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Homological coalgebra

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The purpose of this paper is to give an introduction to the homological theory of comodules over coalgebras and Hopf algebras. Section 1 is a self-contained exposition of basic concepts such as cotensor product, injective comodules and change of coalgebras. Some results analogous to the results (Cline-Parshall-Scott [2], Hochschild [5]) on rational modules over affine algebraic groups are proved. Section 2 deals with the representation theory of co-Frobenius coalgebras and coseparable coalgebras. We reproduce in this section some of Lin's results [7] and Larson's results [6], partly with simplified proof. Section 3 deals with the cohomology theory of coalgebras.

Throughout this paper, the field k is fixed. Vector spaces over k are called k-spaces, and linear maps between k-spaces are called k-maps. We freely use the terminology and results of Sweedler [9].

§1. Coalgebras and comodules.

A coalgebra over k is a k-space C together with k-maps $\Delta: C\to C\otimes C$ and $\varepsilon: C\to k$ such that $(I\otimes\Delta)\Delta=(\Delta\otimes I)\Delta$ and $(I\otimes\varepsilon)\Delta=(\varepsilon\otimes I)\Delta=I$. If C is a coalgebra, a left C-comodule is a k-space M together with a k-map $\rho_M: M\to C\otimes M$ such that $(I\otimes\rho_M)\rho_M=(\Delta\otimes I)\rho_M$ and $(\varepsilon\otimes I)\rho_M=I$. If M and N are left C-comodules, a comodule map from M to N is a k-map $f: M\to N$ such that $(I\otimes f)\rho_M=\rho_N f$. The k-space of all comodule maps from M to N is denoted by $\mathrm{Com}_C(M, N)$ and the category of left C-comodules is denoted by ${}^C\mathbf{M}$. Similarly, we define $\mathbf{M}^C$, the category of right C-comodules.

1.1. Cotensor products and injective comodules.

If M is a right C-comodule and N is a left C-comodule, the *cotensor product* $M\square_C N$ is the kernel of the k-map

$$\rho_M\otimes I-I\otimes\rho_N: M\otimes N\to M\otimes C\otimes N.$$

Given comodule maps $f: M\to M'$ and $g: N\to N'$, the k-map $f\otimes g: M\otimes N\to M'\otimes N'$ induces a k-map

$$f \square_C g : M \square_C N \to M' \square_C N'.$$

It is clear that $-\square_C-$ is an additive covariant bifunctor from $\mathbf{M}^C \times {}^C\mathbf{M}$ to $\mathbf{M}_k$, the category of k-spaces, and is left exact. The mapping $m \otimes c \to \varepsilon(c)m$ and $c \otimes n \to \varepsilon(c)n$ yield natural isomorphism $M \square_C C \cong M$ and $C \square_C N \cong N$. We shall usually identify these isomorphic k-spaces.

Let C and D be two coalgebras. Suppose that M in addition to being a left C-comodule with structure map $\rho^- : M \to C \otimes M$, is also a right D-comodule with structure map $\rho^+ : M \to M \otimes D$ and that $(I \otimes \rho^+)\rho^- = (\rho^- \otimes I)\rho^+$. We then say that M is a *(C, D)-bicomodule.* If N is a left D-comodule then the map

$$\rho^- \otimes I : M \otimes N \to C \otimes M \otimes N$$

gives a left C-comodule structure to $M \otimes N$ and $M \square_D N$ is a C-subcomodule of $M \otimes N$. Similarly if L is a right C-comodule then $L \square_C M$ becomes a right D-comodule. With the structure described above we have the associativity of cotensor product:

$$(L \square_C M) \square_D N \cong L \square_C (M \square_D N).$$

If N is a left C-comodule which is finite dimensional as a k-space then the dual space N^* is a right C-comodule with structure map

$$N^* \to \mathrm{Hom}(N, C) \cong N^* \otimes C, \quad n^* \to (I \otimes n^*)\rho_N.$$

If M is a right C-comodule we have canonically

$$M \square_C N \cong \mathrm{Com}_C(N^*, M).$$

Since every comodule is the union of its finite dimensional subcomodules, this implies that the functor $M \square_C -$ from ${}^C\mathbf{M}$ to $\mathbf{M}_k$ is exact if and only if so is the functor $\mathrm{Com}_C(-, M)$ from $\mathbf{M}^C$ to $\mathbf{M}_k$ (cf. Takeuchi [**11**]). A right C-comodule M is called *injective* (or *C-injective*) if the functor $\mathrm{Com}_C(-, M)$ is exact, and is called *projective* (or *C-projective*) if the functor $\mathrm{Com}_C(M, -)$ is exact.

By the flatness of injective comodules and the associativity of cotensor products we have:

PROPOSITION 1. *Let L be a right C-comodule and M be a (C, D)-bicomodule. If L is C-injective and M is D-injective then $L \square_C M$ is D-injective.*

We use the opposite coalgebra C^{op} to convert a left (or right) C-comodule V into a right (or left) C^{op}-comodule. Every (C, D)-bicomodule M becomes a left $C \otimes D^{op}$-comodule. Similarly M may be regarded as a left $D^{op} \otimes C$-comodule, a right $C^{op} \otimes D$-comodule and a right $D \otimes C^{op}$-comodule.

Let C, D and E be coalgebras. For a (D^{op}, C)-bicomodule L, a (C, E)-bicomodule M and a (E, D^{op})-bicomodule N, we have a natural isomorphism

$$(L \square_C M) \square_{D \otimes E} N \cong L \square_{C \otimes D} (M \square_E N).$$

PROPOSITION 2. (1) *Let L be a (D^{op}, C)-bicomodule. If L is injective as a right $C\otimes D$-comodule then it is injective as a right D-comodule.*

(2) *Let L be a right D-comodule and M be a right E-comodule. If L is D-injective and M is E-injective then $M\otimes N$ is injective as a right $D\otimes E$-comodule.*

PROOF. (1) Setting $M=C$ and $E=k$ in the above isomorphism, yields the isomorphism for every left D-comodule N,

$$L\,\square_D\,N\cong L\,\square_{C\otimes D}(C\otimes N)\,.$$

This shows that the functor $L\,\square_D -$ is exact.

(2) Setting $C=k$, yields the isomorphism for every left $D\otimes E$-comodule N,

$$(L\otimes M)\,\square_{D\otimes E}\,N\cong L\,\square_D(M\,\square_E\,N)\,.$$

This shows that the functor $(L\otimes M)\,\square_{D\otimes E}-$ is exact. Q.E.D.

Let W be a right C-comodule. For every k-space X, $X\otimes W$ is a right C-comodule with structure map

$$I\otimes\rho_W:\ X\otimes W\rightarrow X\otimes W\otimes C\,,$$

which we denote $(X)\otimes W$. $(X)\otimes W$ is a direct sum of copies of W. The next well-known result is fundamental.

PROPOSITION 3. *Let V be a right C-comodule and X be a k-space. Then the map*

$$\phi:\ \mathrm{Com}_C(V, (X)\otimes C)\rightarrow \mathrm{Hom}(V, X)$$

given by $\phi(F)=(I\otimes\varepsilon)F$ for each $F\in\mathrm{Com}_C(V, (X)\otimes C)$ is a k-isomorphism. The inverse ψ of ϕ is given by $\psi(f)=(f\otimes I)\rho_V$ for each $f\in\mathrm{Hom}(V, X)$.

PROOF. Straightforward.

A right C-comodule M is called *free* if there exists a k-space X such that $M\cong(X)\otimes C$ as right C-comodules.

COROLLARY 1. *Every free comodule is injective.*

Note that an injective comodule need not be free. For example, take $C=C_1\oplus C_2$, the direct sum of coalgebras C_1 and C_2. Then C_1 is clearly not free, but is injective as a C-comodule. In [11], Takeuchi showed that if C is cocommutative and irreducible then every injective comodule is free.

