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Immersions of Lie groups

Dedicated to the late Hsien-Chung Wang

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§ 1. Introduction.

Let G and L be topological groups. A group-homomorphism $f: G \rightarrow L$ is said to be an *immersion* if f is one-one and continuous. When the image $f(G)$ is dense in L the immersion f is called *dense*, and if f is a homeomorphism to $f(G)$ we shall call f an *imbedding*.

In this paper we are mainly interested in the case when G is an analytic group (=connected Lie group). First suppose that L is also a Lie group. Immersions of this kind have been studied extensively since Yosida [**19**], 1937, in which he proved that any (finite-dimensional) irreducible faithful representation of an analytic group is an imbedding. In particular, A. Malcev in [**14**], 1945, proved the following theorem.

THEOREM A. *Let G and L be analytic groups, and let $f: G \rightarrow L$ be a dense immersion. Then there exists a one-parameter subgroup (=analytic subgroup of dimension one) A of G such that*

$$L=\overline{f(A)}f(G).$$

Theorem A was also obtained in Goto [**4**], and related subjects to this theorem have been discussed in Hochschild [**11**], Djoković [**2**] and others.

Next in [**17**], 1951, van Est defined an analytic group G to be a *(CA)-group* if the group $\mathrm{Ad}(G)$ of all inner automorphisms of G is closed in the group $Aut(G)$ composed of all bicontinuous automorphisms of G, and proved the following theorem among other things:

THEOREM B. *Let G be a (CA)-group with center Z, and let L be a Lie group. If $f: G \rightarrow L$ is an immersion, then*

(i) $\overline{f(G)}=\overline{f(Z)}f(G)$.

(ii) *If $f|Z$ is an imbedding, then f is an imbedding.*

It is easy to see that (i) implies (ii), which extends some results in Yosida [**20**] and Goto [**4**]. Immersions into a more general topological group have been studied by Goto, Gleason-Palais, Lee-Wu, Ōmori, Zerling and so on. In particular, Ōmori in [**16**], 1966, generalized some part of Theorem B, and the result was supplemented by the author in [**6**] as follows:

THEOREM C. *For an analytic group G, the following conditions are all equivalent.*

1) *G is a (CA)-group with compact center.*

2) *Every immersion of G into a topological group is an imbedding.*

3) *Every immersion of G into a Lie group is an imbedding.*

4) *For every immersion f of G, the image $f(G)$ is closed.*

We shall call G *absolutely closed* if the equivalent conditions in Theorem C are satisfied.

The main purpose of this paper is to extend Theorem A for an arbitrary topological group L, by developing the theory of immersions in a systematic manner. The first step to our goal is to establish the following theorem, which was first given by Yosida in [**20**] for a Lie group L.

THEOREM 1. *Let G be an analytic group, L a topological group, and let $f: G \to L$ be a dense immersion.*

(i) *If N is a normal analytic subgroup of G, then $f(N)$ is a normal subgroup of L.*

(ii) *We put $\gamma(a)x = f^{-1}(af(x)a^{-1})$ for $a \in L$, $x \in G$. Then $\gamma(a) \in Aut(G)$, and the homomorphism*

$$L \ni a \mapsto \gamma(a) \in Aut(G)$$

is continuous.

Let $\mathfrak{g}$ be the Lie algebra of G. Then $Aut(G)$ can be identified with a closed subgroup of the general linear group $GL(\mathfrak{g})$. Hence we shall call γ the *canonical representation* of L (with respect to the immersion f). The canonical representation γ extends the adjoint representation of G, and the kernel of γ coincides with the center of L.

Next, in case $L \subset GL(n, \boldsymbol{R})$ in Theorem A, the result was strongly sharpened by the author in [**7**], 1973. Here we shall improve it further as follows:

THEOREM 2. *Let G be an analytic group, and let $f: G \to GL(n, \boldsymbol{R})$ be an immersion. Then we can find a closed subgroup $V \cong \boldsymbol{R}^r$ of G and a closed normal subgroup N of G such that G is a semi-direct product*

$$G = VN, \quad V \cap N = \{e\},$$

$\overline{f(V)}$ is a torus, $f(N)$ is closed, and $\overline{f(G)}$ is a semi-direct product of $\overline{f(V)}$ and $f(N)$:

$$\overline{f(G)} = \overline{f(V)} f(N), \quad \overline{f(V)} \cap f(N) = \{1\}.$$

Roughly speaking, Theorem 2 insists that we can decompose $f(G)$ completely into a closed part and a non-closed part. Changing the notation, let G be an arbitrary analytic group. We shall explain an excellent idea of Zerling in [**21**] to find a semi-direct product decomposition of G, by applying Theorem 2 to the adjoint group $\mathrm{Ad}(\mathfrak{g})$.

Since $\mathrm{Ad}(\mathfrak{g})$ is an analytic subgroup of $GL(\mathfrak{g})$, we get a semi-direct product decomposition in Theorem 2:

$$\mathrm{Ad}(\mathfrak{g})=V'N', \quad V'\cap N'=\{1\}.$$

Denoting the adjoint representation by α, we put $\alpha^{-1}(N')=N$. Then we can find a suitable subgroup V such that

$$G=VN, \quad V\cap N=\{e\}, \quad \alpha(V)=V'.$$

Let us call the decomposition a *Z-decomposition* of G.

Notice that in order to prove Theorem A it suffices to find an abelian analytic subgroup B such that $L=\overline{f(B)}f(G)$. Indeed, then there is a one-parameter subgroup A of B with $\overline{f(B)}=\overline{f(A)}f(B)$. Also the statement of Theorem A is not true for $G=\boldsymbol{R}^2$ and for some compact group L, see Goto [5]. The author tried to find a "non-closed part" of G, i.e. an abelian analytic subgroup B with the property $\overline{f(G)}=\overline{f(B)}f(G)$ for *all* immersions f of G, and a solution was obtained as follows.

Let T' be a maximal torus in $\overline{\mathrm{Ad}(\mathfrak{g})}$. We call the subgroup $H=\alpha^{-1}(T'\cap\mathrm{Ad}(\mathfrak{g}))$ a *gm-torus* (*generalized maximal torus*) of G. It turns out that H is a closed connected abelian subgroup containing the center Z, a maximal torus T of G, and V in a suitable Z-decomposition of G. Furthermore all gm-tori are conjugated to each other with respect to inner automorphisms. For a gm-torus H and the unique maximal torus T in H, we take a vector part $v(H)$:

$$H=Tv(H), \quad T\cap v(H)=\{e\}, \quad v(H)\cong\boldsymbol{R}^s.$$

THEOREM 3. *Let G be an analytic group, H a gm-torus of G, and let $v(H)$ be a vector part of H.*

(i) *If L is a topological group, and $f: G\to L$ is a dense immersion, then $L=\overline{f(H)}f(G)=\overline{f(v(H))}f(G)$.*

(ii) *For an immersion $f: G\to L$, if $f|v(H)$ is an imbedding, then so is $f|H$ and f is an imbedding.*

(iii) *If $W\neq v(H)$ is an analytic subgroup of $v(H)$, then there exists a non-trivial dense immersion f from G into a suitable analytic group such that $f|W$ is an imbedding.*

We shall see that Theorem 3 extends not only Theorem A but also Theorem B. First notice that if G is a (CA)-group then H/Z is compact (and conversely), by definition of gm-torus. This implies easily that $\overline{f(H)}=\overline{f(Z)}f(H)$ and if $f|Z$ is an imbedding then $f|H$ is also an imbedding. Thus for a (CA)-

group G it is equivalent to replace H in Theorem 3 (i) and (ii) by Z, see (4.1).

Now it is easy to see that Theorem C is an extreme case of Theorem 3. Indeed, that G is (CA) and Z is compact implies that a gm-torus is a maximal torus and conversely.

As applications of Theorem 3, we give three theorems in the end of the paper.

Let G be an analytic group, and let L be a topological group. Let $f: G \to L$ be a dense immersion.

I. Let $G = G^1 \supset G^2 \supset G^3 \supset \cdots$ and $L = L^1 \supset L^2 \supset L^3 \supset \cdots$ be descending central series of the groups G and L, respectively. Then $G^k = L^k$ for $k = 2, 3, \cdots$.

II. If $\overline{f(v(H))}$ is locally compact, then so is L.

III. If $\overline{f(v(H))}$ is an analytic group, then so is L.

Some special case of II was discussed in Ōmori [**16**].

Problems. We shall list problems along the line.

P1. Characterize an abelian topological group (say with complete metric) which contains a dense one-parameter subgroup.

P2. Let G be an analytic group. Describe the set of all dense immersions of G into analytic groups in terms of fixed gm-torus H of G.

P3. Let G be an analytic group and W an abelian analytic subgroup of G, and suppose that for any immersion f of G, $\overline{f(G)} = \overline{f(W)} f(G)$, i.e. W is a "non-closed part" of G. Does W contain some $v(H)$?

In concluding the introduction, the author is pleased to acknowledge his gratitude to D. Zerling for all valuable suggestions and discussions during the preparation of the present paper.

