_e-1)n_{k/Q}(c_e^2). \tag{3.15}$$

Now we need the following

LEMMA 3.7. *Let* $c_e=2-2\cos(\pi/e)$. *Then the following assertions hold:*

(i) *If* $e\neq 2^m$, *then* c_e *is a unit of the ring of integers in the field* $\boldsymbol{Q}(\cos(\pi/e))$.

(ii) *If* $e=2^m$, *then* $n_{\boldsymbol{Q}(\cos(\pi/e))/\boldsymbol{Q}}(c_e)=2$.

The proof of this lemma is referred to Lehmer [**8**] and Liang [**9**].

Let $k_0=\boldsymbol{Q}(\cos(\pi/e))$, $n_1=[k:k_0]$. Then we have $n=n_1\cdot\varphi(2e)/2$, where $\varphi(\ )$ is the Euler function. We divide into two cases: $e=2^m$ and $e\neq 2^m$. Firstly consider the case $e\neq 2^m$. By (3.15) and Lemma 3.7 we have

$$2^{\varphi(2e)/2}<9/(1-\cos(\pi/e)). \tag{3.16}$$

Since $t^2/2-t^4/24<1-\cos(t)$ $(0<t)$, we have

$$2^{\varphi(2e)/2}<18e^2/(\pi^2(1-12^{-1}(\pi/e)^2)). \tag{3.17}$$

It is known that for an arbitrary $\delta>0$, $\lim_{m\to\infty}\varphi(m)/m^{1-\delta}=\infty$ (cf. Hardy-Wright [**6**] Theorem 3.27). Using this result we can prove that there exist only a finite number of such numbers e. By (3.15) we see that there are also finitely many such numbers n. In order to determine the pair (e, n) more precisely we need the following

LEMMA 3.8. *If* $43\leqq m$, *then* $m^{2/3}\leqq\varphi(m)$.

PROOF. Let $m=p_1^{e_1}\cdots p_r^{e_r}$ be the prime divisors decomposition, where p_i is a prime number such that $p_1<p_2<\cdots<p_r$ and $e_i\geqq 1$. Let p be a prime number. If $e\geqq 3$, then $p^{e-3}(p-1)\geqq 1$. Let $\psi(m)=\varphi(m)^3/m^2$. Then we have $\psi(m)=\prod_{1\leq i\leq r}p_i^{e_i-3}(p_i-1)^3$. It suffices to prove that $\psi(m)\geqq 1$ for $m\geqq 43$. By an easy calculation we have $\psi(2)=1/4$, $\psi(3)=8/9$, $\psi(2^2)=1/2$, $\psi(3^2)=8/3$. Furthermore, for

an arbitrary prime number p such that $p\geqq 5$ we see that $\psi(p)=p-3+(3-1/p)/p>1$ and $\psi(p^2)=p^2-3p+3-1/p>1$. Therefore, we see easily that if there is a p_i such that $p_i\geqq 11$, then $\psi(m)>1$. Now we may assume that $m=2^{e_1}3^{e_2}5^{e_3}7^{e_4}$ $(0\leqq e_i)$. We distinguish several cases. If $e_4\geqq 2$, then $\psi(m)\geqq(1/4)(8/9)(6^3/7)>1$. Consider the case $e_4=1$. If $e_1=1$, $e_2=1$, then by the condition $43\leqq m$ we have $e_3\geqq 1$. Hence $\psi(m)\geqq(1/4)(8/9)(4^3/5^2)(6^3/7^2)>1$. If $e_1\geqq 2$ or $e_2\geqq 2$, then $\psi(m)\geqq(1/2)(8/9)(6^3/7^2)>1$ or $\geqq(1/4)(6^3/7^2)>1$. We may consider the case $m=2^{e_1}3^{e_2}5^{e_3}$. If $e_3\geqq 2$, then $\psi(m)\geqq(1/4)(8/9)(4^3/5)>1$. If $e_3=1$, then by the condition $43\leqq m$ we have $e_1\geqq 2$ or $e_2\geqq 2$. Hence we see easily that $\psi(m)>1$. It remains the case $m=2^{e_1}3^{e_2}$. By the similar argument as above we can verify our assertion. Q.E.D.

Now we return to the proof of Theorem 3.6. We assume that $e\geqq 32$. Then by Lemma 3.8 we have $(2e)^{2/3}\leqq\varphi(2e)$. Hence by (3.17) and by the inequality $\pi/32<1/10$ we have $2^{(2e)^{2/3}/2}<18(1200/1199)\pi^{-2}e^2$. Let $t=(2e)^{2/3}/2$. Then we have $8\leqq t$ and $2^t<36(1200/1199)\pi^{-2}t^3$. We note that the approximate value of $36(1200/1199)\pi^{-2}$ is 3.6506. Let $f(t)=2^t/t^3$. Then $f(t)$ is monotone increasing on $8\leqq t$. Since $f(13)\fallingdotseq 3.7287$, we see that $t<13$. Hence we have $e\leqq 66$. For each $e\leqq 66$ such that $e\neq 2^m$ we examine (3.16) and by (3.15) we obtain the pairs listed in Theorem 3.6.

Next let us consider the case $e=2^m$. Assume that $m\geqq 2$. In this case we denote $d=2^{m-1}=\varphi(2e)/2$. By (3.14) and Lemma 3.7 we have

$$4^{n-n_1-1}<(9/c_e)(9/c_e-1)\,. \tag{3.18}$$

By the assumption $2\leqq m$ we have $2\leqq d$. By (3.18) we have $2^{d-1}\leqq 2^{(d-1)n_1}<9/(1-\cos(\pi/e))$. Hence $2^{e/2}<36e^2/(\pi^2(1-(\pi/e)^2/12))$. Assume that $e\geqq 32$. Then by the same argument as in the case $e\neq 2^m$ we have $e<22$. This is a contradiction. Thus we see that $e=4, 8, 16$. For each $e=4, 8, 16$ by (3.18) we can determine all n.

Let us consider the case $e=2$. In this case the above argument does not work. By (3.13) we have $8<X(X-2)(X-1)^2<63\cdot 64(=4032)$, $0<\varphi_i(X(2-X)(X-1)^2)\leqq 1/4$ $(2\leqq i\leqq n)$. Since X is an algebraic integer, we have $1\leqq|n_{k/Q}(X(2-X)(X-1)^2)|$. Hence we have $4^{n-1}<4032$. Therefore, we have $n\leqq 6$. This completes the proof of Theorem 3.6.

§ 4. Determination of all arithmetic Fuchsian groups with signature (1; e).

4.1. In this section we shall determine explicitly all arithmetic Fuchsian groups Γ with signature $(1; e)$. In order to do this it suffices to give a fundamental triple (α, β, γ) of generators of Γ. Let $x=\mathrm{tr}(\alpha)$, $y=\mathrm{tr}(\beta)$, $z=\mathrm{tr}(\alpha\beta)$. Then (α, β, γ) is uniquely determined by (x, y, z) up to $GL_2(\boldsymbol{R})$-conjugation. The conditions for Γ to be arithmetic are given in terms of (x, y, z) in Theorem 3.4

§3. In the following theorem we shall give a complete list of all (x, y, z) such that the group generated by (α, β, γ) obtained from (x, y, z) is an arithmetic Fuchsian group with signature $(1; e)$. We can also determine the quaternion algebra A over k associated with each Γ. We shall give the discriminant $D(A)$ of A explicitly.

THEOREM 4.1. *The complete list of all* (x, y, z) *such that the group* Γ *generated by* (α, β, γ) *obtained from* (x, y, z) *is an arithmetic Fuchsian group with signature* $(1; e)$ *is as follows:*

(i) $e=\infty$.

k	(x, y, z)	$D(A)$
$\boldsymbol{Q}$	$(\sqrt{5}, 2\sqrt{5}, 5)$	(1)
$\boldsymbol{Q}$	$(\sqrt{6}, 2\sqrt{3}, 3\sqrt{2})$	(1)
$\boldsymbol{Q}$	$(2\sqrt{2}, 2\sqrt{2}, 4)$	(1)
$\boldsymbol{Q}$	$(3, 3, 3)$	(1)

(ii) $e=2$.

