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Rate of decay of local energy and wave operators for symmetric systems

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§ 1. Introduction.

A large class of wave propagation phenomena of classical physics and quantum mechanics are governed by "symmetric systems" of partial differential equations of the form

$$\frac{1}{i}\frac{\partial u}{\partial t}=M(x)(P(D)+\sum_{j=1}^{K}q_j(x)Q_j(D))u\,. \tag{1.1}$$

Here $x\in \boldsymbol{R}^n$, $t\in \boldsymbol{R}$, $D=-i\partial/\partial x$, $u(x, t)$ is a $\boldsymbol{C}^m$-valued function, $P(D)+\sum_{j=1}^{K}q_j(x)Q_j(D)$ is a self-adjoint differential operator in $[L_2(\boldsymbol{R}^n)]^m$, and $M(x)$ is an $m\times m$ Hermitian matrix with

$$C|\xi|^2\leqq(M(x)\xi, \xi)\leqq C^{-1}|\xi|^2, \qquad x, \xi\in \boldsymbol{R}^n \tag{1.2}$$

for some positive constant C.

In this paper we study the asymptotic behavior as $t\rightarrow\infty$ of the solution of the system (1.1) with initial value having finite energy. In doing so, we compare the system (1.1) with the unperturbed system

$$\frac{1}{i}\frac{\partial u}{\partial t}=P(D)u\,, \tag{1.3}$$

assuming that for some $s>1$ and $C>0$

$$|M(x)-I|+\sum_{j=1}^{K}|q_j(x)|\leqq C(1+|x|^2)^{-s/2}, \qquad x\in \boldsymbol{R}^n\,. \tag{1.4}$$

Here I is the unit matrix, and $|A|$ denotes the norm of an $m\times m$ matrix A: $|A|=(\sum_{i,j=1}^{m}|A_{ij}|^2)^{1/2}$.

Let $\boldsymbol{H}_0$ and $\boldsymbol{H}$ be Hilbert spaces with inner products

$$(f, g)_{\boldsymbol{H}_0}=\int_{\boldsymbol{R}^n}f(x)\overline{g(x)}dx, \qquad f, g\in[L_2(\boldsymbol{R}^n)]^m \tag{1.5}$$

and

$$(f, g)_{\boldsymbol{H}}=\int_{\boldsymbol{R}^n}M(x)^{-1}f(x)\overline{g(x)}dx\,, \qquad f, g\in[L_2(\boldsymbol{R}^n)]^m\,, \tag{1.6}$$

respectively. By virtue of (1.2) these spaces are equal as vector spaces and have equivalent norms. Let J be the identification operator from $\boldsymbol{H}_0$ to $\boldsymbol{H}$ defined by $(Jf)(x)=f(x)$. Let H be a self-adjoint realization in $\boldsymbol{H}$ of the formal differential operator appearing on the right side of (1.1), and H_0 be the natural self-adjoint realization in $\boldsymbol{H}_0$ of the differential operator $P(D)$. The wave operators $W_\pm$ are defined by

$$W_\pm = s\text{-}\lim_{t\to\pm\infty} e^{-itH} J e^{itH_0} E_{0,ac}, \tag{1.7}$$

where $E_{0,ac}$ denotes the projection of $\boldsymbol{H}_0$ onto the subspace of absolute continuity for H_0 (c.f. [7]). Assuming (1.3), we shall show that the wave operators exist, and are partially isometric with initial set $E_{0,ac}\boldsymbol{H}_0$. (Thus $W_\pm$ are unitary operators from $E_{0,ac}\boldsymbol{H}_0$ onto the ranges $R(W_\pm)$ of $W_\pm$.) Furthermore, we shall give an estimation of the rate of convergence of the limit (1.7).

We see easily that the local energy of the solution $e^{itH}\varphi$ with $\varphi\in R(W_\pm)$ decays as $t\to\pm\infty$. The main purpose of this paper is to investigate the rate of decay of such solution. We shall show, roughly speaking, that the local energy of $e^{itH}\varphi$ decays as $t\to\pm\infty$ so fast as $(W_\pm^{-1}\varphi)(x)$ decays when $|x|$ approaches to infinity. The algebraic decay will be dealt under the assumption (1.3), and the exponential decay will be dealt under the assumption

$$|M(x)-I| + \sum_{j=1}^{K} |q_j(x)| \leqq Ce^{-a|x|}, \qquad x\in\boldsymbol{R}^n, \tag{1.8}$$

where a and C are positive constants.

But what are the ranges $R(W_\pm)$? Since $W_\pm$ are partially isometric, $R(W_\pm)$ are contained in the subspace of absolute continuity for the operator H. We could not show, in general, that $R(W_\pm)$ are equal to it. (If they are, $W_\pm$ are said to be complete.) But in many cases of practical importance it can be shown that

$$R(W_+) = R(W_-) = (\boldsymbol{H}^p)^\perp, \tag{1.9}$$

where $(\boldsymbol{H}^p)^\perp$ is the orthogonal complement of the closed subspace spanned by eigenfunctions of the operator H. We shall show, for example, that (1.9) holds for systems of constant deficit.

The symmetric hyperbolic system

$$\frac{1}{i}\frac{\partial u}{\partial t} = M(x)P(D) = M(x)\sum_{j=1}^{n} A_j D_j \tag{1.10}$$

is called a system of constant deficit if the rank of the matrix $P(\xi)$ is constant for all $\xi\neq 0$ (see [19]). If it satisfies the additional condition that every root of the equation $\det(\lambda I - P(\xi))=0$ has constant multiplicity, then it is called a uniformly propagative system (see [23]). Systems of constant deficit include

Maxwell's equations in crystals, the equations of acoustics, and the equation of elasticity. Later we shall give a detailed discussion on these systems in order to illustrate the scope of the main theorems.

So far as scalar partial differential equations are concerned, spectral and scattering theory has been developed to a satisfactory extent (see [**8**, **10**]). As for systems of partial differential equations, however, it seems that some important problems are left open. Wilcox [**23**] introduced the notion of uniformly propagative system, and proved the existence of the wave operators for the system under (1.4). The completeness for the system was proved by Suzuki [**20**] and Yajima [**24**]. Ikebe [**4**] introduced the notion of system of constant deficit, and proved the existence of the wave operators for the system, roughly speaking, under the condition (1.4) with $s>n/2+2$. In [**5**], the completeness has been proved by himself under the additional condition that all roots of $\det(\lambda I-P(\xi))=0$ are smooth. The completeness for the system was also proved by Schulenberger [**19**] under the condition (1.4) with $s>n$ (see also [**11**]). In this paper the existence and completeness of the wave operators for the system will be proved for $s>1$. Avilla [**1**] treated the system (1.10) without assuming that the rank of $P(\xi)$ is constant for all $\xi\neq 0$, and showed the existence of the wave operators for the system assuming (1.4) for $s>n/2+2$. In this paper that for the system (1.1), which includes the one treated in [**1**], will be proved for $s>1$. But the problem of completeness is still open for the systems such that the rank of $P(\xi)$ is not constant for all $\xi\neq 0$, which include, for example, the equation of magnetgasdynamics (see [**1**]). It should be remarked that the existence and completeness of the wave operators for the symmetric hyperbolic systems including Dirac's equation with compact potential has been shown by Lax-Phillips [**12**]. For Dirac's equation with potential decreasing as $|x|^{-1-\varepsilon}$, that was proved by Yamada [**25**]. In this paper his result will be extended to more general systems.

The rate of decay of the local energy of solutions of initial value problems has been studied by many mathematicians (see [**12**], [**17**], [**21**], and references there). Using the abstract theory developed in [**12**], Lax and Phillips gave sufficient conditions under which the solutions of some hyperbolic equations in an odd dimensional Euclidean space decay exponentially. Vainberg [**21**] treated the strongly hyperbolic systems which are homogeneous hyperbolic systems with constant coefficients in the exterior of a compact set, and gave the rate of decay of the solution with initial value having compact support, assuming that each bicharacteristic curve of the system tends to infinity. In [**17**], the author has treated (1.1) in the special case that H is a self-adjoint scalar elliptic partial differential operator with coefficients satisfying (1.4). Assuming that there exist no "generalized" eigenvalues of $\{H,$

he has investigated the algebraic decay of the operator norm

$$\|(1+|x|^2)^{-s\theta/2}e^{itH}E_{ac}(1+|x|^2)^{-s\theta/2}\|_{B(L_2(R^n))}, \tag{1.11}$$

where $0\leqq\theta\leqq 1$, $s>1$, E_{ac} is the projection of $\boldsymbol{H}$ onto the subspace of absolute continuity for H, and $\boldsymbol{B}(L_2(\boldsymbol{R}^n))$ is the set of all bounded linear operators in $L_2(\boldsymbol{R}^n)$. It was shown there, for example, that the operator norm (1.11) decays as $t^{-(n/2)\theta}$ when $s=(n+1)/2$ and H is the Schrödinger operator in $\boldsymbol{R}^n$ $(n\geqq 3)$ with potential decreasing like $(1+|x|^2)^{-(n+1)/2}$. In this paper we shall investigate the rate of decay of the solution of (1.1) without such assumption that there exist no generalized eigenvalues of H and that each bicharacteristic curve tends to infinity. Instead of studying (1.11), we shall investigate the algebraic or exponential decay of the operator norm

$$\|A^\theta e^{itH}W_\pm(1-\Delta)^{-\gamma\theta}A^\theta\|_{B(H_0,H)},$$

where $0\leqq\theta\leqq 1$, γ is a positive constant, $-\Delta=D_1^2+\cdots+D_n^2$, and A is a mutiplication operator $(1+|x|^2)^{-s/2}$ or $\exp(-a|x|)$.