COROLLARY 2. *Every comodule can be embedded in a free comodule.*

PROOF. For every right C-comodule M, its structure map ρ_M is a C-comodule map from M to $(M)\otimes C$. Since $(I\otimes\varepsilon)\rho_M=I$, ρ_M is a monomorphism.

Q.E.D.

We note that a C-comodule V is injective if and only if it is a direct summand of a free C-comodule.

If C is a coalgebra, then $C^*=\mathrm{Hom}(C, k)$ is an algebra, with multiplication defined by $\alpha\beta=(\alpha\otimes\beta)\Delta:\ C\rightarrow k$, where α, $\beta\in C^*$. If V is a right C-comodule, then defining $c^*\rightharpoonup v=(I\otimes c^*)\rho_V(v)$ for $c^*\in C^*$, $v\in V$, makes V into a left C^*-module. In a similar fashion, left C-comodules have a right C^*-module structure. For right C-comodules M, N we have

$$\mathrm{Com}_C(M, N)=\mathrm{Mod}_{C^*}(M, N).$$

Thus $\mathbf{M}^C$ may be regarded as a full subcategory of the category of left C^*-modules. It follows that if a right C-comodule M is injective (resp. projective) as a left C^*-module then it is injective (resp. projective) as a right C-comodule.

PROPOSITION 4. *Let M be a finite dimensional right C-comodule. Then M is injective* (*resp. projective*) *as a left C^*-module if and only if it is injective* (*resp. projective*) *as a right C-comodule.*

PROOF. We need to show the "if" part. Suppose that M is C-injective. Then the map

$$0\longrightarrow M\overset{\rho}{\longrightarrow}(M)\otimes C\cong C\oplus\cdots\oplus C \quad \text{(finite times)}$$

splits as right C-comodules, and so as left C^*-modules. Taking the dual, the map

$$C^*\oplus\cdots\oplus C^*\longrightarrow M^*\longrightarrow 0$$

splits as right C^*-modules. This means that M^* is projective as a right C^*-module. Therefore $M=M^{**}$ is injective as a left C^*-module.

Next we show that if M is C-projective then it is projective as a left C^*-module. Since M^* is injective as a left C-comodule, it follows from the above that M^* is injective as a right C^*-module. Therefore $M=M^{**}$ is projective as a right C^*-module. This completes the proof.

1.2. Change of coalgebras.

We shall consider two coalgebras C and D, and a coalgebra map $\pi: C\rightarrow D$. Every right C-comodule V may be treated as a right D-comodule with structure map

$$(\pi\otimes I)\rho:\ V\longrightarrow V\otimes C\longrightarrow V\otimes D,$$

which we denote V_π. Similarly for left comodules. In particular C itself may be regarded as a left or a right D-comodule. Regarding C as a (D, C)-bicomodule, we form the right C-comodule

$$W^\pi=W\square_D C, \text{ where } W \text{ is a right } D\text{-comodule},$$

which we call the *induced* comodule for W.

PROPOSITION 5. *The following are equivalent:*

(i) *The functor $-\square_D C$ from $\mathbf{M}^D$ to $\mathbf{M}^C$ is exact.*

(ii) *C is injective as a left D-comodule.*

(iii) *Every injective left C-comodule is injective as a left D-comodule.*

PROOF. The equivalence of (i) and (ii) has already been proved in 1.1. (ii) implies (iii) by Proposition 1, and (iii) implies (ii) since C is an injective left C-comodule. Q. E. D.

The next result is a generalization of Proposition 3.

PROPOSITION 6. *Let V be a right C-comodule and W be a right D-comodule. Then the map*

$$\phi : \mathrm{Com}_C(V, W^{\pi}) \longrightarrow \mathrm{Com}_D(V_{\pi}, W)$$

given by $\phi(F)=(I \square \pi)F$, for each $F \in \mathrm{Com}_C(V, W^{\pi})$, is a k-isomorphism. The inverse ψ of ϕ is given by $\psi(f)=(f \otimes I)\rho_V$ for each $f \in \mathrm{Com}_D(V_{\pi}, W)$.

PROOF. For $F \in \mathrm{Com}_C(V, W^{\pi})$, the following diagram is commutative:

$$\begin{array}{ccccccc}
V & \xrightarrow{\quad F \quad} & W \square_D C & \xrightarrow{\quad I \square \pi \quad} & W \square_D D & \cong & W \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle I \otimes \Delta} & & & & \\
V \otimes C & \xrightarrow{\quad F \otimes I \quad} & W \square_D C \otimes C & & \downarrow{\scriptstyle I \otimes \Delta} & & \downarrow{\scriptstyle \rho} \\
\downarrow{\scriptstyle I \otimes \pi} & & \downarrow{\scriptstyle I \square I \otimes \pi} & & & & \\
V \otimes D & \xrightarrow{\quad F \otimes I \quad} & W \square_D C \otimes D & \xrightarrow{\quad I \square \pi \otimes I \quad} & W \square_D D \otimes D & \cong & W \otimes D .
\end{array}$$

This implies that $\phi(F) \in \mathrm{Com}_D(V_{\pi}, W)$.

Next we show that $\psi(f) \in \mathrm{Com}_C(V, W^{\pi})$ for $f \in \mathrm{Com}_D(V_{\pi}, W)$. We have

$$\begin{aligned}
(\rho_W \otimes I)\psi(f) &= (\rho_W f \otimes I)\rho_V = ((f \otimes I)(I \otimes \pi)\rho_V \otimes I)\rho_V \\
&= (f \otimes I \otimes I)(I \otimes \pi \otimes I)(I \otimes \Delta)\rho_V \\
&= (I \otimes (\pi \otimes I)\Delta)(f \otimes I)\rho_V = (I \otimes (\pi \otimes I)\Delta)\psi(f) .
\end{aligned}$$

This concludes that the image of the map $\psi(f)$ is contained in $W \square_D C$. So $\psi(f)$ is clearly a C-comodule map from V into W^{π}. It is easily checked that $\psi\phi = I$ and $\phi\psi = I$. Q. E. D.

COROLLARY. *If a right D-comodule W is injective then W^{π} is injective as a right C-comodule.*

A right C-comodule V is said to be *π-injective* if for every exact sequence of C-comodules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

which splits as D-comodules, the sequence

$$0 \longrightarrow \mathrm{Com}_C(M'', V) \longrightarrow \mathrm{Com}_C(M, V) \longrightarrow \mathrm{Com}_C(M', V) \longrightarrow 0$$

is also exact. Proposition 6 shows that W^{π} is π-injective for every right D-comodule W. Let $V=V_1\oplus V_2$, where V_1 and V_2 are subcomodules of V. Then V is π-injective if and only if V_1 and V_2 are π-injective.

For every right C-comodule V, the structure map ρ may be regarded as a C-comodule map from V into $(V_{\pi})^{\pi}=V\square_D C$. The composition

$$V \xrightarrow{\rho} V\square_D C \xrightarrow{I\square\pi} V\square_D D=V$$

is the identity, which shows that V map be treated as a direct summand of $(V\square_D C)_{\pi}$ as a D-comodule, since $I\square\pi$ is a D-comodule map. This observation leads us to the following result.

PROPOSITION 7. *The following statements concerning a right C-comodule V are equivalent:*

(i) *V is π-injective.*

(ii) *Every exact sequence of C-comodules*

$$0 \longrightarrow V \longrightarrow M \longrightarrow N \longrightarrow 0$$

which splits as D-comodules, also splits as C-comodules.

(iii) *There exists a C-comodule map $g:(V_{\pi})^{\pi}\rightarrow V$ such that $\rho g=I$, that is, V is a direct summand of $(V_{\pi})^{\pi}$ as a C-comodule.*

PROPOSITION 8. *Let V be a right C-comodule. If V is π-injective and D-injective, then it is C-injective.*

PROOF. By Corollary of Proposition 6, $V\square_D C$ is C-injective. Since V is π-injective, V is a direct summand of $V\square_D C$ as a C-comodule. Therefore V is C-injective. Q. E. D.