$\boldsymbol{Q}$	$(\sqrt{5}, 2\sqrt{3}, \sqrt{15})$	(2)(3)
$\boldsymbol{Q}$	$(\sqrt{6}, 2\sqrt{2}, 2\sqrt{3})$	(2)(3)
$\boldsymbol{Q}$	$(\sqrt{7}, \sqrt{7}, 3)$	(2)(7)
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{2w_5+2}, \sqrt{4w_5+4}, \sqrt{6w_5+4})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{3w_5+2}, \sqrt{3w_5+2}, \sqrt{4w_5+4})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{3w_5+2}, \sqrt{3w_5+3}, \sqrt{3w_5+3})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{w_8+3}, \sqrt{8w_8+12}, \sqrt{9w_8+13})$	$\mathfrak{p}_7$ $(=(3+\sqrt{2}))$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{2w_8+3}, \sqrt{3w_8+5}, \sqrt{3w_8+5})$	$\mathfrak{p}_7'$ $(=(3-\sqrt{2}))$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{2w_8+4}, \sqrt{2w_8+4}, \sqrt{4w_8+6})$	$\mathfrak{p}_2$ $(=(\sqrt{2}))$
$\boldsymbol{Q}(\sqrt{3})$	$(\sqrt{w_{12}+3}, \sqrt{4w_{12}+8}, \sqrt{5w_{12}+9})$	$\mathfrak{p}_3$ $(=(\sqrt{3}))$
$\boldsymbol{Q}(\sqrt{3})$	$(\sqrt{2w_{12}+4}, \sqrt{2w_{12}+4}, \sqrt{2w_{12}+4})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{13})$	$(\sqrt{w_{13}+2}, \sqrt{8w_{13}+12}, \sqrt{9w_{13}+12})$	$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_3'$
$\boldsymbol{Q}(\sqrt{13})$	$(\sqrt{w_{13}+3}, \sqrt{3w_{13}+4}, \sqrt{3w_{13}+4})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{17})$	$(\sqrt{w_{17}+2}, \sqrt{4w_{17}+8}, \sqrt{5w_{17}+8})$	$\mathfrak{p}_2'$ $(=(w_{17}'+2))$
$\boldsymbol{Q}(\sqrt{17})$	$(\sqrt{w_{17}+3}, \sqrt{2w_{17}+4}, \sqrt{3w_{17}+5})$	$\mathfrak{p}_2$ $(=(w_{17}+2))$
$\boldsymbol{Q}(\sqrt{21})$	$(\sqrt{w_{21}+2}, \sqrt{3w_{21}+6}, \sqrt{3w_{21}+7})$	$\mathfrak{p}_2$

$\boldsymbol{Q}(\sqrt{6})$	$(\sqrt{w_{24}+3},\ \sqrt{2w_{24}+5},\ \sqrt{2w_{24}+6})$	$\mathfrak{p}_2\ (=(w_{24}+2))$
$\boldsymbol{Q}(\sqrt{33})$	$(\sqrt{w_{33}+3},\ \sqrt{w_{33}+4},\ \sqrt{2w_{33}+5})$	$\mathfrak{p}_2\ (=(w_{33}-3))$

We define w_d for the discriminant d of a quadratic field $\boldsymbol{Q}(\sqrt{d})$ as follows:

$$w_d=\begin{cases}(1+\sqrt{d})/2 & \text{if } d\equiv 1 \pmod 4, \\ \sqrt{d}/2 & \text{if } d\equiv 0 \pmod 4.\end{cases} \tag{4.1}$$

$f(t)$	$d(k)$	ρ		
t^3-t^2-2t+1	49	$2\cos(\pi/7)$	$(\sqrt{\rho^2+\rho},\ \sqrt{3\rho^2+2\rho-1},\ \sqrt{3\rho^2+2\rho-1})$	$\mathfrak{p}_2\mathfrak{p}_7$
t^3-3t-1	81	$\rho\doteqdot 1.8794$	$(\sqrt{\rho^2+\rho+1},\ \rho+1,\ \rho+1)$	(1)
t^3-t^2-3t+1	148	$\rho\doteqdot 2.1700$	$(\sqrt{\rho^2+\rho},\ \sqrt{\rho^2+\rho},\ \sqrt{\rho^2+2\rho+1})$	(1)
t^3-t^2-3t+1	148	$\rho\doteqdot 0.3111$	$(\sqrt{-\rho^2+\rho+4},\ \sqrt{-12\rho^2+8\rho+40},\ \sqrt{-13\rho^2+9\rho+42})$	(1)
t^3-t^2-3t+1	148	$\rho\doteqdot -1.4811$	$(\sqrt{\rho^2-2\rho+1},\ \sqrt{\rho^2-3\rho+2},\ \sqrt{\rho^2-3\rho+2})$	(1)
t^3-4t-1	229	$\rho\doteqdot 2.1149$	$(\sqrt{\rho+2},\ \sqrt{8\rho^2+16\rho+4},\ \sqrt{8\rho^2+17\rho+4})$	$\mathfrak{p}_2\mathfrak{p}_2'$
t^3-4t-1	229	$\rho\doteqdot -0.2541$	$(\sqrt{-\rho^2+5},\ \sqrt{-3\rho^2+\rho+13},\ \sqrt{-4\rho^2+\rho+16})$	$\mathfrak{p}_2\mathfrak{p}_2'$
t^3-4t-1	229	$\rho\doteqdot -1.8608$	$(\sqrt{\rho^2-2\rho},\ \sqrt{\rho^2-2\rho},\ \sqrt{\rho^2-2\rho+1})$	$\mathfrak{p}_2\mathfrak{p}_2'$
$t^4-t^3-3t^2+t+1$	725	$\rho\doteqdot -1.3556$	$(x=\sqrt{-\rho^3+2\rho^2+\rho},\ y=z=\sqrt{-2\rho^3+5\rho^2-\rho-1})$	$\mathfrak{p}_2$
$t^4-t^3-3t^2+t+1$	725	$\rho\doteqdot -0.4772$	$(x=\sqrt{\rho^3-2\rho^2-2\rho+4},\ y=z=\sqrt{9\rho^3-13\rho^2-21\rho+19})$	$\mathfrak{p}_2$
$t^4-t^3-4t^2+4t+1$	1125	$\rho\doteqdot -1.9562$	$(\sqrt{\rho^2-\rho},\ \sqrt{\rho^2-2\rho+1},\ \sqrt{-\rho^3+\rho^2+\rho+1})$	$\mathfrak{p}_2$

where $f(t)$ denotes the irreducible polynomial of ρ over $\boldsymbol{Q}$ such that $k=\boldsymbol{Q}(\rho)$.

(iii) $e=3$.

$\boldsymbol{Q}$	$(\sqrt{5},\ 4,\ 2\sqrt{5})$	(3)(5)
$\boldsymbol{Q}$	$(\sqrt{6},\ \sqrt{10},\ \sqrt{15})$	(2)(5)
$\boldsymbol{Q}$	$(\sqrt{7},\ 2\sqrt{2},\ \sqrt{14})$	(2)(3)

$\boldsymbol{Q}$	$(2\sqrt{2}, 2\sqrt{2}, 3)$	(2)(3)
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{2w_5+2}, \sqrt{6w_5+4}, \sqrt{8w_5+5})$	$\mathfrak{p}_3$
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{3w_5+2}, \sqrt{4w_5+3}, \sqrt{4w_5+3})$	$\mathfrak{p}_5$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{2w_8+3}, \sqrt{4w_8+6}, \sqrt{4w_8+6})$	$\mathfrak{p}_3$
$\boldsymbol{Q}(\sqrt{3})$	$(\sqrt{2w_{12}+4}, \sqrt{2w_{12}+4}, \sqrt{4w_{12}+7})$	$\mathfrak{p}_3$
$\boldsymbol{Q}(\sqrt{13})$	$(\sqrt{2w_{13}+3}, \sqrt{2w_{13}+3}, \sqrt{3w_{13}+4})$	$\mathfrak{p}_3'$ $(=(w_{13}'))$
$\boldsymbol{Q}(\sqrt{13})$	$(\sqrt{w_{13}+2}, \sqrt{12w_{13}+16}, \sqrt{13w_{13}+17})$	$\mathfrak{p}_3$ $(=(w_{13}))$
$\boldsymbol{Q}(\sqrt{17})$	$(\sqrt{w_{17}+2}, \sqrt{6w_{17}+10}, \sqrt{7w_{17}+11})$	$\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_3$
$\boldsymbol{Q}(\sqrt{21})$	$(\sqrt{w_{21}+2}, \sqrt{4w_{21}+8}, \sqrt{5w_{21}+9})$	$\mathfrak{p}_3$
$\boldsymbol{Q}(\sqrt{7})$	$(\sqrt{w_{28}+3}, \sqrt{2w_{28}+6}, \sqrt{3w_{28}+8})$	$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_3'$