Notation and Terminology.

If f is a map defined on A and B is a subset of A, then $f|B$ denotes the restriction of f into B.

For a subset C of a topological space, $\bar{C}$ denotes the closure of C.

A topological space is called *σ-compact* if it is a countable union of compact subsets.

Let L be a topological group. By an *automorphism* of L we shall mean an imbedding from L onto L. The group of all automorphisms of L is denoted by $Aut(L)$. We adopt the notation $\mathrm{Ad}(L)$ for the subgroup of $Aut(L)$ composed of all inner automorphisms.

The identity element in a general linear group, or an automorphism group will be denoted by 1. Otherwise, we let e denote the identity element of the group in question, unless specified otherwise.

For a subset D containing e of a topological group, the identity component ($=$connected component containing e) of D will be denoted by D^0.

A subset P of a topological group is said to be *symmetric* if $P = P^{-1} =$

$\{p^{-1}; p \in P\}$.

If there is an imbedding from a topological group L onto a topological group L^*, we adopt the notation $L \cong L^*$.

Unless specified otherwise, an analytic group and its Lie algebra shall be denoted by the same capital Roman and capital script letters, respectively.

Let $\boldsymbol{Z}$ denote the additive group of all integers with discrete topology. For $n=0, 1, 2, \cdots$, an analytic group $\cong \boldsymbol{R}^n$ (or $(\boldsymbol{R}/\boldsymbol{Z})^n$) is called a *vector group* (or a *torus*), respectively. For a connected, locally compact abelian group Q, a maximal vector group in Q is called a *vector part* of Q, and is denoted by $v(Q)$.

By a Lie algebra we shall always mean a Lie algebra of finite dimension over $\boldsymbol{R}$. For a Lie algebra $\mathcal{G}$, we denote the group of all automorphisms of $\mathcal{G}$ by $Aut(\mathcal{G})$. For X in $\mathcal{G}$, we adopt the notation $(\mathrm{ad}\, X)Y=[X, Y]$ for $Y \in \mathcal{G}$, and let $\mathrm{Ad}(\mathcal{G})$ denote the analytic subgroup of the Lie group $Aut(\mathcal{G})$ corresponding to the Lie algebra $\mathrm{ad}(\mathcal{G})=\{\mathrm{ad}\, X; X \in \mathcal{G}\}$. Let G be an analytic group, and $\mathcal{G}$ its Lie algebra. We identify $Aut(G)$ with a subgroup of $Aut(\mathcal{G})$, and $\mathrm{Ad}(G)$ with $\mathrm{Ad}(\mathcal{G})$ whenever it is convenient, see § 3. The adjoint representation $G \to \mathrm{Ad}(G) = \mathrm{Ad}(\mathcal{G})$ will be denoted by α.

§ 2. Lemmas on locally compact groups.

Many of the results in this section are more or less known. For the sake of convenience we shall give proofs for most of them.

(2.1) *Let L be a topological group, and let H be a subgroup of L. If H is locally compact (with respect to the topology as a subspace of L), then H is closed in L.*

PROOF. The closure $\bar{H}$ of H is also a subgroup. Because H is locally compact, H is an open subset of $\bar{H}$. On the other hand, an open subgroup of a topological group is closed. Hence H is closed in $\bar{H}$, i.e. $H=\bar{H}$. Q.E.D.

(2.2) *If a topological group L has a closed subgroup H such that both H and the factor space L/H are locally compact, then L is also locally compact.*

See Montgomery-Zippin [**15**], pp. 52-53.

(2.3) *Let G be a locally compact, locally connected group, G^* a topological group, and let $f: G \to G^*$ be a surjective immersion. Let $\mathcal{V}$ be a base of neighborhoods of the identity e^* of G^*. Then $\{(f^{-1}(V))^0; V \in \mathcal{V}\}$ is a base of neighborhoods of the identity e in G.*

PROOF. Let K be a compact neighborhood of e, and B the boundary of K. Since the underlying space of any topological group is regular, the set $\{\bar{V}; V \in \mathcal{V}\}$ is also a base. Then $\{f^{-1}(\bar{V}) \cap B; V \in \mathcal{V}\}$ is a family of closed sets in the compact space B, and

$$\bigcap_{V\in\mathcal{V}}(f^{-1}(\bar{V})\cap B)=(\bigcap_{V\in\mathcal{V}}f^{-1}(\bar{V}))\cap B=\{e\}\cap B=\emptyset\,.$$

Hence there exists a finite subset $\{V_1, V_2, \cdots, V_m\}$ of $\mathcal{V}$ such that $\bigcap_i f^{-1}(V_i)\cap B=\emptyset$. Picking $V_0\in\mathcal{V}$ with $V_0\subset V_1\cap\cdots\cap V_m$, we have $f^{-1}(V_0)\cap B=\emptyset$ and $(f^{-1}(V_0))^0\subset K-B$. Q. E. D.

(2.4) CATEGORY THEOREM. *Let G_1 and G_2 be locally compact, σ-compact groups, and G^* a topological group. If $f_1: G_1\to G^*$ and $f_2: G_2\to G^*$ are surjective immersions, then $f_2^{-1}\circ f_1$ is an imbedding.*

PROOF. Let U be a neighborhood of the identity in G_1. There exists a compact symmetric neighborhood K with $K^2\subset U$. Since G_1 is σ-compact, there is a countable subset $\{a_1, a_2, \cdots\}$ in G_1 such that $a_1K\cup a_2K\cup\cdots=G_1$. Because K is compact, so is $f_1(K)$, and $\varphi(K)$ is closed, where $\varphi=f_2^{-1}\circ f_1$. Hence each $\varphi(a_iK)$ is closed, and G_2 is a union of these closed sets. Then by the category argument, some of $\varphi(a_iK)$ contains an interior point, and so does $\varphi(K)$. Since K is symmetric, $\varphi(K)^2$ is a neighborhood of the identity in G_2. This proves that the map $\varphi=f_2^{-1}\circ f_1$ is open. In a similar way, $f_1^{-1}\circ f_2$ is open, and $f_2^{-1}\circ f_1$ is a homeomorphism. Q. E. D.

The category theorem (2.4) is due to Goto [**5**]. The following (2.5) has been known before (2.4).

(2.5) *Let G be a locally compact σ-compact group, and G^* a locally compact group. Any surjective immersion $f: G\to G^*$ is an imbedding.*

PROOF. By the assumption, G^* is also σ-compact. Put f_2=the identity map in (2.4). Q. E. D.

(2.6) *Let G be a locally compact, σ-compact group, and G^* a topological group. Let $f: G\to G^*$ be a surjective immersion. If ϕ is an automorphism of G^*, then $f^{-1}\circ\phi\circ f$ is an automorphism of G. That is $Aut(G^*)$ is naturally isomorphic with a subgroup of $Aut(G)$.*

PROOF. Both $\phi\circ f$ and f are immersions from G onto G^*, and we can apply (2.4). Q. E. D.

(2.7) *Let G be a locally compact, σ-compact group, L a topological group, and let $f: G\to L$ be an immersion. If K is a subgroup of G such that $f(K)$ is (locally) compact, then K is (locally) compact.*

PROOF. $K=f^{-1}(f(K))$ is a closed subgroup of G, by (2.1), and is locally compact and σ-compact. By (2.5) then, $f|K$ is an imbedding. Q. E. D.

A closed subgroup A of a topological group B is called *uniform* if the factor space B/A is compact.

(2.8) *Let G be a locally compact, σ-compact group and let A be a uniform subgroup of G. Let L be a topological group. If $f: G\to L$ is an immersion and if $f|A$ is an imbedding, then f is an imbedding.*

PROOF. As a closed subgroup of G, the group A is locally compact and so is $f(A)$. Hence $f(A)$ is a closed subgroup of L. The map $G \ni x \mapsto f(x)f(A) \in f(G)/f(A)$ is continuous and induces a continuous one-one map ξ from G/A onto $f(G)/f(A)$. Since G/A is compact, ξ is a homeomorphism, and $f(G)/f(A)$ is compact. Then by (2.2), $f(G)$ is locally compact, and by (2.5), f is an imbedding. Q. E. D.