1.3. Comodules over Hopf algebras.

A *Hopf algebra* over k is a k-space H together with k-maps $\Delta: H\rightarrow H\otimes H$, $\varepsilon: H\rightarrow k$, $m: H\otimes H\rightarrow H$, $u: k\rightarrow H$ and $S: H\rightarrow H$ such that (H, Δ, ε) is a coalgebra over k, (H, m, u) is an algebra over k, m and u are coalgebra maps and $m(I\otimes S)\Delta=u\varepsilon=m(S\otimes I)\Delta$. The map S is called the *antipode* of the Hopf algebra. Let V_i $(i=1, 2)$ be right H-comodules with the structure map $\rho_i: V_i\rightarrow V_i\otimes H$ $(i=1, 2)$. Then the composition

$$\rho=(I\otimes I\otimes m)(I\otimes t\otimes I)(\rho_1\otimes\rho_2): V_1\otimes V_2\rightarrow V_1\otimes V_2\otimes H$$

gives $V_1\otimes V_2$ the structure of a right H-comodule, where t denotes the twist map, which we call the *tensor product comodule* of V_1 and V_2.

Now we shall consider two Hopf algebras H and L, and a Hopf algebra map $\pi: H\rightarrow L$ (i. e. π is both a coalgebra map and algebra map, and $\pi S_L=S_H\pi$). Using the fact that the antipode is an anti-coalgebra map and an anti-algebra map, we have the next result.

PROPOSITION 9. *Let V be a right H-comodule and W be a right L-comodule. Then the map*

$$\phi:\ V\otimes W^{\pi}\rightarrow(V_{\pi}\otimes W)^{\pi}$$

given by $\phi(\sum v\otimes w\otimes h)=\sum v_{(0)}\otimes w\otimes v_{(1)}h$ *(we write* $\rho_V(v)=\sum v_{(0)}\otimes v_{(1)}$*) is an isomorphism of H-comodules. The inverse ψ of ϕ is given by* $\psi(\sum v\otimes w\otimes h)=\sum v_{(0)}\otimes w\otimes S(v_{(1)})h$.

Taking $L=k$, $\pi=\varepsilon_H$ and $W=k$, we have:

COROLLARY 1. *Let V be a right H-comodule. Then there exists an isomorphism*

$$V\otimes H\cong(V)\otimes H$$

as H-comodules.

COROLLARY 2. *Let V be a right H-comodule and W be an injective right H-comodule. Then the tensor product comodule $V\otimes W$ is H-injective.*

PROOF. Since W is injective, W is a direct summand of $(W_{\varepsilon})^{\varepsilon}=(W)\otimes H$. Hence $V\otimes W$ is a direct summand of $V\otimes(W_{\varepsilon})^{\varepsilon}$. By the above Proposition, $V\otimes(W_{\varepsilon})^{\varepsilon}\cong(V_{\varepsilon}\otimes W_{\varepsilon})^{\varepsilon}$, and this implies that $V\otimes W$ is H-injective. Q. E. D.

An algebra map $\omega: L\rightarrow H$ is called a (right) *cross-section* of $\pi: H\rightarrow L$ if it is a right L-comodule map, that is, $(I\otimes\pi)\Delta_H\omega=(\omega\otimes I)\Delta_L$. Assume that there exists a cross-section. Then, defining $h\leftharpoonup l=h\omega(l)$ for $h\in H$, $l\in L$, H makes into a right L-module. We compute

$$\begin{aligned}(I\otimes\pi)\Delta(h\leftharpoonup l)&=(I\otimes\pi)\Delta(h)\cdot(I\otimes\pi)\Delta(\omega(l))\\&=(\sum h_{(1)}\otimes\pi(h_{(2)}))\cdot(\sum\omega(l_{(1)})\otimes l_{(2)})\\&=\sum h_{(1)}\leftharpoonup l_{(1)}\otimes\pi(h_{(2)})l_{(2)}\,.\end{aligned}$$

This shows that H is a Hopf module. So we can apply the structure Theorem of Hopf modules (Sweedler [**9**], p. 84) to obtain an isomorphism of H to $(H')\otimes L$ as L-comodules, where $H'=\{h\in H\mid(I\otimes\pi)\Delta(h)=h\otimes 1\}$. Thus we have proved:

PROPOSITION 10. *Let $\pi: H\rightarrow L$ be a Hopf algebra map. If there exists a right cross-section of π, then H is free as a right L-comodule.*

§2. A bilinear form for coalgebras.

2.1. Co-Frobenius coalgebras.

We shall consider a coalgebra C and a bilinear form $b: C\times C\rightarrow k$. Then b induces two k-maps $\tau: C\otimes C\rightarrow k$ and $\theta: C\rightarrow C^*$ by setting $\tau(c\otimes d)=b(c, d)$ and $\theta(d)(c)=b(c, d)$, for $c, d\in C$. The next Lemma is clear.

LEMMA 1. *In the above situation, the following are equivalent:*

(i) $\sum c_{(1)}b(c_{(2)}, d)=\sum b(c, d_{(1)})d_{(2)}$, *for all* $c, d\in C$.

(ii) $b(c\leftharpoonup c^*, d)=b(c, c^*\rightharpoonup d)$, *for all* $c, d\in C$, $c^*\in C^*$.

(iii) $(I\otimes\tau)(\Delta\otimes I)=(\tau\otimes I)(I\otimes\Delta)$.

(iv) $\theta: C\to C^*$ *is a left* C^**-module map.*

A bilinear form $b: C\times C\to k$ is called *C-balanced* if the above conditions hold.

LEMMA 2. *Let* $b: C\times C\to k$ *be a C-balanced bilinear form and* X *be a subspace of* C. *Then we have:*

(1) *If* X *is a left coideal (i.e.* $\Delta(X)\subset C\otimes X$*), then* $X^\perp=\{d\in C\,|\,b(x, d)=0$ *for all* $x\in X\}$ *is a right coideal.*

(2) *If* X *is a right coideal of* C, *then* ${}^\perp X=\{c\in C\,|\,b(c, x)=0$ *for all* $x\in X\}$ *is a left coideal.*

PROOF. Let X be a left coideal. Note that $\Delta(X)\subset C\otimes X$ and $X\leftharpoonup C^*\subset X$ are equivalent. Now we have

$$b(X, C^*\rightharpoonup X^\perp)=b(X\leftharpoonup C^*, X^\perp)\subset b(X, X^\perp)=0\,.$$

Hence $C^*\rightharpoonup X^\perp\subset X^\perp$, and so $X^\perp$ is a right coideal. This completes the proof of (1). In the similar way we have the proof of (2). Q. E. D.

A bilinear form $b: C\times C\to k$ is called *left non-degenerated* if $C^\perp=\{0\}$, equivalently $\theta: C\to C^*$ is injective. A coalgebra C is called *left co-Frobenius* if there exists a bilinear form $b: C\times C\to k$ which is left non-degenerated and C-balanced, i.e. if there exists a *left* C^*-monomorphism from C to C^*. We note that if a coalgebra C is co-semi-simple then it is left (and right) co-Frobenius. For we let $C=\oplus_\lambda C_\lambda$, where C_λ are simple subcoalgebras of C. Since $A_\lambda=C_\lambda^*$ is a simple algebra, we have $A_\lambda\cong A_\lambda^*$ as left A_λ-modules. Hence we have $C_\lambda\cong C_\lambda^*$ as left C_λ^*-modules, and so as left C^*-modules. Thus we have

$$C=\oplus_\lambda C_\lambda\cong\oplus_\lambda C_\lambda^*\hookrightarrow \Pi_\lambda C_\lambda^*=C^*$$

as left C^*-modules.