The remainder of this paper is organized as follows. §2 is a preparatory section. There we shall give a spectral representation of the operator H_0, an algebraic lemma concerning the critical values of the root $\lambda_j(\xi)$ of the equation $\det(\lambda I-P(\xi))=0$, interpolation theorems for some functionspaces, and the characterization of the Fourier image of a function decreasing exponentially at infinity. Propositions 2.10, 2.14. and 2.15 concerning the characterization of such function may be of independent interest. We note that the similar results have already been given in [**16**]. In §3 we shall show the existence of the wave operators. In §4 we shall study the rate of decay of the solution of (1.1). There the rate of convergence of the limit (1.7) will also be discussed. In §5 we shall study the completeness of the wave operators and the spectral properties of the operator H. In Appendix we shall show an interpolation theorem for weighted Sobolev spaces.

§2. Preliminaries.

In this section we present interpolation theorems and lemmas concerning the spectral representation of the operator H_0.

Throughout this paper the following notations will be used. For a Banach space X and an open set Ω in $\boldsymbol{R}^n$, $C_0^\infty(\Omega; X)$ denotes the space of all X-valued infinitely differentiable functions on $\boldsymbol{R}^n$ with compact support contained in Ω. $\mathcal{D}'(\Omega; X)$ denotes the space of X-valued distributions on Ω. $\mathcal{S}(\boldsymbol{R}^n; X)$ denotes the space of X-valued rapidly decreasing functions on $\boldsymbol{R}^n$, and $\mathcal{S}'(\boldsymbol{R}^n; X)$ denotes the space of X-valued tempered distributions. For a Hilbert space-valued tempered distribution f, the Fourier transform of f will be denoted by $\mathcal{F}f$ or $\hat{f}$, and $\mathcal{F}^{-1}f$ will be denoted by $\tilde{f}$. For $\sigma\in\boldsymbol{R}$, we put

$$H^{\sigma}(\boldsymbol{R}^n)=\{f\in\mathcal{S}'(\boldsymbol{R}^n\,;\,\boldsymbol{C}^m)\,;\ \|f\|_{H^{\sigma}(\boldsymbol{R}^n)}=\|(1-\Delta)^{\sigma/2}f\|_{L_2(\boldsymbol{R}^n;\,\boldsymbol{C}^m)}<\infty\}\ .$$

For a Banach space X and an open set $\Omega\subset\boldsymbol{R}^n$, $\mathcal{O}(\boldsymbol{R}^n\times i\Omega\,;\,X)$ denotes the space of all X-valued holomorphic functions on $\boldsymbol{R}^n\times i\Omega$. We shall sometimes write $\boldsymbol{R}^n\times i\Omega=T_{\Omega}$. For $\sigma\in\boldsymbol{R}$ and an open convex bounded set Ω with $0\in\Omega$, we put

$$\mathcal{H}^{\sigma}(T_{\Omega})=\{f\in\mathcal{O}(T_{\Omega}\,;\,\boldsymbol{C}^m)\,;\ \|f\|_{\mathcal{H}^{\sigma}(T_{\Omega})}=\|(1-\Delta_{\xi})^{\sigma/2}f(\xi+i\eta)\|_{L_2(\boldsymbol{R}^n\times\Omega)}<\infty\}\ .$$

We observe that $\mathcal{H}^{\sigma}(T_{\Omega})$ is a Banach space since $\bar{\partial}$ is an elliptic system. For Banach spaces X and Y, $\boldsymbol{B}(X, Y)$ denotes the space of all bounded linear operators from X to Y. X' denotes the dual space of X. For a densely defined closed linear operator T from X to Y, T^* denotes the adjoint operator of T. $D(T)$, $R(T)$, $\sigma_p(T)$, and $\rho(T)$ denote the domain, range, point spectrum, and resolvent set of T, respectively. For $\zeta\in\rho(T)$, $R(\zeta\,;\,T)=(T-\zeta)^{-1}$. For $0<\theta<1$ and $1\leqq q\leqq\infty$, $(X, Y)_{\theta,q}$ and $[X, Y]_{\theta}$ denote the mean interpolation space and the complex one, respectively (see [**2**]).

Now we state some spectral properties of the operator H_0. For the $m\times m$ Hermitian matrix $P(\xi)$, we put $p(\lambda, \xi)=\det(\lambda I-P(\xi))$. Decompose it into irreducible factors R_j $(j=1, \cdots, l)$: $p=R_1^{m_1}\cdots R_l^{m_l}$ $(R_i\not\equiv R_j$ if $i\neq j)$. By requiring that the coefficient of highest power of λ in each R_j be 1, we can determine the factors R_j $(j=1, \cdots, l)$ uniqeely apart from their order. Put

$$R=R_1\times\cdots\times R_l\,. \tag{2.1}$$

We denote the discriminant of $R(\lambda, \xi)$ by $S(\xi)$, which is not identically zero. We ennumerate the roots of the equation $p(\lambda, \xi)=0$ which are not identically constants as

$$\lambda_1(\xi)\leqq\cdots\leqq\lambda_r(\xi)\,,$$

where $\lambda_i\not\equiv\lambda_j$ if $i\neq j$. The remaining roots $\lambda_j(\xi)$ $(j=r+1, \cdots, k)$ are identically constants: $\lambda_j(\xi)=a_j$, $j=r+1, \cdots, k$. It is easily seen that

(2.2) for any $j=r+1, \cdots, k$ there exists j' such that $R_{j'}(\lambda, \xi)=\lambda-a_j$;

(2.3) $\lambda_j(\xi)$ is locally Hölder continuous on $\boldsymbol{R}^n$ and real analytic in $\{\xi\in\boldsymbol{R}^n\,;\,S(\xi)\neq 0\}$;

(2.4) if $i\neq j$, $\lambda_i(\xi)\neq\lambda_j(\xi)$ for any $\xi\in\{S(\xi)\neq 0\}$.

We put

$$\Lambda_e=\{\lambda\in\boldsymbol{R}\,;\,\det(\lambda I-P(\xi))=S(\xi)=0 \text{ for some } \xi\in\boldsymbol{R}^n\}\cup\{a_j\,;\ j=r+1, \cdots, k\}. \tag{2.5}$$

The set Λ_e, which is called the set of all exceptional values of $P(\xi)$, will be used in §5. Set

$$
F_j(\xi)=\begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_j(\xi)}(P(\xi)-\lambda)^{-1}d\lambda\,, & \xi\in\{S(\xi)\neq 0\} \\ 0\,, & \xi\in\{S(\xi)=0\}\,, \end{cases} \tag{2.6}
$$

where $\gamma_j(\xi)=\{\lambda\in \boldsymbol{C}\,;\ |\lambda-\lambda_j(\xi)|=\min_{i\neq j}|\lambda_i(\xi)-\lambda_j(\xi)|/2\}$. Then we see that

(2.7) $F_j(\xi)$ is a bounded measurable matrix-valued function, and real analytic in $\{S(\xi)\neq 0\}$;

(2.8) for each $\xi\in\{S(\xi)\neq 0\}$, $\sum_{j=1}^{k} F_j(\xi)=I$, $F_iF_j(\xi)=\delta_{ij}F_j(\xi)$ and $F_j^*(\xi)=F_j(\xi)$;

$$
P(\xi)F_j(\xi)=\lambda_j(\xi)F_j(\xi)\,. \tag{2.9}
$$

Hence the spectral measure E_0 associated with H_0 is represented by

$$
E_0(B)=\mathcal{F}^{-1}\Big(\sum_{j=1}^{k}\chi(\lambda_j^{-1}(B)\,;\ \xi)F_j(\xi)\Big)\mathcal{F} \tag{2.10}
$$

for any Borel set B in $\boldsymbol{R}$, where $\chi(\lambda_j^{-1}(B)\,;\ \xi)$ is the characteristic function of the set $\lambda_j^{-1}(B)\subset \boldsymbol{R}^n$. As for the projection $E_{0,ac}$ of $\boldsymbol{H}_0$ onto the subspace of absolute continuity for H_0, we have

PROPOSITION 2.1. $\quad E_{0,ac}=\mathcal{F}^{-1}\Big(\sum_{j=1}^{r} F_j(\xi)\Big)\mathcal{F}\,.$

For the proof we prepare a lemma. Set

$$
T(\xi)=f_1(\xi)\times\cdots\times f_r(\xi)\,,\quad f_j(\xi)=\sum_{i=1}^{n}\Big(\frac{\partial R}{\partial\xi_i}(\lambda_j(\xi),\,\xi)\Big)^2. \tag{2.11}
$$

Then we have

LEMMA 2.2. *$T(\xi)$ is a polynomial which is not identically zero. Furthermore,*

$$
\bigcup_{j=1}^{r}\{\xi\in\boldsymbol{R}^n\,;\ S(\xi)\neq 0,\ \mathrm{grad}\,\lambda_j(\xi)\neq 0\}=\{\xi\in\boldsymbol{R}^n\,;\ S(\xi)\neq 0,\ T(\xi)=0\}\,. \tag{2.12}
$$

PROOF. We have for any $\xi\in\{S(\xi)\neq 0\}$

$$
\frac{\partial\lambda_j}{\partial\xi_i}(\xi)=-\frac{\partial R}{\partial\xi_i}(\lambda_j(\xi),\,\xi)\Big/\frac{\partial R}{\partial\lambda}(\lambda_j(\xi),\,\xi)\,. \tag{2.13}
$$

This implies (2.12). Since $\lambda_j(\xi)$ $(j=1,\cdots,r)$ are not identically constants, it follows from (2.11) and (2.13) that $T(\xi)\not\equiv 0$. We see that T can be extended to $\{\zeta\in\boldsymbol{C}^n\,;\ S(\zeta)\neq 0\}$ as a holomorphic function, which is continuous up to the boundary $\{S(\zeta)=0\}$ and has polynomial growth at infinity. This concludes that T is a polynomial. q.e.d.