(2.9) *Let N be a locally compact, σ-compact group, M a locally compact group, and let L be a topological group. Let $f: N \to L$ be a dense immersion, and let $\gamma: L \to M$ be a continuous homomorphism with $\gamma(f(N)) = M$. Let Z be the kernel of $\gamma \circ f$. Then the kernel of γ is $\overline{f(Z)}$ and*

$$L/\overline{f(Z)} \cong N/Z \cong f(N)/f(Z) \cong M.$$

PROOF. Let K denote the kernel of γ. By $\gamma(f(N)) = M$, we have $L = f(N)K$. Then the homomorphism

$$N \ni x \mapsto f(x)K \in L/K$$

is continuous and surjective, and induces a surjective immersion $\tilde{f}: N/Z \to L/K$. Also $\gamma: L \to M$ induces a surjective immersion $\tilde{\gamma}: L/K \to M$. Hence $\tilde{\gamma} \circ \tilde{f}$ is an immersion from N onto M and is an imbedding by (2.5). Hence $\tilde{\gamma}$ and $\tilde{f}$ are imbeddings. Next applying the result obtained to $L = f(N)$, we have

$$N/Z \cong f(N)/f(Z) \cong M.$$

Now it suffices to prove that $K = \overline{f(Z)}$. Let W be a neighborhood of e in L. Then $V = W \cap f(N)$ is a neighborhood of e in $f(N)$, and $\gamma(V)$ is a neighborhood of the identity in M. Hence we can find a neighborhood W' of e in L such that $\gamma(W') \subset \gamma(V)$ and $W' \subset W$. Let a be in K. We pick $x \in aW' \cap f(N)$. Then $\gamma(x) \in \gamma(aW') = \gamma(W') \subset \gamma(V)$. Hence $x \in f(Z)V \subset f(Z)W$ and $f(Z)W \cap aW \neq \emptyset$. Hence $a \in \overline{f(Z)}$. Q. E. D.

§ 3. Automorphism groups.

Let G be a locally compact group. For a compact subset K and a neighborhood U of e in G, we put

$$[K, U] = \{\varphi \in Aut(G);\ k^{-1}\varphi(k) \in U \text{ for all } k \in K\}.$$

(3.1) *If K is a compact connected subset of G containing e, then for any neighborhood U of e, we have*

$$[K, U] = [K, U^0].$$

PROOF. For a fixed φ in $Aut(G)$, the set $K^{\varphi-1} = \{k^{-1}\varphi(k);\ k \in K\}$ is connected and contains e. Hence if $K^{\varphi-1} \subset U$ then $K^{\varphi-1} \subset U^0$. Q. E. D.

(3.2) *Let G be a connected, locally compact group. We can introduce a* *topology in $Aut(G)$ such that $Aut(G)$ becomes a topological group, and for any compact neighborhood K of e, the totality of $[K, U]$ where U runs through a base of neighborhoods of e becomes a base of neighborhoods of 1 in $Aut(G)$.*

See e. g. Goto-Kimura [**9**]. After this, we assume that $Aut(G)$ is a topological group with this topology.

(3.3) *Let G be a connected, locally compact, locally connected group, L a topological group, and let $f: G \rightarrow L$ be an immersion. Suppose that $f(G)$ is a normal subgroup of L. Then defining*

$$\gamma(a)x = f^{-1}(af(x)a^{-1}) \qquad \text{for } a \in L,\ x \in G,$$

we have that $\gamma(a) \in Aut(G)$ and the homomorphism

$$\gamma : L \rightarrow Aut(G)$$

is continuous.

Proof. For a in L, the map $f(G) \ni y \mapsto aya^{-1}$ is an automorphism of the topological group $f(G)$. Since a connected, locally compact group is σ-compact, by (2.6), $\gamma(a)$ is an automorphism of G.

Next, in order to prove the continuity of γ, let us fix a compact connected neighborhood K of e in G. Let U be a neighborhood of e in G. By (2.3), there is a neighborhood V of the identity e^* in $f(G)$ such that $(f^{-1}(V))^0 \subset U$. Let us define a map $g: K \times L \rightarrow f(G)$ by

$$g(k, a) = f(k)^{-1} af(k)a^{-1} \quad \text{for } k \in K,\ a \in L.$$

Then g is continuous and $g(k, e^*) = e^*$ for all $k \in K$. Because K is compact we can find a neighborhood W of e^* in L such that $g(K, W) \subset V$. Then for a in W, $K^{\gamma(a)-1} = \{k^{-1}\gamma(a)k\,;\ k \in K\}$ is a connected set containing e and $K^{\gamma(a)-1} = f^{-1}(g(K, a)) \subset f^{-1}(g(K, W)) \subset f^{-1}(V)$. Hence we have $K^{\gamma(a)-1} \subset U$, and we get that $\gamma(W) \subset [K, U]$. Q. E. D.

Next, let G be an analytic group. Then the group $Aut(\mathfrak{g})$ of all automorphisms of the Lie algebra $\mathfrak{g}$ is a closed subgroup of the general linear group $GL(\mathfrak{g})$ and is a Lie group. For any φ in $Aut(G)$, there corresponds a unique $d\varphi$ in $Aut(\mathfrak{g})$ such that

$$\varphi(\exp X) = \exp(d\varphi X) \qquad \text{for } X \in G\,,$$

and the map $\varphi \mapsto d\varphi$ is an imbedding from $Aut(G)$ onto a closed subgroup in $Aut(\mathfrak{g})$. In particular, $Aut(G)$ is a Lie group. For X and Y in $\mathfrak{g}$, we adopt the notation

$$(\operatorname{ad} X)Y = [X, Y]\,.$$

Then $\{\operatorname{ad} X\,;\ X \in \mathfrak{g}\}$ forms a subalgebra of the Lie algebra of $Aut(\mathfrak{g})$, and the corresponding analytic subgroup is called the adjoint group of G (or $\mathfrak{g}$), and is denoted by $\mathrm{Ad}(\mathfrak{G})$. Then $\mathrm{Ad}(\mathfrak{G})$ is the image of $\mathrm{Ad}(G)$ in the imbedding $\varphi \mapsto d\varphi$. About $Aut(G)$ the reader may refer Chevalley [**1**] and Hochschild [**10**].

(3.4) *Let G be an analytic group, and let K be a compact neighborhood of e. Let $\{\varphi_1, \varphi_2, \cdots\}$ be a sequence in $Aut(G)$. If for any neighborhood U of e, there exists a natural number $n(U)$ such that*

$$\varphi_l^{-1}\varphi_m \in [K, U] \quad \textit{for } l \geqq n(U) \textit{ and } m \geqq n(U),$$

then the sequence $\{\varphi_n\}$ converges to a suitable $\varphi_0 \in Aut(G)$.

PROOF. First we shall see that $Aut(G)$ has a complete left-invariant metric. Notice that $Aut(G)$ can be identified with a closed subgroup of $GL(r, \boldsymbol{C})$ where $r = \dim G$. $GL(r, \boldsymbol{C})$ is an analytic group, and in any analytic group, a left-invariant Riemannian metric gives rise to a complete metric. Let d be a complete, left-invariant metric in $Aut(G)$, and for $\varepsilon > 0$ let $B(\varepsilon)$ denote the open ball of radius ε about 1 with respect to d.

For a given $\varepsilon > 0$, there exists a neighborhood U of e such that $[K, U] \subset B(\varepsilon)$. Then for $l, m \geqq n(U)$ we have $\varphi_l^{-1}\varphi_m \in B(\varepsilon)$, i. e. $d(\varphi_l, \varphi_m) = d(1, \varphi_l^{-1}\varphi_m) < \varepsilon$. Hence $\{\varphi_n\}$ is a Cauchy sequence. Q. E. D.

§ 4. Canonical representations.

THEOREM 1. *Let G be an analytic group, L a topological group, and let $f: G \to L$ be a dense immersion.*

(i) *If N is a normal analytic subgroup of G, then $f(N)$ is a normal subgroup of L.*

(ii) *We put $\gamma(a)x = f^{-1}(af(x)a^{-1})$ for $a \in L$, $x \in G$. Then $\gamma(a) \in Aut(G)$, and the homomorphism*

$$L \ni a \mapsto \gamma(a) \in Aut(G)$$

is continuous.

PROOF. Let $\mathcal{U}$ and $\mathcal{W}$ denote the set of all open neighborhoods of the identity in the analytic group N and the topological group L, respectively. Then $\mathcal{V} = \{W \cap f(N);\ W \in \mathcal{W}\}$ is the set of all open neighborhoods of the identity in the topological group $f(N)$. For the sake of simpleness, after this in the proof, we identify G with the image $f(G)$ by the map f. Then in particular $\mathcal{V} \subset \mathcal{U}$.

We pick a countable base $U_1 \supset U_2 \supset \cdots$ in $\mathcal{U}$. For each $n = 1, 2, \cdots$, by (2.3), we can pick $V_n \in \mathcal{V}$ such that $V_n^0 \subset U_n$, where V_n^0 is the identity component of V_n in the analytic group N. We take $W'_n \in \mathcal{W}$ with $W'_n \cap N = V_n$.

Next, we fix a compact connected neighborhood K of e in the analytic group N. Then K is a compact connected subset of L. Let us define a continuous map $g: K \times L \to L$ by

$$g(k, a)=k^{-1}aka^{-1} \qquad \text{for } k\in K,\ a\in L.$$

Since K is compact and $g(K, e)=e$, for the above W'_n we can pick a symmetric $W''_n\in\mathcal{W}$ such that $g(K, (W''_n)^2)\subset W'_n$. We put $W_n=W''_1\cap\cdots\cap W''_n$. Then the sequence $\{W_n\}$ in $\mathcal{W}$ is decreasing and

$$W_n=W_n^{-1}, \quad g(K, W_n^2)\subset W'_n.$$

For x in G and y in N, we put $\varphi(x)y=xyx^{-1}$. Then $\varphi(x)$ is an automorphism of N, e.g. by (3.3). Let a be in L. For each $n=1, 2, \cdots$ we pick $x_n\in aW_n\cap G$. Then if $l\geqq n$ and $m\geqq n$, $x_l^{-1}x_m\in W_n^2$, and for $k\in K$

$$k^{-1}\cdot\varphi(x_l^{-1})\varphi(x_m)(k)=k^{-1}(x_l^{-1}x_m)k(x_l^{-1}x_m)^{-1}\in W'_n\cap N=V_n.$$

By (3.1), $\varphi(x_l)^{-1}\varphi(x_m)\in[K, V_n^0]\subset[K, U_n]$. Then by (3.4), $\{\varphi(x_n)\}$ converges to a suitable $\varphi_0\in Aut(N)$.