THEOREM 1 (I-p. Lin). *Let C be a left co-Frobenius coalgebra. Then we have:*

(1) *An injective cover of every finite dimensional right C-comodule is finite dimensional.*

(2) *Every injective right C-comodule is C-projective.*

PROOF. (1) Let M be a finite dimensional right C-comodule and let $\sigma(M)=\bigoplus_{i=1}^{n} S_i$ be the socle of M (i.e. the sum of all simple right C-subcomodules of M). For the notion of socles and injective covers, we refer to Green [4]. It is easy to see that an injective cover $J(M)$ of M is isomorphic to $\bigoplus_{i=1}^{n} J(S_i)$, where $J(S_i)$ denotes an injective cover of S_i. Therefore in order to prove (1) it suffices to prove that $J(S)$ is finite dimensional for each simple right C-subcomodules S of M. We may assume that S is a minimal right coideal of C and $J(S)\subset C$. Let x be a non-zero element in S. Then we have $S=C^*\rightharpoonup x$. Since C is left co-Frobenius, there exists an element c in C such that $b(c, x)\neq 0$. We claim that $(c\leftharpoonup C^*)^{\perp}\cap S=\{0\}$. Suppose that there exists a non-zero element y in S such that y lies in $(c\leftharpoonup C^*)^{\perp}$. Since $S=C^*\rightharpoonup x=C^*\rightharpoonup y$ there exists an element c^* in C^* such that $c^*\rightharpoonup y=x$. Then

$$b(c\leftharpoonup c^*, y)=b(c, c^*\rightharpoonup y)=b(c, x)\neq 0.$$

But $y\in(c\leftharpoonup C^*)^{\perp}$ implies $b(c\leftharpoonup c^*, y)=0$. This is a contradiction.

Since $c\leftharpoonup C^*$ is a left coideal, $(c\leftharpoonup C^*)^{\perp}$ is a right coideal, by Lemma 2. It follows that $(c\leftharpoonup C^*)^{\perp}\cap J(S)=\{0\}$. In generally, if X is a finite dimensional subspace of C, $X^{\perp}$ is cofinite dimensional since $X^{\perp}$ is the kernel of the map $C\rightarrow X^*$ defined by $c\rightarrow\theta(c)|X$. Thus we have that $(c\leftharpoonup C^*)^{\perp}$ is cofinite dimensional. It follows that $J(S)$ is finite dimensional. Thus (1) is proved.

(2) Let V be an injective right C-comodule and let $\sigma(V)=\bigoplus_{\lambda}S_{\lambda}$ be the socle of V. Then we have $V\cong\bigoplus_{\lambda}J(S_{\lambda})$. Since $J(S_{\lambda})$ is finite dimensional it follows from Proposition 4 that $J(S_{\lambda})$ is an injective left C^*-module. The embedding

$$J(S_{\lambda})\subset C\xrightarrow{\theta} C^*$$

yields that $J(S_{\lambda})$ is a direct summand of C^* as a left C^*-module. Therefore $J(S_{\lambda})$ is a projective left C^*-module, and so is V. Thus V is a projective right C-comodule. This completes the proof.

COROLLARY 1. *If C is a left co-Frobenius coalgebra then C is projective as a right C-comodule.*

COROLLARY 2. *Let C be a left co-Frobenius coalgebra. Then the category of left C-comodules has enough projectives.*

PROOF. We have to show that for each left C-comodule N there exists an epimorphism $P\rightarrow N\rightarrow 0$ with P projective. Without loss of generality, we may assume that N is finite dimensional. Then we consider a monomorphism of finite dimensional right C-comodules $0\rightarrow N^*\rightarrow J(N^*)$. Taking the dual, we have an epimorphism of left C-comodules $J(N^*)^*\rightarrow N\rightarrow 0$. Q. E. D.

2.2. Integrals.

An *augmented coalgebra* is a coalgebra C together with a coalgebra map $u: k\rightarrow C$. Clearly $u(1)$ is a grouplike element of C. Using $u: k\rightarrow C$ we may convert any k-space X into a left (or right) C-comodule ${}_uX$ (or X_u) by setting $\rho(x)=u(1)\otimes x$ (or $\rho(x)=x\otimes u(1)$). In particular k has a left (or right) C-comodule structure. Every Hopf algebra H may be regarded as an augmented coalgebra with unit map $u: k\rightarrow H$.

$x \in C^*$ is called a *left integral* if x is a left C-comodule map from C to k, i.e. $\sum c_{(1)}\langle x, c_{(2)}\rangle = \langle x, c\rangle u(1)$ for all $c \in C$. We note that $x \in C^*$ is a left integral if and only if $c^* \cdot x = \langle c^*, u(1)\rangle x$ for all $c^* \in C^*$. An augmented coalgebra need not have a non-zero integral. However, if C is left co-Frobenius then C has a non-zero left integral. In fact, it is easily checked that $b(-, u(1)) = \theta(u(1))$ is a non-zero left integral.

PROPOSITION 11. *Let C be an augmented coalgebra. If C is finite dimensional and left co-Frobenius then the k-space of left integrals is one dimensional.*

PROOF. We have that $C \cong C^*$ as right C-comodules. Therefore

$$\mathrm{Com}_C(C, k) \cong C^* \square_C k \cong C \square_C k \cong k . \qquad \text{Q.E.D.}$$

LEMMA 3. *Let H be a Hopf algebra. If J is a non-zero right ideal and a right coideal, then J is equal to H.*

PROOF. If $\varepsilon(J) = \{0\}$ then for all $h \in J$, $h = \sum \varepsilon(h_{(1)}) h_{(2)} = 0$ (since $\Delta(J) \subset J \otimes H$). Hence we must have $\varepsilon(J) \neq \{0\}$. Thus there exists an element h in J such that $\varepsilon(h) = 1$. Since $1 = \varepsilon(h) = \sum h_{(1)} S(h_{(2)})$ and $J \cdot H \subset J$, we have $1 \in J$. Q.E.D.

THEOREM 2 (Lin-Larson-Sweedler-Sullivan). *The following statements concerning a Hopf algebra H are equivalent:*

(i) *H has a non-zero left integral.*

(ii) *H is left co-Frobenius.*

(iii) *H has a non-zero right integral.*

(iv) *H is right co-Frobenius.*

PROOF. (i)$\Rightarrow$(ii). Let x be a non-zero left integral. We define a bilinear form $b: H \times H \to k$ as follows;

$$b(c, d) = \langle x, cS(d)\rangle, \quad \text{for all } c, d \in H.$$

Then we compute

$$\sum b(c, d_{(1)}) d_{(2)} = \sum \langle x, cS(d_{(1)})\rangle d_{(2)}$$
$$= \sum c_{(1)} S(d_{(2)}) \langle x, c_{(2)} S(d_{(1)})\rangle d_{(3)}$$
$$= \sum c_{(1)} \varepsilon(d_{(2)}) \langle x, c_{(2)} S(d_{(1)})\rangle$$
$$= \sum c_{(1)} \langle x, c_{(2)} S(d)\rangle = \sum c_{(1)} b(c_{(2)}, d) .$$

This shows that $b: H \times H \to k$ is C-balanced. Next we show that $H^\perp$ ($= \{d \in H \mid b(c, d) = 0$ for all $c \in H\}$) is zero. Let $d \in H^\perp$ and $h \in H$. For all $c \in H$, we have

$$b(c, dh) = \langle x, cS(dh)\rangle = \langle x, cS(h)S(d)\rangle = b(cS(h), d) = 0 .$$

Hence $dh \in H$, so $H^\perp$ is a right ideal of H. Since $x \neq 0$, $H^\perp$ is a proper right ideal. Also $H^\perp$ is a right coideal, by Lemma 2. Therefore we have $H^\perp = \{0\}$, by Lemma 3.

(ii)$\Rightarrow$(iii). In the proof of Theorem 1, (1), we obtained that H contains a proper right coideal of finite codimension. Therefore, by (2.14) in Sweedler [**10**], H has a non-zero right integral.