PROOF OF PROPOSITION 2.1. We set

$$
P_1=\mathcal{F}^{-1}\Big(\sum_{j=1}^{r}F_j(\xi)\Big)\mathcal{F}\,,\quad P_2=\mathcal{F}^{-1}\Big(\sum_{j=r+1}^{k}F_j(\xi)\Big)\mathcal{F}\,.
$$

It follows easily from (2.7)~(2.9) that $P_1\boldsymbol{H}_0$ and $P_2\boldsymbol{H}_0$ are closed subspaces, are orthogonal complements to each other, and reduce the operator H_0. Since $P_2\boldsymbol{H}_0$ is spanned by eigenfunctions of H_0, it is included in $(E_{0,ac}\boldsymbol{H}_0)^{\perp}$ (see [7]). Thus the proposition follows if we show the inclusion

$$P_1\boldsymbol{H}_0\subset E_{0,ac}\boldsymbol{H}_0.$$

Hence we have only to show that $E_0(B)P_1=0$ for any Borel set B in $\boldsymbol{R}$ whose measure is zero. We have by Lemma 2.2 that λ_j $(j=1, \cdots, r)$ are smooth and $\operatorname{grad}\lambda_j(\xi)\neq 0$ outside the set $\{\xi\in\boldsymbol{R}^n; S(\xi)T(\xi)=0\}$, of which measure is zero. Thus the measure of $\lambda_j^{-1}(B)$ is zero. On the other hand, we obtain by (2.10)

$$\|E_0(B)P_1u\|_{H_0}=\sum_{j=1}^{r}\int_{\lambda_j^{-1}(B)}|F_j(\xi)\hat{u}(\xi)|^2d\xi.$$

Hence if the measure of B is zero, then $E_0(B)P_1=0$. q.e.d.

Next we shall show the following lemma concerning the finiteness of the critical values of the function $\lambda_j|_{\{S(\xi)\neq 0\}}$.

LEMMA 2.3. *The set Λ_c defined by*

$$\Lambda_c=\bigcup_{j=1}^{r}\{\lambda_j(\xi); S(\xi)\neq 0, \operatorname{grad}\lambda_j(\xi)=0\} \tag{2.14}$$

is a finite set.

For the proof we review some facts about real algebraic varieties and elementary symmetric polynomials. The i-th elementary symmetric polynomial of r-variables will be denoted by σ_i.

LEMMA 2.4 ([**22**, Theorems 1 and 4]). *Let $V\subset\boldsymbol{R}^n$ be a real algebraic variety, and V' be a subvariety of V. Then there exist finite number of connected manifolds M^ν such that*

$$V\backslash V'=\bigcup_{\nu} M^\nu.$$

LEMMA 2.5 ([**14**, Lemma 1, p. 111]). *Let f_j $(j=1, \cdots, r)$ be continuous functions on a connected topological space E. If $\sigma_l(f_1(x), \cdots, f_r(x))\neq 0$ on E and $\sigma_i(f_1(x), \cdots, f_r(x))\equiv 0$ on E for $i>l$, then for any j*

$$f_j(x)\neq 0 \quad \text{on } E \text{ or } \quad f_j(x)\equiv 0 \quad \text{on } E.$$

PROOF OF LEMMA 2.3. Set

$$T_i=\sigma_i(f_1, \cdots, f_r),$$

where f_j is the function defined by (2.11). We obtain in the same way as in the proof of Lemma 2.2 that T_i is a polynomial which is not identically zero. Decompose the set $V=\{\xi\in\boldsymbol{R}^n; S(\xi)\neq 0, T(\xi)=0\}$ as follows:

$$V=V_1\cup\cdots\cup V_r, \quad V_i=\{\xi\in\boldsymbol{R}^n; T_r(\xi)=\cdots=T_{r+1-i}(\xi)=0, S(\xi)T_{r-i}(\xi)\neq 0\}.$$

Then we have by Lemma 2.4 that there exist finite number of connected manifolds M_i^ν such that $V_i=\bigcup_\nu M_i^\nu$. Lemma 2.5 implies that for each M_i^ν and j

$$\operatorname{grad}\lambda_j(\xi)\neq 0 \quad \text{on } M_i^\nu \text{ or } \quad \operatorname{grad}\lambda_j(\xi)\equiv 0 \quad \text{on } M_i^\nu .$$

Since M_i^ν is a connected manifold and $\lambda_j(\xi)$ is smooth on M_i^ν, $\lambda_j(\xi)$ must be identically constant on M_i^ν if $\operatorname{grad}\lambda_j(\xi)\equiv 0$ on M_i^ν. This proves the lemma.

Finally we state some interpolation theorems and a characterization of the Fourier image of a function decreasing exponentially at infinity. For a Banach space X, $r\in \boldsymbol{R}$, $1\leqq p, q\leqq\infty$, and a positive measurable function ρ on a measure space $(M, d\mu)$, $L_{p,q}^{r,\rho}(M, d\mu; X)$ denotes the Banach space of all X-valued strongly measurable functions with

$$\|f\|_{L_{p,q}^{r,\rho}(M,d\mu;X)}=\left[\sum_{j=-\infty}^{\infty}\left(\int_{M_j}\|\rho^r f\|_X^p d\mu\right)^{q/p}\right]^{1/q}<\infty ,$$

where $M_j=\{x\in M;\ 2^j\leqq\rho(x)<2^{j+1}\}$. We note here that when $p=q$ it is equal to the usual weighted L_p-space. Furthermore, it is an intermediate space of the weighted L_p-spaces.

PROPOSITION 2.6 ([**17**, Proposition 3.1]). *For any $r_0<r_1$, $0<\theta<1$, and $1\leqq p, q, q_0, q_1\leqq\infty$*

$$(L_{p,q_0}^{r_0,\rho}(M, d\mu; X), L_{p,q_1}^{r_1,\rho}(M, d\mu; X))_{\theta,q}=L_{p,q}^{r_0(1-\theta)+r_1\theta,\rho}(M, d\mu; X).$$

When dx is the usual Lebesgue measure on $\boldsymbol{R}^n$ and $\rho(x)=(1+|x|^2)^{1/2}$, we write $L_{p,q}^{r,\rho}(\boldsymbol{R}^n, dx; X)=L_{p,q}^r(\boldsymbol{R}^n; X)$. We write $L_{p,p}^r(\boldsymbol{R}^n; X)=L_p^r(\boldsymbol{R}^n; X)$. Denoting by $B_{p,q}^\tau(\boldsymbol{R}^n; X)$ the Besov space of X-valued distributions on $\boldsymbol{R}^n$ (see [**2**] and [**15**]), we have

PROPOSITION 2.7 ([**17**, Proposition 3.2]). *Let X be a Hilbert space, $\sigma\in\boldsymbol{R}$, $1\leqq p\leqq 2$, $p'=p(p-1)^{-1}$, and $1\leqq q\leqq\infty$. Then the Fourier transform is a bounded linear operator from $B_{p,q}^\sigma(\boldsymbol{R}^n; X)$ to $L_{p',q}^\sigma(\boldsymbol{R}^n; X)$. Furthermore, it is an isomorphism when $p=p'=2$.*

For σ, $s\in\boldsymbol{R}$, and a positive measurable function ρ on $\boldsymbol{R}^n$, we put

$$(2.15)\qquad H_{s,\rho}^\sigma=\{f\in\mathcal{D}'(\boldsymbol{R}^n; \boldsymbol{C}^m);\ \|f\|_{H_{s,\rho}^\sigma}=\|\rho(x)^s(b-\Delta)^{\sigma/2}f\|_{L_2(\boldsymbol{R}^n;\boldsymbol{C}^m)}<\infty\},$$

where b is a positive constant. Consider the following two cases:

$$(1)\ \rho(x)=(1+|x|^2)^{1/2};\qquad (2)\ \rho(x)=(1+|x|^2)^{r/2}\left(\int_\Omega e^{2x\eta}d\eta\right)^{1/2},$$

where $r\in\boldsymbol{R}$ and Ω is a bounded open convex set in $\boldsymbol{R}^n$ which contains the origin. Then the space $H_{s,\rho}^\sigma$ does not depend on b if $b\geqq 1$ in the first case, and $b\geqq\sup\{|\eta|^2+1;\ \eta\in\Omega\}$ in the second case. In these cases we have

PROPOSITION 2.8. $(L_2(\boldsymbol{R}^n; \boldsymbol{C}^m), H_{s,\rho}^\sigma)_{\theta,2}=H_{s\theta,\rho}^{\sigma\theta}$, σ, $s\geqq 0$, $0<\theta<1$.

