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Centers of Chevalley algebras*

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1. Introduction.

Let L be a finite dimensional simple Lie algebra over the complex field, H an n-dimensional Cartan subalgebra, Φ the set of (nonzero) roots of L relative to H, and $\Pi=\{r_1, \cdots, r_n\}$ a simple system of roots. Chevalley [2] showed that L has a basis $B=\{e_r|r\in\Phi\}\cup\{h_1, \cdots, h_n\}$, $h_i\in H$, such that the constants of structure of L relative to this basis are all integers.

Let L_Z be the free abelian group on a given Chevalley basis. Let R be a commutative ring with identity. The *Chevalley algebra* of L over R is defined to be $L_R=R\otimes_Z L_Z$. Similarly, we define $H_R=R\otimes_Z H_Z$, where H_Z is the free abelian group on $\{h_1, \cdots, h_n\}$. In [5] and [11], the general ideal structure of L_R has been worked out, subject to restrictions on certain integers, for instance that 2 and 3 not be zero or zero divisors in R. In the present paper, we show how the center of L_R can be easily and quickly determined from the Cartan matrix of L, for any commutative ring R with identity, even when 2 or 3 may be zero in R. Calculation of the center of Chevalley algebras over general commutative rings with identity does not seem to appear in the literature, but specialization of our main result here to the case of a field of positive characteristic yields immediately results of other authors ([1], [3], [4], [9], and, implicitly, [7, 8]) about centers of Lie algebras of Chevalley type over fields.

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2. Background and preliminary results.

In this section, we set forth the notation and information needed for calculating the center of L_R. First, we recall the equations of structure of L

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relative to the Chevalley basis B.

(1) $[e_r, e_{-r}]=h_r$, a certain [**10**, Lemma 1] integral linear combination of $h_1, \cdots, h_n$.

(2) $[h_i, h_j]=0$ for all $i, j=1, 2, \cdots, n$.

(3) $[e_r, e_s]=N_{rs}e_{r+s}$, if $r+s\neq 0$, where N_{rs} is zero if $r+s\notin\Phi$ and otherwise is $\pm(p+1)$, where p is the largest integer i such that $s-ir\in\Phi$.

(4) $[h_i, e_s]=c(s, r_i)e_s$, where $c(s, r_i)=2(s, r_i)/(r_i, r_i)$ is the Cartan integer $p-q$, q the largest j such that $s+jr_i\in\Phi$.

Note that for any root r, we have in fact,

$$[h_r, e_s]=c(s, r)e_s\,. \tag{4'}$$

Next, recall that if $h=\sum_{i=1}^{n}k_ih_i\in L_R$, then for any $j=1, 2, \cdots, n$,

$$[h, e_j]=\sum_{i=1}^{n}k_ic(r_j, r_i)e_j\,. \tag{5}$$

We now can make the following basic definition.

2.1 DEFINITION. Let $f: H_R\to H_R$ be the R-homomorphism defined by $f(h_j) =\sum_{i=1}^{n}c(r_i, r_j)h_i$.

Since H_R is a free R-module, it is natural to represent f by its matrix relative to the basis $\{h_1, \cdots, h_n\}$. This matrix is the Cartan matrix $C=(c(r_i, r_j))$. We note that we can rewrite (5) as

$$[h, e_j]=f(h)_je_j\,, \tag{5'}$$

where $f(h)_j$ is the j-th coordinate of $f(h)$. We can describe the center Z of L_R easily in terms of f.

