J. Math. Soc. Japan
Vol. 35, No. 3, 1983

A characterization of the hermitian quadrics

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(Received Feb. 19, 1982)

Introduction.

Let M be a $(2n-1)$-dimensional manifold and S a subbundle of the complexified tangent bundle $\boldsymbol{C}T(M)$ of M. Then S is called a PC (or CR) structure if it satisfies the following conditions: 1) $S\cap\bar{S}=\{0\}$, 2) $[\Gamma(S), \Gamma(S)]\subset\Gamma(S)$ and 3) $\dim_{\boldsymbol{C}} S=n-1$. The manifold M equipped with the PC structure S is called a PC (or CR) manifold. Furthermore following Tanaka [**5**], we say that a PC manifold M is normal if there is given an infinitesimal automorphism ξ which is transversal to the subbundle $S+\bar{S}\subset\boldsymbol{C}T(M)$.

For example consider the hermitian quadric Q_r of the n-dimensional complex projective space $P_n(\boldsymbol{C})$ defined by the equation

$$\sum_{j=0}^{r}|z_j|^2-\sum_{k=r+1}^{n}|z_k|^2=0\,,$$

where $0\leqq r\leqq\frac{n-1}{2}$. For a given positive number c, let $\xi_{(c)}$ be the vector field on Q_r induced from the 1-parameter group of transformations

$$\tau_t([z_0, \cdots, z_n])=[z_0, \cdots, z_r, e^{\sqrt{-1}ct}z_{r+1}, \cdots, e^{\sqrt{-1}ct}z_n]\,,$$

where $[z_0, \cdots, z_n]\in Q_r$. Then we can see that Q_r is endowed with a PC structure and that the PC manifold Q_r is normal with respect to the vector field $\xi_{(c)}$.

The main purpose of the present paper is to characterize the hermitian quadrics in terms of normal PC structures.

Let us now proceed to the description of the main results in the present paper. Let (M, ξ) be a normal PC manifold. We assume that M is compact and non-degenerate of index r and that (M, ξ) satisfies Condition (C) (This condition requires that the PC structure S is suitably decomposed into subbundles S^1 and S^2. For the details, see §1). Then we know that to the normal PC manifold (M, ξ) together with the decomposition of S, there are naturally associated a Riemannian metric g, the canonical affine connection ∇, and two kinds of scalar curvatures σ_1 and σ_2. Furthermore let $\mathfrak{a}(M)$ be the Lie algebra of all infinitesimal automorphisms of the PC manifold M, and let $\mathfrak{c}(M)$ be the centralizer of ξ in $\mathfrak{a}(M)$, which is nothing but the Lie algebra of all infinitesimal automorphisms of the normal PC manifold (M, ξ).

Now we notice that the hermitian quadric $(Q_r, \xi_{(c)})$ satisfies the conditions above, that the scalar curvatures σ_1 and σ_2 coincide and equal to the constant c, and that $\mathfrak{a}(Q_r)\neq\mathfrak{c}(Q_r)$. Conversely we can show that these properties characterize the normal PC manifold $(Q_r, \xi_{(c)})$. More precisely we have the following

THEOREM A. *Let (M, ξ) be a normal* PC *manifold. Assume the following conditions:* 1) *M is compact and non-degenerate of index r,* 2) *(M, ξ) satisfies Condition* (C), 3) *both the scalar curvatures σ_1 and σ_2 are constant, and* 4) $\mathfrak{a}(M)\neq\mathfrak{c}(M)$. *Then $\sigma_1=\sigma_2>0$, and the normal* PC *manifold (M, ξ) is isomorphic to the hermitian quadric $(Q_r, \xi_{(c)})$ with the constant $c=\sigma_1=\sigma_2$.*

Let us consider the special case where M is non-degenerate of index 0, i.e., M is strongly pseudo-convex. We first remark that Condition (C) is automatically satisfied and that the scalar curvature σ_1 vanishes. Furthermore we remark that the hermitian quadric Q_0 may be naturally identified with the unit sphere S^{2n-1} of $\boldsymbol{C}^n$ and that the vector field $\xi_{(c)}$ is induced from the 1-parameter group of transformations, $\tau_t(z)=e^{\sqrt{-1}ct}z$, $z\in S^{2n-1}$. Therefore Theorem A yields the following

THEOREM B. *Let (M, ξ) be a normal* PC *manifold. Assume the following conditions:* 1) *M is compact and strongly pseudo-convex,* 2) *the scalar curvature σ $(=\sigma_2)$ is constant, and* 3) $\mathfrak{a}(M)\neq\mathfrak{c}(M)$. *Then $\sigma>0$, and the normal* PC *manifold (M, ξ) is isomorphic to the normal* PC *manifold $(S^{2n-1}, \xi_{(c)})$ with the constant $c=\sigma$.*

As an immediate consequence of Theorem B we obtain

COROLLARY C (Markowitz [**3**]). *Let (M, ξ) be a normal* PC *manifold, and let H be the group of all automorphisms of the normal* PC *manifold (M, ξ). Assume the following conditions:* 1) *M is compact and strongly pseudo-convex,* 2) *the group H acts transitively on M, and* 3) $\mathfrak{a}(M)\neq\mathfrak{c}(M)$. *Then the normal* PC *manifold (M, ξ) is isomorphic to the normal* PC *manifold $(S^{2n-1}, \xi_{(c)})$ with a suitable positive constant c.*

In § 1 we first recall several known results on normal PC manifolds and give the definition of Condition (C). We also recall several known results on Lie algebras $\mathfrak{a}(M)$. In § 2 we introduce certain differential equations which are closely related to infinitesimal automorphisms of PC manifolds. Then we show that there exists a non-trivial solution of these differential equations under the conditions in Theorem A. In § 3, by using the method of S. Tanno [**6**], we determine the curvature tensor of the canonical affine connection ∇. § 4 is devoted to the considerations of the normal PC structures of the hermitian quadrics Q_r and the Lie algebras $\mathfrak{a}(Q_r)$ and $\mathfrak{c}(Q_r)$. In § 5 we completes the proof of Theorem A. As an application of Theorem B, in § 6, we show a different proof of Theorem 6.5 in [**4**] which characterizes the hyperplane section bundles over the complex projective spaces.

The author would like to express his sincere thanks to Professor N. Tanaka who

has encouraged him and kindly read through the manuscript during the preparation of this paper.

PRELIMINARY REMARKS.

1) Throughout this paper we always assume the differentiability of class C^∞ and assume that the manifolds to be considered are connected.

2) Given a manifold M, $C^\infty(M)$ denotes the space of all complex valued differentiable functions on M, and given a vector field X, $\mathcal{L}_X$ denotes the Lie derivation with respect to X. Let E be a vector bundle over M. $\Gamma(E)$ denotes the space of all differentiable cross sections of E.

§1. Normal PC manifolds and Condition (C).

Let M be a differentiable manifold of dimension $2n-1$ $(n \geqq 2)$. A PC (or CR) structure on M is a subbundle S of the complexified tangent bundle $\boldsymbol{C}T(M)$ which satisfies the following conditions:

(PC. 1) $\dim_C S = n-1$ and $S \cap \bar{S} = \{0\}$.

(PC. 2) $[\Gamma(S), \Gamma(S)] \subset \Gamma(S)$.

The manifold M equipped with the PC structure S is called a PC (or CR) manifold.

Let M and M' be PC manifolds with PC structures S and S' respectively. A diffeomorphism $\phi: M \to M'$ is said to be an isomorphism of the PC manifold M onto the PC manifold M' if the differential ϕ_* of ϕ sends S onto S'. In particular an isomorphism of M onto itself is called an automorphism of M.

Let ζ be a real vector field on M, and let ϕ_t be the local 1-parameter group of local transformations generated by ζ. Then ζ is called an infinitesimal automorphism if each ϕ_t is a local automorphism. Note that ζ is an infinitesimal automorphism if and only if $[\zeta, \Gamma(S)] \subset \Gamma(S)$. We denote by $\mathfrak{a}(M)$ the Lie algebra of all infinitesimal automorphisms of the PC manifold M.

By a normal PC manifold, we mean a PC manifold M equipped with an infinitesimal automorphism ξ which is transversal to the subbundle $S+\bar{S}$ of $\boldsymbol{C}T(M)$. We remark that a normal PC manifold is called a PC manifold satisfying Condition (C. 1) in the paper [4]. Let (M, ξ) and (M', ξ') be normal PC manifolds. A diffeomorphism $\phi: M \to M'$ is called an isomorphism of the normal PC manifold (M, ξ) onto the normal PC manifold (M', ξ') if it is an isomorphism as PC manifolds and $\phi_*(\xi) = \xi'$. A real vector field ζ on a normal PC manifold (M, ξ) is called an infinitesimal automorphism of (M, ξ), if it generates a local 1-parameter group of local automorphisms of (M, ξ). We denote by $\mathfrak{c}(M)$ the Lie algebra of all infinitesimal automorphisms of (M, ξ). We notice that $\mathfrak{c}(M)$ is the centralizer of ξ in $\mathfrak{a}(M)$.