We shall prove that φ_0 is independent of the choice of $\{x_n\}$. Indeed, if $x_n, x'_n\in aW_n\cap G$ then $x_n^{-1}x'_n\in W_n^2$ and $\varphi(x_n)^{-1}\varphi(x'_n)\in[K, U_n]$, which implies that $\lim\varphi(x_n)=\lim\varphi(x'_n)$.

Now in order to prove (i) it suffices to show that

$$\varphi_0(k)=aka^{-1} \qquad \text{for all } k\in K.$$

Let W be in $\mathcal{W}$. For each n we pick $W_n^*\in\mathcal{W}$ such that

$$W_n^*\subset W_n \quad\text{and}\quad g(K, W_n^*)\subset W'_n\cap a^{-1}Wa,$$

and take $x_n\in aW_n^*\cap G$. Then $a^{-1}x_n\in W_n^*$ and

$$g(k, a^{-1}x_n)=k^{-1}(a^{-1}x_n)k(a^{-1}x_n)^{-1}\in a^{-1}Wa,$$

i.e. $\varphi(x_n)(k)\in aka^{-1}W$ for all $k\in K$. On the other hand $\lim\varphi(x_n)(k)=\varphi_0(k)$ in the analytic group N and so in L. Hence $\varphi(x_n)(k)\in\varphi_0(k)W$ for a sufficiently large n, and we have $aka^{-1}W\cap\varphi_0(k)W\neq\emptyset$. This implies that $\varphi_0(k)=aka^{-1}$.

By (i), in particular, $f(G)$ is a normal subgroup of L, and (ii) follows directly from (3.3). Q. E. D.

Definition. We shall call γ the *canonical representation* of L (with respect to the immersion $f: G\to L$).

The kernel of γ coincides with the center $Z(L)$ of L, and

$$\mathrm{Ad}(G)\subset\gamma(L)\subset\overline{\mathrm{Ad}(G)}.$$

If in particular G is a (CA)-group, i.e. $\mathrm{Ad}(G)=\overline{\mathrm{Ad}(G)}$, then $\gamma(L)=\gamma(f(G))=\mathrm{Ad}(G)$, and we can apply (2.9) to this case. The result is the following extension of Theorem B. We shall see later that (4.1) is a special case of the fundamental Theorem 3.

(4.1) *Let G be a (CA)-group with center Z, and let L be a topological group* *with center $Z(L)$. If $f: G \to L$ is a dense immersion, then*

(i) $L=\overline{f(Z)}f(G)$, $\overline{f(Z)}=Z(L)$.

(ii) *If $f|Z$ is an imbedding, then f is an imbedding.*

PROOF. (2.9) implies (i), and also $f(G)/f(Z) \cong \mathrm{Ad}(G)$. Hence if $f(Z)$ is locally compact, then so is $f(G)$ by (2.2), and f is an imbedding by (2.5). Q. E. D.

Many of the theorems on an analytic group G can be extended to those on connected, locally compact G. Here we shall give an example.

(4.2) *Let G be a connected, locally compact group, and L a topological group. If $f: G \to L$ is a dense immersion, then $f(G)$ is a normal subgroup of L.*

PROOF. By Yamabe [**18**], G contains a compact normal subgroup C such that the factor group G/C is an analytic group. Then $f(C)$ is closed in L, and the normalizer of $f(C)$ is a closed subgroup containing $f(G)$, that is L. Hence $f(C)$ is a normal subgroup of L. In the factor group $L/f(C)$ the subgroup $f(G)/f(C)$ is dense and the natural map $G/C \to f(G)/f(C)$ is an immersion. Hence $f(G)/f(C)$ is a normal subgroup of $L/f(C)$, and so is $f(G)$ in L. Q. E. D.

§ 5. Proof of Theorem 2.

First we recall results in Iwasawa [**12**] on maximal compact subgroups of an analytic group. Let G be an analytic group. Any compact subgroup of G is contained in a suitable maximal compact subgroup. Let K be a maximal compact subgroup of G. Then K is connected and there is a finite subset $\{X_1, \cdots, X_n\}$ in $\mathfrak{g}$ such that

$$K \times \boldsymbol{R}^n \ni (k, (t_1, \cdots, t_n)) \mapsto k \exp t_1 X_1 \cdots \exp t_n X_n \in G$$

is a surjective homeomorphism. All maximal compact subgroups of G are conjugated to each other with respect to inner automorphisms. If N is a closed connected normal subgroup of G, then $K \cap N$ and KN/N are maximal compact subgroups of N and G/N, respectively.

Next about maximal tori in a compact analytic group K the following theorems are known mainly by H. Weyl, see e. g. Hochschild [**11**], XIII and Goto-Grosshans [8], Chapter 6. All maximal tori in K are conjugated to each other with respect to inner automorphisms. Any element in K is contained in some maximal torus. A maximal torus T is a maximal abelian subgroup in K and contains the center, in particular. Also the factor space K/T is simply connected.

Now we shall consider maximal tori in an analytic group G.

(5.1) (i) *All maximal tori in an analytic group G are conjugated to each other with respect to inner automorphisms.*

Let T be a maximal torus in G.

(ii) *T is a maximal compact abelian subgroup of G.*

(iii) *The factor space G/T is simply connected.*

(iv) *If N is a closed connected normal subgroup of G, then $T\cap N$ is a maximal torus in N.*

PROOF. (i), (ii) and (iii) are obvious. In order to prove (iv), we pick a maximal torus T_N in N. Then there is a maximal torus T_0 in G such that $T_N\subset T_0$. Since $T_0\cap N$ is a compact abelian subgroup containing T_N, we have $T_0\cap N=T_N$ by (ii). We pick x in G with $xT_0x^{-1}=T$. Then $T\cap N=xT_Nx^{-1}$ is a maximal torus in N. Q.E.D.

(5.2) *Let L be an analytic group, and N a closed connected normal subgroup of L. If L/N is a torus, then there exists a torus T_1 in L such that*

$$L=T_1N, \quad T_1\cap N=\{e\}.$$

PROOF. Let K be a maximal compact subgroup of L. Then KN/N is a maximal compact subgroup of L/N and coincides with L/N, i.e. $KN=L$. Let T be a maximal torus in K, and let $[K, K]$ denote the commutator subgroup of K. Since G/N is abelian, $[K, K]$ is contained in N. Also we know that $K=T[K, K]$. Hence we have $L=KN=TN$. By (5.1) (iv), $T\cap N$ is a torus and we can pick a complementary torus T_1 in T such that $T=T_1(T\cap N)$, $T_1\cap(T\cap N)=\{e\}$. Then $L=T_1N$ and $T_1\cap N=\{e\}$. Q.E.D.

REMARK. (5.2) extends Proposition 3.1 in A. Borel, Sous-groupes commutatifs et torsion des groupes de Lie compact connexes, Tôhoku Math. J., **13** (1961), pp. 216-240, and Lemma in [**7**].

We add one obvious lemma.

(5.3) *Let G and L be abelian analytic groups, and let $f: G\to L$ be a dense immersion. If the image $f(A)$ of each one-parameter subgroup $A\neq\{e\}$ is non-closed in L, then G is a vector group and L is a torus.*

PROOF. Notice that the closure of a non-closed one-parameter subgroup is a torus. Q.E.D.

Now we are ready to prove the following theorem.

THEOREM 2. *Let G and L be analytic groups, and let $f: G\to L$ be an immersion. Suppose that the commutator subgroup $f([G, G])$ of $f(G)$ is closed in L. (This condition is satisfied for $L=GL(n, \boldsymbol{R})$, see Goto [**4**].) Then there is a vector subgroup V and a closed connected normal subgroup N, of G, such that G is a semi-direct product*

$$G=VN, \quad V\cap N=\{e\},$$

$\overline{f(V)}$ is a torus, $f(N)$ is closed and

$$\overline{f(G)}=\overline{f(V)}f(N), \quad \overline{f(V)}\cap f(N)=\{e\}.$$

In particular, $f(G)$ is a topological direct product of $f(V)$ and N in a natural way.

