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Two local ergodic theorems on L_∞

Dedicated to Professor Shisanji Hokari on his 70th birthday

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Introduction.

Let $(X, \mathfrak{F}, \mu)$ be a σ-finite measure space and let $L_p(X)=L_p(X, \mathfrak{F}, \mu)$, $1\leqq p\leqq\infty$, be the usual Banach spaces of real or complex functions on $(X, \mathfrak{F}, \mu)$. In this paper we first prove that if $(T_t: t>0)$ is a strongly continuous one-parameter semigroup of linear contractions on $L_1(X)$ such that $\mu(A)>0$ implies $\int_A |T_t f| d\mu>0$ for some $f\in L_1(X)$ and $t>0$, then the local ergodic theorem holds for the adjoint semigroup $(T_t^*: t>0)$ acting on $L_\infty(X)$, i.e. for any $f\in L_\infty(X)$ there exists a scalar function $T_t^*f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathfrak{F}$, such that for each fixed $t>0$, $T_t^*f(x)$ as a function of x belongs to the equivalence class of T_t^*f, and the following local ergodic limit

$$\lim_{b\to 0}\frac{1}{b}\int_0^b T_t^*f(x)dt$$

exists and is finite a.e. on X; in particular, $\lim_{t\to 0}\|T_t v-v\|_1=0$ for all $v\in L_1(X)$ if and only if

$$\lim_{b\to 0}\frac{1}{b}\int_0^b T_t^*f(x)dt=f(x) \quad \text{a.e.}$$

for all $f\in L_\infty(X)$. This generalizes Krengel's local ergodic theorem [7] for semigroups of nonsingular point transformations on $(X, \mathfrak{F}, \mu)$. For another related result we refer the reader to the author [13], in which positive semigroups on $L_1(X)$ are considered and a similar result is obtained, only assuming that the positive semigroup $(T_t: t>0)$ is strongly integrable over every finite interval. We next prove that if $(T_t: t>0)$ is a strongly continuous one-parameter semigroup of bounded linear operators on $L_p(X)$ for some fixed p, $1\leqq p<\infty$, such that $(T_t: t>0)$ is also simultaneously a semigroup of linear contractions on $L_\infty(X)$ with $T_t^*L_1(X)\subset L_1(X)$ for all $t>0$, then under one of the following two conditions (I) and (II), the local ergodic theorem holds for $(T_t: t>0)$ on

$L_\infty(X)$: (I) $\mu(A)>0$ implies $\lim_{t\to 0}\|T_t 1_A\|_\infty>0$; (II) T_t on $L_p(X)$ converges strongly to a linear contraction T_0 on $L_p(X)$ as $t\to 0$.

Definitions and preliminaries.

Let $(X, \mathfrak{F}, \mu)$ be a σ-finite measure space and let $L_p(X)=L_p(X, \mathfrak{F}, \mu)$, $1\leqq p\leqq\infty$, be the usual Banach spaces of real or complex functions on $(X, \mathfrak{F}, \mu)$. All sets and functions introduced below are assumed to be measurable; all relations are assumed to hold modulo sets of measure zero. If A is a subset of X, then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X-A$. If $f\in L_p(X)$, then $\operatorname{supp} f$ is defined to be the set of all $x\in X$ at which $f(x)\neq 0$. A linear operator T on $L_p(X)$ is called *positive* if $Tf\geqq 0$ for all $f\geqq 0$, and a *contraction* if $\|T\|_p\leqq 1$.

Let $(T_t: t>0)$ be a one-parameter semigroup of bounded linear operators on $L_p(X)$, i.e. $\|T_t\|_p<\infty$ and $T_tT_s=T_{t+s}$ for all t, $s>0$. $(T_t: t>0)$ is called *strongly continuous* (on $(0, \infty)$) if $\lim_{t\to s}\|T_tf-T_sf\|_p=0$ for every $f\in L_p(X)$ and all $s>0$, and *strongly integrable over every finite interval* if for each $f\in L_p(X)$ the vector valued function $t\mapsto T_tf$ is Bochner integrable with respect to the Lebesgue measure on every finite interval.

It is well known (cf. [5, Chapter VIII]) that if $(T_t: t>0)$ is strongly integrable over every finite interval, then it is strongly continuous and so for any $f\in L_p(X)$ there exists a scalar function $T_tf(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathfrak{F}$, such that for each fixed $t>0$, $T_tf(x)$ as a function of x belongs to the equivalence class of T_tf. Moreover, by Fubini's theorem, there exists a set $N(f)\subset X$ with $\mu(N(f))=0$, dependent on f but independent of t, such that if $x\notin N(f)$ then the function $t\mapsto T_tf(x)$ is Lebesgue integrable on every finite interval $(a, b)\subset(0, \infty)$ and the integral $\int_a^b T_tf(x)dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_tfdt$ $(\in L_p(X))$.