In the following we fix a normal PC manifold (M, ξ). We define a real 1-

form θ on M by $\theta(\xi)=1$ and $\theta(X)=0$, $X\in(S+\bar{S})_x$. For $x\in M$, we define a hermitian form L_x on S_x by

$$L_x(X, Y)=-\sqrt{-1}(d\theta)(X, \bar{Y}) \qquad X, Y\in S_x.$$

The hermitian form L_x is called the Levi form of M at x corresponding to θ.

The manifold M is called non-degenerate if L_x is non-degenerate at any $x\in M$, and M is called of index r if $r=\mathrm{Min}(\lambda_+(x), \lambda_-(x))$ at any $x\in M$, where $\lambda_+(x)$ (resp. $\lambda_-(x)$) stands for the number of the positive (resp. negative) eigenvalues of L_x. In particular, M is called strongly pseudo-convex if M is non-degenerate of index 0, i.e., L_x is definite at any $x\in M$.

PROPOSITION 1.1 ([**5**], see also [**4**]). *Assume that M is non-degenerate. There exists a unique affine connection*

$$\nabla: \Gamma(T(M)) \longrightarrow \Gamma(T(M)\otimes T(M)^*)$$

on M satisfying the following conditions.

1) *S is parallel with respect to* ∇.
2) ξ, θ *and* $d\theta$ *are all parallel.*
3) *The torsion tensor T of* ∇ *is parallel and possesses the following properties:*

$$T(X, Y)=T(\bar{X}, \bar{Y})=0, \qquad X, Y\in S_x.$$

$$T(X, \bar{Y})=(d\theta)(X, \bar{Y})\xi_x(=\sqrt{-1}L_x(X, Y)\xi_x), \qquad X, Y\in S_x.$$

$$T(\xi_x, X)=T(\xi_x, \bar{X})=0, \qquad X\in S_x.$$

The above affine connection ∇ is called the canonical affine connection.

In the following we assume that M is non-degenerate of index r. Moreover we assume the following condition which was called Condition (C.2) in the paper [**4**].

(C) There exist subbundles S^1 and S^2 of S satisfying the following:

1) $\dim_C S^1=r$ and $\dim_C S^2=s$, where $r+s=n-1$.

2) $S=S^1+S^2$ (direct sum).

3) Both S^1 and S^2 are parallel with respect to the canonical affine connection ∇.

4) At any point x of M, L_x is negative definite (resp. positive definite) on S_x^1 (resp. on S_x^2), and S_x^1 and S_x^2 are mutually orthogonal with respect to L_x.

Note that if M is a strongly pseudo-convex manifold, then M automatically satisfies Condition (C), by setting $S^1=0$ and $S^2=S$.

Using the direct sum decomposition $\boldsymbol{C}T(M)=\boldsymbol{C}\xi+S^1+S^2+\bar{S}^1+\bar{S}^2$, we define a Riemannian metric g on M as follows:

1) $g_x(X, \bar{Y})=-L_x(X, Y), \quad X, Y\in S_x^1.$
2) $g_x(X, \bar{Y})=L_x(X, Y), \quad X, Y\in S_x^2.$

3) $g_x(\xi_x, \xi_x)=1$.

4) The other components of g_x are zero.

The Riemannian metric g induces a hermitian inner product in the space S_x in a natural manner.

Let R be the curvature tensor of the canonical affine connection ∇.

PROPOSITION 1.2 (cf. [4] and [5]). *Let* $X, Y, Z, W\in S_x$.

(1) $R(\xi, X)=R(\xi, \bar{X})=R(X, Y)=R(\bar{X}, \bar{Y})=0$.

(2) $R(X, \bar{Y})Z=R(Z, \bar{Y})X$.

(3) $R(X, \bar{Y})\xi=0, \quad R(X, \bar{Y})S_x^1\subset S_x^1, \quad R(X, \bar{Y})S_x^2\subset S_x^2$.

(4) $R(X, \bar{Y})Z=R(Z, \bar{Y})X=0$, *if* $X\in S_x^1$ *and* $Z\in S_x^2$.

(5) $R(X, \bar{Y})=R(Y, \bar{X})=0$, *if* $X\in S_x^1$ *and* $Y\in S_x^2$.

(6) $g(R(X, \bar{Y})Z, \bar{W})+g(Z, R(X, \bar{Y})\bar{W})=0$.

PROOF. (1) and (2) were proved in Proposition 1.3 in [4]. (3) follows from the fact that ξ, S^1 and S^2 are all parallel with respect to ∇. By (2) we have

$$R(X, \bar{Y})Z=R(Z, \bar{Y})X, \qquad X\in S_x^1, \quad Y\in S_x, \quad Z\in S_x^2.$$

Therefore it follows from (3) that $R(X, \bar{Y})Z=R(Z, \bar{Y})X\in S_x^1\cap S_x^2=\{0\}$. This proves (4).

Since $d\theta$ is parallel with respect to ∇, we have

$$(d\theta)(R(X, \bar{Y})Z, \bar{W})+(d\theta)(Z, R(X, \bar{Y})\bar{W})=0. \tag{1.1}$$

In particular if $X, Z, W\in S_x^1$ and $Y\in S_x^2$, then by (4) we have

$$(d\theta)(R(X, \bar{Y})Z, \bar{W})=0.$$

Since $R(X, \bar{Y})Z\in S_x^1$, we have $R(X, \bar{Y})Z=0$. Similarly we can show that if $X\in S_x^1$ and $Y, Z\in S_x^2$, then $R(X, \bar{Y})\bar{Z}=0$. These facts combined with (3) and (4) imply (5). Finally (6) is an immediate consequence of (3) and (1.1). q.e.d.

From now on the two indices a, b range over the integers $1, \cdots, r$ while the two indices α, β range over the integers $r+1, \cdots, n-1$. Let $e_1, \cdots, e_r$ (resp. $e_{r+1}, \cdots, e_{n-1}$) be a basis of S_x^1 (resp. of S_x^2) such that $g(e_a, \bar{e}_b)=\delta_{ab}$ (resp. $g(e_\alpha, \bar{e}_\beta)=\delta_{\alpha\beta}$). By using these bases, we will express various tensor fields in terms of their components.

We define a linear operator $R_*: S_x\to S_x$ by

$$R_*X=\sum_{i=1}^{n-1}R(e_i, \bar{e}_i)X, \qquad X\in S_x,$$

which is called the Ricci operator. From (5) of Proposition 1.2, we see that R_* is a hermitian operator on S_x with respect to the hermitian inner product associated with g. Moreover from (3) and (4) of Proposition 1.2, it is also shown that if $X\in S_x^1$ (resp. $X\in S_x^2$), then $R_*X\in S_x^1$ and $R_*X=\sum_\alpha R(e_\alpha, \bar{e}_\alpha)X$ (resp. $R_*X \in S_x^2$ and $R_*X=\sum_\alpha R(e_\alpha, \bar{e}_\alpha)X$).

We also define the scalar curvatures σ_1 and σ_2 respectively by

$$\sigma_1=\frac{1}{r(r+1)}\sum_a g(R_*e_a, \bar{e}_a) \quad\text{and}\quad \sigma_2=\frac{1}{s(s+1)}\sum_\alpha g(R_*e_\alpha, \bar{e}_\alpha).$$

In particular, if M is a strongly pseudo-convex manifold, σ_2 will be simply denoted by σ.

We will frequently use the following equalities.

LEMMA 1.3 (The Ricci formulas cf. Lemma 2.3 in [4]). *Let $f\in C^\infty(M)$ and $X, Y, Z\in T(M)_x$.*

(1) $\nabla_X\nabla_Y f=\nabla_Y\nabla_X f-\nabla_{T(X,Y)}f$.

(2) $\nabla_X\nabla_Y\nabla_Z f=\nabla_Y\nabla_X\nabla_Z f-\nabla_{T(X,Y)}\nabla_Z f-\nabla_{R(X,Y)Z}f$.