(iii)$\Rightarrow$(iv). The proof is the same as (i)$\Rightarrow$(ii).

(iv)$\Rightarrow$(i). The proof is the same as (ii)$\Rightarrow$(iii).

2.3. Coseparable coalgebras.

Let C be a coalgebra. For every right C-comodule V, we have $\mathrm{Com}_C(V, C) \cong V^*$, by Proposition 3. If in addition V is a (C, C)-bicomodule then we have an isomorphism

$$\begin{array}{ccc} \mathrm{Com}_{C,C}(V, C) & \cong & \{\gamma\in V^* \mid (I\otimes\gamma)\rho^- = (\gamma\otimes I)\rho^+\}, \\ \tau & \longrightarrow & \varepsilon\tau \\ (I\otimes\gamma)\rho^- & \longleftarrow & \gamma\,. \end{array}$$

A coalgebra C is called *coseparable* if there exists a k-map $\tau: C\otimes C\to k$ such that $(I\otimes\tau)(\varDelta\otimes I)=(\tau\otimes I)(I\otimes\varDelta)$ and $\tau\varDelta=\varepsilon$. We have immediately from the above isomorphism that C is coseparable if and only if there exists a (C, C)-bicomodule map $\pi: C\otimes C\to C$ such that $\pi\varDelta=I$. We note that $\varDelta$ may be viewed as a (C, C)-bicomodule map from C to $C\otimes C$. Thus we may conclude that C is coseparable if and only if C is injective as a $C\otimes C^{op}$-comodule.

Let C and D be coalgebras and let $\tau: C\otimes C\to k$ be a k-map such that $(I\otimes\tau)(\varDelta\otimes I)=(\tau\otimes I)(I\otimes\varDelta)$. For any (C, D)-bicomodule M, N and for each $f\in \mathrm{Com}_D(M, N)$, we define a k-map

$$f_C: M\to N$$

by setting $f_C=(\tau\otimes I)(I\otimes\rho_N^-)(I\otimes f)\rho_M^-$.

LEMMA 4. *In the above situation, f_C is a (C, D)-bicomodule map.*

PROOF. We can construct the following commutative diagram:

$$\begin{array}{ccccccccc} M & \xrightarrow{\ \rho\ } & C\otimes M & \xrightarrow{\ I\otimes f\ } & C\otimes N & \xrightarrow{\ I\otimes\rho\ } & C\otimes C\otimes N & \xrightarrow{\ \tau\otimes I\ } & N \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle I\otimes\rho} & & \downarrow{\scriptstyle I\otimes\rho} & & \downarrow{\scriptstyle I\otimes I\otimes\rho} & & \downarrow{\scriptstyle \rho} \\ M\otimes D & \xrightarrow{\ \rho\otimes I\ } & C\otimes M\otimes D & \xrightarrow{\ I\otimes f\otimes I\ } & C\otimes N\otimes D & \xrightarrow{\ I\otimes\rho\otimes I\ } & C\otimes C\otimes N\otimes D & \xrightarrow{\ \tau\otimes I\otimes I\ } & N\otimes D\,. \end{array}$$

This shows that f_C is a right D-comodule map. We also have a commutative diagram:

$$\begin{array}{ccccccccc} M & \xrightarrow{\ \rho\ } & C\otimes M & \xrightarrow{\ I\otimes f\ } & C\otimes N & \xrightarrow{\ I\otimes\rho\ } & C\otimes C\otimes N & \xrightarrow{\ \tau\otimes I\ } & N \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \varDelta\otimes I} & & \downarrow{\scriptstyle \varDelta\otimes I} & & & & \downarrow{\scriptstyle \rho} \\ C\otimes M & \xrightarrow{\ I\otimes\rho\ } & C\otimes C\otimes M & \xrightarrow{\ I\otimes I\otimes f\ } & C\otimes C\otimes N & \xrightarrow{\ I\otimes I\otimes\rho\ } & C\otimes C\otimes C\otimes N & \xrightarrow{\ I\otimes\tau\otimes I\ } & C\otimes N\,. \end{array}$$

This shows that f_C is a left C-comodule. Q. E. D.

LEMMA 5. *Let L, M, N and P be (C, D)-bicomodules. For each $g \in \mathrm{Com}_{C,D}(L, M)$, $f \in \mathrm{Com}_D(M, N)$ and $h \in \mathrm{Com}_{C,D}(N, P)$, we have $(hfg)_C = hf_C g$.*

PROOF.
$$(hfg)_C = (\tau \otimes I)(I \otimes \rho_P^-)(I \otimes hfg)\rho_L^-$$
$$= (\tau \otimes I)(I \otimes I \otimes h)(I \otimes \rho_N^- f)(I \otimes g)\rho_L^-$$
$$= h(\tau \otimes I)(I \otimes \rho_N^- f)\rho_M^- g = hf_C g\,.$$
Q. E. D.

LEMMA 6. *Let C be a coseparable coalgebra. Let M and N be (C, D)-bicomodules. If $f: M \to N$ is a (C, D)-bicomodule map, then $f_C = f$.*

PROOF.
$$f_C = (\tau \otimes I)(I \otimes \rho_N^- f)\rho_M^-$$
$$= (\tau \otimes I)(I \otimes I \otimes f)(I \otimes \rho_M^-)\rho_M^-$$
$$= f(\tau \otimes I)(\Delta \otimes I)\rho_M^-$$
$$= f(\varepsilon \otimes I)\rho_M^- = f\,.$$
Q. E. D.

PROPOSITION 12. *If C is a coseparable coalgebra and D is a co-semi-simple coalgebra then $C \otimes D$ is a co-semi-simple coalgebra.*

PROOF. It suffices to prove that every (C, D)-bicomodule M is completely reducible. Let N be a (C, D)-subcomodule of M. Since D is co-semi-simple, there exists a D-comodule map $f: M \to N$ such that $fi = I$, where $i: N \to M$ is the inclusion. We then have $f_C i = I$, by Lemma 4 and 5. Since f_C is a (C, D)-bicomodule map, it follows that N is a direct summand of M as a (C, D)-bicomodule. Q. E. D.

COROLLARY. *If a coalgebra C is coseparable then it is co-semi-simple.*

§ 3. Cohomology.

Since ${}^C\mathbf{M}$ is an abelian category and has enough injectives, we can define the functor $\mathrm{Ext}_C^n(M, N)$ as the n-th right derived functor of the functor $\mathrm{Com}_C(-, N)$. Explicitly, we take an injective resolution $\mathbf{X}$ of a left C-comodule N:

$$0 \longrightarrow N \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots\,.$$

Then $\mathrm{Ext}_C^n(M, N)$ is defined as the n-th cohomology group of the complex $\mathrm{Com}_C(M, \mathbf{X})$.

3.1. Cohomology of coalgebras.

Let C be a coalgebra and N be a (C, C)-bicomodule. Let $C^e = C \otimes C^{op}$ be the enveloping coalgebra of C. Then we regard N as a left (or right) C^e-comodule. In particular we regard C as a left C^e-comodule. Now we define the n-th *cohomology group of C with coefficients in N* as

$$\mathrm{H}^n(N, C)=\mathrm{Ext}^n_{C^e}(N, C).$$

Thus we have $\mathrm{H}^n(N, C)=\mathrm{H}^n(\mathrm{Com}_{C,C}(N, \mathbf{X}))$, where $\mathbf{X}$ is an injective resolution of C as a left C^e-comodule. On the other hand, consider the complex $N\square_{C^e}\mathbf{X}$ and we define another n-th cohomology group as

$$\mathrm{Hoch}^n(N, C)=\mathrm{H}^n(N\square_{C^e}\mathbf{X}).$$

We note that if N is finite dimensional then $\mathrm{H}^n(N, C)\cong\mathrm{Hoch}^n(N^*, C)$.