PROOF. The commutator subgroup of $\overline{f(G)}$ coincides with the commutator subgroup $f([G, G])$ of $f(G)$, see Goto [**4**] and Hochschild [**11**], p. 190, also cf. §10. Let N be a maximal analytic subgroup of G such that $[G, G] \subset N$ and $f(N)$ is closed. Then the dense immersion

$$G/N \ni xN \mapsto f(x)f(N) \in \overline{f(G)}/f(N)$$

satisfies the conditions in (5.3) and we have $G/N \cong \boldsymbol{R}^r$ and $\overline{f(G)}/f(N)$ is a torus. By (5.2), there is a torus T_1 in $\overline{f(G)}$ such that

$$\overline{f(G)} = T_1 f(N), \quad T_1 \cap f(N) = \{e\}.$$

Since $T_1 \cap f(G)$ is a closed subgroup of $f(G)$, and $f(G) = (T_1 \cap f(G))f(N)$, putting $V = f^{-1}(T_1 \cap f(G))$ we have that V is closed in G, and

$$G = VN, \quad V \cap N = \{e\}. \qquad \text{Q.E.D.}$$

REMARK. In [**7**], the author gave Theorem 2 with a weaker conclusion that $\overline{f(V)} \cap f(N)$ is a finite group.

§6. Z-decompositions of an analytic group.

First, we give a lemma.

(6.1) *Let W be an analytic group.*

(i) *If the adjoint group* Ad($\mathcal{W}$) *is completely reducible, then the Lie algebra $\mathcal{W}$ is a direct sum of a semisimple ideal and the center.*

(ii) *If* Ad($\mathcal{W}$) *is completely reducible and abelian, then W is abelian.*

(iii) *If* Ad($\mathcal{W}$) *is compact, then W is a direct product of a compact group and a vector group.*

PROOF. (i) Let $\mathcal{J}$ be an ideal of $\mathcal{W}$. Then $\mathrm{Ad}(\mathcal{W})\mathcal{J} = \mathcal{J}$, and there exists a subspace $\mathcal{J}^{\perp}$ of $\mathcal{W}$ such that

$$\mathcal{W} = \mathcal{J} + \mathcal{J}^{\perp}, \quad \mathcal{J} \cap \mathcal{J}^{\perp} = \{0\},$$

and $\mathrm{Ad}(\mathcal{W})\mathcal{J}^{\perp} = \mathcal{J}^{\perp}$, which implies that $\mathcal{J}^{\perp}$ is an ideal. Hence $\mathcal{W}$ is a direct sum of simple ideals and one-dimensional ideals.

(ii) Let $Z(W)$ be the center of W. Then $W/Z(W)$ is abelian and W is nilpotent. Hence $\mathcal{W}$ cannot contain a simple ideal.

(iii) This follows easily from the known fact that the adjoint group of a non-compact semisimple analytic group is non-compact. Q.E.D.

Let G be an analytic group, and Ad($\mathcal{G}$) the adjoint group of G. Let α denote the adjoint representation $G \to \mathrm{Ad}(\mathcal{G})$. Let Z be the center of G. By Theorem 2, the analytic group $\mathrm{Ad}(\mathcal{G}) \cong G/Z$ has a semi-direct product decomposition

$$\mathrm{Ad}(\mathcal{G})=V'N', \quad V'\cap N'=\{1\}, \quad V'\cong \boldsymbol{R}^r,$$

N' is a normal subgroup of $\mathrm{Ad}(\mathcal{G})$ and is closed in $GL(\mathcal{G})$, $\bar{V}'$ is a torus, and

$$\overline{\mathrm{Ad}(\mathcal{G})}=\bar{V}'N', \quad \bar{V}'\cap N'=\{1\}.$$

We put $\alpha^{-1}(N')=N$. Then N is a closed normal subgroup of G and

$$G/N\cong \mathrm{Ad}(\mathcal{G})/N'\cong V',$$

where we regard $\mathrm{Ad}(\mathcal{G})$ as an analytic group, is simply connected. Hence N is connected. Also because the kernel of α is Z, we have $Z\subset N$.

Next, let W denote the identity component of $\alpha^{-1}(V')$. Then $\mathrm{Ad}(\mathcal{W})=\{\xi|\mathcal{W}\,;\,\xi\in V'\}$. Since V' is completely reducible and abelian, so is $\mathrm{Ad}(\mathcal{W})$, and W is an abelian group, by (6.1) (ii). We pick a subspace $\mathcal{V}$ of $\mathcal{W}$ such that

$$\mathcal{W}=\mathcal{V}+\mathcal{Z}, \quad \mathcal{V}\cap\mathcal{Z}=\{0\}.$$

Then

$$\mathcal{G}=\mathcal{V}+\mathcal{N}, \quad \mathcal{V}\cap\mathcal{N}=\{0\},$$

and in particular $G=VN$.

Let us prove that $V\cap N=\{e\}$ and $V\cong \boldsymbol{R}^r$. Let A be a non-trivial one-parameter subgroup of V. By the isomorphism $G/N\cong V'$, we have $AN/N\cong \boldsymbol{R}$, and so $A\cong \boldsymbol{R}$ and $A\cap N=\{e\}$. Thus we have the following theorem due to Zerling [21].

(6.2) *Let G be an analytic group with center Z, and let*

$$\mathrm{Ad}(\mathcal{G})=V'N', \quad V'\cap N'=\{1\}$$

be a decomposition of $\mathrm{Ad}(\mathcal{G})$ *as in Theorem* 2. *Then we can find a closed subgroup V and a closed normal subgroup N, in G, such that*

$$G=VN, \quad V\cap N=\{e\},$$

$$Z\subset N, \quad \alpha(N)=N' \quad \textit{and} \quad \alpha(V)=V'.$$

Let us call the above decomposition $G=VN$ a *Z-decomposition* of G (compatible to the decomposition $\mathrm{Ad}(\mathcal{G})=V'N'$).

§ 7. Generalized maximal torus (gm-torus).

(7.1) *Let G and G' be analytic groups, and let β be a continuous homomorphism from G onto G'. Let T' be a maximal torus of G', and H' a closed connected subgroup of G' containing T'.* (*In fact any analytic subgroup containing T' is closed.*) *Then $\beta^{-1}(H')$ is a* (*closed*) *connected subgroup of G.*

PROOF. First we shall prove that $\beta^{-1}(T')$ is connected. Indeed, the factor space $G/\beta^{-1}(T')$ is homeomorphic to G'/T' and is simply connected.

Next, the factor space $\beta^{-1}(H')/\beta^{-1}(T')$ is homeomorphic to H'/T' and is connected. Since $\beta^{-1}(T')$ is connected so is $\beta^{-1}(H')$. Q. E. D.

DEFINITION. Let G be an analytic group, and let $\alpha: G\rightarrow \mathrm{Ad}(\mathcal{G})$ be the adjoint representation. For a maximal torus T' in $\overline{\mathrm{Ad}(\mathcal{G})}$, the group $\alpha^{-1}(T'\cap \mathrm{Ad}(\mathcal{G}))$ is called a *generalized maximal torus* (*gm-torus*) of G.

We fix a Z-decomposition $G=VN$ compatible to $\mathrm{Ad}(\mathcal{G})=V'N'$.

Let T' be a maximal torus of $\overline{\mathrm{Ad}(\mathcal{G})}$ with $V'\subset T'$. By (5.1) (iv), $T_{N'}=T'\cap N'$ is a maximal torus in N'. Since $\overline{\mathrm{Ad}(\mathcal{G})}=\bar{V}'N'$ and $T'\supset \bar{V}'$, we have

$$T'=\bar{V}'T_{N'}, \quad \bar{V}'\cap T_{N'}=\{1\},$$

and

$$H'=T'\cap \mathrm{Ad}(\mathcal{G})=V'T_{N'}.$$

We put $H=\alpha^{-1}(H')$. Since $\mathrm{Ad}(\mathcal{G})/N'\cong \boldsymbol{R}^r$, the torus $T_{N'}$ is maximal in $\mathrm{Ad}(\mathcal{G})$, and by (7.1), H is a closed connected subgroup of G. Because $\mathrm{Ad}(\mathcal{H})=\{\varphi|\mathcal{H}; \varphi\in H'\}$ is completely reducible and abelian, H is an abelian group by (6.1) (ii). Since $V'\subset H'$, we have $V\subset H$. Since the kernel of α is the center Z, we have $Z\subset H$. Next, let T be a maximal torus of G. Then $\alpha(T)$ is a torus, and there exists $\eta\in N'$ such that $\eta\alpha(T)\eta^{-1}\subset T_{N'}$. Pick y in N with $\alpha(y)=\eta$. Then $yTy^{-1}\subset H$. Hence H contains a maximal torus of G.

Now the conjugacy of maximal tori in $\overline{\mathrm{Ad}(\mathcal{G})}$ easily implies that all gm-tori are conjugated to each other with respect to inner automorphisms.

Thus we have

(7.2) *Let G be an analytic group with center Z.*

(i) *A gm-torus H of G is a closed connected abelian subgroup of G containing Z and a maximal torus of G.*

(ii) *All gm-tori are conjugated to each other with respect to inner automorphisms.*

(iii) *Let H be a gm-torus. Then for a suitable Z-decomposition $G=VN$ we have*

$$H=VH_N, \quad (H_N=H\cap N), \quad V\cap H_N=\{e\},$$

where $\alpha(H_N)$ is a maximal torus of $\mathrm{Ad}(\mathcal{G})$.