Let $C\mathfrak{a}(M)$ be the complexification of $\mathfrak{a}(M)$, which is a Lie algebra of complex vector fields on M. We define a subspace $\mathfrak{F}(M)$ of $C^\infty(M)$ by

$$\mathfrak{F}(M)=\{f\in C^\infty(M)\,|\,\nabla_X\nabla_Y f=\nabla_{\bar{X}}\nabla_{\bar{Y}}f=0,\quad X, Y\in S_x\},$$

and a linear mapping $\zeta\to f_\zeta$ of $C\mathfrak{a}(M)$ to $C^\infty(M)$ by

$$f_\zeta=\theta(\zeta).$$

PROPOSITION 1.4 (cf. Proposition 1.5 in [4]). *The assignment $\zeta\to f_\zeta$ gives a linear isomorphism of $C\mathfrak{a}(M)$ onto $\mathfrak{F}(M)$, and $\zeta\in\mathfrak{a}(M)$ if and only if f_ζ is a real valued function. The correspondence $f_\zeta\to\zeta$ is given by*

$$\zeta=f_\zeta\xi+U+\bar{U},$$

where U is the cross section of S defined by

$$\bar{Y}f_\zeta+(d\theta)(U, \bar{Y})=0, \qquad Y\in S_x.$$

Now we define differential operators N, $\Box_1$, $\Box_2$, A_1, A_2 and A_3 on $C^\infty(M)$ respectively by

$$\begin{aligned}
&Nf=\sqrt{-1}\,\xi f,\\
&\Box_1 f=-\sum\nabla_a\nabla_{\bar{a}}f,\\
&\Box_2 f=-\sum\nabla_\alpha\nabla_{\bar{\alpha}}f,\\
&A_1 f=\Box_1^2 f-\Box_1 Nf+\sum R_{*\bar{a}}^{\,b}\nabla_b\nabla_{\bar{a}}f+\overline{W}_1 f,\\
&A_2 f=\Box_2^2 f+\Box_2 Nf+\sum R_{*\bar{\alpha}}^{\,\beta}\nabla_\beta\nabla_{\bar{\alpha}}f+\overline{W}_2 f,\\
&A_3 f=\Box_1\Box_2 f,
\end{aligned}$$

where $f\in C^\infty(M)$, and $R_{*\bar{a}}^{\,b}$ and $R_{*\bar{\alpha}}^{\,\beta}$ stand for the components of R_*, that is,

$R_{*}e_{a}=\sum R_{*}{}_{\bar{\alpha}}^{b}e_{b}$ and $R_{*}e_{\alpha}=\sum R_{*}{}_{\bar{\alpha}}^{\beta}e_{\beta}$, and W_{1} (resp. W_{2}) is the cross section of S^{1} (resp. of S^{2}) defined by

$$W_{1}=\sum\nabla_{\bar{\alpha}}R_{*}{}_{\bar{\alpha}}^{b}e_{b}\qquad(\text{resp. } W_{2}=\sum\nabla_{\bar{\alpha}}R_{*}{}_{\bar{\alpha}}^{\beta}e_{\beta}).$$

In particular, in the case where M is a strongly pseudo-convex manifold, $\square_{2}$ and A_{2} will be simply denoted by $\square$ and A.

LEMMA 1.5 (cf. Propositions 2.4 and 2.8 in [4]). *Let $\bar{N}$, $\bar{\square}_{1}$, $\bar{\square}_{2}$, $\bar{A}_{1}$, $\bar{A}_{2}$ and $\bar{A}_{3}$ be the conjugate operators of N, $\square_{1}$, $\square_{2}$, A_{1}, A_{2} and A_{3} respectively. Then,*

(1) $\bar{N}=-N$.

(2) $\bar{\square}_{1}=\square_{1}+rN$ *and* $\bar{\square}_{2}=\square_{2}-sN$.

(3) $\bar{A}_{1}=A_{1}+(r+1)(rN^{2}-r\sigma_{1}N+2\square_{1}N)+W_{1}-\bar{W}_{1}$.

$\bar{A}_{2}=A_{2}+(s+1)(sN^{2}+s\sigma_{2}N-2\square_{2}N)+W_{2}-\bar{W}_{2}$.

$\bar{A}_{3}=A_{3}-s\square_{1}N+r\square_{2}N-rsN^{2}$.

LEMMA 1.6 (cf. Propositions 2.2 and 2.6 in [4]). *Assume that M is compact. Let $f\in C^{\infty}(M)$.*

(1) $\square_{1}f=0$ *(resp. $\bar{\square}_{1}f=0$) if and only if $\bar{X}f=0$ (resp. $Xf=0$) for all $X\in S_{x}^{1}$. $\square_{2}f=0$ (resp. $\bar{\square}_{2}f=0$) if and only if $\bar{X}f=0$ (resp. $Xf=0$) for all $X\in S_{x}^{2}$.*

(2) *$f\in\tilde{\mathfrak{F}}(M)$ if and only if $A_{i}f=\bar{A}_{i}f=0$, $i=1, 2, 3$. In particular, in the case where $r=0$, $f\in\tilde{\mathfrak{F}}(M)$ if and only if $Af=\bar{A}f=0$.*

§ 2. Differential equations $(\mathrm{D})_{c}$.

For each $\nu\in\boldsymbol{R}$, we define a subspace $\tilde{\mathfrak{g}}_{(\nu)}$ of $\boldsymbol{C}\mathfrak{a}(M)$ by

$$\tilde{\mathfrak{g}}_{(\nu)}=\{\zeta\in\boldsymbol{C}\mathfrak{a}(M)\mid\sqrt{-1}[\xi, \zeta]=\nu\zeta\},$$

and a subspace $\tilde{\mathfrak{F}}_{(\nu)}$ of $\tilde{\mathfrak{F}}(M)$ by

$$\tilde{\mathfrak{F}}_{(\nu)}=\{f\in\tilde{\mathfrak{F}}(M)\mid Nf=\nu f\}.$$

One should note that the assignment $\zeta\rightarrow f_{\zeta}$ gives a linear isomorphism of $\tilde{\mathfrak{g}}_{(\nu)}$ onto $\tilde{\mathfrak{F}}_{(\nu)}$, and $\tilde{\mathfrak{g}}_{(0)}$ coincides with the complexification $\boldsymbol{C}\mathfrak{c}(M)$ of $\mathfrak{c}(M)$. We also remark that $\dim\tilde{\mathfrak{F}}_{(\nu)}=\dim\tilde{\mathfrak{F}}_{(-\nu)}$, because $\bar{N}=-N$. If M is compact, we can decompose $\tilde{\mathfrak{F}}(M)$ into the eigenspaces of N. Hence we have $\tilde{\mathfrak{F}}(M)=\sum_{\nu}\tilde{\mathfrak{F}}_{(\nu)}$ (direct sum) (cf. Propositions 3.1 and 5.3 in [4]).

In this section we will prove the following two propositions.

PROPOSITION 2.1. *Assume that M is compact and both the scalar curvatures σ_{1} and σ_{2} are constant. If $\mathfrak{a}(M)\neq\mathfrak{c}(M)$, then $\sigma_{1}=\sigma_{2}>0$, and $\tilde{\mathfrak{F}}_{(c)}\neq 0$ for the constant $c=\sigma_{1}=\sigma_{2}$.*

PROPOSITION 2.2. *Assume that M is compact and both the scalar curvatures σ_{1} and σ_{2} are equal to a positive constant c. Let $f\in\tilde{\mathfrak{F}}_{(c)}$. Then f satisfies the* *following differential equations:*

$$(\mathrm{D})_c \quad \begin{aligned} &Nf=cf. \\ &\nabla_{\bar{X}}f=0, \quad X\in S_x^1 \quad \text{and} \quad \nabla_Y f=0, \quad Y\in S_x^2. \\ &\nabla_X\nabla_Y f=0, \qquad X, Y\in S_x^1. \\ &\nabla_{\bar{X}}\nabla_{\bar{Y}}f=0, \qquad X, Y\in S_x^2. \end{aligned}$$

From these propositions we have the following

COROLLARY 2.3. *Assume that M is compact and both the scalar curvatures σ_1 and σ_2 are constant. If $\mathfrak{a}(M)\neq\mathfrak{c}(M)$, then $\sigma_1=\sigma_2=c$ for a positive constant c and there exists a non-trivial solution of equations* $(\mathrm{D})_c$.

PROOF OF PROPOSITION 2.1. We begin with the following lemma.

LEMMA 2.4 (cf. Propositions 3.1 and 5.3 in [4]).

(1) *Assume that $r\geqq 1$. In the case where $\sigma_1=\sigma_2=c$ for a positive constant c, $\tilde{\mathfrak{F}}_{(\nu)}=0$ for $\nu\neq 0, -c, c$. In other cases, $\tilde{\mathfrak{F}}_{(\nu)}=0$ for $\nu\neq 0$.*

(2) *Assume that $r=0$. In the case where $\sigma=c$ for a positive constant c, $\tilde{\mathfrak{F}}_{(\nu)}=0$ for $|\nu|>c$. In other cases, $\tilde{\mathfrak{F}}_{(\nu)}=0$ for $\nu\neq 0$.*

Since $\mathfrak{a}(M)\neq\mathfrak{c}(M)$, we have $\tilde{\mathfrak{F}}(M)\neq\tilde{\mathfrak{F}}_{(0)}$. If $r\geqq 1$, then the assertion of Proposition 2.1 is immediate from Lemma 2.4.