$f(t)$	$d(k)$	ρ		
t^3-t^2-2t+1	49	$2\cos(\pi/7)$	$(x=\sqrt{\rho^2+\rho}, y=z=\sqrt{4\rho^2+3\rho-2})$	(1)
t^3-3t^2+1	81	$-1/(2\cos(5\pi/9))$	(ρ, ρ, ρ)	(1)

where $f(t)$ *denotes the irreducible polynomial of* ρ *over* $\boldsymbol{Q}$ *such that* $k=\boldsymbol{Q}(\rho)$.

(iv) $e=4$.

$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{3+\sqrt{2}}, \sqrt{20+12\sqrt{2}}, \sqrt{21+14\sqrt{2}})$	$\mathfrak{p}_2\mathfrak{p}_7\mathfrak{p}_7'$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{4+\sqrt{2}}, \sqrt{8+4\sqrt{2}}, \sqrt{10+6\sqrt{2}})$	$\mathfrak{p}_7$ $(=(3-\sqrt{2}))$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{3+2\sqrt{2}}, \sqrt{7+4\sqrt{2}}, \sqrt{7+4\sqrt{2}})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{3+2\sqrt{2}}, \sqrt{6+4\sqrt{2}}, \sqrt{9+4\sqrt{2}})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{4+2\sqrt{2}}, \sqrt{6+2\sqrt{2}}, \sqrt{8+5\sqrt{2}})$	$\mathfrak{p}_7$ $(=(3+\sqrt{2}))$
$\boldsymbol{Q}(\sqrt{2})$	$(\sqrt{5+2\sqrt{2}}, \sqrt{5+2\sqrt{2}}, \sqrt{6+4\sqrt{2}})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{7-2\sqrt{2}})$	$d(k)=2624$ $\rho=(1+\sqrt{13+8\sqrt{2}})/2$	
	$(x=\sqrt{\rho+2}, y=z=\sqrt{(1+2\sqrt{2})\rho+5+2\sqrt{2}})$	$\mathfrak{p}_2$ $(=(\sqrt{2}))$
$\boldsymbol{Q}(\sqrt{7+2\sqrt{2}})$	$d(k)=2624$ $\rho=(1+\sqrt{2}+\sqrt{7+2\sqrt{2}})/2$	
	$(x, y, z)=(\rho, \rho, \rho+1)$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{2}, \sqrt{3})$	$d(k)=2304$ $\rho=(2+\sqrt{2}+\sqrt{6})/2$	
	$(x, y, z)=(\rho, \rho, \rho)$	$\mathfrak{p}_2$

(v) $e=5$.

$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{w_5+3}, \sqrt{12w_5+8}, \sqrt{14w_5+9})$	$\mathfrak{p}_5\ (=(\sqrt{5}))$
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{2w_5+2}, \sqrt{6w_5+6}, \sqrt{9w_5+6})$	$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5$
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{2w_5+3}, \sqrt{4w_5+4}, \sqrt{7w_5+5})$	$\mathfrak{p}_5$
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{3w_5+2}, \sqrt{4w_5+4}, \sqrt{4w_5+4})$	$\mathfrak{p}_5$
$\boldsymbol{Q}(\sqrt{5})$	$(\sqrt{3w_5+3}, \sqrt{3w_5+3}, \sqrt{5w_5+5})$	$\mathfrak{p}_2$
$\boldsymbol{Q}(\sqrt{13w_5+9})$	$d(k)=725 \quad \rho=(w_5+\sqrt{13w_5+9})/2$	
	$(x=\sqrt{\rho+2}, y=z=\sqrt{(w_5+1)\rho+2w_5+2})$	$\mathfrak{p}_5$
$\boldsymbol{Q}(\sqrt{7w_5+6})$	$d(k)=725 \quad \rho=(w_5+3+\sqrt{7w_5+6})/2$	
	$(x=\sqrt{\rho}, y=z=\sqrt{(5w_5+2)\rho-2w_5+1})$	$\mathfrak{p}_5$
$\boldsymbol{Q}(\sqrt{33w_5+21})$	$d(k)=1125 \quad \rho=(1+w_5+(2-w_5)\sqrt{33w_5+21})/2$	
	$(x, y, z)=(\rho, \rho, \rho)$	$\mathfrak{p}_5$

(vi) $e=6$.

$\boldsymbol{Q}(\sqrt{3})$	$(\sqrt{3+\sqrt{3}}, \sqrt{14+6\sqrt{3}}, \sqrt{15+8\sqrt{3}})$	$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}$
$\boldsymbol{Q}(\sqrt{3})$	$(\sqrt{5+\sqrt{3}}, \sqrt{6+2\sqrt{3}}, \sqrt{9+4\sqrt{3}})$	$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}$

(vii) $e=7$.

$k=\boldsymbol{Q}(\cos(\pi/7)) \quad d(k)=49 \quad \rho=2\cos(\pi/7)$

$(x, y, z)=(\sqrt{\rho^2+1}, \sqrt{16\rho^2+12\rho-8}, \sqrt{17\rho^2+13\rho-9})$	$\mathfrak{p}_7\mathfrak{p}_{13}$
$(\sqrt{\rho^2+\rho}, \sqrt{5\rho^2+3\rho-2}, \sqrt{5\rho^2+3\rho-2})$	$\mathfrak{p}_7\mathfrak{p}'_{13}$
$(\sqrt{2\rho^2+\rho}, \sqrt{2\rho^2+\rho}, \sqrt{3\rho^2+\rho-1})$	$\mathfrak{p}_7\mathfrak{p}''_{13}$
$(\sqrt{2\rho^2}, \sqrt{2\rho^2+2\rho}, \sqrt{4\rho^2+3\rho-2})$	(1)

(viii) $e=9$.

$k=\boldsymbol{Q}(\cos(\pi/9)) \quad d(k)=81 \quad \rho=2\cos(\pi/9)$

$(x, y, z)=(\sqrt{\rho^2+1}, \sqrt{4\rho^2+8\rho+4}, \sqrt{5\rho^2+9\rho+3})$	$\mathfrak{p}_3\mathfrak{p}_{17}$
$(\sqrt{\rho^2+\rho+1}, \sqrt{2\rho^2+2\rho+1}, \sqrt{2\rho^2+2\rho+1})$	$\mathfrak{p}_3\mathfrak{p}'_{17}$
$(\sqrt{\rho^2+2\rho+1}, \sqrt{\rho^2+2\rho+2}, \sqrt{\rho^2+2\rho+2})$	$\mathfrak{p}_3\mathfrak{p}''_{17}$

(ix) $e=11$.

$k=\boldsymbol{Q}(\cos(\pi/11)) \quad d(k)=11^4 \quad \rho=2\cos(\pi/11)$

$(x, y, z)=(\rho^2-1, \rho^3-2\rho, \rho^3-2\rho)$ (1)

where we denote by $\mathfrak{p}_p$ *the prime ideal of* k *dividing* (p) *for a prime number* p.