In Appendix, we shall give a proof of this proposition in the first case.

In order to treat the second case, we have only to use the following proposition instead of Proposition 2.7.

PROPOSITION 2.9. *Let* $\sigma\in\boldsymbol{R}$ *and* $a_{\Omega,2}(x)=\left(\int_{\Omega}e^{2x\eta}d\eta\right)^{1/2}$. *Then one has for any* $f\in L_2(\boldsymbol{R}^n)$

$$\|a_{\Omega,2}(x)(1+|x|^2)^{\sigma/2}f\|_{L_2(R^n)}=\|\hat{f}\|_{\mathcal{H}^{\sigma}(R^n\times i\Omega)}.$$

We shall show below a precise version of this proposition. For a Banach space X, $1\leqq p$, $q\leqq\infty$, and $\sigma>0$, we put

$$\mathcal{B}^{\sigma}_{p,q}(T_{\Omega};X)=\{f\in\mathcal{O}(T_{\Omega};X);\ \|f\|_{\mathcal{B}^{\sigma}_{p,q}(T_{\Omega};X)}=\|f\|_{B^{\sigma}_{p,q}(R^n\times\Omega;X)}<\infty\}.$$

For $\sigma\leqq 0$,

$$\mathcal{B}^{\sigma}_{p,q}(T_{\Omega};X)=\{(1-\Delta)^k g;\ g\in\mathcal{B}^{\sigma+2k}_{p,q}(T_{\Omega};X)\},\ \sigma+2k>0.$$

Note that $\mathcal{H}^{\sigma}(T_{\Omega})=\mathcal{B}^{\sigma}_{2,2}(T_{\Omega};\boldsymbol{C}^m)$. For $1\leqq p\leqq\infty$, we put $a_{\Omega,p}(x)=\|e^{x\eta}\|_{L_p(\Omega)}$. Now we can state the precise version of Proposition 2.9.

PROPOSITION 2.10. *Let* $1\leqq p\leqq 2$, $p'=p(p-1)^{-1}$, $1\leqq q\leqq\infty$, $\sigma\in\boldsymbol{R}$, *and* X *be a Hilbert space. Then there exists a constant* C *such that*

$$\|a_{\Omega,p}(x)\tilde{f}(x)\|_{L^{\sigma}_{p',q}(R^n;X)}\leqq C\|f\|_{\mathcal{B}^{\sigma}_{p,q}(T_{\Omega};X)},\qquad f\in\mathcal{B}^{\sigma}_{p,q}(T_{\Omega};X).\tag{2.16}$$

Furthermore, for $p=p'=2$

$$\|\hat{f}\|_{\mathcal{B}^{\sigma}_{2,q}(T_{\Omega};X)}\leqq C\|a_{\Omega,2}f\|_{L^{\sigma}_{2,q}(R^n;X)},\qquad f\in L^{\sigma}_{2,q}(\boldsymbol{R}^n;X).\tag{2.17}$$

Before proceeding to the proof we give a comment on the function $a_{\Omega,p}(x)$. We see

$$a_{\Omega,\infty}(x)=\exp(\sup_{\eta\in\Omega}x\eta).$$

Since Ω is an open convex bounded set, geometric consideration yields

$$\begin{aligned}C(1+|x|^2)^{-n/2p}\exp(\sup_{\eta\in\Omega}x\eta)&\leqq a_{\Omega,p}(x)\\&\leqq C^{-1}(1+|x|^2)^{-1/2p}\exp(\sup_{\eta\in\Omega}x\eta),\qquad x\in\boldsymbol{R}^n,\ 1\leqq p<\infty,\end{aligned}\tag{2.18}$$

where C is a positive constant. In general, this is the best possible estimate (consider the case that $\Omega=(-1,1)^n$). But when $\partial\Omega$ is smooth and its Gaussian curvature never vanishes, we can get a more precise estimate. That is, we obtain by the method of stationary phase

$$a_{\Omega,p}(x)=C\exp(\sup_{\eta\in\Omega}x\eta)|x|^{-(n+1)/2p}(1+O(|x|^{-1}))\qquad\text{as }|x|\to\infty,\tag{2.19}$$

where C is a positive constant.

For the proof of Proposition 2.10 we prepare some lemmas. Since the functions treated in the remainder of this section are all X-valued ones, we shall abbreviate, for example, $\mathcal{B}^{\sigma}_{p,q}(T_{\Omega};X)$ to $\mathcal{B}^{\sigma}_{p,q}(T_{\Omega})$. For a non-negative integer k and $1\leqq p\leqq\infty$, we put

$$\mathcal{W}^{k}_{p}(T_{\Omega})=\{f\in\mathcal{O}(T_{\Omega});\ \|f\|_{\mathcal{W}^{k}_{p}(T_{\Omega})}=\|f\|_{W^{k}_{p}(\boldsymbol{R}^{n}\times\Omega)}<\infty\}\ ,$$

where $W^{k}_{p}(\boldsymbol{R}^{n}\times\Omega)$ is the usual Sobolev space of functions on $\boldsymbol{R}^{n}\times\Omega$. We write $\mathcal{W}^{0}_{p}(T_{\Omega})=\mathcal{L}_{p}(T_{\Omega})$. The following lemma can be shown in the same way as in the proof of the interpolation theorem for the usual Besov space (see [**2**, p. 258]).

LEMMA 2.11. *For $1\leqq p,\ q\leqq\infty$ and $0<\sigma<k$,*

$$\mathcal{B}^{\sigma}_{p,q}(T_{\Omega})=(\mathcal{L}_{p}(T_{\Omega}),\ \ \mathcal{W}^{k}_{p}(T_{\Omega}))_{\sigma/k,q}\,.$$

As with the L_p-space, the space $\mathcal{L}_{p}(T_{\Omega})$ $(1<p<2)$ is an intermediate space of $\mathcal{L}_{2}(T_{\Omega})$ and $\mathcal{L}_{1}(T_{\Omega})$.

LEMMA 2.12. *For $1<p<2$, $\mathcal{L}_{p}(T_{\Omega})=(\mathcal{L}_{2}(T_{\Omega}),\ \mathcal{L}_{1}(T_{\Omega}))_{2/p-1,\,p}$.*

PROOF. We see that the interpolation space considered is a closed subspace of $\mathcal{L}_{p}(T_{\Omega})$. But the Fourier transform of the space $C^{\infty}_{0}(\boldsymbol{R}^{n})$ is included in it and dense in $\mathcal{L}_{p}(T_{\Omega})$. This proves the lemma.

Cauchy's theorem yields

LEMMA 2.13. *For $\sigma\in\boldsymbol{R}$, $1\leqq p,\ q\leqq\infty$, a non-negative integer k, and a compact set K in Ω, there exists a constant C such that*

$$\sup_{\eta\in K}\|f(\cdot+i\eta)\|_{W^{k}_{p}(\boldsymbol{R}^{n})}\leqq C\|f\|_{\mathcal{B}^{\sigma}_{p,q}(T_{\Omega})}\,.$$

Now we can give a proof of Proposition 2.10.

PROOF OF PROPOSITION 2.10. Parseval's equality shows that for any $f\in\mathcal{L}_{2}(T_{\Omega})$

$$\int|f(\xi+i\eta)|^{2}d\xi=\int|\tilde{f}(x)e^{x\eta}|^{2}dx\,. \tag{2.20}$$

Here we used the fact that $f(\cdot+i\eta)\in L_{2}(\boldsymbol{R}^{n})$ for each $\eta\in\Omega$, which is a conclusion of Lemma 2.13. Integrating (2.20) with respect to η, we obtain

$$\|a_{\Omega,2}(x)\tilde{f}(x)\|_{L_{2}(\boldsymbol{R}^{n})}=\|f\|_{\mathcal{L}_{2}(T_{\Omega})}\,. \tag{2.21}$$

Similarly,

$$\|a_{\Omega,1}(x)\tilde{f}(x)\|_{L_{\infty}(\boldsymbol{R}^{n})}\leqq C\|f\|_{\mathcal{L}_{1}(T_{\Omega})}\,. \tag{2.22}$$

Combining (2.21) and (2.22), we obtain by Lemma 2.12 and Lemma B.2 in [**15**]

$$\|(a_{\Omega,2})^{2/p'}(a_{\Omega,1})^{2/p-1}\tilde{f}\|_{L_{p'}(\boldsymbol{R}^{n})}\leqq C\|f\|_{\mathcal{L}_{p}(T_{\Omega})}\,,\qquad 1\leqq p\leqq 2\,.$$

On the other hand,

$$a_{\Omega,p}(x)\leqq\|e^{x\eta}\|^{2/p'}_{L_{2}(\Omega)}\|e^{x\eta}\|^{2/p-1}_{L_{1}(\Omega)}=(a_{\Omega,2}(x))^{2/p'}(a_{\Omega,1}(x))^{2/p-1}\,.$$

Hence

$$\|a_{\Omega,p}\tilde{f}\|_{L_{p'}(R^n)} \leqq C\|f\|_{\mathcal{L}_p(T_\Omega)}.$$

Since $\mathcal{F}^{-1}((1-\Delta)^k f)=(1+|x|^2)^k \tilde{f}(x)$ for any non-negative integer k, this implies

$$\|a_{\Omega,p}(x)(1+|x|^2)^k \tilde{f}(x)\|_{L_{p'}(R^n)} \leqq C\|f\|_{\mathcal{W}_p^{2k}(T_\Omega)}.$$

Proposition 2.6 and Lemma 2.11 now show the inequality (2.16) for $\sigma>0$, from which that for $\sigma\leqq 0$ follows easily. The inequality (2.17) is now obvious.

q. e. d.