Now let us consider the case where $r=0$. From Lemma 2.4, we see that $\sigma=c$ for a positive constant c and there is a real number μ such that $0<\mu\leqq c$ and $\tilde{\mathfrak{F}}_{(\mu)}\neq 0$. We will show that $\mu=c$. Let f be a non-trivial function which belongs to $\tilde{\mathfrak{F}}_{(\mu)}$. By Lemma 1.6, we have $Af=\bar{A}f=0$. We also have $Nf=\mu f$. Since

$$g(X, \overline{W})=g(X, \overline{\sum\nabla_{\bar{\alpha}}R_*{}^{\bar{\beta}}_{\alpha}e_{\beta}})=g(X, \overline{\sum\nabla_{\bar{\alpha}}R_*{}^{\bar{\beta}}_{\beta}e_{\alpha}})=n(n-1)X\sigma=0,$$

for every X, we have $W=0$. It follows from Lemma 1.5 that

$$n\{(n-1)N^2+(n-1)cN-2\Box N\}f=Af-\bar{A}f=0.$$

This means that

$$\Box f=\frac{(n-1)(c+\mu)}{2}f \quad \text{and} \quad \bar{\Box}f=\frac{(n-1)(c-\mu)}{2}f.$$

Let C (resp. D) be the cross section of S defined by

$$g(C, \bar{Y})=\sqrt{-1}\,\bar{Y}f \quad (\text{resp. } g(D, \bar{Y})=\sqrt{-1}\,\overline{Yf}), \qquad Y\in S_x.$$

By putting $\gamma=\{(n-1)c+(n+1)\mu\}/2$ and $\delta=\{(n-1)c-(n+1)\mu\}/2$, we have

LEMMA 2.5. *$R_*C=\gamma C$ and $R_*D=\delta D$.*

PROOF. First we remark that both R_*C and R_*D are cross sections of S, because S is parallel with respect to ∇. We have

$$\bar{Y}(\Box f)=\frac{(n-1)(c+\mu)}{2}\bar{Y}f\,, \qquad Y\in S_x\,.$$

On the other hand, by using the Ricci formula (Lemma 1.3), we obtain

$$\bar{Y}(\Box f)=-\sum_\alpha \nabla_{\bar{Y}}\nabla_\alpha\nabla_{\bar{\alpha}}f=-\sum_\alpha \nabla_\alpha\nabla_{\bar{Y}}\nabla_{\bar{\alpha}}f-\bar{Y}(Nf)+(\sum_\alpha R(\bar{e}_\alpha,\, e_\alpha)\bar{Y})f$$
$$=-\mu\bar{Y}f+(\overline{R_*Y})f\,.$$

Therefore we have $(\overline{R_*Y})f=\gamma\bar{Y}f$. Since R_* is hermitian, it follows that

$$g(R_*C,\, \bar{Y})=g(C,\, \overline{R_*Y})=\sqrt{-1}(\overline{R_*Y})f=\sqrt{-1}\gamma\bar{Y}f=g(\gamma C,\, \bar{Y})\,,$$

and hence $R_*C=\gamma C$. Similarly we obtain $R_*D=\delta D$. q. e. d.

Now let p (resp. q) be the real part of f (resp. the imaginary part of f). Since $Nf=\mu f$, it follows that

$$\xi p=\mu q \quad\text{and}\quad \xi q=-\mu p\,.$$

In view of these differential equations, we may conclude that p does not vanish identically and there is a point m of M such that $(dp)_m=0$ and $p(m)\neq 0$. We define a linear transformation Φ_C (resp. Φ_D) of S_m by $\Phi_C(X)=\nabla_X C$ (resp. by $\Phi_D(X)=\nabla_X D$) for $X\in S_m$.

LEMMA 2.6. (1) $C_m=D_m=0$.

(2) $R_*|\mathrm{Im}\,\Phi_C=\gamma\,\mathrm{Id}$ *and* $R_*|\mathrm{Im}\,\Phi_D=\delta\,\mathrm{Id}$, *where* Id *stands for the identity transformation.*

(3) $S_m=\mathrm{Im}\,\Phi_C+\mathrm{Im}\,\Phi_D$ (*orthogonal decomposition with respect to* L_m).

PROOF. Let $X\in S_m$.

(1) We have $g(C+D,\, \bar{X})=2\sqrt{-1}\bar{X}p=0$. This means that

$$C_m+D_m=0\,.$$

By applying the Ricci operator R_*, we have

$$\gamma C_m+\delta D_m=0\,.$$

Since $\gamma\neq\delta$, it follows that $C_m=D_m=0$.

(2) By differentiating the both sides of the equality $R_*C=\gamma C$, we have

$$(\nabla_X R_*)(C_m)+R_*(\nabla_X C)_m=\gamma(\nabla_X C)_m\,.$$

From (1), it follows that

$$R_*(\nabla_X C)_m=\gamma(\nabla_X C)_m\,.$$

Hence we have $R_*(\Phi_C(X))=\gamma\Phi_C(X)$. Similarly we obtain $R_*(\Phi_D(X))=\delta\Phi_D(X)$.

(3) From (2), it follows that $\mathrm{Im}\,\Phi_C$ (resp. $\mathrm{Im}\,\Phi_D$) is contained in the eigenspace of R_* corresponding to the eigenvalue γ (resp. δ). Since R_* is hermitian with respect to L_m, $\mathrm{Im}\,\Phi_C$ and $\mathrm{Im}\,\Phi_D$ are mutually orthogonal with respect to L_m.

We now show that $\operatorname{Ker}\Phi_C\cap\operatorname{Ker}\Phi_D=\{0\}$. Suppose that $X\in\operatorname{Ker}\Phi_C\cap\operatorname{Ker}\Phi_D$. We have

$$\nabla_X\nabla_{\bar{X}}f=-\sqrt{-1}\,g(\nabla_X C,\bar{X})=0 \quad\text{and}\quad \nabla_X\nabla_{\bar{X}}\bar{f}=-\sqrt{-1}\,g(\nabla_X D,\bar{X})=0\,.$$

Since $T(X,\bar{X})=\sqrt{-1}\,g(X,\bar{X})\xi$, it follows from the Ricci formula that

$$\begin{aligned}0&=\nabla_{\bar{X}}\nabla_X f-\nabla_X\nabla_{\bar{X}}f=(T(X,\bar{X}))f\\&=g(X,\bar{X})(Nf)(m)=g(X,\bar{X})\mu f(m)\,.\end{aligned}$$

Since $\mu\neq 0$ and $f(m)\neq 0$, we obtain $g(X,\bar{X})=0$ and hence $X=0$. This means that $\operatorname{Ker}\Phi_C\cap\operatorname{Ker}\Phi_D=\{0\}$. Hence we obtain $\dim S_m=\dim\operatorname{Im}\Phi_C+\dim\operatorname{Im}\Phi_D$. Therefore we have

$$S_m=\operatorname{Im}\Phi_C+\operatorname{Im}\Phi_D\,. \qquad \text{q.e.d.}$$

We are now in a position to complete the proof of Proposition 2.1. By Lemma 2.6, we have

$$\begin{aligned}\operatorname{trace}R_*&=\gamma\dim\operatorname{Im}\Phi_C+\delta\dim\operatorname{Im}\Phi_D\\&=\frac{(n-1)c}{2}(\dim\operatorname{Im}\Phi_C+\dim\operatorname{Im}\Phi_D)\\&\quad+\frac{(n+1)\mu}{2}(\dim\operatorname{Im}\Phi_C-\dim\operatorname{Im}\Phi_D)\\&=\frac{(n-1)^2c}{2}+\frac{(n+1)\mu}{2}(\dim\operatorname{Im}\Phi_C-\dim\operatorname{Im}\Phi_D)\,.\end{aligned}$$

From the definition of the scalar curvature σ, it follows that $\operatorname{trace}R_*=n(n-1)\sigma=n(n-1)c$. Hence we have

$$\mu(\dim\operatorname{Im}\Phi_C-\dim\operatorname{Im}\Phi_D)=(n-1)c\,.$$

Since $0<\mu\leqq c$ and $\dim\operatorname{Im}\Phi_C-\dim\operatorname{Im}\Phi_D\leqq n-1$, it follows that $\mu=c$. This proves our assertion. q.e.d.

PROOF OF PROPOSITION 2.2. By (2) of Lemma 1.6, we have

$$A_i f=\bar{A}_i f=0\,,\qquad i=1,2,3\,.$$

We also have $Nf=cf$. Since $W_1=W_2=0$, it follows from (3) of Lemma 1.5 that

$$\begin{aligned}&2(r+1)c\,\square_1 f=\bar{A}_1 f-A_1 f=0\,,\\&2(s+1)c(scf-\square_2 f)=\bar{A}_2 f-A_2 f=0\,.\end{aligned}$$

Therefore we have $\square_1 f=0$ and $\square_2 f=scf$. By (2) of Lemma 1.5, we have $\bar{\square}_2 f=0$. From these equations and Lemma 1.6, it follows that f satisfies equations $(\mathrm{D})_c$. q.e.d.

§ 3. The curvature R.

In this section we will determine the curvature tensor R, assuming the existence of a non-trivial solution of equations $(\mathrm{D})_c$. In view of Proposition 1.2, it suffices to compute the components $R(X, \bar{Y})Z$, where $X, Y, Z \in S_x^1$ or $X, Y, Z \in S_x^2$. We will show the following proposition.

PROPOSITION 3.1. *Assume that M is compact and there is a non-trivial solution f of equations $(\mathrm{D})_c$, with a positive constant c. Then*

$$R(X, \bar{Y})Z = c\{g(X, \bar{Y})Z + g(Z, \bar{Y})X\},$$

where $X, Y, Z \in S_x^1$ or $X, Y, Z \in S_x^2$.