We now assume that $1\leqq p<\infty$, and denote by $(T_t^*: t>0)$ the adjoint semigroup of a strongly continuous one-parameter semigroup $(T_t: t>0)$ on $L_p(X)$. Thus $(T_t^*: t>0)$ acts on $L_q(X)$, where $1/p+1/q=1$, and $\langle T_tv, f\rangle=\langle v, T_t^*f\rangle$ for all $v\in L_p(X)$, $f\in L_q(X)$ and $t>0$, where we let $\langle v, f\rangle=\int vfd\mu$ for $v\in L_p(X)$ and $f\in L_q(X)$. In case $1<p<\infty$, $L_p(X)$ is a reflexive Banach space and hence the adjoint semigroup $(T_t^*: t>0)$ is also strongly continuous; furthermore T_t converges strongly to T_0 as $t\to 0$ if and only if T_t^* converges strongly to T_0^* as $t\to 0$. In case $p=1$, this is not the case. But we then have the following proposition, originally due to Lin [9].

PROPOSITION. *Let $(T_t : t>0)$ be a strongly continuous one-parameter semigroup of bounded linear operators on $L_1(X)$ such that $\sup_{0<t<1} \|T_t\|_1<\infty$. Then for any $f\in L_\infty(X)$ there exists a scalar function $T_t^*f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathfrak{F}$, and a set $N(f)\subset X$ with $\mu(N(f))=0$, dependent on f but independent of t, such that if $x\notin N(f)$ then the function $t\mapsto T_t^*f(x)$ is Lebesgue integrable on every finite interval $(a, b)\subset(0, \infty)$ and the integral $\int_a^b T_t^*f(x)\,dt$, as a function of x, is in $L_\infty(X)$ and satisfies*

$$\langle v(x), \int_a^b T_t^*f(x)dt\rangle=\langle\int_a^b T_t v\,dt, f\rangle$$

for all $v\in L_1(X)$.

PROOF. A minor change of the argument given in [**12**, Lemma A] is sufficient for the proof of the proposition and we omit the details.

Theorems.

THEOREM 1. *Let $(T_t : t>0)$ be a strongly continuous one-parameter semigroup of linear contractions on $L_1(X)$ such that $\mu(A)>0$ implies $\int_A |T_t f|\,d\mu>0$ for some $f\in L_1(X)$ and $t>0$. Then for any $f\in L_\infty(X)$ there exists a scalar function $T_t^*f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathfrak{F}$, such that for each fixed $t>0$, $T_t^*f(x)$ as a function of x belongs to the equivalence class of T_t^*f, and further the following local ergodic limit*

$$\lim_{b\to 0} \frac{1}{b}\int_0^b T_t^*f(x)dt \tag{1}$$

exists and is finite a.e. on X.

In particular, we have $\lim_{t\to 0} \|T_t v-v\|_1=0$ for all $v\in L_1(X)$ if and only if

$$\lim_{b\to 0} \frac{1}{b}\int_0^b T_t^*f(x)dt=f(x) \quad a.e. \tag{2}$$

for all $f\in L_\infty(X)$.

REMARK. It is well known (cf. [8]) that $\lim_{t\to 0} \|T_t v-v\|_1=0$ for all $v\in L_1(X)$ if and only if

$$\lim_{b\to 0} \frac{1}{b}\int_0^b T_t v(x)dt=v(x) \quad \text{a.e.} \tag{3}$$

for all $v\in L_1(X)$.

PROOF. Let τ_t denote the linear modulus of T_t in the sense of Chacon and Krengel [**3**]. Therefore τ_t is a positive linear contraction on $L_1(X)$ such that

$$\tau_t f = \sup\{|T_t g| : g \in L_1(X) \text{ and } |g| \leqq f\}$$

for all $0 \leqq f \in L_1(X)$. If $0 \leqq f \in L_1(X)$, define

$$S_t f = \sup\left\{\tau_{t_1} \cdots \tau_{t_n} f : \sum_{i=1}^{n} t_i = t,\ t_i > 0,\ n \geqq 1\right\}.$$