Next we shall describe a construction of a standard complex. For each integer $n\geqq -1$, let $S^n(C)$ denote the $(n+2)$-fold tensor product of C. We convert $S^n(C)$ into a (C, C)-bicomodule by setting $\rho^-(c_0\otimes c_1\otimes\cdots\otimes c_{n+1})=\Delta(c_0)\otimes c_1\otimes\cdots\otimes c_{n+1}$ and $\rho^+(c_0\otimes c_1\otimes\cdots\otimes c_{n+1})=c_0\otimes\cdots\otimes c_n\otimes\Delta(c_{n+1})$. Clearly $S^n(C)$ is injective as a left C^e-comodule. We now define for each $n\geqq 0$ a C^e-comodule map

$$d^n: S^n(C)\to S^{n+1}(C)$$

by $d^n(c_0\otimes c_1\otimes\cdots\otimes c_{n+1})=\sum_{i=0}^{n+1}(-1)^i c_0\otimes\cdots\otimes\Delta(c_i)\otimes\cdots\otimes c_{n+1}$. We define for each $n\geqq 1$ a right C-comodule map

$$s^n: S^n(C)\to S^{n-1}(C)$$

by $s^n(c_0\otimes c_1\otimes\cdots\otimes c_{n+1})=\varepsilon(c_0)c_1\otimes\cdots\otimes c_{n+1}$. One verifies directly that

$$s^{n+1}d^n+d^{n-1}s^n=I \quad (n\geqq 1).$$

This shows that

$$C=S^{-1}(C)\xrightarrow{\Delta}S^0(C)\xrightarrow{d^0}S^1(C)\xrightarrow{d^1}\cdots$$

is an injective resolution of C as a left C^e-comodule. We observe that $S^0(C)=C\otimes C$ coincides with $C^e=C\otimes C^{op}$ as a C^e-comodule. More generally we have $S^n(C)\cong C^e\otimes C^{[n]}$ as a C^e-comodule, where $C^{[n]}$ is the n-fold tensor product of C for each $n>0$, and $C^{[0]}=k$.

In computing the cohomology groups we use the identifications:

$$\mathrm{Com}_{C^e}(N, S^n(C))=\mathrm{Com}_{C^e}(N, C^e\otimes C^{[n]})=\mathrm{Hom}(N, C^{[n]})$$

$$N\square_{C^e}S^n(C)=N\square_{C^e}(C^e\otimes C^{[n]})=N\otimes C^{[n]}.$$

Thus $\mathrm{H}^n(N, C)$ are the cohomology groups of the complex $\{\mathrm{Hom}(N, C^{[n]})\}_{n\geqq 0}$ with differentiation

$$\delta^n: \mathrm{Hom}(N, C^{[n]})\to\mathrm{Hom}(N, C^{[n+1]})$$

by

$$\delta^n(f)=(I\otimes f)\rho_N^- -(\Delta\otimes I\otimes\cdots)f+(I\otimes\Delta\otimes\cdots)f$$
$$-\cdots\pm(I\otimes\cdots\otimes\Delta)f\mp(f\otimes I)\rho_N^+.$$

And $\mathrm{Hoch}^n(N, C)$ are the cohomology groups of the complex $\{N\otimes C^{[n]}\}_{n\geq 0}$ with differentiation

$$D^n: N\otimes C^{[n]}\to N\otimes C^{[n+1]}$$

by
$$D^n(v\otimes c_1\otimes\cdots\otimes c_n)=\rho^+(v)\otimes c_1\otimes\cdots\otimes c_n$$
$$+\sum_{i=1}^{n}(-1)^i v\otimes c_1\otimes\cdots\otimes\Delta(c_i)\otimes\cdots\otimes c_n$$
$$+(-1)^{n+1}\sum v_{(0)}\otimes c_1\otimes\cdots\otimes c_n\otimes v_{(-1)},$$

where we write $\rho^-(v)=\sum v_{(-1)}\otimes v_{(0)}\in C\otimes N$. We obtain that

$$\mathrm{H}^0(N, C)=\{\gamma\in N^*\mid (I\otimes\gamma)\rho^-=(\gamma\otimes I)\rho^+\}\cong\mathrm{Com}_{C,C}(N, C)$$

and
$$\mathrm{Hoch}^0(N, C)=\{n\in N\mid t\rho^-(n)=\rho^+(n)\}.$$

A k-map $f: N\to C$ from a (C, C)-bicomodule N into C with the property $\Delta f=(I\otimes f)\rho^-+(f\otimes I)\rho^+$ is called a *coderivation* from N into C. The coderivation f is called an *inner* coderivation provided that there exists a $\gamma\in N^*$ such that $f=(I\otimes\gamma)\rho^--(\gamma\otimes I)\rho^+$. Thus we have an exact sequence

$$0\longrightarrow\mathrm{H}^0(N, C)\longrightarrow N^*\longrightarrow\mathrm{Coder}(N, C)\longrightarrow\mathrm{H}^1(N, C)\longrightarrow 0,$$

where $\mathrm{Coder}(N, C)$ denotes the k-space of all coderivations from N into C.

We now introduce a universal coderivation. Let L be the cokernel of $\Delta: C\to C\otimes C$. Then we have an exact sequence of (C, C)-bicomodules

$$0\longrightarrow C\xrightarrow{\Delta} C\otimes C\xrightarrow{\omega} L\longrightarrow 0.$$

We denote $c\circ c'=\omega(c\otimes c')$ and we define a map

$$\lambda: L\to C$$

by $\lambda(c\circ c')=c\varepsilon(c')-\varepsilon(c)c'$. It is easily checked that λ is a coderivation from L into C. Moreover λ is a universal coderivation in the following sense:

PROPOSITION 13. *For any (C, C)-bicomodule N, the map*

$$\mathrm{Com}_{C,C}(N, L)\longrightarrow\mathrm{Coder}(N, C)$$

sending σ to $\lambda\sigma$, is a k-isomorphism.

PROOF. Let $f\in\mathrm{Coder}(N, C)$. Then $\omega(f\otimes I)\rho_N^+\in\mathrm{Com}_{C,C}(N, L)$. For any $n\in N$, we have

$$\lambda\omega(f\otimes I)\rho_N^+(n)=\sum\lambda\omega(f(n_{(0)})\otimes n_{(1)})$$
$$=\sum f(n_{(0)})\varepsilon(n_{(1)})-\sum\varepsilon(f(n_{(0)}))n_{(1)}=f(n),$$

since $\varepsilon f=0$ for any coderivation f. Hence we have $\lambda\omega(f\otimes I)\rho_N^+=f$.

Conversely, let $\sigma \in \mathrm{Com}_{C,C}(N, L)$. Then we have $\omega(\lambda\sigma\otimes I)\rho_N^+=\omega(\lambda\otimes I)\rho_L^+\sigma=\sigma$, since $\omega(\lambda\otimes I)\rho_L^+=I$. Thus the correspondence $\sigma\to\lambda\sigma$ gives a k-isomorphism, and this completes the proof.

THEOREM 3. *The following statements concerning a coalgebra C are equivalent:*

(i) *C is coseparable.*

(ii) *For every (C, C)-bicomodule N, we have* $\mathrm{H}^n(N, C)=\{0\}$ *for all* $n\geqq 1$.