Combining Propositions 1.2, 3.1 and Corollary 2.3, we obtain the following

COROLLARY 3.2. *Assume that M is compact and both the scalar curvatures σ_1 and σ_2 are constant. If $\mathfrak{a}(M) \neq \mathfrak{c}(M)$, then*

$$R(X, \bar{Y})Z = c\{g(X, \bar{Y})Z + g(Z, \bar{Y})X\},$$

where $X, Y, Z \in S_x^1$ or $X, Y, Z \in S_x^2$, and c is a positive constant. In particular, $\nabla R = 0$ and $\sigma_1 = \sigma_2 = c$.

PROOF OF PROPOSITION 3.1. The essential idea of the proof is similar to that of S. Tanno (Theorem 5.1 in [6]). We only consider the case where $X, Y, Z \in S_x^2$. By replacing f by $\bar{f}$ the proof goes through with several changes of signs, even in the case where $X, Y, Z \in S_x^1$.

Let p (resp. q) be the real part (resp. the imaginary part) of f. We define a real vector field η by

$$\eta = 2p\xi + C + \bar{C} = (f + \bar{f})\xi + C + \bar{C},$$

where C is the cross section of S^2 defined by

$$g(C, \bar{Y}) = \sqrt{-1}\,\bar{Y}f, \qquad Y \in S_x^2.$$

Note that, in the case where $r=0$, η is an infinitesimal automorphism of M.

LEMMA 3.3. *Let $X \in S_x^2$.*

(1) $\nabla_{\bar{X}} C = 0$.

(2) $\nabla_X C = -\sqrt{-1}\,c f(x) X$.

PROOF. First we remark that both $\nabla_{\bar{X}} C$ and $\nabla_X C$ belong to S_x^2. Let $Y \in S_x^2$. We have

$$g(\nabla_{\bar{X}} C, \bar{Y}) = \sqrt{-1}\,\nabla_{\bar{X}} \nabla_{\bar{Y}} f = 0.$$

Hence we have $\nabla_{\bar{X}} C = 0$, proving (1). Using the Ricci formula, we have

$$g(\nabla_X C,\ \bar{Y})=\sqrt{-1}\nabla_X\nabla_{\bar{Y}}f=\sqrt{-1}\nabla_{\bar{Y}}\nabla_X f-\sqrt{-1}T(X,\ \bar{Y})f$$
$$=-\sqrt{-1}T(X,\ \bar{Y})f=-\sqrt{-1}g(X,\ \bar{Y})(Nf)(x)$$
$$=-\sqrt{-1}g(X,\ \bar{Y})cf(x).$$

This means that $\nabla_X C=-\sqrt{-1}cf(x)X$. q.e.d.

LEMMA 3.4. (1) $[\eta,\ \Gamma(S^2)]\subset\Gamma(S^2)$.

(2) $[\eta,\ \xi]=-2cq\xi+\sqrt{-1}cC-\sqrt{-1}c\bar{C}$.

(3) $(\mathcal{L}_\eta g)(X,\ \bar{Y})=2cq(x)g(X,\ \bar{Y})$, $\quad X,\ Y\in S_x^2$.

(4) $\eta p=2cpq$.

(5) $\eta q=-2cp^2-g(C,\ \bar{C})$.

PROOF. (1) Let $X^*\in\Gamma(S^2)$. Using Lemma 3.3 and the fact that S^2 is parallel, we obtain

$$[\eta,\ X^*]=\nabla_\eta X^*-\nabla_{X^*}\eta-T(\eta,\ X^*)$$
$$\equiv-(X^*\bar{f})\xi+T(X^*,\ \bar{C}),\qquad (\mathrm{mod}\ \Gamma(S^2)).$$

Since $(X^*\bar{f})\xi=\sqrt{-1}g(X^*,\ \bar{C})\xi=T(X^*,\ \bar{C})$, we have $[\eta,\ X^*]\in\Gamma(S^2)$.

(2) Since $\xi f=-\sqrt{-1}cf$, we have $[\xi,\ C]=-\sqrt{-1}cC$. Hence we have

$$[\eta,\ \xi]=-(\xi f+\overline{\xi f})\xi+[C,\ \xi]+[\bar{C},\ \xi]$$
$$=-2cq\xi+\sqrt{-1}cC-\sqrt{-1}c\bar{C}.$$

(3) We have

$$(\mathcal{L}_\eta g)(X,\ \bar{Y})=(\nabla_\eta g)(X,\ \bar{Y})+g(\nabla_X\eta,\ \bar{Y})+g(X,\ \nabla_{\bar{Y}}\eta)$$
$$+g(T(\eta,\ X),\ \bar{Y})+g(X,\ T(\eta,\ \bar{Y})).$$

Since $\nabla_\eta g=0$, $T(\eta,\ X)\in\boldsymbol{C}\xi_x$ and $T(\eta,\ \bar{Y})\in\boldsymbol{C}\xi_x$, it follows that

$$(\mathcal{L}_\eta g)(X,\ \bar{Y})=g(\nabla_X\eta,\ \bar{Y})+g(X,\ \nabla_{\bar{Y}}\eta).$$

By Lemma 3.3, we have

$$g(\nabla_X\eta,\ \bar{Y})=-\sqrt{-1}cf(x)g(X,\ \bar{Y}),$$
$$g(X,\ \nabla_{\bar{Y}}\eta)=\sqrt{-1}c\bar{f}(x)g(X,\ \bar{Y}).$$

Therefore we obtain

$$(\mathcal{L}_\eta g)(X,\ \bar{Y})=-\sqrt{-1}cf(x)g(X,\ \bar{Y})+\sqrt{-1}c\bar{f}(x)g(X,\ Y)$$
$$=2cq(x)g(X,\ \bar{Y}).$$

(4) Since $Cf=0$, we have

$$\eta p=p(\xi f+\overline{\xi f})+\frac{1}{2}(C\bar{f}+\bar{C}f).$$

Since $\xi f=-\sqrt{-1}cf$ and $\bar{C}f=-\sqrt{-1}\,g(C, \bar{C})$, it follows that

$$\eta p=2cpq.$$

(5) By similar calculations, we have

$$\eta q=p(-\sqrt{-1}\xi f+\sqrt{-1}\xi\bar{f})+\frac{\sqrt{-1}}{2}C\bar{f}-\frac{\sqrt{-1}}{2}\bar{C}f=-2cp^2-g(C, \bar{C}).$$

q.e.d.

Now we define a tensor field H of type $(1,1)$ on M by

$$HX=\nabla_X\eta+T(\eta, X), \qquad X\in T(M)_x,$$

and a tensor field K of type $(1,2)$ on M by

$$K(X, Y)=(\nabla_X H)Y+R(\eta, X)Y, \qquad X, Y\in T(M)_x.$$

Note that K is nothing but the Lie derivative of ∇ with respect to η:

$$K(X^*, Y^*)=[\eta, \nabla_{X^*}Y^*]-\nabla_{[\eta, X^*]}Y^*-\nabla_{X^*}([\eta, Y^*]),$$

where X^* and Y^* are vector fields on M. By direct calculations, we have the following formulas.

LEMMA 3.5. *Let* $X, Y, Z\in T(M)_x$.

(1) $(\mathcal{L}_\eta T)(X, Y)=K(X, Y)-K(Y, X)$.

(2) $(\mathcal{L}_\eta R)(X, Y)=(\nabla_X K)(Y, Z)-(\nabla_Y K)(X, Z)+K(T(X, Y), Z)$.

Now we compute the tensor fields K and $\mathcal{L}_\eta R$.

LEMMA 3.6. *Let* $X, Y, Z\in S^2_x$.

(1) $K(X, Y)\in S^2_x$ *and* $K(X, \bar{Y})\in \bar{S}^2_x$.

(2) $K(X, \bar{Y})=cg(X, \bar{Y})\bar{C}$ *and* $K(\bar{Y}, X)=cg(X, \bar{Y})C$.

(3) $K(X, Z)=-cg(X, \bar{C})Z-cg(Z, \bar{C})X$.

(4) $K(\xi, X)=-c^2f(x)X$.

(5) $(\mathcal{L}_\eta R)(X, \bar{Y})Z=2c^2q(x)\{g(X, \bar{Y})Z+g(Z, \bar{Y})X\}$.

PROOF. We extend the vectors X, Y and Z to cross sections of S^2, say X^*, Y^* and Z^* respectively.