We shall give the proof of Theorem 4.1 in 4.2-4.10. We have only to deal with the cases (e, n) listed in Theorem 3.6 and the case $e=\infty$. Let Γ be an arithmetic Fuchsian group with signature $(1; e)$. Then by Proposition 3.1 and (2.5) we have

$$(2\pi)^{-2n}d(k)^{3/2}\cdot\zeta_k(2)\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)=d_2^{-1}(1-1/e), \tag{4.2}$$

where $d_2=[\Gamma^{(1)}(A, O_1):\Gamma^{(2)}\cdot\{\pm 1_2\}]$. In solving the simultaneous linear inequalities we used a programable electronic calculator YHP-67. In view of Theorem 3.4 the general procedure to obtain all solutions (x, y, z) over k is as follows. Let $\{w_1, \cdots, w_n\}$ be a $\boldsymbol{Z}$-basis of O_k. Then we have the expressions $x^2=m_1w_1+\cdots+m_nw_n$, $y^2=r_1w_1+\cdots+r_nw_n$, $z^2=s_1w_1+\cdots+s_nw_n$ $(m_i, r_i, s_i\in\boldsymbol{Z})$. From (3.10), (3.12) we have the simultaneous inequalities for $(m_1, \cdots, m_n)$. For each solution x^2 by (3.10), (3.11) we have the inequalities for $(r_1, \cdots, r_n)$. For each (x^2, y^2) we have the inequalities for $(s_1, \cdots, s_n)$. Finally for each (x, y, z) we check the condition (3.9). We note here that x, y, z are not necessarily contained in k.

Let us consider the case $e=\infty$. Since Γ contains a parabolic element in this case, it is well-known that $k=\boldsymbol{Q}$, $A\cong M_2(\boldsymbol{Q})$. From (3.10) we see that x^2 is a rational integer such that $5\leqq x^2\leqq 9$. For each x^2 we can easily solve the inequalities for y^2, z^2. Thus, we can obtain all solutions in the case $e=\infty$.

4.2. The case $e=2$.

Let us consider the case $e=2$. In this case by Theorem 3.6 we have $1\leqq n\leqq 6$. Furthermore, from (3.10), (3.12) we have

$$4<x^2\leqq 4+2\sqrt{3},\quad 0<\varphi_i(x^2)<2\quad (2\leqq i\leqq n). \tag{4.3}$$

In the case $n=1$ it is easy to obtain all solutions. Let us consider the case $n=2$. Let w_d be the same as in (4.1). Then $\{1, w_d\}$ is a $\boldsymbol{Z}$-basis of the ring O_k of integers in k. Put $x^2=a+bw_d$ $(a, b\in\boldsymbol{Z})$. Then from (4.3) we have $2<b\sqrt{d}\leqq 4+2\sqrt{3}$. Hence $d\leqq(4+2\sqrt{3})^2=55.7\cdots$. Thus we have $d=d(k)\leqq 55$. For each d we can obtain all solutions x^2 in O_k satisfying (4.3). For each x^2 we can solve the inequalities of Theorem 3.4 for y, z. Hence we obtain all solutions in the case $n=2$.

Let us consider the case $n=3$. Since $\zeta_k(2)>1$ and $\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)\geqq 1$, from (4.2) we have $d(k)<((2\pi)^6/2)^{2/3}=981.822\cdots$. Hence we have $d(k)\leqq 981$. A list of the totally real algebraic number fields k of degree 3 with small $d(k)$ can be found in K. K. Billevich [1] p. 134 and in B. N. Delone-D. K. Faddeev [2] p. 159. In view of these lists we obtain the following 25 cases:

$d(k)=49, 81, 148, 169, 229, 257, 316, 321, 361, 404, 469, 473, 564, 568,$
$621, 697, 733, 756, 761, 785, 788, 837, 892, 940, 961.$

For each $d(k)$ listed above the defining equation for k and a $\boldsymbol{Z}$-basis of O_k are given in [**1**], [**2**]. Using those data we can compute the relative degrees $f_{\mathfrak{p}}$ and the ramification indexes $e_{\mathfrak{p}}$ for the prime ideals $\mathfrak{p}$ of k dividing the prime numbers $p=2, 3, 5$. Thus, we can compute the $\mathfrak{p}$-factor of the Euler product $\zeta_k(2)=\prod_{\mathfrak{p}}(1-n_{k/Q}(\mathfrak{p})^{-2})^{-1}$. It implies that the cases $d(k)=788, 837, 892, 940, 961$ are excluded because the left hand side of (4.2) is greater than 1/2 in these cases. In each remaining case a $\boldsymbol{Z}$-basis of O_k is given. Therefore following the general procedure we can obtain the solutions for x^2 and then for y, z.

Let us consider the case $n=4$. From (4.2) we have $d(k)<((2\pi)^8/2)^{2/3}=11383.416\cdots$. Hence we have $d(k)\leqq 11383$. A list of the totally real algebraic number fields k with $d(k)\leqq 11664$ is given by H. J. Godwin [**4**]. A list of such fields k with $d(k)\leqq 8112$ (resp. 7168) is also given in [**2**] (resp. [**1**]). A $\boldsymbol{Z}$-basis of O_k for each k is also given there. Using these data we can obtain all solutions. However, in order to avoid the extensive numerical computations we make the following arguments.

We distinguish two cases $2<y^2-x^2$ and $0\leqq y^2-x^2\leqq 2$. First let us consider the former case. From (3.10) we have $x^2+2<(x^2-2)/(x-2)$. Solving this inequality numerically we have

(4.4) $$4<x^2<6.4, \quad 0<\varphi_i(x^2)<2 \quad (2\leqq i\leqq 4).$$

It follows from (4.4) that $|n_{k/Q}(x^2(2-x^2)(1-x^2)^2)|<12.83\cdots$. Hence we have

(4.5) $$-12\leqq n_{k/Q}(x^2(2-x^2)(1-x^2)^2)\leqq -1\,.$$

Let $f(t)=t^4+a_3t^3+a_2t^2+a_1t+a_0$ $(a_i\in\boldsymbol{Z})$ be the irreducible polynomial of x^2 over $\boldsymbol{Q}$. Let $b_i=f(i)$ $(0\leqq i\leqq 2)$. Then from (4.4), (4.5) we have

(4.6) $$b_0\geqq 1\,,\quad b_2\leqq -1\,,\quad -12\leqq b_0b_1^2b_2\leqq -1\,.$$

We can determine easily all triples (b_0, b_1, b_2) satisfying (4.6). Since $a_3=-\mathrm{tr}_{k/Q}(x^2)$, from (4.4) we have

(4.7) $$-12\leqq a_3\leqq -5\,.$$

Using the expressions: $a_0=b_0$, $a_1=(-b_2+4b_1-3b_0+4a_3+12)/2$, $a_2=(b_2-2b_1+b_0-6a_3-14)/2$, we can obtain all (a_3, a_2, a_1, a_0) such that x^2 satisfies (4.4), which are as follows:

a_3	a_2	a_1	a_0	$d(f)$
-7	13	-7	1	725 $(=5^2\cdot 29)$
-8	18	-13	1	725
-8	14	-7	1	1125 $(=3^2\cdot 5^3)$
-9	20	-14	3	1957 $(=19\cdot 103)$

-10	27	-26	7	2624 $(=2^6\cdot 41)$
-9	22	-18	3	3981 $(=3\cdot 1327)$
-8	15	-8	1	4752 $(=2^4\cdot 3^3\cdot 11)$
-8	16	-9	1	8069 (prime)
-9	19	-12	2	11324 $(=2^2\cdot 19\cdot 149)$
-9	21	-15	1	14197 (prime)
-9	19	-11	1	36677 (prime),

where we denote by $d(f)$ the discriminant of $f(t)$. It is known that 725 is the smallest discriminant of the totally real algebraic number fields of degree 4. Since $d(f)=d(k)m^2$ $(m\in\boldsymbol{Z})$, in view of the list in Godwin [4] we see that $d(f)=d(k)$ and $O_k=\boldsymbol{Z}[x^2]$ in each case listed above. Since $n=4$ is even, by (2.1) and the Hasse's principle we see that the number of the prime ideals of k dividing $D(A)$ is odd. In particular, $D(A)\neq(1)$. In the cases: $d(f)=8069$, 11324, 14197, 36677 we can see easily that the left hand side of (4.2) is greater than 1/2. Hence these cases are excluded. In the remaining cases we can obtain all solutions following the general procedure.