Then, as observed by Kipnis [6] and the author [14], S_t can be extended to a positive linear contraction on $L_1(X)$, $S_t S_s = S_{t+s}$ for all t, $s>0$, and the semigroup $(S_t : t>0)$ is strongly continuous. Since $\mu(A)>0$ implies

$$\int_A S_t |f| d\mu \geqq \int_A |T_t f| d\mu > 0$$

for some $f \in L_1(X)$ and $t>0$ by hypothesis, an approximation argument shows that if $f_0 \in L_1(X)$, $f_0 > 0$ a.e. on X and

$$h = \int_0^\infty e^{-s} S_s f_0 ds \quad (\in L_1(X)),$$

then $h>0$ a.e. on X, and for all $t>0$ we have

$$S_t h = \int_0^\infty e^{-s} S_{t+s} f_0 ds \leqq e^t \int_0^\infty e^{-s} S_s f_0 ds = e^t h.$$

Hence, if we let for all $t>0$

$$\begin{cases} P_t f = (1/h) S_t(fh) \\ Q_t f = (1/h) T_t(fh) \end{cases} \quad (f \in L_1(X, \mathfrak{F}, h d\mu)),$$

then $(P_t : t>0)$ and $(Q_t : t>0)$ are strongly continuous one-parameter semigroups of linear contractions on $L_1(X, \mathfrak{F}, h d\mu)$ such that

$$|Q_t f| \leqq P_t |f| \quad \text{a.e.}$$

for all $f \in L_1(X, \mathfrak{F}, h d\mu)$, and thus $\|Q_t\|_\infty \leqq \|P_t\|_\infty \leqq e^t$ for all $t>0$. We now apply the Riesz convexity theorem and an approximation argument to observe that $(Q_t : t>0)$ can be regarded as a strongly continuous one-parameter semigroup of bounded linear operators on $L_2(X, \mathfrak{F}, h d\mu)$. Then the adjoint semigroup $(Q_t^* : t>0)$ on $L_2(X, \mathfrak{F}, h d\mu)$ is again strongly continuous. It is now easy to see that

$$Q_t^* = T_t^* \quad \text{on } L_\infty(X, \mathfrak{F}, h d\mu) = L_\infty(X, \mathfrak{F}, \mu)$$

for all $t>0$, that Q_t^* is extended to a bounded linear operator on $L_1(X, \mathfrak{F}, h d\mu)$ with $\|Q_t^*\|_1 = \|Q_t\|_\infty \leqq e^t$, and that the semigroup $(Q_t^* : t>0)$ on $L_1(X, \mathfrak{F}, h d\mu)$ is strongly continuous. Therefore for any $f \in L_\infty(X, \mathfrak{F}, h d\mu)$ $(\subset L_1(X, \mathfrak{F}, h d\mu))$

there exists a scalar function $T_t^*f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathfrak{F}$, such that for each fixed $t>0$, $T_t^*f(x)$ as a function of x belongs to the equivalence class of $T_t^*f=Q_t^*f$.

Since $\|e^{-t}Q_t^*\|_1\leqq 1$ and $\|e^{-t}Q_t^*\|_\infty\leqq\|T_t^*\|_\infty\leqq 1$, we can now apply the local ergodic theorem in [11] and obtain that the limit

$$\lim_{b\to 0}\frac{1}{b}\int_0^b e^{-t}T_t^*f(x)dt$$

exists and is finite a. e. on X. This establishes the first half of the theorem, because $\lim_{t\to 0} e^{-t}=1$.

To complete the proof, suppose $\lim_{t\to 0}\|T_tv-v\|_1=0$ for all $v\in L_1(X)$. Obviously this condition implies that

$$\sup\left\{\int_A |T_tf|\,d\mu : f\in L_1(X) \text{ and } t>0\right\}>0$$

for all $A\in\mathfrak{F}$ with $\mu(A)>0$, and therefore

$$\lim_{b\to 0}\frac{1}{b}\int_0^b T_t^*f(x)dt$$

exists and is finite a. e. on X for all $f\in L_\infty(X)$. Given a function v in $L_1(X)$, we then have (cf. Proposition), by Lebesgue's dominated convergence theorem, that

$$\begin{aligned}\left\langle v(x),\ \lim_{b\to 0}\frac{1}{b}\int_0^b T_t^*f(x)dt\right\rangle&=\lim_{b\to 0}\left\langle v(x),\ \frac{1}{b}\int_0^b T_t^*f(x)dt\right\rangle\\&=\lim_{b\to 0}\left\langle\frac{1}{b}\int_0^b T_tv\,dt,\ f\right\rangle\\&=\langle v, f\rangle .\end{aligned}$$

This shows that $f(x)=\lim_{b\to 0}\frac{1}{b}\int_0^b T_t^*f(x)dt$ a. e. on X.