(1) Since S^2 is parallel with respect to ∇, by (1) of Lemma 3.4, we have

$$K(X^*, Y^*)\in\Gamma(S^2) \quad\text{and}\quad K(X^*, \bar{Y}^*)\in\Gamma(\bar{S}^2).$$

(2) Since $T(X^*, \bar{Y}^*)=\sqrt{-1}\,g(X^*, \bar{Y}^*)\xi$, we have

$$(\mathcal{L}_\eta T)(X, \bar{Y})=\sqrt{-1}(\mathcal{L}_\eta g)(X, \bar{Y})\xi+\sqrt{-1}\,g(X, \bar{Y})[\eta, \xi].$$

Hence, by using (2) and (3) of Lemma 3.4 and (1) of Lemma 3.5, we have

$$K(X, \bar{Y})-K(\bar{Y}, X)=(\mathcal{L}_\eta T)(X, \bar{Y})=-cg(X, \bar{Y})C+cg(X, \bar{Y})\bar{C}.$$

Therefore by (1), we obtain

$$K(X, \bar{Y})=cg(X, \bar{Y})\bar{C} \quad \text{and} \quad K(\bar{Y}, X)=cg(X, \bar{Y})C.$$

(3) We have

$$\nabla_{X^*}(\mathcal{L}_\eta \bar{Y}^*)-\mathcal{L}_\eta(\nabla_{X^*}\bar{Y}^*)=-\nabla_{[\eta, X^*]}\bar{Y}^*-K(X^*, \bar{Y}^*),$$

$$\nabla_{X^*}(\mathcal{L}_\eta Z^*)-\mathcal{L}_\eta(\nabla_{X^*}Z^*)=-\nabla_{[\eta, X^*]}Z^*-K(X^*, Z^*).$$

These equations yield

$$(\nabla_{X^*}(\mathcal{L}_\eta g))(\bar{Y}^*, Z^*)-(\mathcal{L}_\eta(\nabla_{X^*}g))(\bar{Y}^*, Z^*)$$
$$=-(\nabla_{[\eta, X^*]}g)(\bar{Y}^*, Z^*)+g(K(X^*, Z^*), \bar{Y}^*)+g(Z^*, K(X^*, \bar{Y}^*)).$$

Since g is parallel with respect to ∇, we have

$$(\mathcal{L}_\eta(\nabla_{X^*}g))(\bar{Y}^*, Z^*)=\nabla_{[\eta, X^*]}g(\bar{Y}^*, Z^*)=0.$$

By (3) of Lemma 3.4, we also have

$$(\nabla_{X^*}(\mathcal{L}_\eta g))(Z^*, \bar{Y}^*)=2c(X^*q)g(Z^*, \bar{Y}^*).$$

Therefore we obtain

$$2c(Xq)g(Z, \bar{Y})=g(K(X, Z), \bar{Y})+g(Z, K(X, \bar{Y})).$$

Since $2(Xq)=\sqrt{-1}X\bar{f}=-g(X, \bar{C})$, it follows that

$$g(K(X, Z), \bar{Y})=-cg(X, \bar{C})g(Z, \bar{Y})-cg(X, \bar{Y})g(Z, \bar{C}),$$

which implies (3).

(4) Since $\nabla_\xi=\mathcal{L}_\xi$, we have

$$K(\xi, X^*)=[\eta, [\xi, X^*]]-\nabla_{[\eta, \xi]}X^*-[\xi, [\eta, X^*]]$$
$$=\mathcal{L}_{[\eta, \xi]}X^*-\nabla_{[\eta, \xi]}X^*$$
$$=-\nabla_{X^*}([\eta, \xi])-T([\eta, \xi], X^*).$$

By (2) of Lemma 3.4, we have

$$K(\xi, X)=-\nabla_X(-2cq\xi+\sqrt{-1}cC-\sqrt{-1}c\bar{C})+\sqrt{-1}cT(\bar{C}, X)$$
$$=2c(Xq)\xi-\sqrt{-1}c\nabla_X C+cg(X, \bar{C})\xi$$
$$=-cg(X, \bar{C})\xi-c^2f(x)X+cg(X, \bar{C})\xi$$
$$=-c^2f(x)X.$$

(5) Combining (2) of Lemma 3.5 and (2), (3) and (4) above, we obtain

$$\begin{aligned}(\mathcal{L}_\eta R)(X, \bar{Y})Z &= (\nabla_X K)(\bar{Y}, Z) - (\nabla_{\bar{Y}} K)(X, Z) + K(T(X, \bar{Y}), Z) \\ &= cg(Z, \bar{Y})\nabla_X C + cg(X, \overline{\nabla_Y C})Z + cg(Z, \overline{\nabla_Y C})X \\ &\quad -\sqrt{-1}c^2 f(x) g(X, \bar{Y})Z \\ &= 2c^2 q(x)\{g(Z, \bar{Y})X + g(X, \bar{Y})Z\}. \end{aligned}$$

q.e.d.

We define a tensor field Θ of type (1, 3) on M by

(i) $\Theta(X, \bar{Y})Z = R(X, \bar{Y})Z - c\{g(X, \bar{Y})Z + g(Z, \bar{Y})X\}$, $X, Y, Z \in S_x^2$.

(ii) The other components of Θ vanish.

LEMMA 3.7. *$\Theta_x = 0$ at any $x \in M$ such that $p(x) \neq 0$.*

PROOF. Let ϕ_t, $t \in \boldsymbol{R}$, be the 1-parameter group of transformations generated by η and $\gamma(t)$, $t \in \boldsymbol{R}$, the curve defined by $\gamma(t) = \phi_t(x)$. By (3) of Lemma 3.4, we have $(\phi_t^* g)_x = \rho(t) g_x$, where $\rho(t)$ is a positive function satisfying the ordinary differential equation

$$\frac{d}{dt}\rho(t) = 2cq(\gamma(t))\rho(t)$$

with the initial condition $\rho(0) = 1$. By (3) of Lemma 3.4 and (5) of Lemma 3.6, we also have $\mathcal{L}_\eta \Theta = 0$, and hence $(\phi_t^* \Theta)_x = \Theta_x$.

Let $e_{r+1}, \cdots, e_{n-1}$ be a basis of S_x^2 such that $g(e_\alpha, \bar{e}_\beta) = \delta_{\alpha\beta}$. To prove the assertion, it suffices to show that $g(\Theta(e_\alpha, \bar{e}_\beta)e_\gamma, \bar{e}_\delta) = 0$ for any quadruplet $(\alpha, \beta, \gamma, \delta)$ of indices. Let $\{e_\alpha(t)\}$ be the frame of S^2 along $\gamma(t)$ defined by $e_\alpha(t) = (\phi_t)_*(e_\alpha)$. Then it follows that

$$\begin{aligned} g(e_\alpha(t), \overline{e_\beta(t)}) &= g((\phi_t)_* e_\alpha, (\phi_t)_* \bar{e}_\beta) = (\phi_t^* g)(e_\alpha, \bar{e}_\beta) \\ &= \rho(t) g(e_\alpha, \bar{e}_\beta) = \rho(t)\delta_{\alpha\beta}. \end{aligned}$$

Put $X_\alpha(t) = (1/\sqrt{\rho(t)})e_\alpha(t)$ so that $g(X_\alpha(t), \overline{X_\beta(t)}) = \delta_{\alpha\beta}$. Then it follows that

$$\begin{aligned} (3.2) \qquad & g(\Theta(X_\alpha(t), \overline{X_\beta(t)})X_\gamma(t), \overline{X_\delta(t)}) \\ &= \rho(t)^{-2} g(\Theta(e_\alpha(t), \overline{e_\beta(t)})e_\gamma(t), \overline{e_\delta(t)}) \\ &= \rho(t)^{-2} g(\Theta((\phi_t)_* e_\alpha, (\phi_t)_* \bar{e}_\beta)(\phi_t)_* e_\gamma, (\phi_t)_* \bar{e}_\delta) \\ &= \rho(t)^{-2} g((\phi_t)_* \Theta(e_\alpha, \bar{e}_\beta)e_\gamma, (\phi_t)_* \bar{e}_\delta) \\ &= \rho(t)^{-2} (\phi_t^* g)(\Theta(e_\alpha, \bar{e}_\beta)e_\gamma, \bar{e}_\delta) \\ &= \rho(t)^{-1} g(\Theta(e_\alpha, \bar{e}_\beta)e_\gamma, \bar{e}_\delta). \end{aligned}$$

If we put $\chi(t) = p(\gamma(t))/p(x)$, then, by using (4) of Lemma 3.4, we have $\chi(0) = 1$ and

$$\frac{d\chi}{dt}(t)=(\eta p)(\gamma(t))/p(x)=2cp(\gamma(t))q(\gamma(t))/p(x)=2cq(\gamma(t))\chi(t).$$

From the uniqueness of the solutions of ordinary differential equation (3.1), we have $\rho(t)=\chi(t)$.

By (5) of Lemma 3.4, we see that the function $q(\gamma(t))$ is a bounded decreasing function. Hence we can choose a sequence $\{t_i\}$, $t_i\in \boldsymbol{R}$, such that

$$\lim_{i\to+\infty}\frac{d}{dt}q(\gamma(t))|_{t=t_i}=0.$$

By using (5) of Lemma 3.4 again, we obtain $\lim_{i\to+\infty}p(\gamma(t_i))=0$. Therefore we have $\rho(t_i)=\chi(t_i)\to 0$ $(i\to+\infty)$.