Now let us consider the second case $0\leqq y^2-x^2\leqq 2$. Let $a=y^2-x^2$. Then from (3.10), (3.12) we have

$$0\leqq a<2, \quad -2<\varphi_i(a)<2 \quad (2\leqq i\leqq 4). \tag{4.8}$$

We need the following lemma (cf. Pólya-Szegö [12] p. 145).

LEMMA 4.2. *Let a be a totally real algebraic integer such that all conjugates $\varphi_i(a)$ of a satisfy the inequalities $-2\leqq\varphi_i(a)\leqq 2$ $(1\leqq i\leqq n)$. Then $a=2\cos(2\pi r)$ $(r\in\boldsymbol{Q})$.*

From this lemma we have $a=2\cos(2\pi r)$ $(r\in\boldsymbol{Q})$. By (3.10), (3.11) we have

$$\begin{cases} (8-a+\sqrt{a^2+32})/2\leqq x^2<4+2\sqrt{3}, \\ 0<\varphi_i(x^2)\leqq(8-\varphi_i(a)-\sqrt{\varphi_i(a)^2+32})/2 \quad (2\leqq i\leqq n). \end{cases} \tag{4.9}$$

Moreover,

$$\text{If } \varphi_i(a)<0, \text{ then } -\varphi_i(a)<\varphi_i(x^2) \quad (2\leqq i\leqq n). \tag{4.10}$$

Since a is contained in k, we see that $[\boldsymbol{Q}(a):\boldsymbol{Q}]=1, 2, 4$. Assume that $a\in\boldsymbol{Q}$. Then $a=0$ or 1. In these cases from (4.9) we see that $0<\varphi_i(x^2)\leqq 4-2\sqrt{2}$ or $(7-\sqrt{33})/2$. Hence we have $|n_{k/\boldsymbol{Q}}(x^2(1-x^2))|<1$, which is a contradiction.

Let us consider the case $[\boldsymbol{Q}(a):\boldsymbol{Q}]=2$. Then we see that $a=2\cos(\pi/4)$, $2\cos(\pi/5)$, $2\cos(2\pi/5)$ or $2\cos(\pi/6)$. Let $v_1=\mathrm{tr}_{k/\boldsymbol{Q}}(x^2)$, $v_0=n_{k/\boldsymbol{Q}}(x^2)$. Then by (4.9) and (4.10) we obtain the inequalities for v_0, v_1 and their $\boldsymbol{Q}$-conjugates v_0', v_1'. Solving these inequalities for each case, we see that there exist no solutions in each case.

Let us consider the case $[\boldsymbol{Q}(a):\boldsymbol{Q}]=4$. We have $k=\boldsymbol{Q}(a)=\boldsymbol{Q}(\cos(\pi/8))$, $\boldsymbol{Q}(\cos(\pi/10))$, $\boldsymbol{Q}(\cos(\pi/12))$ or $\boldsymbol{Q}(\cos(\pi/15))$. Since we know that $\zeta_k(2)=2^{-5}3^{-1}5(2\pi)^8 d(k)^{-3/2}$ for $k=\boldsymbol{Q}(\cos(\pi/8))$ (cf. [18] p. 208), from (4.2) we have $\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)d_2=2^4 3/5$, which is not an integer. This is a contradiction.

In order to deal with the remaining cases we need the following

LEMMA 4.3. *Let $\rho=2\cos(2\pi/m)$ $(m\in\boldsymbol{Z})$ and $k=\boldsymbol{Q}(\rho)$. Then $\{1, \rho, \rho^2, \cdots, \rho^{d-1}\}$ is a $\boldsymbol{Z}$-basis of the ring O_k of integers in k, where $d=[k:\boldsymbol{Q}]$.*

The proof of this lemma is referred to Liang [9]. Let $\rho_r=2\cos(\pi/r)$ for each case $k=\boldsymbol{Q}(\cos(\pi/r))$, $r=10$, 12 or 15. By (3.10) we have the inequality $4(x^2-2)/(x^2-4)<x^2+2$ in this case. Solving this inequality numerically, we have

$$6<x^2<4+2\sqrt{3},\quad 0<\varphi_i(x^2)<2\quad (2\leqq i\leqq 4). \tag{4.10}$$

By Lemma 4.3 we have the expression $x^2=\sum_{0\leqq i\leqq 3} m_i\rho_r^i$ $(m_i\in\boldsymbol{Z})$. Solving the inequalities for (m_0, m_1, m_2, m_3) given by (4.10) numerically, we see that there exist no solutions for x^2.

Let us consider the case $n=5$. Solving the inequality numerically $1\leqq |n_{k/Q}(x^2(2-x^2)(1-x^2)^2)|\leqq x^2(x^2-2)(x^2-1)^2/4^4$, we have

$$5.06<x^2<4+2\sqrt{3},\quad 0<\varphi_i(x^2)<2\quad (2\leqq i\leqq 5). \tag{4.11}$$

We shall show that if $0\leqq y^2-x^2\leqq 2$ or $0\leqq z^2-y^2\leqq 2$, then $k=\boldsymbol{Q}(\cos(\pi/11))$. Assume that $0\leqq y^2-x^2\leqq 2$. Then by (3.12) and Lemma 4.2 we have $y^2-x^2=2\cos(2\pi r)$, $r\in\boldsymbol{Q}$. Since $n=5$, we have $y^2-x^2=0$, 1 or $2\cos(2\pi s/11)$. If $y^2-x^2=0$ or 1, then by (4.9) we have $|n_{k/Q}(x^2(1-x^2))|<1$, which is a contradiction. Assume that $0\leqq z^2-y^2\leqq 2$. Then we have $z^2-y^2=0$, 1 or $2\cos(2\pi s/11)$. If $z^2-y^2=0$ or 1, then by the fact that the function $(x^2-2)/(x-2)$ is monotone-decreasing on $\sqrt{5.06}<x<3$, from (3.10), (3.11) we have

$$y^2<12.268,\quad 0<\varphi_i(y^2)<4-2\sqrt{2}\quad (\text{resp. } (7-\sqrt{33})/2)\quad (2\leqq i\leqq 5).$$

It follows that $|n_{k/Q}(y^2(1-y^2))|<1$, which is a contradiction. Hence we see that $k=\boldsymbol{Q}(\cos(\pi/11))$. Since $\zeta_k(2)=2^{-3}\cdot 3^{-1}\cdot 5\cdot 11^{-1}(2\pi)^{10}d(k)^{-3/2}$ for $k=\boldsymbol{Q}(\cos(\pi/11))$ (cf. [18] p. 208), we have $\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)d_2=2^2\cdot 3\cdot 11/5$, which is not an integer. This is a contradiction.

Now we must consider the case:

$$x^2+2<y^2,\quad y^2+2<z^2. \tag{4.12}$$

From (3.10) and the second inequality of (4.12) we have

$$y^2<x^2(1+\sqrt{x^2-3})/(x^2-4). \tag{4.13}$$

Combining the first inequality of (4.12) with (4.13), we have

$$5.06<x^2<6.071,\quad 0<\varphi_i(x^2)<2\quad (2\leqq i\leqq 5). \tag{4.14}$$

From (4.14) we have $|n_{k/Q}(x^2(2-x^2)(1-x^2)^2)|<2.48\cdots$. Hence we have $n_{k/Q}(1-x^2)=\pm 1$, $n_{k/Q}(x^2(2-x^2))=-1$ or -2. Since $x^2+(2-x^2)=2$, $n_{k/Q}(x^2)$ is divisible by 2 if and only if $n_{k/Q}(2-x^2)$ is so. Therefore we have

$$(4.15) \qquad n_{k/Q}(-x^2)=-1, \quad n_{k/Q}(2-x^2)=-1, \quad n_{k/Q}(1-x^2)=\pm 1.$$

Let $f(t)=t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0$ $(a_i\in \boldsymbol{Z})$ be the irreducible polynomial of x^2 over $\boldsymbol{Q}$. Then from (4.15) we have $f(0)=-1$, $f(2)=-1$, $f(1)=\pm 1$.