Conversely, if (2) holds for all $f\in L_\infty(X)$, then again we apply Lebesgue's dominated convergence theorem to obtain that

$$\lim_{b\to 0}\left\langle\frac{1}{b}\int_0^b T_tv\,dt,\ f\right\rangle=\langle v, f\rangle$$

for all $v\in L_1(X)$ and $f\in L_\infty(X)$. It now follows that $v=\text{weak-}\lim_{b\to 0}\frac{1}{b}\int_0^b T_tv\,dt$, and thus v is in the closed linear hull of the set $\{T_tv : t>0\}$. Since $\|T_t\|_1\leqq 1$ for all $t>0$, this together with an easy approximation argument shows that

$\lim_{t\to 0} \|T_t v - v\|_1 = 0$ for all $v \in L_1(X)$.

The proof is completed.

Before stating the next theorem we shall remark the following fact: Let T be a bounded linear operator on $L_p(X)$ for some fixed p, $1 \leqq p < \infty$, such that $\|Tf\|_\infty \leqq \|f\|_\infty$ for all $f \in L_p(X) \cap L_\infty(X)$. Then there exists a (unique) linear contraction S on $L_1(X)$ such that $Tf = S^* f$ for all $f \in L_p(X) \cap L_\infty(X)$.

To see this, let q be such that $1/p + 1/q = 1$. Then for any $f \in L_q(X) \cap L_1(X)$ we have

$$\int |T^* f| \, d\mu = \langle k, T^* f \rangle = \lim_n \langle k_n, T^* f \rangle$$

$$= \lim_n \langle T k_n, f \rangle \leqq \int |f| \, d\mu$$

where we let $k(x) = \operatorname{sgn} \overline{T^* f(x)}$ and $k_n(x) = k(x) 1_{A_n}(x)$ with $\mu(A_n) < \infty$ for each $n \geqq 1$, $A_1 \subset A_2 \subset \cdots$, and $\lim_n A_n = X$. Hence it follows that T^* can be extended to a linear contraction S on $L_1(X)$, and clearly we have

$$T = S^* \quad \text{on} \quad L_p(X) \cap L_\infty(X).$$

Thus, in the sequel, we may assume that such an operator T is also simultaneously a linear contraction on $L_\infty(X)$ with $T^* L_1(X) \subset L_1(X)$; for any $f \in L_\infty(X)$ we may define Tf by

$$Tf(x) = \lim_n T(f 1_{A_n})(x) \quad \text{a. e.}$$

where $\mu(A_n) < \infty$ for each $n \geqq 1$, $A_1 \subset A_2 \subset \cdots$, and $\lim_n A_n = X$.

THEOREM. 2. *Let $(T_t : t > 0)$ be a strongly continuous one-parameter semigroup of bounded linear operators on $L_p(X)$ for some fixed p, $1 \leqq p < \infty$, such that $(T_t : t > 0)$ is simultaneously a semigroup of linear contractions on $L_\infty(X)$ with $T_t^* L_1(X) \subset L_1(X)$ for all $t > 0$. Assume either that $\mu(A) > 0$ implies $\lim_{t\to 0} \|T_t 1_A\|_\infty > 0$, or that T_t on $L_p(X)$ converges strongly to a linear contraction T_0 on $L_p(X)$ as $t \to 0$. Then for any $f \in L_\infty(X)$ there exists a scalar function $T_t f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathfrak{F}$, such that for each fixed $t > 0$, $T_t f(x)$ as a function of x belongs to the equivalence class of $T_t f$, and the following local ergodic limit*

$$\lim_{b\to 0} \frac{1}{b} \int_0^b T_t f(x) \, dt \tag{4}$$

exists and is finite a. e. on X.