Suppose that $g(\Theta(e_\alpha, \bar{e}_\beta)e_\gamma, \bar{e}_\delta)\neq 0$. From (3.2), we have

$$\lim_{i\to+\infty}|g(\Theta(X_\alpha(t_i), \overline{X_\beta(t_i)})X_\gamma(t_i), \overline{X_\delta(t_i)})|=+\infty.$$

On the other hand, since M is compact and $g(X_\alpha(t), \overline{X_\beta(t)})=\delta_{\alpha\beta}$, it follows that $|g(\Theta(X_\alpha(t), \overline{X_\beta(t)})X_\gamma(t), \overline{X_\delta(t)})|$, $t\in\boldsymbol{R}$, are bounded, which is a contradiction. Hence we have

$$g(\Theta(e_\alpha, \bar{e}_\beta)e_\gamma, \bar{e}_\delta)=0. \qquad \text{q.e.d.}$$

To complete the proof of Proposition 3.1, it suffices to show that $\{x\in M|p(x)\neq 0\}$ is an open dense subset of M. To see this fact, we remark that

$$(\Box_1+\overline{\Box}_1+\Box_2+\overline{\Box}_2+N^2)f=(n+c-1)cf.$$

We also have $(\Box_1+\overline{\Box}_1+\Box_2+\overline{\Box}_2+N^2)p=(n+c-1)cp$, because $\Box_1+\overline{\Box}_1+\Box_2+\overline{\Box}_2+N^2$ is a real operator. Thus we see that p satisfies a strongly elliptic differential equation of order two, and hence $\{x\in M|p(x)\neq 0\}$ is an open dense subset of M. q.e.d.

§4. The hermitian quadrics Q_r.

Let $P_n(\boldsymbol{C})$ be the n-dimensional complex projective space with homogeneous coordinate system $z_0, \cdots, z_n$. Consider the hermitian quadric Q_r of $P_n(\boldsymbol{C})$ defined by the equation

$$\sum_{j=0}^{r}|z_j|^2-\sum_{k=r+1}^{n}|z_k|^2=0, \qquad 0\leqq r\leqq\frac{n-1}{2}.$$

Let S' be the induced PC structure of Q_r, that is, $S'=\boldsymbol{C}T(Q_r)\cap T^{1,0}(P_n(\boldsymbol{C}))$. Let τ_t be the 1-parameter group of transformations defined by

$$\tau_t([z_0, \cdots, z_n])=[z_0, \cdots, z_r, e^{\sqrt{-1}ct}z_{r+1}, \cdots, e^{\sqrt{-1}ct}z_n],$$

where c is a positive constant, and $\xi_{(c)}$ be the vector field on Q_r induced from

τ_t. It is easy to see that $\xi_{(c)}$ is an infinitesimal automorphism which is transversal to the subbundle $(S'+\bar{S}')$ and hence Q_r together with $\xi_{(c)}$ is a normal PC manifold.

Let π be the natural projection of $C^{n+1}-\{0\}$ onto $P_n(C)$. For $z\in\pi^{-1}(Q_r)$, let T_z be the subspace of the holomorphic tangent space $T^{1,0}(C^{n+1}-\{0\})_z$ defined by

$$T_z=\left\{\sum_{i=0}^{n} t_i\frac{\partial}{\partial z_i}\;\middle|\;\sum_{j=0}^{r} t_j\bar{z}_j=\sum_{k=r+1}^{n} t_k\bar{z}_k\right\},$$

where $z=(z_0, \cdots, z_n)$. Then π_* induces an isomorphism of T_z onto S'_x, $x=\pi(z)$ To consider the Levi form of Q_r, we define a 1-form θ' on $\pi^{-1}(Q_r)$ by

$$\theta'=\sqrt{-1}\iota^*\partial\log\left(\sum_{j=0}^{r}|z_j|^2\right)-\sqrt{-1}\iota^*\partial\log\left(\sum_{k=r+1}^{n}|z_k|^2\right)$$

where ι is an injection of $\pi^{-1}(Q_r)$ to $C^{n+1}-\{0\}$. We easily see that there is a unique real 1-form θ on Q_r such that $\theta'=\pi^*\theta$. It is easy to see that θ satisfies $\theta(\xi_{(c)})=1$ and $\theta(X)=0$ for every $X\in(S'+\bar{S}')_x$. Hence the Levi form L_x at $x\in Q_r$ is given by

$$L_x(X, Y)=-\sqrt{-1}(d\theta)(X, \bar{Y}), \qquad X, Y\in S'_x.$$

Now we claim that Q_r is a non-degenerate PC manifold and satisfies Condition (C). To show this, we define subbundles $S^{1\prime}$ and $S^{2\prime}$ of S' as follows. For a point z of $\pi^{-1}(Q_r)$, we define subspaces T_z^1 and T_z^2 of T_z respectively by

$$T_z^1=\left\{\sum t_i\frac{\partial}{\partial z_i}\in T_z\,|\,t_{r+1}=\cdots=t_n=0\right\},$$

$$T_z^2=\left\{\sum t_i\frac{\partial}{\partial z_i}\in T_z\,|\,t_0=\cdots=t_r=0\right\}.$$

For $x\in Q_r$, we put $S_x^{1\prime}=\pi_*(T_z^1)$ and $S_x^{2\prime}=\pi_*(T_z^2)$, where $z\in\pi^{-1}(x)$. It is clear that the definitions of $S_x^{1\prime}$ and $S_x^{2\prime}$ do not depend on the choice of z, and $S'_x=S_x^{1\prime}+S_x^{2\prime}$ (direct sum). We define $S^{1\prime}$ and $S^{2\prime}$ respectively by $S^{1\prime}=\bigcup_x S_x^{1\prime}$ and $S^{2\prime}=\bigcup_x S_x^{2\prime}$. From the equality

$$\begin{aligned}\sqrt{-1}d\theta'=&-\{(\sum|z_j|^2)\iota^*(\sum dz_j\wedge d\bar{z}_j)-\iota^*(\sum\bar{z}_jdz_j)\wedge\iota^*(\sum z_jd\bar{z}_j)\}/(\sum|z_j|^2)^2\\&+\{(\sum|z_k|^2)\iota^*(\sum dz_k\wedge d\bar{z}_k)-\iota^*(\sum\bar{z}_kdz_k)\wedge\iota^*(\sum z_kd\bar{z}_k)\}/(\sum|z_k|^2)^2,\end{aligned}$$

where j (resp. k) range over $0, \cdots, r$ (resp. $r+1, \cdots, n$), we can see that Q_r satisfies Condition (C) with respect to the decomposition $S'=S^{1\prime}+S^{2\prime}$. It is also verified that

$$R'(X, \bar{Y})Z=c\{g'(X, \bar{Y})Z+g'(Z, \bar{Y})X\},$$

where R' is the curvature and $X, Y, Z\in S_x^{1\prime}$ or $X, Y, Z\in S_x^{2\prime}$. In particular, we have $\nabla'R'=0$ and $\sigma'_1=\sigma'_2=c$.

Now we consider the space $\mathfrak{a}(Q_r)$. First we define an $(n+1)\times(n+1)$-matrix J_r by

$$J_r=\begin{pmatrix} -I_{r+1} & 0 \\ 0 & I_{s+1} \end{pmatrix},$$

I_{r+1} (resp. I_{s+1}) being the identity matrix of degree $r+1$ (resp. of degree $s+1$). Let G be the subgroup of $GL(n+1, \boldsymbol{C})$ defined by

$$G=\{U\in GL(n+1, \boldsymbol{C})\,|\,{}^t\bar{U}J_rU=J_r\}\,.$$

For $U\in G$, we denote by U' the projective transformation of $P_n(C)$ associated with U. It is easy to see that U' keeps Q_r invariant, and hence induces an automorphism of Q_r as a PC manifold. It is known that the mapping $U\to U'|Q_r$ induces an isomorphism of the quotient group of G by its center onto the group of all automorphisms of Q_r. Therefore the Lie algebra $\mathfrak{a}(Q_r)$ is naturally isomorphic to the Lie algebra

$$\{U\in \mathfrak{sl}(n+1, \boldsymbol{C})\,|\,{}^t\bar{U}J_r+J_rU=0\}\,,$$

and hence $\boldsymbol{C}\mathfrak{a}(Q_r)$ is naturally isomorphic to the simple Lie algebra $\mathfrak{sl}(n+1, \boldsymbol{C})$. In particular, under this isomorphism, $\xi_{(c)}$ corresponds to the matrix

$$\begin{pmatrix} \gamma I_{r+1} & 0 \\ 0 & \delta I_{s+1} \end{pmatrix}$$

where $\gamma=-\sqrt{-1}\dfrac{s+1}{n+1}c$ and $\delta=\sqrt{-1}\dfrac{r+1}{n+1}c$, and hence $\mathfrak{c}(Q_r)$ is naturally isomorphic to the Lie algebra

$$\left\{U=\begin{pmatrix} U^1 & 0 \\ 0 & U^2 \end{pmatrix} \middle|\; U^1\in\mathfrak{u}(r+1),\ U^2\in\mathfrak{u}(s+1),\ U\in\mathfrak{sl}(n+1, \boldsymbol{C})\right\}.$$

§ 5. Proof of Theorem A.

In this section we will complete the proof of Theorem A.

First by Proposition 1.1 and Corollary 3.2, we have

$$\nabla T=\nabla R=0\,.$$

Therefore M is a real analytic manifold and ∇ is a real analytic connection (Theorem 7.7 in [2], p. 263).