2.2 PROPOSITION. $Z=\operatorname{Ker} f$.

PROOF. First, observe that $Z\subseteq H_R$. For if $z=\sum_{r\in\Phi}k_re_r+\sum_{i=1}^{n}k_ih_i\in Z$, then for any root s we have

$$0=[z, e_{-s}]=k_sh_s+\sum_{r\neq -s}k_rN_{rs}e_{r+s}+s\Big(\sum_{i=1}^{n}k_ih_i\Big)e_s\,.$$

Hence, $k_s=0$, so that $z\in H_R$. Next, since $z\in Z$, the expression $s\Big(\sum_{i=1}^{n}k_ih_i\Big)$ is zero for all roots s. This holds in particular if s is any simple root r_j, in which case by (5′) we have $f(z)_j=0$. Hence, $f(z)=0$ which shows $Z\subseteq\operatorname{Ker} f$. But, the converse inclusion follows easily from (5). For suppose that $f(h)=0$ for $h=\sum_{i=1}^{n}k_ih_i$. Then $f(h)_j=\sum_{i=1}^{n}c(r_j, r_i)k_i=0$ for each j, so that $[h, e_j]=0$ for all j. Since H_R is abelian, $h \in Z$. Q. E. D.

The final piece of information needed to calculate Z is a list of the elementary divisors of C. The matrix of integers C can be diagonalized over Z, and so over R, by elementary row and column operations. By simple calculation, or reference to [**5**], we have the following.

2.3 LEMMA. *For L of the type indicated, the Cartan matrix can be reduced by elementary row and column operations to the corresponding diagonal matrix.*

A_n: diag $(1, 1, \cdots, 1, n+1)$; B_n *and* C_n: diag $(1, 1, \cdots, 1, 2)$;
D_n, $n \geqq 4$, n *even*: diag $(1, 1, \cdots, 1, 2, 2)$;
D_n, $n \geqq 5$, n *odd*: diag $(1, 1, \cdots, 1, 4)$;
E_6: diag $(1, 1, 1, 1, 1, 3)$; E_7: diag $(1, 1, 1, 1, 1, 1, 2)$;
E_8: I_8; F_4: I_4; G_2: I_2.

3. Center of L_R. Our main result now follows directly from those in Section 2.

3.1 THEOREM. *Let Z be the center of L_R. Then*

(a) *If L is of type E_8, F_4, or G_2, then $Z=0$.*
(b) *If L is of type A_n, then $Z=0$ unless $n+1=0$ in R. If $n+1=0$ in R, then Z is a 1-dimensional subspace of H_R.*
(c) *If L is of type B_n, C_n, or E_7, then $Z=0$ unless R has characteristic 2. If R has characteristic 2, then Z is a 1-dimensional subspace of H_R.*
(d) *If L is of type D_n, n odd, then $Z=0$ unless $4=0$ in R, in which case Z is a 1-dimensional subspace of H_R.*
(e) *If L is of type D_n, n even, then $Z=0$ unless R has characteristic 2. If R does have characteristic 2, then Z is a 2-dimensional subspace of H_R.*
(f) *If L is of type E_6, then $Z=0$ unless R has characteristic 3, in which case Z is a 1-dimensional subspace of H_R.*

PROOF. From Lemma 2.3, we see that the Cartan matrix has full rank, and then by Proposition 2.2 $Z=0$, unless one or more elementary divisors are zero in R. In case not all elementary divisors are nonzero, C has rank $n-1$ in all cases, except for D_n, n even, when it has rank $n-2$. Thus, by Proposition 1.2, Z is 1-dimensional in H_R in all cases except (e) of the statement, when it is two-dimensional. Q. E. D.

As we noted in the Introduction, Theorem 3.1 generalizes some results of other authors on centers of Lie algebras of classical type. While generally requiring more work than the foregoing, and being carried out only over fields, their methods have the advantage of generally yielding more information than just determination of the center. In [4] for instance, the centers of more general Chevalley algebras (cf. [**6**]) are also determined. The method used in the present paper does not apply directly to the extended Chevalley algebras, because of the role played by the Cartan matrix. Modifying our method along the lines of [**6**] will yield the centers of the more general Chevalley algebras, but aside from applying over general commutative rings rather than just fields, it is no more efficient than the calculations used in [**4**].

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