The following proposition is a precise version of Lemma 2.13, which describes the rate of divergence at the boundary $\partial\Omega$ of a function $f\in\mathcal{B}_{p,q}^{\sigma}(T_\Omega)$.

PROPOSITION 2.14. *For any* $\sigma<k$ *there exists a constant* C *such that*

$$\left\{\sum_{j=-\infty}^{\infty}((2^j)^{k-\sigma}\|f\|_{W_p^k(R^n\times\Omega_j)})^q\right\}^{1/q} \leqq C\|f\|_{\mathcal{B}_{p,q}^{\sigma}(T_\Omega)}, \qquad 1\leqq p,\ q\leqq\infty, \tag{2.23}$$

where $\Omega_j=\{\eta\in \boldsymbol{R}^n;\ 2^j<\mathrm{dist}(\eta,\partial\Omega)<2^{j+1}\}$.

PROOF. For any $f\in\mathcal{L}_p(T_\Omega)$, we obtain by Cauchy's theorem

$$\|D^\alpha f\|_{L_p(R^n\times\Omega_j)} \leqq C(2^j)^{-|\alpha|}\|f\|_{L_p(R^n\times(\Omega_{j-1}\cup\Omega_j\cup\Omega_{j+1}))}.$$

Denoting $\rho(\eta)=\mathrm{dist}(\eta,\partial\Omega)$, we have

$$\|\rho(\eta)^{|\alpha|}D^\alpha f\|_{L_p(R^n\times\Omega)} \leqq C\|f\|_{\mathcal{L}_p(T_\Omega)}.$$

On the other hand,

$$\|D^\alpha f\|_{L_p(R^n\times\Omega)} \leqq C\|f\|_{\mathcal{W}_p^{|\alpha|}(T_\Omega)}.$$

Thus application of Proposition 2.6 and Lemma 2.11 gives

$$\|D^\alpha f\|_{L_{p,q}^{|\alpha|-\sigma,\rho}(R^n\times\Omega)} \leqq C\|f\|_{\mathcal{B}_{p,q}^{\sigma}(T_\Omega)}, \qquad 0<\sigma<|\alpha|.$$

This proves the inequality (2.23) for $\sigma>0$. The one for $\sigma\leqq 0$ can be shown in the same way as above. q. e. d.

The inequality opposite (2.23) does not hold for all $1\leqq p\leqq\infty$. For example, $|(\eta-1)(d/dz)\log(z-i)|\leqq 1$, but $\log(z-i)\notin\mathcal{L}_\infty(\boldsymbol{R}\times i(-1,1))$. For $p=2$, however, the reverse one is valid because of (2.17) and the following proposition. In order to state it we introduce a function space. For $1\leqq p,\ q\leqq\infty$ and $s\in\boldsymbol{R}$, $\mathcal{L}_{p,q}^s(T_\Omega)$ denotes the Banach space of all holomorphic functions f on T_Ω with

$$\|f\|_{\mathcal{L}_{p,q}^s(T_\Omega)}=\left\{\sum_{j=-\infty}^{\infty}((2^j)^s\|f\|_{L_p(R^n\times\Omega_j)})^q\right\}^{1/q}<\infty,$$

where $\Omega_j=\{\eta\in\boldsymbol{R}^n;\ 2^j<\mathrm{dist}(\eta,\partial\Omega)<2^{j+1}\}$.

PROPOSITION 2.15. *Let* $1\leqq p\leqq 2$, $p'=p(p-1)^{-1}$, $1\leqq q\leqq\infty$, *and* $\sigma<2k$. *Then there exists a constant* C *such that for any* $f\in\mathcal{O}(T_\Omega)$ *with* $(1-\Delta)^k f\in\mathcal{L}_{p,q}^{2k-\sigma}(T_\Omega)$

$$\|a_{\Omega,p}\tilde{f}\|_{L_{p',q}^{\sigma}(R^n)} \leqq C\|(1-\Delta)^k f\|_{\mathcal{L}_{p,q}^{2k-\sigma}(T_\Omega)}.$$

PROOF. Since

$$\mathcal{L}^{s}_{p,q}(T_\Omega)=(\mathcal{L}_p(T_\Omega),\ \mathcal{L}^{j}_{p}(T_\Omega))_{s/j,q}\,, \qquad 0<s<j\,,\ \ 1\leqq p,\ q\leqq\infty\,,$$

we have only to show the inequality

$$\|a_{\Omega,p}(x)(1+|x|^2)^{-j/2}\tilde{f}(x)\|_{L_{p'}(R^n)}\leqq C\|f\|_{\mathcal{L}^{j}_{p}(T_\Omega)}\,, \qquad 1\leqq p\leqq 2\,. \tag{2.24}$$

First we claim that there exists a constant C such that

$$(1+|x|^2)^{-j/2}\int_\Omega e^{x\eta}d\eta\leqq C\int_\Omega e^{x\eta}(\mathrm{dist}(\eta,\ \partial\Omega))^j d\eta\,. \tag{2.25}$$

Choosing $\delta>0$ so small as $\{|\eta|\leqq\delta\}\subset\Omega$, we put

$$m=[\min\{\delta^2(|\eta|^2+\delta^2)^{-1/2}\,;\ \eta\in\partial\Omega\}]^{-1}\,.$$

We see that $\{(1-m|x|^{-1})\eta\,;\ \eta\in\Omega\}\subset\{\eta\,;\ \mathrm{dist}(\eta,\ \partial\Omega)>|x|^{-1}\}$ for any sufficiently large x. Thus

$$\int_\Omega e^{x\eta}(\mathrm{dist}(\eta,\ \partial\Omega))^j d\eta\geqq|x|^{-j}\|e^{x\eta}\|_{L_1(\{\mathrm{dist}(\eta,\partial\Omega)>|x|^{-1}\})}$$

$$\geqq|x|^{-j}\|e^{x\eta}\|_{L_1(\{(1-m|x|^{-1})\eta;\ \eta\in\Omega\})}\geqq C|x|^{-j}\int_\Omega e^{x\eta}d\eta\,.$$

This proves (2.25). Using it we obtain

$$\begin{aligned}&\|a_{\Omega,1}(x)(1+|x|^2)^{-j/2}\tilde{f}(x)\|_{L_\infty(R^n)}\\&\leqq C\Big\|\int_\Omega e^{x\eta}(\mathrm{dist}(\eta,\ \partial\Omega))^j d\eta\tilde{f}(x)\Big\|_{L_\infty(R^n)}\leqq C\|f\|_{\mathcal{L}^{j}_{1}(T_\Omega)}\,.\end{aligned} \tag{2.26}$$

Similarly,

$$\|a_{\Omega,2}(x)(1+|x|^2)^{-j/2}\tilde{f}(x)\|_{L_2(R^n)}\leqq C\|f\|_{\mathcal{L}^{j}_{2}(T_\Omega)}\,. \tag{2.27}$$

Since $\mathcal{L}^{j}_{p}(T_\Omega)=(\mathcal{L}^{j}_{2}(T_\Omega),\ \mathcal{L}^{j}_{1}(T_\Omega))_{2/p-1,p}$ for each $p\in(1,\ 2)$, (2.26) and (2.27) imply (2.24). q. e. d.

§ 3. Existence of the wave operators.

In this section we shall prove the existence of the wave operators defined by (1.7). We assume throughout the present paper that the differential operator

$$M(x)(P(D)+\sum_{j=1}^{K}q_j(x)Q_j(D)) \tag{3.1}$$

satisfies the following conditions (A.I) and (A.II). The condition (A.II), which is essential in our argument, means that the perturbation is a short range one.

(A.I) (i) $M(x)$ is an $m\times m$ Hermitian matrix-valued measurable function such that for some positive constant C

$$CI\leqq M(x)\leqq C^{-1}I, \qquad x\in \boldsymbol{R}^n;$$

(ii) $P(\xi)$ is a polynomial of degree N with $m\times m$ matrix coefficients, and is Hermitian symmetric for each $\xi\in \boldsymbol{R}^n$; (iii) $\sum_{j=1}^{K} q_j(x)Q_j(D)$ is a differrential operator with bounded measurable $m\times m$ matrix coefficients, and its restriction to $\mathcal{S}(\boldsymbol{R}^n; \boldsymbol{C}^m)$ is symmetric in the Hilbert space $\boldsymbol{H}_0$.