We distinguish two cases: $f(1)=1$ and -1. Let us consider first the case $f(1)=-1$. In this case we have the expression: $f(t)=t(t-1)(t-2)(t^2+c_1t+c_0)-1$ $(c_i\in \boldsymbol{Z})$. Since $\mathrm{tr}_{k/Q}(x^2)=3-c_1$, by (4.14) we have

$$(4.16) \qquad -11\leqq c_1\leqq -3.$$

From (4.14) we have $f(5.06)<0<f(6.071)$. This gives the inequalities for (c_0, c_1). Solving these inequalities numerically, we have a finite set of solutions for (c_0, c_1). We check the condition (4.14) for each case (c_0, c_1). Hence we have only one case: $f(t)=t^5-10t^4+29t^3-32t^2+12t-1$, $d(f)=24217$. However, solving the inequalities (3.10), (3.11) for y^2 in this case, we see that there exist no solutions.

For the case $f(1)=1$ by the same way as in the case $f(1)=-1$ we see that there exist no solutions.

Let us consider the case $n=6$. From the inequality:

$$1\leqq |n_{k/Q}(x^2(2-x^2)(1-x^2)^2)|<-x^2(2-x^2)(1-x^2)^2/4^5,$$

we have

$$(4.17) \qquad 6.7<x^2<4+2\sqrt{3}, \quad 0<\varphi_i(x^2)<2 \quad (2\leqq i\leqq 6).$$

Let $a=y^2-x^2$. Then from (3.10) we have

$$0\leqq a\leqq (x^2-2)/(x-2)-x^2.$$

Since the function $(x^2-2)/(x-2)-x^2$ is monotone-decreasing on $\sqrt{6.7}\leqq x$, we have

$$0\leqq a\leqq 1.2873, \quad -2<\varphi_i(a)<2 \quad (2\leqq i\leqq 6).$$

By Lemma 4.2 we have $a=2\cos(2\pi s/r)$, where $1\leqq r$, $s\in \boldsymbol{Z}$ such that $\varphi(r)=2$, 4, 6 or 12 and $(r, s)=1$ and $0<s/r\leqq 1/4$.

Assume that $a=0$ or 1. Then by (4.9) we have $|n_{k/Q}(x^2(1-x^2))|<1$, which is a contradiction. There remain the cases $\varphi(r)=4$, 6 or 12. In these cases for each pair (r, s) we can show that $|n_{k/Q}(x^2)|<1$ or $|n_{k/Q}(y^2)|<1$. This is a contradiction. We have finished the case $e=2$.

4.3. The case $e=3$.

Let us consider the case $e=3$. In this case we have $c_e=1$. By Theorem 3.6 we have $1\leqq n\leqq 4$. From (3.10) we have $x^2\leqq (x^2-1)/(x-2)$. Solving this inequality numerically, we have

$$(4.18) \qquad 4<x^2<8.291, \quad 0<\varphi_i(x^2)<1 \quad (2\leqq i\leqq n).$$

By the inequalities: $1\leqq|n_{k/Q}(x^2(1-x^2))|<8.291\cdot 7.291/4^{n-1}$, we have $4^{n-1}<60.449$. Hence we see that $n=1$, 2 or 3. In the cases $n=1$, 2 we can obtain easily all solutions.

Now let us consider the case $n=3$. We distinguish two cases: $3<y^2-x^2$ and $0\leqq y^2-x^2\leqq 3$. Let us consider the first case $3<y^2-x^2$. By (3.10) we have $x^2+3<(x^2-1)/(x-2)$. Solving this inequality, we have $x^2<6.6947$. On the other hand, since $1\leqq|n_{k/Q}(x^2(1-x^2))|<x^2(x^2-1)/4^2$, we have

(4.19) $$(1+\sqrt{65})/2<x^2<6.6947,\quad 0<\varphi_i(x^2)<1\quad (2\leqq i\leqq 3).$$

From (4.19) we see easily that

(4.20) $$n_{k/Q}(x^2(1-x^2))=-1,\ -2.$$

Let $f(t)=t^3+a_2t+a_1t+a_0$ $(a_i\in \boldsymbol{Z})$ be the irreducible polynomial of x^2 over $\boldsymbol{Q}$. By (4.19) we have $-8\leqq a_2\leqq -5$. By (4.20) we have $f(0)\cdot f(1)=1$ or 2. From these relations we obtain a finite set of solutions for (a_0, a_1, a_2). Checking the condition (4.19) for each (a_0, a_1, a_2), we obtain $f(t)=t^3-6t^2+5t-1$, $d(f)=d(k)=49$. For this x^2 we have a solution such that $y=z$.

Let us consider the case $0\leqq y^2-x^2\leqq 3$. Let $a=y^2-x^2$. Then from (3.12) we have

(4.21) $$-1\leqq a-1\leqq 2,\quad -2<\varphi_i(a-1)<0\quad (2\leqq i\leqq 3).$$

Since $[\boldsymbol{Q}(a):\boldsymbol{Q}]=1$ or 3, from (4.21) we have $a=0$, $1+2\cos(2\pi/7)$ or $1+2\cos(\pi/9)$. In the case $a=0$ by (3.10), (3.11), (4.18) we have

(4.22) $$4+2\sqrt{3}\leqq x^2<8.291,\quad 0<\varphi_i(x^2)\leqq 4-2\sqrt{3}\quad (2\leqq i\leqq 3).$$

From this we obtain a solution such that $x=y=z$, $d(k)=81$. For two other cases we see that there exist no solutions.

4.4. The case $e=4$.

Let us consider the case $e=4$. In this case we have $c_e=2-\sqrt{2}$. By Theorem 3.4 (i) we see that k contains $k_0=\boldsymbol{Q}(\sqrt{2})$. From (3.10), (3.12) we have

(4.23) $$4<x^2<9,\quad 0<\varphi_i(x^2)<\varphi_i(2-\sqrt{2})\quad (2\leqq i\leqq n).$$

Let $u=1+\sqrt{2}$. Since $0<\varphi_i(ux^2(\sqrt{2}-ux^2))<1/2$, we have

$$1\leqq|n_{k/Q}(ux^2(\sqrt{2}-ux^2)(\sqrt{2}ux^2-1)^2)|<9u(9u-\sqrt{2})(9\sqrt{2}u-1)^2/8^{n-1}.$$

Hence $8^{n-1}<390064.58\cdots$. It follows that $n=2$, 4 or 6. Since $x^2\leqq y^2\leqq(x^2-2+\sqrt{2})/(x-2)$, we have

(4.24) $$4<x^2<8.596,\quad 0<\varphi_i(x^2)<\varphi_i(2-\sqrt{2})\quad (2\leqq i\leqq n).$$

In the case $n=2$ we have $k=k_0$ and it is easy to obtain all solutions.

Let us consider the case $n=4$. Then k is a quadratic extension of k_0. Let

$a_0=n_{k/k_0}(x^2)$, $a_1=\mathrm{tr}_{k/k_0}(x^2)$. From (4.24) we have the inequalities for a_0, a_1 and their $\boldsymbol{Q}$-conjugates. We obtain 11 cases for (a_0, a_1). For each case we can calculate $d(k)$ and obtain an O_{k_0}-basis $\{1, \rho\}$ of O_k. Using the expressions $y^2=b_0+b_1\rho$, $z^2=c_0+c_1\rho$ $(b_i, c_i\in O_{k_0})$, we obtain the inequalities for b_i, c_i and their $\boldsymbol{Q}$-conjugates. Solving these inequalities we obtain three solutions for (x, y, z).

Let us consider the case $n=6$. Let $g(t)=t^3+b_2t^2+b_1t+b_0$ $(b_i\in O_{k_0})$ be the irreducible polynomial of ux^2 over k_0. By (4.24) we have

$$4u<ux^2<8.596u,\quad 0<\pm\varphi_i(ux^2)<\sqrt{2}\quad (2\leqq i\leqq 6), \tag{4.25}$$

where the sign $\pm$ is determined according to $\varphi_i(\sqrt{2})=\pm\sqrt{2}$. From (4.25) we have inequalities for b_i and their $\boldsymbol{Q}$-conjugates. For each solution for (b_i) we check the condition (4.25) and we see that there exist no solutions.