We fix a point y of Q_r and a point x of M and take a linear isomorphism Φ of $T(Q_r)_y$ onto $T(M)_x$ such that

(i) $\Phi(\xi_{(c)y})=\xi_x$,

(ii) $\Phi(S_y^{1\prime})=S_x^1$ and $\Phi(S_y^{2\prime})=S_x^2$,

(iii) $g(\Phi X, \overline{\Phi Y})=g'(X, \overline{Y}), \quad X, Y\in S'_y,$

where we extend Φ to a linear isomorphism of $CT(Q_r)_y$ onto $CT(M)_x$ in a natural manner. By Proposition 1.1 and Corollary 3.2, we see that Φ maps T'_y (resp. R'_y) to T_x (resp. to R_x).

LEMMA 5.1. *There exists an affine mapping ϕ of Q_r to M such that $(\phi_*)_y=\Phi$.*

PROOF. Since T, R, T' and R' are all parallel and Φ maps T'_y and R'_y to T_x and R_x respectively, it follows that there is an affine mapping ϕ' of a connected neighborhood of y onto a connected neighborhood of x such that $(\phi'_*)_y=\Phi$ (Theorem 7.4 in [2], p. 261).

Here we remark that the canonical affine connection ∇ is complete, because $\nabla g=0$ and M is compact. Since Q_r is simply connected, we see that ϕ' can be uniquely extended to an affine mapping ϕ of Q_r to M (Theorem 5.1 in [2], p. 252). q.e.d.

LEMMA 5.2. *$\phi_*(S^{1\prime})=S^1$, $\phi_*(S^{2\prime})=S^2$ and $\phi_*(\xi_{(c)})=\xi$.*

PROOF. These assertions follow immediately from the fact that ξ, S^1, S^2, $\xi_{(c)}$, $S^{1\prime}$ and $S^{2\prime}$ are all parallel. q.e.d.

LEMMA 5.3. *ϕ is a diffeomorphism.*

PROOF. Since Q_r is compact, ϕ is a covering map. Let ψ be a covering transformation of the covering space (Q_r, ϕ). To prove the assertion, it suffices to show that ψ has a fixed point. By Proposition 2.1, we have a non-trivial element ζ which belongs to $\mathfrak{z}_{(c)}$. Let ζ' be the lift of ζ to Q_r. Since $[\xi, \zeta]=-\sqrt{-1}\,c\zeta$, we also have $[\xi_{(c)}, \zeta']=-\sqrt{-1}\,c\zeta'$.

Under the identification of $C\mathfrak{a}(M)$ with $\mathfrak{sl}(n+1, C)$, we easily see that ζ' is of the form

$$\zeta'=\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where $A=(a_{ij})$, $0\leqq i\leqq r$, $r+1\leqq j\leqq n$, is an $(r+1)\times(s+1)$-matrix. Since ψ is an automorphism of Q_r, it follows that $\psi=U'|Q_r$ for some $U\in G$. Since $\psi_*\xi_{(c)}=\xi_{(c)}$ and $\psi_*\zeta'=\zeta'$, we have

$$U=\begin{pmatrix} U^1 & 0 \\ 0 & U^2 \end{pmatrix},$$

where $U^1\in U(r+1)$ and $U^2\in U(s+1)$ such that $U^1A=AU^2$.

Without loss of generality, we may assume that U^1 and U^2 are diagonal matrices so that

$$U^1_{ij}=\lambda_i\delta_{ij} \quad \text{and} \quad U^2_{ij}=\mu_i\delta_{ij},$$

where U^1_{ij} and U^2_{ij} are the components of U^1 and U^2 respectively and λ_i and μ_j are complex numbers such that $|\lambda_i|=|\mu_j|=1$. Then we have

$$\lambda_i a_{ij}=a_{ij}\mu_j\,.$$

Take a pair of indices (i, j) such that $a_{ij}\neq 0$, then we have $\lambda_i=\mu_j$. Let p be the point such that $z_i=z_j=1$ and $z_k=0$, $k\neq i, j$. Then we have $\phi(p)=p$, which means our assertion. q.e.d.

§6. A characterization of the complex projective spaces.

In [4] we have shown a characterization of the complex projective space:

THEOREM 6.1 (Theorem 6.5 in [4]). *Let $\tilde{M}$ be an $(n-1)$-dimensional compact complex manifold and F a holomorphic line bundle over $\tilde{M}$ with a hermitian metric h. Assume the following conditions:*

[a] *The first Chern form Φ of the hermitian holomorphic line bundle F is a positive form.*

[b] *The Kählerian metric $\tilde{g}$ associated with Φ is an Einstein metric.*

[c] *The vector bundle $T^{1,0}(\tilde{M})\otimes F^{-1}$ admits a non-trivial holomorphic cross section, where F^{-1} stands for the dual line bundle of F.*

Then $\tilde{M}$ and F must be one of the following

(1) *$\tilde{M}$ is the $(n-1)$-dimensional complex projective space $P_{n-1}(\boldsymbol{C})$ and F is the hyperplane section bundle over $P_{n-1}(\boldsymbol{C})$.*

(2) *$n=2$ and $\tilde{M}$ is the 1-dimensional complex projective space $P_1(\boldsymbol{C})$ and F is the holomorphic tangent bundle of $P_1(\boldsymbol{C})$.*

Let χ be the Ricci form associated with $\tilde{g}$. By Condition [b], we have $\chi=nc\Phi$ for some real constant c. In Theorem 6.5 in [4], we showed that if $nc\neq 1$, then (1) holds and if $nc=1$, then (2) holds. In the proof of the case where $nc\neq 1$, we used the results of Kobayashi-Ochiai. Here we show (1) by using Theorem B.

As in [4], let P be the principal $\boldsymbol{C}^*$-bundle associated with F^{-1}. Let M be the $U(1)$-reduction of P with respect to the hermitian metric induced from h, which is a real hypersurface of P. Let ξ be the vector field on M induced from the 1-parameter group of the right translations $R_{e^{\sqrt{-1}t}}$, $t\in\boldsymbol{R}$. By Condition [a], M together with ξ is a normal strongly pseudo-convex manifold. Furthermore, since $\chi=nc\Phi$, it follows that $R_*=nc\,\mathrm{Id}$ and hence Condition (C.3) in [4] is satisfied (cf. [4] p. 96). In particular we have $\sigma=c$.

Now we will show that $c=1$ and $\mathfrak{a}(M)\neq\mathfrak{c}(M)$. For $f\in C^\infty(M)$, let ζ_f be the complex vector field defined by

$$\zeta_f=f\xi+U+\bar{U},$$

where U is the cross section of S satisfying

$$\bar{Y}f+(d\theta)(U, \bar{Y})=0 \qquad \text{for every} \quad Y\in\Gamma(S).$$

For $\nu\in\boldsymbol{R}$, let $\mathfrak{g}^1_{(\nu)}$ and $\mathfrak{g}^2_{(\nu)}$ the sets of the complex vector fields on M defined respectively by

$$\mathfrak{g}^1_{(\nu)} = \{\zeta_f \mid \Box f = 0,\ Nf = \nu f\},$$

$$\mathfrak{g}^2_{(\nu)} = \{\zeta_f \mid \Box f = (nc-\nu)f,\ Nf = \nu f\},$$

(cf. §5 in [4]). In Theorem 5.6 in [4], we showed that $\mathfrak{g}^2_{(\nu)}=0$ for $\nu>0$, unless $\nu=c>0$.

By Proposition 6.4 in [4], we see that $\mathfrak{g}^2_{(1)}$ is isomorphic to the space of all holomorphic cross sections of $T^{1,0}(\tilde{M})\otimes F^{-1}$ (the space $\mathfrak{g}^2(P)_{(-m)}$ in Proposition 6.4 in [4] is isomorphic to $\mathfrak{g}^2_{(m)}$ by the definition of $\mathfrak{g}^2(P)_{(-m)}$). Consequently we have $\mathfrak{g}^2_{(1)}\neq 0$ and hence $c=1$. By Theorem 5.7 in [4], we have

$$C\mathfrak{a}(M) = \mathfrak{g}^1_{(0)} + \mathfrak{g}^2_{(0)} + \mathfrak{g}^1_{(-1)} + \mathfrak{g}^2_{(1)},$$

and

$$C\mathfrak{c}(M) = \mathfrak{g}^1_{(0)} + \mathfrak{g}^2_{(0)}.$$

Since $\mathfrak{g}^2_{(1)}\neq 0$, we have $\mathfrak{a}(M)\neq\mathfrak{c}(M)$.

Now we apply Theorem B. Then we obtain an isomorphism ϕ of $(S^{2n-1}, \xi_{(1)})$ onto (M, ξ). Clearly ϕ is equivariant with respect to the $U(1)$-actions. Hence ϕ can be extended uniquely to a bundle isomorphism ϕ' of $\boldsymbol{C}^n-\{0\}$ onto P as $\boldsymbol{C}^*$-bundles. It is easy to see that ϕ' induces a biholomorphic mapping of $P_{n-1}(\boldsymbol{C})$ onto $\tilde{M}$ and also induces an isomorphism of the hyperplane section bundle onto F.

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