(A.II) There exists a constant $s>1$ such that

$$(3.2) \qquad (|M(x)-I|+\sum_{j=1}^{K}|q_j(x)|)(1+|x|^2)^{s/2}\in L_\infty(\boldsymbol{R}^n).$$

Furthermore we assume the existence of a self-adjoint realization of the operator (3.1), which follows from (A.I) under such additional condition that all q_j are identically zero or (3.1) is elliptic. That is, we assume

(A.III) There exists a self-adjoint extension in $\boldsymbol{H}$ of the restriction L of the differential operator (3.1) to $\mathcal{S}(\boldsymbol{R}^n; \boldsymbol{C}^m)$.

Under the assumptions (A.I), (A.II), and (A.III), we have

THEOREM 3.1. *For any self-adjoint extension H of the operator L, the wave operators*

$$(3.3) \qquad W_\pm = s\text{-}\lim_{t\to\pm\infty} e^{-itH}Je^{itH_0}E_{0,ac}$$

exist and are isometric on $E_{0,ac}\boldsymbol{H}_0$:

$$(3.4) \qquad \|W_\pm u\|_{H}=\|u\|_{H_0}, \qquad u\in E_{0,ac}\boldsymbol{H}_0.$$

REMARK 3.2. It follows from the proof below that the conclusion of the theorem is valid even if $q_j(x)$ has singularity. Precisely, (A.II) can be weakened to the following condition: There exists a constant $s>1$ such that

$$(|M(x)-I|+\sum_{j=1}^{K}|q_j(x)|)(1+|x|^2)^{s/2}\in L_\infty(\boldsymbol{R}^n)+L_2(\boldsymbol{R}^n).$$

For the proof of Theorem 3.1 we prepare a lemma. Put $\Omega=\{\xi\in\boldsymbol{R}^n; S(\xi)T(\xi)\neq 0\}$, where S is the discriminant and T is the polynomial defined by (2.11). Then we have

LEMMA 3.3 *For any* $\varphi\in\mathcal{S}(\boldsymbol{R}^n; \boldsymbol{C}^m)$ with $\hat{\varphi}\in C_0^\infty(\Omega)$ *and* $s>0$, *there exists a constant C such that*

$$(3.5) \qquad \|(1+|x|^2)^{-s/2}e^{itH_0}E_{0,ac}\varphi\|_{L_p(\boldsymbol{R}^n)}\leqq C(1+|t|)^{-s}, \quad 2\leqq p\leqq\infty, \ t\in\boldsymbol{R}.$$

PROOF. We obtain by Proposition 2.1

$$e^{itH_0}E_{0,ac}\varphi(x)=\sum_{j=1}^{r}(2\pi)^{-n/2}\int e^{it\lambda_j(\xi)}e^{ix\xi}F_j(\xi)\hat{\varphi}(\xi)d\xi.$$

Since λ_j and F_j are smooth and grad $\lambda_j(\xi)\neq 0$ in Ω, we can reduce the proof by partition of unity to the estimate of the function

$$f(t, x)=\int e^{it\lambda(\xi)}e^{ix\xi}\psi(\xi)d\xi ,$$

where $\psi\in C_0^\infty(\boldsymbol{R}^n)$ and $\lambda(\xi)$ is a C^∞-function such that $\partial\lambda/\partial\xi_1\neq 0$ in a neighborhood of Supp ψ. Choose a positive integer k such that $s<k$. Using

$$\left[\left(it\frac{\partial\lambda}{\partial\xi_1}\right)^{-1}\frac{\partial}{\partial\xi_1}\right]^k \exp(it\lambda(\xi))=\exp(it\lambda(\xi)) ,$$

we obtain by integration by parts

$$f=t^{-k}\int e^{it\lambda(\xi)}e^{ix\xi}\sum_{j=0}^{k} x_1^j\psi_j(\xi)d\xi ,$$

where $\psi_j\in C_0^\infty(\boldsymbol{R}^n)$. Since there exists a constant C such that

$$\|e^{it\lambda(\xi)}\psi_j(\xi)\|_{L_q(\boldsymbol{R}^n)}\leqq C , \qquad 1\leqq q\leqq 2 ,\ t\in\boldsymbol{R} ,$$

it follows that

$$\|(1+|x|^2)^{-k/2}f(t, x)\|_{L_p}\leqq Ct^{-k} , \qquad 2\leqq p\leqq\infty ,\ t\in\boldsymbol{R} .$$

On the other hand, $\|f(t, \cdot)\|_{L_p}$ is bounded on $\boldsymbol{R}$. Thus we obtain by Proposition 2.6

$$\|(1+|x|^2)^{-s/2}f(t, x)\|_{L_p}\leqq Ct^{-s} ,$$

which proves the lemma.

PROOF OF THEOREM 3.1. Let φ be a rapidly decreasing function with $\hat{\varphi}\in C_0^\infty(\Omega)$. We have

$$e^{-itH}Je^{itH_0}E_{0,ac}\varphi=JE_{0,ac}\varphi-i\int_0^t e^{-iyH}(HJ-JH_0)e^{iyH_0}\varphi dy . \tag{3.6}$$

We obtain by (3.5) and (A.II)

$$\begin{aligned}\|(HJ-JH_0)e^{iyH_0}\varphi\|_H &\leqq \|(M(x)-I)e^{iyH_0}P(D)\varphi\|_H \\ &+\|M(x)\sum_{j=1}^{K} q_j(x)e^{iyH_0}Q_j(D)\varphi\|_H\leqq C(1+|y|)^{-s} .\end{aligned} \tag{3.7}$$

Since $s>1$, (3.6) and (3.7) imply the existence of the limit

$$\lim_{t\to\pm\infty} e^{-itH}Je^{itH_0}E_{0,ac}\varphi .$$

Since $C_0^\infty(\Omega)$ is dense in $L_2(\boldsymbol{R}^n)$, Banach-Steinhaus' theorem shows the existence of the wave operators.

In order to show (3.4) we introduce the unitary operator J_1 from $\boldsymbol{H}_0$ onto $\boldsymbol{H}$ defined by $(J_1f)(x)=M(x)^{1/2}f(x)$. We obtain in the same way as above that for any φ with $\hat{\varphi}\in C_0^\infty(\Omega)$

$$\|(J_1-J)e^{itH_0}E_{0,ac}\varphi\|_H \leqq C(1+|t|)^{-s}.$$

Thus

$$\underset{t\to\pm\infty}{s\text{-lim}}\, e^{-itH}Je^{itH_0}E_{0,ac}=\underset{t\to\pm\infty}{s\text{-lim}}\, e^{-itH}J_1e^{itH_0}E_{0,ac}.$$

Since J_1 is unitary, this implies (3.4) (see [**6**, Theorem 6.2]).

§ 4. Rate of decay of local energy.

In this section we shall study the rate of decay of local energy of the solution of the equation (1.1). In particular, we shall treat systems of constant deficit thoroughly in order to show applicability of the general criteria Theorems 4.1 and 4.2.

For $\gamma\geqq 0$ and a positive measurable function $\rho(x)$ on $\boldsymbol{R}^n$, we put

$$X_0=L_2(\boldsymbol{R}^n;\boldsymbol{C}^m),\quad X_1=H^{\gamma}_{1,\rho};\qquad X_\theta=(X_0,X_1)_{\theta,2},\ 0<\theta<1.$$

(See (2.15).) Note that $X_\theta=H^{\gamma\theta}_{\theta,\rho}$ when $\rho(x)=(1+|x|^2)^{s/2}$ or $a_{\Omega,2}(x)$. When we discuss the exponential decay of the solution, we shall need the following condition (A.II′), which is a generalization of (A.II) (see § 3).

(A.II′) There exists a positive measurable function $a(x)$ such that $a(x)^{-1}\in L_\infty(\boldsymbol{R}^n)$ and

$$(|M(x)-I|+\sum_{j=1}^{K}|q_j(x)|)a(x)\in L_\infty(\boldsymbol{R}^n).$$

For simplicity of notation we shall sometimes write $P(D)=Q_0(D)$.

We begin with the following theorem.

THEOREM 4.1. *Assume* (A.I)$\sim$(A.III) *and* (A.II′). *Assume further that there exist constants* $R\geqq 0$, $\sigma\in\boldsymbol{R}$, $\gamma\geqq 0$, $C>0$, *and a positive measurable function* $\rho(x)$ *on* $\boldsymbol{R}^n$ *such that*

$$\text{(4.1)}\quad \sum_{j=0}^{K}\int_t^\infty \|a(x)^{-1}Q_j(D)e^{\pm iyH_0}E_{0,ac}(1-\Delta)^{-\gamma/2}\rho(x)^{-1}\|_{B(H_0)}dy\leqq Ce^{-Rt}t^{-\sigma},\qquad t>1.$$

Then for $0\leqq\theta\leqq 1$ *and any self-adjoint extension* H *of the operator* L

$$\text{(4.2)}\qquad \|e^{\pm itH}W_\pm-Je^{\pm itH_0}E_{0,ac}\|_{B(X_\theta,H)}\leqq Ce^{-R\theta t}t^{-\sigma\theta},\qquad t>1.$$

PROOF. Since the left hand side of (4.2) for $\theta=1$ is equal to

$$\left\|\int_{\pm t}^{\pm\infty}e^{i(\pm t-y)H}(HJ-JH_0)e^{iyH_0}E_{0,ac}dy\right\|_{B(X_1,H)},$$

we obtain by (4.1) and (A.II′) the inequality (4.2) for $\theta=1$. Since the inequality (4.2) for $\theta=0$ is obviously valid, the interpolation theorem shows the one for $0<\theta<1$. q.e.d.