REMARK. It is known (cf. [2], [10], [11]) that if $\|T_t\|_p \leqq 1$ and $\|T_t\|_\infty \leqq 1$ for all $t>0$, then T_t on $L_p(X)$ converges strongly to a linear contraction T_0 on $L_p(X)$ as $t \to 0$, and furthermore for every $f \in L_p(X)$ we have

$$T_0 f(x) = \lim_{b \to 0} \frac{1}{b} \int_0^b T_t f(x) dt \quad \text{a.e.} \tag{5}$$

PROOF. We shall first prove that the semigroup $(T_t^*: t>0)$ on $L_1(X)$ is strongly continuous. To do this, by the Riesz convexity theorem we may assume that $1<p<\infty$. Let q be such that $1/p+1/q=1$. Let $f \in L_1(X) \cap L_\infty(X)$ and $s>0$ be given. Since we have

$$\lim_{t \to s+0} \|T_t^* f - T_s^* f\|_q = 0,$$

any strictly decreasing sequence (s_n) of positive reals, with $\lim_n s_n = s$, has a subsequence (t_n) such that

$$\lim_n T_{t_n}^* f(x) = T_s^* f(x) \quad \text{a.e.}$$

Since $\|T_{t_n}^* f\|_1 = \|T_{t_n-s}^*(T_s^* f)\|_1 \leqq \|T_s^* f\|_1$, we then have, by Fatou's lemma, that

$$\lim_n \|T_{t_n}^* f\|_1 = \|T_s^* f\|_1$$

and consequently that

$$\lim_n \|T_{t_n}^* f - T_s^* f\|_1 = 0.$$

This shows that $\lim_{t \to s+0} \|T_t^* f - T_s^* f\|_1 = 0$, and hence we can apply a standard approximation argument to infer that the semigroup $(T_t^*: t>0)$ on $L_1(X)$ is strongly continuous.

Since the adjoint semigroup acting on $L_\infty(X)$ of the semigroup $(T_t^*: t>0)$ on $L_1(X)$ is identical with the original semigroup $(T_t: t>0)$ on $L_\infty(X)$, if we assume that $\mu(A)>0$ implies $\lim_{t \to 0} \|T_t 1_A\|_\infty > 0$, then the desired conclusion follows immediately from Theorem 1.

Next let us assume that T_t on $L_p(X)$ converges strongly to a linear contraction T_0 on $L_p(X)$ as $t \to 0$. (Again we may assume here that $1<p<\infty$.) Then T_t^* on $L_q(X)$ converges strongly to T_0^* on $L_q(X)$ as $t \to 0$, and therefore the semigroup $(T_t^*: t \geqq 0)$ on $L_q(X)$ is strongly continuous on $[0, \infty)$. So we observe, as above, that the semigroup $(T_t^*: t \geqq 0)$ on $L_1(X)$ is again strongly continuous on $[0, \infty)$.

Since $T_0^{*2} = T_0^*$ and $\|T_0^*\|_1 \leqq 1$, we can choose a function f_0 in $L_1(X)$, with $T_0^* f_0 = f_0$, so that $\operatorname{supp} g \subset \operatorname{supp} f_0$ for all $g \in L_1(X)$ with $T_0^* g = g$ (cf. [1], [4]). Put

$$C = \operatorname{supp} f_0 \quad \text{and} \quad D = X - C.$$

Since $\|T_0^*\|_q \leqq 1$, for any $f \in L_1(X) \cap L_q(X)$ with $\operatorname{supp} f \subset D$ and any $\varepsilon > 0$ we have, as in [1], that

$$(1+\varepsilon)^q \int |T_0^* f|^q d\mu = \int |T_0^*(T_0^* f + \varepsilon f)|^q d\mu$$

$$\leqq \int |T_0^* f + \varepsilon f|^q d\mu$$

$$= \int |T_0^* f|^q d\mu + \varepsilon^q \int |f|^q d\mu ,$$

because $\operatorname{supp} T_0^* f$ is contained in C and hence disjoint from $\operatorname{supp} f \subset D$. Thus, letting $\varepsilon \to 0$, we must conclude that $\|T_0^* f\|_q = 0$. Hence it follows that

$$T_0^* L_1(D) = \{0\} \quad \text{and} \quad T_0^* L_1(C) \subset L_1(C) ,$$

so that $T_0 L_\infty(D) = \{0\}$ and $T_0 L_\infty(C) \subset L_\infty(C)$. Since $T_t T_0 = T_0 T_t = T_t$ on $L_\infty(X)$ for all $t \geqq 0$, it then follows that

$$T_t L_\infty(D) = \{0\} \quad \text{and} \quad T_t L_\infty(C) \subset L_\infty(C) \quad (t \geqq 0) .$$

Therefore without loss of generality we may assume that $X = C$. Then for any $A \in \mathfrak{F}$ with $\mu(A) > 0$ we have

$$\lim_{t \to 0} \int_A |T_t^* f_0| \, d\mu = \int_A |T_0^* f_0| \, d\mu = \int_A |f_0| \, d\mu > 0 ,$$

and thus Theorem 1 completes the proof.

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