(iii) *Every coderivation from any (C, C)-bicomodule into C is an inner coderivation.*

(iv) $\lambda: L\to C$ *is an inner coderivation.*

PROOF. (i)⇒(ii) is immediate from the fact that a coseparable coalgebra C is injective as a C^e-comodule. (ii)⇒(iii) and (iii)⇒(iv) are obvious. Now we prove (iv)⇒(i). Suppose that λ is inner. Then there exists a $\gamma\in L^*$ such that $\lambda=(I\otimes\gamma)\rho_L^- -(\gamma\otimes I)\rho_L^+$. We define a (C, C)-bicomodule map $\xi: L\to C\otimes C$ by $\xi=(I\otimes\gamma\otimes I)(\rho_L^-\otimes I)\rho_L^+$. Then we have

$$
\begin{aligned}
\xi&=((\lambda+(\gamma\otimes I)\rho_L^+)\otimes I)\rho_L^+\\
&=(\lambda\otimes I)\rho_L^+ +(\gamma\otimes I\otimes I)(I\otimes\Delta)\rho_L^+\\
&=(\lambda\otimes I)\rho_L^+ +\Delta(\gamma\otimes I)\rho_L^+ .
\end{aligned}
$$

Hence we have $\omega\xi=\omega(\lambda\otimes I)\rho_L^+=I$. This means that the exact sequence

$$0\longrightarrow C\xrightarrow{\ \Delta\ } C\otimes C\xrightarrow{\ \omega\ } L\longrightarrow 0,$$

splits as a (C, C)-bicomodule. Therefore we have that C is coseparable, and the theorem is completely proved.

3.2. Extensions of coalgebras.

Let C be a coalgebra. An *extension* of C is any coalgebra D which contains C as a subcoalgebra.

Now we consider an extension D of C with $D=C\wedge C$, i. e. $\Delta(D)\subset D\otimes C+C\otimes D$ (see Sweedler [9], p. 179). In this case we may regard the quotient space $\bar{D}=D/C$ as a (C, C)-bicomodule by

$$\rho^+: D\xrightarrow{\ \Delta\ } D\otimes D\xrightarrow{\ p\otimes I\ }\bar{D}\otimes D$$

$$\rho^-: D\xrightarrow{\ \Delta\ } D\otimes D\xrightarrow{\ I\otimes p\ } D\otimes\bar{D}$$

where $p: D\to\bar{D}$ denotes the natural projection, since we have $\mathrm{Im}\,\rho^+\subset\bar{D}\otimes C$ and $\mathrm{Im}\,\rho^-\subset C\otimes\bar{D}$.

Let ϕ be a k-map of $D\to C$ such that $\phi|C=$identity. We then have that the following diagrams are commutative:

$$\begin{array}{ccc} D & \xrightarrow{\Delta} & D\otimes D \\ \downarrow p & & \downarrow p\otimes\phi \\ \bar{D} & \xrightarrow{\rho^{+}} & \bar{D}\otimes C \end{array} \qquad\qquad \begin{array}{ccc} D & \xrightarrow{\Delta} & D\otimes D \\ \downarrow p & & \downarrow \phi\otimes p \\ \bar{D} & \xrightarrow{\rho^{-}} & C\otimes\bar{D} \end{array} .$$

Define a map $f: D\to C\otimes C$ by setting

$$f=(\phi\otimes\phi)\Delta-\Delta\phi .$$

Then $f(C)=0$ and thus f induces a k-map $\bar{f}: \bar{D}\to C\otimes C$ with $\bar{f}p=f$.

LEMMA 6. *$\bar{f}$ is a 2-cocycle in* $\mathrm{Hom}(\bar{D}, C^{[2]})$.

PROOF. We compute

$$\begin{aligned} \delta^{2}(\bar{f})p &=(I\otimes\bar{f})\rho^{-}p-(\Delta\otimes I)\bar{f}p+(I\otimes\Delta)\bar{f}p-(\bar{f}\otimes I)\rho^{+}p \\ &=(\phi\otimes f)\Delta-(\Delta\otimes I)f+(I\otimes\Delta)f-(f\otimes\phi)\Delta \\ &=\{(\phi\otimes\phi\otimes\phi)(I\otimes\Delta)\Delta-(\phi\otimes\Delta\phi)\Delta\}-\{(\Delta\phi\otimes\phi)\Delta-(\Delta\otimes I)\Delta\phi\} \\ &\quad+\{(\phi\otimes\Delta\phi)\Delta-(I\otimes\Delta)\Delta\phi\}-\{(\phi\otimes\phi\otimes\phi)(\Delta\otimes I)\Delta+(\Delta\phi\otimes\phi)\Delta\} \\ &=0 . \end{aligned}$$

Since p is surjective we have $\delta^{2}(\bar{f})=0$. Q. E. D.

Let ϕ_1 and ϕ_2 be k-maps of $D\to C$ such that $\phi_1|C=\phi_2|C=$identity. Construct the maps $\bar{f}_1$ and $\bar{f}_2$ as above.

LEMMA 7. *$\bar{f}_1$ and $\bar{f}_2$ are cohomologous.*

PROOF. Let $g=\phi_1-\phi_2$. Since $g(C)=0$, g induces a k-maps $\bar{g}: \bar{D}\to C$ with $\bar{g}p=g$. Then

$$\begin{aligned} \delta^{1}(\bar{g})p &=(I\otimes\bar{g})\rho^{-}p-\Delta\bar{g}p+(\bar{g}\otimes I)\rho^{+}p \\ &=(\phi_1\otimes g)\Delta-\Delta g+(g\otimes\phi_1)\Delta . \end{aligned}$$

This implies that

$$\begin{aligned} \bar{f}_2p &=(\phi_2\otimes\phi_2)\Delta-\Delta\phi_2 \\ &=((\phi_1-g)\otimes(\phi_1-g))\Delta-\Delta(\phi_1-g) \\ &=\{(\phi_1\otimes\phi_1)\Delta-\Delta\phi_1\}-\{(\phi_1\otimes g)-\Delta g-(g\otimes\phi_1)\Delta\} \\ &=\bar{f}_1p-\delta^{1}(\bar{g})p . \end{aligned}$$

Therefore $\bar{f}_2=\bar{f}_1-\delta^{1}(\bar{g})$, and this shows that $\bar{f}_1$ and $\bar{f}_2$ are cohomologous.

Q. E. D.

Summarizing, we find that an extension D of C with $D=C\wedge C$ defines uniquely an element $[\bar{f}]=$class of $\bar{f}$, in $\mathrm{H}^{2}(\bar{D}, C)$.

THEOREM 4. *Let D be an extension of a coalgebra C with $D=C\wedge C$. Then we have that $[\bar{f}]=0$ in $\mathrm{H}^2(\bar{D}, C)$ if and only if there exists a coalgebra map $\phi: D\rightarrow C$ such that $\phi|C=I$.*

PROOF. Suppose that $[\bar{f}]=0$. Let ϕ be a k-map of $D\rightarrow C$ such that $\phi|C=I$. $\bar{f}$ can be viewed as the 2-cocycle associated with ϕ. Since $[\bar{f}]=0$ there exists a $\bar{g}\in\mathrm{Hom}(\bar{D}, C)$ such that $\bar{f}=\delta^1(\bar{g})$. Set $\phi'=\phi-\bar{g}p$. Then ϕ' is a k-map of $D\rightarrow C$ such that $\phi'|C=I$. Let $\bar{f}'$ be the 2-cocycle associated with ϕ'. The proof of Lemma 7 then implies that

$$\bar{f}'=\bar{f}-\delta^1(\bar{g})=\bar{f}-\bar{f}=0\,,$$

that is, ϕ' is a coalgebra map.

The "if" part of the assertion is clear. Q. E. D.

REMARK. More generally we can show that the second cohomology group $\mathrm{H}^2(M, C)$ for a (C, C)-bicomodule M is in one-to-one correspondence with the set of equivalence classes of extensions over C with cokernel M

$$C \xrightarrow{i} D \xrightarrow{p} M$$

(that is, D is a coalgebra, i is an injective coalgebra map, $i(C)\wedge i(C)=D$, p is a surjective k-map which induces $D/i(C)\cong M$ as a (C, C)-bicomodule.). Two extensions $C \xrightarrow{i} D \xrightarrow{p} M$ and $C \xrightarrow{i'} D' \xrightarrow{p'} M$ over C with cokernel M are *equivalent* if there exists a coalgebra isomorphism $f: D\rightarrow D'$ such that the diagram

$$\begin{array}{ccccc} & \overset{i}{\nearrow} & D & \overset{p}{\searrow} & \\ C & & \downarrow f & & M \\ & \underset{i'}{\searrow} & D' & \underset{p'}{\nearrow} & \end{array}$$

is commutative.