EXAMPLE 1. By $H/Z\cong H'$, we see that G is (CA) if and only if Z is uniform in H. Hence a gm-torus coincides with a maximal torus if and only if G is (CA) and Z is compact, i. e. G is absolutely closed, by Theorem C.

EXAMPLE 2. When G is nilpotent, or more generally when $\mathrm{Ad}(\mathcal{G})$ is closed and is homeomorphic to $\boldsymbol{R}^s$, a gm-torus coincides with the center Z.

Let G be an analytic group, and let A be a closed subgroup of G. A is said to be a *topological direct factor* if there is a closed subset B of G such that

$$A\times B\ni(a, b)\mapsto ab\in G$$

is a surjective homeomorphism.

(7.3) *Let G be an analytic group and $H=VH_N$ a gm-torus of G. Then any vector part $v(H_N)$ of H_N is a topological direct factor in N. (It follows that $v(H)=Vv(H_N)$ is a topological direct factor in G.)*

PROOF. Let K' be a maximal compact subgroup of N' containing $T_{N'}$. We put $\alpha^{-1}(K')=K$. Then K is closed and connected by (7.1). Also $\mathrm{Ad}(\mathcal{K})=\{\varphi|\mathcal{K}\,;\ \varphi\in K'\}$ is compact, and by (6.1) (iii), K contains a compact normal subgroup K_0 and a closed central vector subgroup U such that

$$K=K_0U\,,\quad K_0\cap U=\{e\}\,.$$

Using the conjugacy of maximal compact subgroups, it is easy to see that K_0 is a maximal compact subgroup of N.

Next since $\alpha(U)$ is central in K', and $T_{N'}$ is a maximal torus in K', we have that $\alpha(U)\subset T_{N'}$ and $U\subset\alpha^{-1}(T_{N'})=H_N$. Since $H_N\subset K$, and U is a maximal closed vector subgroup of K, we see that U is a vector part of H_N. It is now obvious that any vector part of H_N is a topological direct factor in K.

Now it suffices to prove that K is a topological direct factor in N. On the other hand, there is a finite subset $\{X_1', \cdots, X_p'\}$ of the Lie algebra of N' such that

$$K'\times\boldsymbol{R}^p\ni(\kappa, (t_1, \cdots, t_p))\mapsto\kappa\exp t_1X_1'\cdots\exp t_pX_p'\in N'$$

is a surjective homeomorphism, by Iwasawa [**12**]. We pick $X_i\in\mathfrak{N}$ with $\mathrm{ad}\,X_i=X_i'$ for $i=1, 2, \cdots, p$. Then it is easy to see that

$$K\times\boldsymbol{R}^p\ni(k, (t_1, \cdots, t_p))\mapsto k\exp t_1X_1\cdots\exp t_pX_p\in N$$

is a surjective homeomorphism. Q. E. D.

§ 8. Linear spans of the center.

Let G be an analytic group with center Z. By a *linear span of the center* we shall mean a minimal analytic subgroup of G containing Z. Cf. Hochschild [**11**], p. 187.

(8.1) *Let S be a linear span of Z.*

(i) *S is closed and abelian.*

(ii) *S/Z is a torus, and there is a gm-torus $H=VH_N$ such that $S\subset H_N$.*

(iii) *A vector part $v(S)$ of S is a vector part of H_N, and so $Vv(S)$ is a vector part of H.*

PROOF. Let J be a gm-torus of S. Since Z is in the center of S, we have $J\subset S$ and $J=S$. Hence S is abelian. We put $\mathcal{P}=\{X\in\mathcal{S}\,;\ \exp\boldsymbol{R}X\cap Z\neq\{e\}\}$. Then $\mathcal{P}$ spans $\mathcal{S}$. Hence there is a basis $X_1, \cdots, X_m$ of $\mathcal{S}$ such that $\exp X_i\in Z$ for $i=1, \cdots, m$. Then $\alpha(\exp\boldsymbol{R}X_i)=\{1\}$ or $\cong\boldsymbol{R}/\boldsymbol{Z}$, and $\alpha(S)=\alpha(\exp\boldsymbol{R}X_1)\cdots$

$\alpha(\exp \boldsymbol{R}X_m)$ is a torus. Let T' be a maximal torus of $\overline{\mathrm{Ad}(G)}$ such that $\alpha(S)\subset T'$. Then there is a decomposition $\mathrm{Ad}(G)=V'N'$ such that

$$H'=T'\cap \mathrm{Ad}(G)=V'T_{N'} \quad \text{where } T_{N'}\supset\alpha(S).$$

Let $G=VN$ be a Z-decomposition compatible to $\mathrm{Ad}(G)=V'N'$. We put $H=\alpha^{-1}(H')=VH_N$. Then $S\subset H_N$. By $S=\alpha^{-1}(\alpha(S))$ we see that S is closed, and $H_N/S\cong T_{N'}/\alpha(S)$ is a torus. Hence there exists a torus U such that $H_N=SU$, $S\cap U=\{e\}$. This implies that any $v(S)$ is a vector part of H_N. Q. E. D.

(8.2) *Let S be a linear span of Z, and let T_S be the maximal torus in S. If* $\dim S-\dim T_S=l$, *then we can find a vector part $v_0(S)$ of S and a uniform subgroup $Z_0\cong \boldsymbol{Z}^l$ of Z such that $Z_0\subset v_0(S)$.*

PROOF. The identity component Z^0 of Z is a divisible group and Z/Z^0 is known to be finitely generated. Hence

$$Z\cong \boldsymbol{R}^a\times(\boldsymbol{R}/\boldsymbol{Z})^b\times \boldsymbol{Z}^c\times F,$$

where $a, b, c=0, 1, 2, \cdots$ and F is a finite group. Therefore we can find a uniform subgroup $Z_0\cong \boldsymbol{Z}^{a+c}$ of Z. Let ϕ denote the natural homomorphism from S onto S/Z_0. Since S/Z is a torus, so is S/Z_0. Then $\phi(T_S)$ is a torus in S/Z_0 and there exists a torus W such that

$$S/Z_0=\phi(T_S)W, \quad \phi(T_S)\cap W=\{e_0\},$$

where e_0 denotes the identity in S/Z_0. Then we have

$$S=T_S\phi^{-1}(W).$$

Let us prove that $T_S\cap\phi^{-1}(W)=\{e\}$. If $x\in T_S$ and $\phi(x)\in W$, then $\phi(x)=e_0$ and $x\in Z_0$. But Z_0 contains no compact subgroup except $\{e\}$, and $x=e$. Hence $\phi^{-1}(W)$ is a vector part of S, $\phi^{-1}(W)\supset Z_0$ and $\phi^{-1}(W)/Z_0$ is a torus. Hence in particular $a+b=l$. Q. E. D.

§ 9. Proof of Theorem 3.

First we shall prove (i) and (ii) of Theorem 3.

THEOREM 3. *Let G be an analytic group, H a gm-torus of G, and let $v(H)$ be a vector part of H. Let $G=VN$ be a Z-decomposition of G. If L is a topological group and $f: G\rightarrow L$ is a dense immersion, then*

(i) $L=\overline{f(v(H))}f(N)$.

(ii) *If $f|v(H)$ is an imbedding, then so is f.*

PROOF. We fix a Z-decomposition $G=VN$ once and for all and take a gm-torus $H=VH_N$. (By the conjugacy of gm-tori if we take a different H then V changes but we can use the same N.) Let Z be the center of G, and let γ

be the canonical representation. Then $\gamma\circ f$ is the adjoint representation of G. We put

$$\alpha(V)=V' \quad \text{and} \quad \alpha(N)=N'.$$

We apply (2.9) to $N\to f(N)\to N'$ and get

$$(*) \qquad f(N)/f(Z)\cong N'.$$

(i) Let us adopt the convention to regard G as a subgroup of L. Then in particular $\gamma(x)=\alpha(x)$ for $x\in G$.

By $V'N'\subset\gamma(L)\subset\bar{V}'N'$ we have

$$\gamma(L)=A'N', \quad A'\cap N'=\{1\} \quad \text{where } A'=\bar{V}'\cap\gamma(L).$$

We put $A=\gamma^{-1}(A')$. Then A is a closed subgroup of L.

Let a be in A, and let W be a neighborhood of e in L. By the continuity of γ we can pick a sequence $x_1, x_2, \cdots$ in $aW\cap G$ such that $\lim\gamma(x_n)=\gamma(a)$. We write $x_n=v_n y_n$ $(v_n\in V,\ y_n\in N)$ uniquely. Then

$$\lim\gamma(v_n)=\gamma(a) \quad \text{and} \quad \lim\gamma(y_n)=1.$$

Since $y_n\in N$ and $\lim\gamma(y_n)=1$, using (*) we have that $y_n\in ZW$ for a sufficiently large n, and then $x_n=v_n y_n\in VZW\subset HW$. Hence $aW\cap HW\neq\emptyset$ and $a\in\bar{H}$. Thus we have proved that $A\subset\bar{H}$.