4.5. The case $e=5$.

Let us consider the case $e=5$. In this case from Theorem 3.6 we see that $n=2$ or 4. For the case $n=2$ we have $k=\boldsymbol{Q}(\sqrt{5})$ and we obtain easily all solutions. Let us consider the case $n=4$. Then k is a quadratic extension of $k_0=\boldsymbol{Q}(\sqrt{5})$. From (3.10) we have the inequality $x^3-3x^2+\frac{3-\sqrt{5}}{2}\leqq 0$. Solving this numerically, we have

$$4<x^2<8.740. \tag{4.26}$$

Let $u=(3+\sqrt{5})/2$. Then we have

$$10.472<ux^2<22.882,\quad 0<\varphi_i(ux^2)<1\quad (2\leqq i\leqq 4). \tag{4.27}$$

It implies that $|n_{k/\boldsymbol{Q}}(ux^2(1-ux^2))|<7.824$. Calculating the relative degree f_p of prime numbers $p=2, 3, 5, 7$ over $k_0/\boldsymbol{Q}$, we have

$$n_{k/\boldsymbol{Q}}(ux^2(1-ux^2))=-1,\ -4,\ -5. \tag{4.28}$$

Let $b_0=n_{k/k_0}(ux^2)$, $b_1=\mathrm{tr}_{k/k_0}(ux^2)$. Then from (4.27), (4.28) we have inequalities for b_i and their $\boldsymbol{Q}$-conjugates. We obtain 5 solutions for (b_i). For each (b_i) we calculate y^2, z^2 and we obtain three solutions.

4.6. The case $e=6$.

Let us consider the case $e=6$. In this case we see that $c_e=2-\sqrt{3}$ and that k contains $k_0=\boldsymbol{Q}(\sqrt{3})$, $n=2$, 4 or 6. Let $u=2+\sqrt{3}$. Then we have

$$4u<ux^2<9u,\quad 0<\varphi_i(ux^2)<1\quad (2\leqq i\leqq n). \tag{4.29}$$

In the case $n=2$ it is easy to obtain all solutions. Let us consider the case $n=4$. Let $g(t)=t^2+b_1t+b_0$ $(b_i\in O_{k_0})$ be the irreducible polynomial of ux^2 over k_0. Solving the inequalities for b_i and their $\boldsymbol{Q}$-conjugates derived from (4.29), we obtain three solutions for (b_0, b_1). For these (b_0, b_1) we have $d(k)=4752$, 27792, 39744. On the other hand, since $n=4$ is even, we have $D(A)\neq(1)$. Hence we have $\zeta_k(2)\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)>4/3$. It implies that $d(k)<13209.28$. Therefore there remains only the case $k=\boldsymbol{Q}(\sqrt{15+8\sqrt{3}})$, $d(k)=4752$. However, by straightforward calculations we see that there exist no solutions. Let us consider the case $n=6$. Let $b=uy^2-ux^2$. Then by (3.10), (3.12) we have

$$0\leqq b,\quad -1<\varphi_i(b)<1\quad (2\leqq i\leqq 6). \tag{4.30}$$

We shall show that $x=y$. Assume that $b\neq 0$. From the inequalities: $1\leqq |n_{k/Q}(b^2(1-b^2))|<b^2(b^2-1)/4^5$, we have $5.701<b$. Since $y^2=x^2+b/u$, we have $x^2+1.5276<(x^2-2+\sqrt{3})/(x-2)$. Solving this inequality, we have $x^2<7.893$. This implies that $|n_{k/Q}(ux^2(1-ux^2))|<1$. This is a contradiction. Therefore we have shown that $x=y$. From (3.10) we have $x^3-3x^2+2-\sqrt{3}\leqq 0$. Solving this numerically, we have

$$ux^2<8.819u. \tag{4.31}$$

On the other hand, from the inequalities: $1\leqq|n_{k/Q}(ux^2(1-ux^2))|<ux^2(ux^2-1)/4^5$, we have

$$8.709u<ux^2. \tag{4.32}$$

If $\varphi_i(ux^2(1-ux^2))<0.243$ for some i, then we have $|n_{k/Q}(ux^2(1-ux^2))|<1$. This is a contradiction. Hence we have

$$32.502<ux^2<32.913,\quad 0.4<\varphi_i(ux^2)<0.6\quad (2\leqq i\leqq 6). \tag{4.33}$$

Let $c=\mathrm{tr}_{k/k_0}(ux^2)$. From (4.33) we have inequalities for c and its $\boldsymbol{Q}$-conjugates. We see easily that there exist no solutions for c in O_{k_0}.

4.7. The cases $e=7$, 9.

Let us consider the case $e=7$. By Theorem 3.6 we have $n=3$ or 6. Let $\rho=2\cos(\pi/7)$, $k_0=\boldsymbol{Q}(\rho)$. If $n=3$, then we have $k=k_0$. Using a $\boldsymbol{Z}$-basis $\{1, \rho, \rho^2\}$, we have inequalities for x^2, $\varphi_i(x^2)$. We obtain four solutions for x^2. For each x we can obtain a solution for (x, y, z).

Let us consider the case $n=6$. In this case k is a quadratic extension of k_0. Let $u=\rho^2+\rho$. Then we have

$$4u<ux^2<9u,\quad 0<\varphi_i(ux^2)<1\quad (2\leqq i\leqq 6). \tag{4.34}$$

From the inequalities: $1\leqq|n_{k/Q}(ux^2(1-ux^2))|<ux^2(ux^2-1)/4^5$ we have

$$(1+\sqrt{4097})/2\leqq ux^2<9u(=45.440\cdots). \tag{4.35}$$

It follows that $|n_{k/Q}(ux^2(1-ux^2))|<1.97$. Hence we have

$$n_{k/Q}(ux^2)=1,\quad n_{k/Q}(1-ux^2)=-1. \tag{4.36}$$

We put $b_0=n_{k/k_0}(ux^2)$, $b_1=\mathrm{tr}_{k/k_0}(ux^2)$. Using the expressions $b_i=\sum_{0\leqq j\leqq 2}b_{ij}\rho^j$

$(b_{ij}\in \boldsymbol{Z})$, by (4.34), (4.35), (4.36) we have inequalities for b_{ij}. For each solution (b_i) we check the condition and we see that there exist no solutions for ux^2.

For the case $e=9$ in the same way as in the case $e=7$ we can obtain all solutions.

4.8. The cases $e=8$, 15.

Let us consider the case $e=8$. Let $\rho=2\cos(\pi/8)$ and $k_0=\boldsymbol{Q}(\rho)$. By Theorem 3.6 we see that $n=4$, 8. If $n=4$, then $k=k_0$ and it is known that $\zeta_{k_0}(2)=2^3\cdot 3^{-1}\cdot 5\cdot \pi^8 d(k_0)^{-3/2}$ (cf. [18] p. 208). Hence we have $\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)d_2=2^2\cdot 3\cdot 7/5$. This is a contradiction because it is not an integer. Let us consider the case $n=8$. Then k is a quadratic extension of k_0. Since $n=8$ is even, we have $D(A)\neq(1)$. Hence we have $\zeta_k(2)\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)>4/3$. By (4.2) we have

$$d(k)<(2^{11}\cdot 3\cdot 7\pi^{16})^{2/3}. \tag{4.37}$$

Since $[k:k_0]=2$, by a theorem of the algebraic number theory we have

$$d(k)=d(k_0)^2 n_{k_0/Q}(D(k/k_0)), \tag{4.38}$$

where $D(k/k_0)$ is the relative discriminant of the extension k/k_0. Since $d(k_0)=2^{11}$, by (4.37) we have

$$n_{k_0/Q}(D(k/k_0))\leqq 58. \tag{4.39}$$

Considering the relative degree f_p for $p=2, 3, \cdots, 57$ over $k_0/\boldsymbol{Q}$, we have

$$n_{k_0/Q}(D(k/k_0))=2^m\cdot q \quad (0\leqq m\leqq 5,\ q=1, 17, 31, 47, 49). \tag{4.40}$$

Now we have the expression $k=k_0(\sqrt{\mu})$, where μ is a totally positive algebraic integer in k_0. Note that the class number of k_0 is 1 and that every totally positive unit of k_0 is a square of some unit of k_0. We obtain six cases for μ satisfying (4.40). Calculating $n_{k_0/Q}(D(k/k_0))$ explicitly for each case μ, we obtain $n_{k_0/Q}(D(k/k_0))=2^9, 2^6\cdot 17, 2^9\cdot 17, 2^8\cdot 31, 2^8\cdot 47, 2^4\cdot 49$, which contradicts (4.40).