Using Theorem 4.1 we can obtain the following theorem.

THEOREM 4.2. *Assume* (A.I)$\sim$(A.III) *and* (A.II$'$). *Assume further that there exist constants* $R\geqq 0$, $\sigma\in\boldsymbol{R}$, $\gamma\geqq 0$, $C>0$, *and a positive measurable function* $\rho(x)$ *on* $\boldsymbol{R}^n$ *such that the estimate*

$$\|a(x)^{-1}e^{\pm itH_0}E_{0,ac}(1-\Delta)^{-\gamma/2}\rho(x)^{-1}\|_{B(H_0)}\leqq Ce^{-Rt}t^{-\sigma}, \qquad t>1 \tag{4.3}$$

and (4.1) *hold. Then for* $0\leqq\theta\leqq 1$ *and any self-adjoint extension* H *of the operator* L

$$\|a(x)^{-\theta}e^{\pm itH}W_{\pm}\|_{B(X_\theta,H)}\leqq Ce^{-R\theta t}t^{-\sigma\theta}, \qquad t>1. \tag{4.4}$$

The assumptions (4.1) and (4.3) are verifiable in many cases of practical importance. First we give an elementary example.

EXAMPLE 4.3. Put $H_0=-\Delta$ in $\boldsymbol{R}^n$. Then for any $\varepsilon>0$ and $0\leqq s\leqq n/2$

$$\|(1+|x|^2)^{-(s+\varepsilon)/2}e^{\pm itH_0}(1+|x|^2)^{-(s+\varepsilon)/2}\|_{B(L_2(R^n))}\leqq Ct^{-s}, \qquad t>1.$$

Next consider the unperturbed system of constant deficit

$$\frac{1}{i}\frac{\partial u}{\partial t}=P(D)u\equiv\sum_{j=1}^{n}A_jD_ju,$$

where A_j is an Hermitian matrix, and the rank of $P(\xi)$ is constant for all $\xi\neq 0$. Let H_0 be the natural self-adjoint realization in $\boldsymbol{H}_0$ of the differential operator $P(D)$. Using the same notation as in § 2, we see that

(4.5) $r=2p$ for some positive integer p;

(4.6) $\lambda_j(-\xi)=-\lambda_{2p-j+1}(\xi)$, $F_j(-\xi)=F_{2p-j+1}(\xi)$, $j=1,\cdots,p$;

(4.7) $\lambda_1(\xi)\leqq\cdots\leqq\lambda_p(\xi)<0<\lambda_{p+1}(\xi)\leqq\cdots\leqq\lambda_{2p}(\xi)$ for each $\xi\neq 0$;

(4.8) λ_j and F_j are positively homogeneous functions of order one and zero, respectively.

Note that λ_j and F_j may not be smooth in $\boldsymbol{R}^n\setminus\{0\}$ without the assumption that λ_j has constant multiplicity for all $\xi\in\boldsymbol{R}^n\setminus\{0\}$ (c.f. [23] and [24]). For $j=p+1,\cdots,2p$, we put $S_j=\{\xi\in\boldsymbol{R}^n;\ \lambda_j(\xi)=1\}$, which is a compact hypersurface. Introduce the generalized polar coordinates

$$\phi_j(\xi)=(\lambda_j(\xi),\ \xi/\lambda_j(\xi))\in\boldsymbol{R}^+\times S_j;\ \phi_j^{-1}(\lambda,\omega)=\lambda\omega,\ \lambda\in\boldsymbol{R}^+,\ \omega\in S_j.$$

Then it is easily seen that there exists a positive measure $d\sigma_j$ on S_j such that

$$d\xi=\lambda^{n-1}d\lambda d\sigma_j, \quad \int_{S_j}1\,d\sigma_j<\infty.$$

Using these facts we obtain

PROPOSITION 4.4. *There exists a constant* C *such that*

(4.9) $$\|(1+|x|^{2})^{-s/2}e^{\pm itH_0}E_{0,ac}(1+|x|^{2})^{-s/2}\|_{B(H_0)}\leqq Ct^{-s}, \qquad t>1,$$

where $s\geqq 0$ *when the space dimension* n *is odd, and* $0\leqq s\leqq n$ *when* n *is even. Furthermore,*

(4.10) $$\|(1+|x|^{2})^{-s/2}P(D)e^{\pm itH_0}E_{0,ac}(1-\Delta)^{-1/2}(1+|x|^{2})^{-s/2}\|_{B(H_0)}\leqq Ct^{-s}, \qquad t>1,$$

where $s\geqq 0$ *when* n *is odd, and* $0\leqq s\leqq n+1$ *when* n *is even.*

PROOF. We have for any $\varphi\in\mathcal{S}(\boldsymbol{R}^n)$

$$e^{itH_0}E_{0,ac}\varphi=\sum_{j=1}^{r}(2\pi)^{-n/2}\int e^{ix\xi}e^{it\lambda_j(\xi)}F_j(\xi)\hat{\varphi}(\xi)d\xi .$$

Changing the variable ξ to $-\xi$ after using (4.6), we obtain for each $j=p+1, \cdots, 2p$

$$\int\exp(ix\xi+it\lambda_{2p-j+1}(\xi))F_{2p-j+1}(\xi)\hat{\varphi}(\xi)d\xi$$
$$=\int\exp(-ix\xi-it\lambda_j(\xi))F_j(\xi)\hat{\varphi}(-\xi)d\xi .$$

Thus

(4.11) $$e^{itH_0}E_{0,ac}\varphi=\sum_{j=p+1}^{2p}(2\pi)^{-n/2}\int_{-\infty}^{\infty}\int_{S_j}e^{i\lambda x\omega}e^{it\lambda}F_j(\omega)\hat{\varphi}(\lambda\omega)|\lambda|^{n-1}d\lambda d\sigma_j .$$

We obtain by integration by parts that the integral appearing in the right hand side of (4.11) is equal to

$$a_j\hat{\varphi}(0)t^{-n}+t^{-n}\sum_{|\alpha+\beta|+\gamma=n}{}' C^n_{\alpha\beta\gamma}x^{\alpha}\int_{-\infty}^{\infty}\int_{S_j}e^{i\lambda x\omega}e^{it\lambda}$$
$$\times F_j(\omega)(D^{\beta}\hat{\varphi})(\lambda\omega)\lambda^{-\gamma}|\lambda|^{n-1}\omega^{\alpha+\beta}d\lambda d\sigma_j ,$$

where $\sum'$ denotes the sum over α, β, γ except for $\gamma>n-1$, a_j is a non-zero constant when n is even and $a_j=0$ otherwise, and $C^n_{\alpha\beta\gamma}$ is a constant. Hence

(4.12) $$e^{itH_0}E_{0,ac}\varphi(x)=a\hat{\varphi}(0)t^{-n}+t^{-n}\sum_{j=1}^{2p}\sum_{|\alpha+\beta|+\gamma=n}{}' C^n_{\alpha\beta\gamma}x^{\alpha}$$
$$\times\mathcal{F}^{-1}[e^{it\lambda_j}F_jD^{\beta}\hat{\varphi}\lambda_j^{-\gamma}(\xi/|\lambda_j(\xi)|)^{\alpha+\beta}](x),$$

where $a=a_{p+1}+\cdots+a_{2p}$. Furthermore, when n is odd, we can obtain for any $k\geqq n$

(4.13) $$e^{itH_0}E_{0,ac}\varphi(x)=t^{-k}\sum_{j=1}^{2p}\sum_{|\alpha+\beta|+\gamma=k}{}' C^k_{\alpha\beta\gamma}x^{\alpha}\mathcal{F}^{-1}[e^{it\lambda_j}F_jD^{\beta}\hat{\varphi}\lambda_j^{-\gamma}(\xi/|\lambda_j(\xi)|)^{\alpha+\beta}](x).$$

Using (4.12) we shall show the inequality (4.9) for even n. We denote by

χ the characteristic function of the set $\{|\xi|<1\}$. Using Hausdorff-Young's inequality, Hölder's inequlity, and the imbedding theorem, we can obtain that for α, β, γ with $|\alpha+\beta|+\gamma=n$ and $\gamma\leqq n-1$

$$(4.14)\qquad \|x^{\alpha}\mathscr{F}^{-1}[e^{it\lambda_j}F_jD^{\beta}\hat{\varphi}\chi\lambda_j^{-\gamma}(\xi/|\lambda_j(\xi)|)^{\alpha+\beta}](x)\|_{L_2^{-n}(R^n)}\leqq C\|\hat{\varphi}\|_{H^n(R^n)},$$

where $L_2^{-n}(\boldsymbol{R}^n)$ is the weighted L_2-space (see § 2). For example, consider the case that $|\alpha|>n/2$. Then

$$\text{(the left hand side of (4.14))}\leqq C\|D^{\beta}\hat{\varphi}\chi\lambda_j^{-\gamma}\|_{L_2}$$
$$\leqq C\|D^{\beta}\hat{\varphi}\|_{L_\infty}\|\chi\lambda_j^{-\gamma}\|_{L_2}\leqq C\|\hat{\varphi}\|_{H^n(R^n)}.$$

We have

$$(4.15)\qquad \|D^{\beta}\hat{\varphi}(1-\chi)\lambda_j^{-\gamma}\|_{L_2}\leqq C\|\hat{\varphi}\|_{H^n(R^n)},\quad |\hat{\varphi}(0)|\leqq C\|\hat{\varphi}\|_{H^n(R^n)}.$$

Since $\|\hat{\varphi}\|_{H^n(R^n)}=\|\varphi\|_{L_2^n(R^n)}$, (4.12), (4.14), and (4.15) imply the inequality (4.9) for $s=n$. Since that for $s=0$ is obviously valid, we obtain by Proposition 2.6 the inequality (4.9) for $0<s<n$.