Because A contains the kernel of γ, so does $\bar{H}$ and $\gamma(\bar{H}N)=A'N'=\gamma(L)$. Hence $\bar{H}N=L$.

Next for any vector part $v(H)$ of H, we have $H=Tv(H)$, where T is a maximal torus of G, and $\bar{H}=T\overline{v(H)}$.

(ii) By (2.8), that $f|v(H)$ is an imbedding implies $f|H$ is an imbedding. Because $Z\subset H$, we have $f(Z)$ is locally compact, and so is $f(N)$, by (*) and (2.2). By (2.5), $f|N$ is an imbedding. Also by (i), f is surjective. After this in this proof, omitting f we suppose that

$$L=VN, \quad V\cap N=\{e\}$$

and V and N are analytic groups.

Since N is an analytic group and $N/Z\cong N'$, for a sufficiently small neighborhood of 1 in N' we can find a cross-section. Therefore we assume that N'_0 is a neighborhood of 1 in N', $\mu: N'_0\to N$ is continuous, and $\gamma\circ\mu$ is the identity map. We put $\mu(N'_0)=C$ and $L_0=\gamma^{-1}(V'N'_0)=VZC$. Then L_0 is a neighborhood of e in L.

Now it suffices to prove that

$$V\times Z\times C\ni(v, z, c)\mapsto vzc\in L_0$$

is a homeomorphism. For $x\in L_0$ we put

$$\gamma(x)=\xi(x)\eta(x) \quad (\xi(x)\in V',\ \eta(x)\in N_0').$$

Then ξ and η are continuous, and so is $c=\mu\circ\eta$. Since $\gamma(xc(x)^{-1})=\xi(x)\in V'$, we have $xc(x)^{-1}\in VZ$. Hence $xc(x)^{-1}=v(x)z(x)$ $(v(x)\in V,\ z(x)\in Z)$, and $v(x)$ and $z(x)$ depend continuously on x. Hence

$$L_0\ni x\mapsto(v(x),\ z(x),\ c(x))\in V\times Z\times C$$

is continuous. Q. E. D.

(iii) *Let* $W\neq v(H)$ *be an analytic subgroup of* $v(H)$. *Then there is an analytic group* L *and a dense immersion* $f: G\rightarrow L$ *such that* $f(G)\neq L$ *and* $f|W$ *is an imbedding.*

PROOF. By (8.2), there is a linear span S of Z in H, such that a vector part $v_0(S)$ contains a subgroup Z_0 of Z as a (uniform) lattice. Then $v_0(H)=Vv_0(S)$ is a vector part of H.

Case 1. Let W be an analytic subgroup of $v_0(H)$, and suppose W does not contain $v_0(S)$. Then W does not contain Z_0 and there is $P=\exp \boldsymbol{R}X\subset v_0(S)$, $z=\exp X\in Z_0$ and $P\cap W=\{e\}$. Then we can pick an analytic subgroup Q of codimension one in $v_0(H)$ such that

$$v_0(S)=PQ\,,\quad P\cap Q=\{e\},\quad Q\supset W.$$

By (7.3), $v_0(H)$ is a topological direct factor in G, and we can find a set R such that $G=PQR$, where $P\times Q\times R\ni(p,\ q,\ r)\mapsto pqr$ is a surjective homeomorphism.

Let π be an irrational number. Then in the direct product group $\boldsymbol{R}/\boldsymbol{Z}\times G$, the subgroup

$$D=\{(\pi m \bmod 1,\ z^m)\,;\ m\in\boldsymbol{Z}\}$$

is discrete and central. We denote $L=(\boldsymbol{R}/\boldsymbol{Z}\times G)/D$. Then the map $f: G\ni x\mapsto D(0,\ x)\in L$ is a dense immersion. We see easily that $\overline{f(P)}$ is the torus $T=(\boldsymbol{R}/\boldsymbol{Z}\times P)/D$, T is a topological direct factor in $L: L=Tf(QR)$, and $f|QR$ is a homeomorphism. Hence in particular $f|W$ is an imbedding.

Case 2. Suppose that $W\supset v_0(S)$. Then we can find an analytic subgroup V_1 of dimension one and an analytic subgroup V_2 of codimension one, in V, such that

$$V=V_1V_2\,,\quad V_2v_0(S)\supset W,\quad V_1\cap V_2=\{e\}.$$

We recall that $\alpha(V_1)$ is a non-closed one-parameter subgroup in $Aut(G)$. We put $V_2N=N_1$ and $A=\{\xi|\mathfrak{N}_1\,;\ \xi\in\overline{\alpha(V_1)}\}$. Then $\overline{\alpha(V_1)}\ni\xi\mapsto\xi|\mathfrak{N}_1\in A\subset Aut(N_1)$ is an imbedding. Let L be the semi-direct product of A and N_1. Then

$$f\in G=V_1N_1\ni vy\mapsto(\alpha(v)|\mathfrak{N}_1,\ y)=A\times N_1=L$$

is a dense immersion, which is an imbedding on N_1.

Thus we have proved (iii) for $v_0(H)$. Let $v(H)$ be a vector part of H, and $W \neq v(H)$ an analytic subgroup of $v(H)$. Then $TW \cap v_0(H) = W_0$ is an analytic subgroup of $v_0(H)$ and $TW = TW_0$. Hence there is a non-trivial immersion $f: G \to L$ such that $f|W_0$ is an imbedding. Since W_0 is uniform in TW, by (2.8), $f|TW$ is an imbedding. Q. E. D.

§ 10. Descending central series.

Let A be a group, and let B and C be normal subgroups of A. Let $[B, C]$ denote the subgroup of A generated by $\{bcb^{-1}c^{-1};\ b \in B,\ c \in C\}$. We put

$$[A, A] = A^2, \quad [A, A^2] = A^3, \cdots, [A, A^j] = A^{j+1}, \cdots.$$

The sequence $A = A^1 \supset A^2 \supset A^3 \supset \cdots$ of normal subgroups is called the *descending central series* of A. When $A^{j-1} \neq A^j = \{e\}$, A is called nilpotent of degree j.

When G is an analytic group, the descending central series $\{G^j\}$ is composed of analytic subgroups, and the Lie algebra $\mathfrak{g}^j$ of G^j is given by

$$\mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}], \cdots, [\mathfrak{g}, \mathfrak{g}^j] = \mathfrak{g}^{j+1}, \cdots.$$

THEOREM 4. *Let G be an analytic group, and L a topological group. If $f: G \to L$ is a dense immersion, then*

$$f(G)^j = L^j \quad \textit{for } j = 2, 3, \cdots.$$

In particular, if G is nilpotent of degree j, then so is L.

PROOF. We adopt the convention to regard G as a subgroup of L. Let γ denote the canonical representation of L. First we shall prove

(1) $$(\gamma(a) - 1)\mathfrak{g}^j \subset \mathfrak{g}^{j+1} \quad \text{for } a \in L,\ j = 1, 2, \cdots.$$

For $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}^j$

$$\alpha(\exp X)Y = \exp(\operatorname{ad} X)Y = \sum_{k=0}^{\infty} (\operatorname{ad} X)^k / k!\, Y,$$

and $(\alpha(\exp X) - 1)Y = [X, Y] + \frac{1}{2!}[X, [X, Y]] + \cdots \in \mathfrak{g}^{j+1}$. For a fixed $Y \in \mathfrak{g}^j$, $(\alpha(x) - 1)Y$ is an analytic function of $x \in G$, and for x sufficiently close to e, we have $(\alpha(x) - 1)Y \in \mathfrak{g}^{j+1}$. Hence $(\alpha(x) - 1)Y \in \mathfrak{g}^{j+1}$ for all $x \in G$. Next for $a \in L$, there is a sequence $\{x_n\}$ in G, such that $\lim \alpha(x_n) = \gamma(a)$. Hence we have (1).

(2) $$[L, G^j] = G^{j+1} \quad (j = 1, 2, \cdots).$$

Let a be in L. For X in $\mathfrak{g}^j$, sufficiently close to 0, we can apply the Campbell-Hausdorff formula to

$$
\begin{aligned}
a \exp X a^{-1} \exp(-X) &= \exp(\gamma(a)X) \exp(-X) \\
&= \exp((\gamma(a)-1)X - \frac{1}{2}[\gamma(a)X, X] + \cdots) \\
&\in \exp \mathfrak{g}^{j+1},
\end{aligned}
$$

by (1), and $a \exp X a^{-1} \exp(-X) \in G^{j+1}$.

Then $G^j \ni x \mapsto axa^{-1}x^{-1} \in G$ is an analytic map and for x sufficiently close to e, we have $axa^{-1}x^{-1} \in G^{j+1}$. Also G^{j+1} is a maximal integral submanifold of the distribution $\mathfrak{g}^{j+1}$. Hence we have $axa^{-1}x^{-1} \in G^{j+1}$ for all $x \in G^j$. This proves (2).