For the case $e=15$ similarly to the case $e=8$ we see that there exist no solutions.

4.9. The cases $e=10$, 12.

Let us consider the cases $e=10$, 12. Let $\rho=2\cos(\pi/e)$ and $k_0=\boldsymbol{Q}(\rho)$ for each case. By Theorem 3.6 we see that $n=4$ (and 8 for $e=12$). Since we know that $\zeta_{k_0}(2)=2^5\cdot 3^{-1}\pi^8 d(k_0)^{-3/2}$, $2^4\cdot \pi^8 d(k_0)^{-3/2}$ for $e=10$, 12 respectively (cf. [18] p. 208), we see that $\prod_{\mathfrak{p}|D(A)}(n_{k_0/Q}(\mathfrak{p})-1)d_2$ is not an integer. Therefore only the case: $e=12$, $n=8$ remains. Now let us consider this case. Put $u=1/(2-\rho)$. Then u is a unit of k_0. By (3.10), (3.12) we have

$$4u<ux^2<9u(=132.06\cdots),\quad 0<\varphi_i(ux^2)<1 \quad (2\leqq i\leqq 8). \tag{4.41}$$

From the inequalities: $1\leqq |n_{k/Q}(ux^2(1-ux^2))|\leqq ux^2(ux^2-1)/4^7$, we have

(4.42) $$(1+\sqrt{65537})/2 \leqq ux^2.$$

Hence we have

(4.43) $$8.757 < x^2 < 9$$

Put $b=uy^2-ux^2$. Since the function $(t^2-2+\rho)/(t-2)-t^2$ is monotone-decreasing on $2<t$, by (3.10), (3.12) we have

(4.44) $$0 \leqq b < 4.4201, \quad -1 < \varphi_i(b) < 1 \quad (2 \leqq i \leqq 8).$$

Since b is an algebraic integer of k, from (4.44) we have $n_{k/Q}(b^2(1-b^2))=0$. Hence $b=0$. This implies that $x=y$. By (3.10) we have $z=(x^2-\sqrt{x^4-8x^2+8-4\rho})/2$. Since the function $t-\sqrt{t^2-8t+8-4\rho}$ is monotone-decreasing on $4 \leqq t$, by (4.43) we have

(4.45) $$z < 3.065.$$

We put $c=uz^2-ux^2$. Then by (3.12), (4.45) we have

(4.46) $$0 \leqq c < 9.351, \quad -1 < \varphi_i(c) < 1 \quad (2 \leqq i \leqq 8).$$

It follows that $0 \leqq |n_{k/Q}(c^2(1-c^2))| < 1$. Since c is an algebraic integer, we have $c=0$. Hence we have $x=y=z$. By (3.10) we have $x^3-3x^2+2-\rho=0$. We can obtain the solution $x=1+2\cos(\pi/36)$. Since $[\boldsymbol{Q}(x):k_0]=3$, we see that $k=k_0(x^2)$ is a cubic extension of k_0. This contradicts the fact $[k:k_0]=2$.

4.10. The remaining cases.

Let us discuss the remaining cases which are as follows by Theorem 3.6: $e=11, 13, 14, 16, 17, 18, 19, 20, 21, 24, 25, 27, 30, 33$.

We put $\rho_e=2\cos(\pi/e)$ for each case e. Let us consider first the case $e=11$. Then we have $k=\boldsymbol{Q}(\rho_{11})$. By Lemma 4.3 we have the expression $x^2=\sum_{0 \leq i \leq 4} a_i \rho_{11}^i$ ($a_i \in \boldsymbol{Z}$). Solving the inequalities for a_i given by (3.10), (3.12), we obtain a solution for (x, y, z).

Let us consider the case $e=13$. Then $k=\boldsymbol{Q}(\rho_{13})$. Let $\eta=\rho_{13}+2\cos(5\pi/13)$ and $k_1=\boldsymbol{Q}(\eta)$. Then we see that $[k_1:\boldsymbol{Q}]=3$, $[k:k_1]=2$. It is easy to see that $\{1, \eta, \eta^2\}$ is a $\boldsymbol{Z}$-basis of O_{k_1} and that $\{1, \rho_{13}\}$ is a O_{k_1}-basis of O_k. Using the expression $x^2=a_0+a_1\rho_{13}$ ($a_i \in O_{k_1}$), we have inequalities for a_i from (3.10), (3.12). We solve these inequalities to see that there exist no solutions for a_i.

For the cases $e=18, 21$ in the similar way to the case $e=13$ we see that there exist no solutions.

Let us consider the case $e=14$. In this case we see that $k=\boldsymbol{Q}(\rho_{14})$, $[k:\boldsymbol{Q}]=6$, $d(k)=2^6 \cdot 7^5$. Since $n=6$ is even, we have $D(A) \neq (1)$. By using the fact that the minimum of $n_{k/Q}(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of k is 7 we have

$$\zeta_k(2) \prod_{\mathfrak{p}|D(A)} (n_{k/Q}(\mathfrak{p})-1) > 7^2/8.$$

Hence we have

$$(2\pi)^{-12}d(k)^{3/2}\zeta_k(2)\prod_{\mathfrak{p}|D(A)}(n_{k/Q}(\mathfrak{p})-1)>1.804>1-1/14.$$

This contradicts (4.2).

For each remaining case e we have $k=\boldsymbol{Q}(\rho_e)$. We can calculate $d(k)$ explicitly and we see that $(2\pi)^{-2n}d(k)^{3/2}>1-1/e$ which contradicts (4.2). We have finished the proof of Theorem 4.1.

4.11. For each triple (x, y, z) listed in Theorem 4.1 we can obtain a triple (α, β, γ) determined by (3.8). This is unique up to $GL_2(\boldsymbol{R})$-conjugation but not $SL_2(\boldsymbol{R})$-conjugation. We have another triple $(g_0^{-1}\alpha g_0, g_0^{-1}\beta g_0, g_0^{-1}\gamma g_0)$ satisfying (3.8), where we denote $g_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. These are complete solutions for (3.8) up to $SL_2(\boldsymbol{R})$-conjugation. Let Γ be the Fuchsian group generated by $\{\alpha, \beta\}$ and let A be the quaternion algebra associated with Γ. For a fixed triple (x, y, z) any Fuchsian group derived from (x, y, z) is $SL_2(\boldsymbol{R})$-conjugate to Γ or $g_0^{-1}\Gamma g_0$. It depends on the case whether these two groups are $SL_2(\boldsymbol{R})$-conjugate or not.

For a fixed e different triples may correspond to the same Γ. Now we shall show that each Γ derived from each triple (x, y, z) listed in Theorem 4.1 is pairwise $GL_2(\boldsymbol{R})$-inconjugate. Let (x', y', z') be another triple for the fixed e. Let Γ' be the Fuchsian group derived from it and A'/k' be the quaternion algebra associated with Γ'. Suppose that $\Gamma'=g^{-1}\Gamma g$ for $g\in GL_2(\boldsymbol{R})$. By a result in [**17**] we see that $\boldsymbol{Q}(\mathrm{tr}(\gamma)\,|\,\gamma\in\Gamma)=\boldsymbol{Q}(x, y, z)$. It follows that $k=k'$, $D(A)=D(A')$, $\boldsymbol{Q}(x, y, z)=\boldsymbol{Q}(x', y', z')$. However, in view of the data in Theorem 4.1 we see that there exist no such triples (x, y, z) and (x', y', z').

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