THEOREM 5 (Sullivan [8]). *For C a coalgebra with coseparable coradical R, there exists a coalgebra map $\phi: C\rightarrow R$ such that $\phi|R=I$.*

PROOF. C has a filtration by subcoalgebras $R=C_0\subset C_1\subset\cdots$ where $C_i=\bigwedge^{i+1}R$ ($i=0, 1, 2, \cdots$). Thus it is enough to construct a sequence $\phi_0, \phi_1, \cdots$ such that ϕ_i is a coalgebra map of $C_i\rightarrow R$ and $\phi_i|C_{i-1}=\phi_{i-1}$, for all $i\geqq 1$. For since $C=\bigcup C_i$ there is a unique coalgebra map $\phi: C\rightarrow R$ which extends all the ϕ_i. It is clear $\phi|R=I$, therefore all is good.

To construct the sequence, assume inductively that we have $\phi_0, \phi_1, \cdots, \phi_n$ for some fixed $n\geqq 1$. Let J_n denote the kernel of ϕ_n. C_{n+1}/J_n can be viewed as an extension coalgebra of C_n/J_n. Then it is easily checked that $C_{n+1}/J_n=C_n/J_n\wedge C_n/J_n$ and $C_n/J_n\cong R$. It follows from Theorem 3 and Theorem 4 that there exists a coalgebra map $f: C_{n+1}/J_n\rightarrow C_n/J_n$ with $f|(C_n/J_n)=I$. Now we define a coalgebra map $\phi_{n+1}: C_{n+1}\rightarrow R$ by the composite

$$C_{n+1} \xrightarrow{\text{proj.}} C_{n+1}/J_n \xrightarrow{f} C_n/J_n \cong R\,.$$

Then we have $\phi_{n+1}|C_n=\phi_n$, and this completes the proof.

REMARK. Given two coalgebra maps ϕ and ϕ' with $\phi|R=\phi'|R=$identity, we can find a relation between ϕ and ϕ'. In fact, C becomes a (R, R)-bicomodule by

$$\rho^- : C \xrightarrow{\Delta} C\otimes C \xrightarrow{\phi\otimes I} R\otimes C$$

$$\rho^+ : C \xrightarrow{\Delta} C\otimes C \xrightarrow{I\otimes\phi'} C\otimes R\,.$$

Since R is a (R, R)-subcomodule of C, C/R is an (R, R)-bicomodule. Since $\phi|R=\phi'|R$, $\phi-\phi' : C\to R$ induces a k-map $\overline{\phi-\phi'} : C/R\to R$. Then it is easy to show that $\overline{\phi-\phi'}$ is a coderivation from a (R, R)-bicomodule C/R into R. It follows from Theorem 3 that there exists an element γ in $(C/R)^*$ such that $\delta^0(\gamma)=\overline{\phi-\phi'}$. Rewriting this equation, we have

$$(\phi\otimes\gamma p)\Delta-(\gamma p\otimes\phi')\Delta=\phi-\phi'$$

where $p : C\to C/R$ denotes the natural projection. Set $d^*=\varepsilon-\gamma p$ (in C^*). Then we obtain

$$\phi(d^*\rightharpoonup c)=\phi'(c\leftharpoonup d^*) \quad \text{for all } c\in C\,.$$

3.3. Cohomology of augmented coalgebras.

Let (C, u) be an augmented coalgebra (see 2.2). Then k has a left C-comodule structure, and cohomology groups $\mathrm{Ext}^n_C(N, k)$ are defined for every left C-comodule N.

THEOREM 6. *For every left C-comodule N, we have*

$$\mathrm{Ext}^n_C(N, k)\cong \mathrm{H}^n(N_u, C)\,.$$

PROOF. We apply Proposition 6 to obtain that for every (C, C)-bicomodule V,

$$\mathrm{Com}_C(N, V\,\square_C\,k)\cong \mathrm{Com}_{C,C}(N_u, V)\,.$$

Therefore it suffices to show that the complex $\{X^n\,\square_C\,k\}$ is an injective resolution of k as a left C-comodule, for each injective resolution of C as a C^e-comodule;

$$C \longrightarrow X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots\,.$$

Taking $V=X^n$ in the above isomorphism, we obtain that $X^n\,\square_C\,k$ is injective as a left C-comodule. Now let $Z^n=\mathrm{Ker}\,d^n=\mathrm{Im}\,d^{n-1}$ $(n\geqq 1)$. Then we have the exact sequences of (C, C)-bicomodules;

$$0 \longrightarrow C \longrightarrow X^0 \longrightarrow Z^1 \longrightarrow 0$$
$$0 \longrightarrow Z^n \longrightarrow X^n \longrightarrow Z^{n+1} \longrightarrow 0 \quad (n \geqq 1).$$

Since C and X^0 are injective as a left C-comodule, so is clearly Z^1. It follows by induction that Z^n $(n \geqq 1)$ is injective as a left C-comodule, since from Proposition 2, (1) X^n $(n \geqq 0)$ is injective as a left C-comodule. Therefore we have the exact sequences;

$$0 \longrightarrow Z^n \square_C k \longrightarrow X^n \square_C k \longrightarrow Z^{n+1} \square_C k \longrightarrow 0.$$

This shows that the complex $\{X^n \square_C k\}$ is an injective resolution of k as a left C-comodule, and completes the proof.

REMARK. Similarly, we can show for every right C-comodule M that $\mathrm{Hoch}^n({}_uM, C)$ coincides with the n-th cohomology group $\mathrm{H}^n(M \square_C X)$, where X is an injective resolution of k as a left C-comodule, since we have that for every (C, C)-bicomodule V,

$$M \square_C (V \square_C k) \cong {}_uM \square_{C^e} V.$$

Now consider the particular case when C is a Hopf algebra. We define a k-map

$$\nabla : C^e = C \otimes C^{op} \longrightarrow C$$

by setting $\nabla(c \otimes d^{op}) = cS(d)$, where S is the antipode of C. It is easily verified that ∇ is a coalgebra map. Given a (C, C)-bicomodule N we shall denote by ${}_\nabla N$ (or N_∇) the k-space N regarded as a left (or right) C-comodule by means of the map ∇. In particular $(C^e)_\nabla$ is a (C^e, C)-bicomodule. Assume that C is involutory, i.e. S^2=identity. Then the map

$$\alpha : (C^e)_\nabla \longrightarrow C \otimes C$$

defined by setting $\alpha(c \otimes d^{op}) = \sum c_{(1)} \otimes c_{(2)} S(d)$ is a right C-comodule isomorphism, where $C \otimes C$ regarded as a right C-comodule by $\rho : C \otimes C \xrightarrow{I \otimes \Delta} C \otimes C \otimes C$. The inverse of α is given by $x \otimes y \to \sum x_{(1)} \otimes (S(y) x_{(2)})^{op}$. Therefore $(C^e)_\nabla$ is free as a right C-comodule. It follows that for each injective resolution of k as a left C-comodule, $k \to X^0 \to X^1 \to \cdots$, we have an exact sequence

$$(C^e)_\nabla \square_C k \longrightarrow (C^e)_\nabla \square_C X^0 \longrightarrow \cdots. \qquad (*)$$

Moreover $(C^e)_\nabla \square_C X^n$ $(n \geqq 1)$ is injective as a left C^e-comodule, by Corollary of Proposition 5. Since $(C^e)_\nabla \square_C k \cong C$, it follows that the sequence $(*)$ is an injective resolution of C as a left C^e-comodule. Thus we have:

THEOREM 7. *Let C be an involutory Hopf algebra. For every (C, C)-bicomodule N, we have*

$$\mathrm{Ext}^n_C({}_\nabla N, k) \cong \mathrm{H}^n(N, C).$$

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