Now it suffices to prove that $G^2 = L^2$. Indeed, then by (2), $G^3 = [L, G^2] = [L, L^2] = L^3, \cdots$. Since G^2 is a normal subgroup of L, let us consider the factor group L/G^2, where G^2 is not closed in general and our discussion is purely algebraic. Then $[L, G] = G^2$ implies that G/G^2 belongs to the center of L/G^2. On the other hand, by Theorem 2, $L = \bar{H}G$ and $\bar{H}$ is abelian. Hence $L/G^2 = \bar{H}G^2/G^2 \cdot G/G^2$ is an abelian group, and $G^2 \supset L^2$. Q. E. D.

Next, for a group A, we define the *derived series* by $A^{(0)} = A$, $A^{(1)} = A^2$, $A^{(2)} = [A^{(1)}, A^{(1)}], \cdots$. Then $A^{(0)} \supset A^{(1)} \supset A^{(2)} \supset \cdots$, and when $A^{(j-1)} \neq A^{(j)} = \{e\}$, we call A solvable of degree j. The following corollary is obvious.

COROLLARY. *Let G be an analytic group, and L a topological group. If $f: G \to L$ is a dense immersion, then*

$$f(G)^{(j)} = L^{(j)} \quad \textit{for } j = 1, 2, \cdots.$$

In particular, if G is solvable of degree j, then so is L.

§ 11. Locally compact closure.

Let B be a connected, locally compact, abelian group. As in the case of analytic groups, B contains a unique maximal compact subgroup K such that

$$B = KV, \quad K \cap V = \{e\}, \quad V \cong \boldsymbol{R}^s.$$

We call V a vector part of B and denote it by $v(B)$. Let A be a one-parameter subgroup of B. Then either A is closed or $\bar{A}$ is compact. We begin by a lemma.

(11.1) *Let A be an abelian analytic group, and B a locally compact group. If $f: A \to B$ is a dense immersion, then there exists a vector subgroup C of A, $C \cong \bar{C} \cong \boldsymbol{R}^m$, such that $f(C)$ is a vector part of B.*

PROOF. Let T be a maximal torus of A. Let D be a maximal analytic subgroup of A such that $f(D)$ is closed. Then $D \supset T$ and in the factor group $B/f(D)$, no non-trivial one parameter subgroup of $f(A)/f(D)$ is closed. Then $B/f(D)$ must be compact. Let C be a vector part of D. Then $f(C)$ is closed and $f(D)/f(C) \cong D/C$ is a torus. Hence $B/f(C)$ is compact, and $f(C)$ is a vector part of B. Q. E. D.

THEOREM 5. *Let G be an analytic group with a gm-torus H. Let L be a topological group, and $f: G \rightarrow L$ an immersion. If $\overline{f(v(H))}$ is locally compact, then $\overline{f(G)}$ is locally compact.*

PROOF. Let T be the maximal torus of G in H. Then $H=Tv(H)$, and $\overline{f(v(H))}$ is uniform in $\overline{f(H)}=\overline{f(v(H))}f(T)$. Hence $\overline{f(H)}$ is locally compact. We fix a Z-decomposition $G=VN$ with $H=VH_N$ once and for all.

Because $\gamma(\overline{f(N)})=N'$, we can apply (2.9) to the sequence $N \rightarrow \overline{f(N)} \rightarrow N'$ and get $\overline{f(N)}/\overline{f(Z)} \cong N'$, where Z is the center of G. Since $Z \subset H$, we have $\overline{f(Z)}$ is locally compact, and so is $\overline{f(N)}$, by (2.2).

Let K be the maximal compact subgroup in $\overline{f(H)}$. Then $P=\overline{f(H_N)}K$ is a closed connected subgroup of $\overline{f(H)}$ containing K. Then by (11.1), there is a vector subgroup Q in H such that $\overline{f(H)}=Pf(Q)$, $P \cap f(Q)=\{e\}$. We put $QH_N \cap V=W$. Then W is a vector subgroup of V and $QH_N=WH_N$, and by $f(H_N) \subset P$ we have

$$\overline{f(H)}=Pf(W), \quad P \cap f(W)=\{e\}.$$

Then $P \times W \ni (p, w) \mapsto pf(w)$ is an immersion from $P \times W$ onto $\overline{f(H)}$, and is an imbedding by (2.5). Hence $f(W)$ is a closed vector group.

We put $M=f(W)\overline{f(N)}$. By (10.1), $L^2=\overline{f(G)^2} \subset \overline{f(N)}$, and M is a normal subgroup of L. First we shall prove that $f(W) \cap \overline{f(N)}=\{e\}$. Indeed, if $x \in f(W) \cap \overline{f(N)}$; then $\gamma(x) \in V' \cap N'=\{1\}$ and $x \in f(W) \cap \gamma^{-1}(1) \subset \overline{f(G)} \cap \gamma^{-1}(1)=\overline{f(Z)}$, but $Z \cap W=\{e\}$, and we have $x=e$. Next, we shall see that the kernel of $\gamma | M$ is $\overline{f(Z)}$. For $w \in f(W)$ and $y \in \overline{f(N)}$, $\gamma(wy)=\gamma(w)\gamma(y)=1$ implies $w=e$ and y is in the kernel of $\gamma | \overline{f(N)}$, i. e. $\overline{f(Z)}$.

Let $\mu_0: N'_0 \rightarrow N$ be a local cross-section of the fibering $N/Z \cong N'$, i. e. N'_0 is an open neighborhood of 1 in N', μ_0 is continuous, and $\alpha \circ \mu_0$ is the identity map. By $N/Z \cong \overline{f(N)}/\overline{f(Z)} \cong N'$, the map $f \circ \mu_0 = \mu$ is a cross-section of $\gamma: \overline{f(N)} \rightarrow N'$. We put $M_0=\{x \in M;\ \gamma(x) \in \alpha(W)N'_0\}$. Then in a similar way as in the proof of Theorem 3 (ii), we have that M_0 is homeomorphic with $f(W) \times \overline{f(Z)} \times N'_0$ and is locally compact. Since M_0 is an open set in M, we see that M is locally compact.

Next we have $\overline{f(G)}=f(K)M$ and M is uniform in $\overline{f(G)}$. Hence $\overline{f(G)}$ is locally compact. Q. E. D.

The following theorem is due to Iwasawa and Kuranishi, see Iwasawa [**12**].

(11.2) *Let A be a topological group, and let B be a closed normal subgroup of A. If both B and A/B are analytic groups, then A is an analytic group.*

Using (11.2), we can easily modify the proof of Theorem 5 to get

THEOREM 6. *If $\overline{f(v(H))}$ is an analytic group, then so is $\overline{f(G)}$.*

Bibliography

[1] C. Chevalley, Theory of Lie groups I, Princeton University Press, 1946.

[2] D. Ž. Djoković, Three notes on real analytic groups, Tôhoku Math. J., **28** (1976), 381-387.

[3] A. M. Gleason and R. S. Palais, On a class of transformation groups, Amer. J. Math., **79** (1957), 631-648.

[4] M. Goto, Faithful representations of Lie groups I, Math. Japonica, **1** (1948), 107-119.

[5] M. Goto, Dense imbeddings of locally compact connected groups, Ann. of Math., **61** (1955), 154-169.

[6] M. Goto, Absolutely closed Lie groups, Math. Ann., **204** (1973), 337-341.

[7] M. Goto, Analytic subgroups of $GL(n, \boldsymbol{R})$, Tôhoku Math. J., **25** (1973), 197-199.

[8] M. Goto and F. Grosshans, Semisimple Lie algebras, Marcel Dekker, 1978.

[9] M. Goto and N. Kimura, Semigroups of endomorphisms of a locally compact group, Trans. Amer. Math. Soc., **87** (1958), 359-371.

[10] G. Hochschild, The automorphism group of a Lie group, Trans. Amer. Math. Soc., **72** (1952), 209-216.

[11] G. Hochschild, The structure of Lie groups, Holden-day Inc., 1965.

[12] K. Iwasawa, On some types of topological groups, Ann. of Math., **50** (1949), 507-558.

[13] D. H. Lee and T. S. Wu, On CA topological groups, Duke Math. J., **37** (1970), 515-521.

[14] A. Malcev, On the theory of the Lie groups in the large, Rec. Math., **16** (1945), 163-190.

[15] D. Montgomery and L. Zippin, Topological transformation groups, Interscience, New York, 1955.

[16] H. Ōmori, Homomorphic images of Lie groups, J. Math. Soc. Japan, **18** (1966), 97-117.

[17] W. T. van Est, Dense imbeddings of Lie groups, Indag. Math., **13** (1951), 321-328.

[18] H. Yamabe, On the conjecture of Iwasawa and Gleason, Ann. of Math., **58** (1953), 48-54.

[19] K. Yosida, A problem concerning the second fundamental theorem of Lie, Proc. Imp. Acad. Japan, **13** (1937), 152-155.

[20] K. Yosida, A theorem concerning the semi-simple Lie groups, Tôhoku Math. J., **43** (1937), 81-84.

[21] D. Zerling, Some theorems on (CA) analytic groups; Trans. Amer. Math. Soc., **205** (1975), 181-192.

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