To prove the inequality (4.9) in the case that n is odd, we have only to use (4.13) instead of (4.12). For the proof of (4.10), use the following identity in place of (4.11):

$$(4.16)\qquad P(D)e^{itH_0}E_{0,ac}(1-\Delta)^{-1/2}\varphi$$
$$=\sum_{j=p+1}^{2p}(2\pi)^{-n/2}\int_{-\infty}^{\infty}\int_{S_j}e^{i\lambda x\omega}e^{it\lambda}F_j(\omega)\hat{\varphi}(\lambda\omega)(1+\lambda^2|\omega|^2)^{-1/2}\lambda|\lambda|^{n-1}d\lambda d\sigma_j.$$

q. e. d.

In order to state the exponential decay of e^{itH_0}, we put for any $R>0$

$$(4.17)\qquad a_R(x)=\exp(\sup_{\eta\in\Omega_R}(-x\eta)),$$

where Ω_R is the convex hull of the set $\{\xi\in\boldsymbol{R}^n;\ \min(\lambda_{p+1}(\xi),\lambda_{p+1}(-\xi))<R\}$. Note that $\bigcup_{1\leq j\leq 2p}\{\mu\omega;\ \omega\in S_j,\ -R<\mu<R\}\subset\Omega_R$. Let b be a positive number such that $b>R^2+2+\sup\{|\eta|^2;\ \eta\in\Omega_R\}$. We have

PROPOSITION 4.5. *Let n be an odd number. Then for any $R>0$, $\alpha\in\boldsymbol{R}$, and $\varepsilon>0$, there exists a constant C such that for all $t>1$*

$$(4.18)\qquad \|Ae^{\pm itH_0}E_{0,ac}A\|_{B(H_0)}\leqq\begin{cases}Ce^{-Rt}t^{-\alpha}, & \alpha\geqq 0\\ Ce^{-Rt}t^{-2\alpha}, & \alpha<0,\end{cases}$$

where $A=a_R(x)^{-1}(1+|x|^2)^{-(\alpha+\varepsilon)/2}$. Furthermore,

$$(4.19)\qquad \|AP(D)e^{\pm itH_0}E_{0,ac}(b-\Delta)^{-1/2}A\|_{B(H_0)}\leqq\begin{cases}Ce^{-Rt}t^{-\alpha}, & \alpha\geqq 0\\ Ce^{-Rt}t^{-2\alpha}, & \alpha<0.\end{cases}$$

For the proof we prepare two lemmas.

LEMMA 4.6. *Let* $-R<\mu<R$ *and* $1\leqq j\leqq 2p$. *Then there exists a constant* C *such that for any* $f\in\mathcal{H}^{(n+1)/2}(\boldsymbol{R}^n\times i\Omega_R)$

$$\|f(\xi+i\mu\xi/|\lambda_j(\xi)|)\|_{L_2(R^n)}\leqq C\|f\|_{\mathcal{H}^{(n+1)/2}(R^n\times i\Omega_R)}.$$

PROOF. We have

$$\|f(\xi+i\cdot)\|_{L_\infty(\Omega_R)}\leqq C\|f(\xi+i\cdot)\|_{H^{(n+1)/2}(\Omega_R)}.$$

Thus

$$\|f(\xi+i\mu\xi/|\lambda(\xi)|)\|^2_{L_2(R^n)}\leqq\int_{R^n}\|f(\xi+i\cdot)\|^2_{L_\infty(\Omega_R)}d\xi$$

$$\leqq C\sum_{|\beta|\leqq(n+1)/2}\int_{R^n}\int_{\Omega_R}|D^\beta f(\xi+i\eta)|^2d\xi d\eta\,.$$

This proves the lemma.

Putting

$$b_R(x)=\left(\int_{\Omega_R}e^{2x\eta}d\eta\right)^{1/2},\ d_j(x,\xi)=\frac{\exp(-\mu x\xi/|\lambda_j(\xi)|)}{b_R(x)(1+|x|^2)^n},\qquad 1\leqq j\leqq 2p,$$

we have

LEMMA 4.7. *There exists a constant* C *such that for any* $f\in L_2(\boldsymbol{R}^n)$

$$\left\|\int_{R^n}e^{ix\xi}d_j(x,\xi)f(\xi)d\xi\right\|_{L_2}\leqq C\|f\|_{L_2}.$$

PROOF. We have by (2.18)

$$\exp\{\sup_{\xi\in R^n}(-\mu x\xi/|\lambda_j(\xi)|)\}\leqq\exp\{\sup_{\eta\in\Omega_R}(-x\eta)\}\leqq(1+|x|^2)^{n/4}b_R(x).\tag{4.20}$$

On the other hand, for each multi-index β there exists a constant C such that $|D^\beta b_R(x)|\leqq Cb_R(x)$. Thus

$$\|D_x^\beta d_j(x,\xi)\|_{L_1(R_x^n)}\leqq C,$$

where C is a constant independent of ξ. Hence, putting

$$K_j(\xi',\xi)=(2\pi)^{-n/2}\int_{R^n}d_j(x,\xi)e^{-ix\xi'}dx,$$

we obtain

$$|K_j(\xi',\xi)|\leqq C(1+|\xi'|^2)^{-(n+1)/2}.$$

Using this we have for any $f,g\in\mathcal{S}(\boldsymbol{R}^n)$

$$\left|\int_{R^n}g(x)\left(\int_{R^n}e^{ix\xi}d_j(x,\xi)f(\xi)d\xi\right)dx\right|$$

$$=\left|\int f(\xi)\left(\int K_j(\xi'-\xi,\xi)\hat{g}(-\xi')d\xi'\right)d\xi\right|$$

$$\leqq C\iint |f(\xi)|(1+|\xi'-\xi|^2)^{-(n+1)/2}|\hat{g}(-\xi')|\,d\xi d\xi'\leqq C\|f\|_{L_2}\|g\|_{L_2}.$$

This proves the lemma.

PROOF OF PROPOSITION 4.5. For the sake of simplicity of notation we shall treat only the operator e^{itH_0} $(t>1)$. First we show the inequality (4.18) for $\alpha=0$. Cauchy's theorem gives that for any $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ and $0\leqq\mu<S$

$$(4.21)\qquad e^{itH_0}E_{0,ac}\varphi=\sum_{j=p+1}^{2p}(2\pi)^{-n/2}\int_{-\infty}^{\infty}\int_{S_j}\exp(i\lambda x\omega-\mu x\omega+it\lambda-t\mu)$$

$$\times F_j(\omega)\hat{\varphi}((\lambda+i\mu)\omega)(\lambda+i\mu)^{n-1}d\lambda d\sigma_j$$

$$=e^{-t\mu}\sum_{j=1}^{2p}(2\pi)^{-n/2}\int_{R^n}\exp(ix\xi-\mu x\xi/|\lambda_j(\xi)|+it\lambda_j(\xi))F_j(\xi)$$

$$\times\hat{\varphi}(\xi+i\mu\xi/|\lambda_j(\xi)|)(1+i\mu/\lambda_j(\xi))^{n-1}d\xi.$$

Denoting by χ the characteristic function of the set $\{|\xi|<1\}$, we have

$$(4.22)\qquad \|\chi(\xi)\hat{\varphi}(\xi+i\mu\xi/|\lambda_j(\xi)|)(1+i\mu/\lambda_j(\xi))^{n-1}\|_{L_1(R^n)}\leqq C\|\hat{\varphi}\|_{\mathcal{H}^{n+1}(R^n\times\Omega_S)}.$$

Combining (4.20), (4.22), Lemmas 4.6 and 4.7, we obtain

$$(4.23)\qquad \|b_S(x)^{-1}(1+|x|^2)^{-n}\int\exp(ix\xi-\mu x\xi/|\lambda_j(\xi)|+it\lambda_j(\xi))F_j(\xi)$$

$$\times\hat{\varphi}(\xi+i\mu\xi/|\lambda_j(\xi)|)[\chi(\xi)+(1-\chi(\xi))](1+i\mu/\lambda_j(\xi))^{n-1}d\xi\|_{L_2}$$

$$\leqq C\|\hat{\varphi}\|_{\mathcal{H}^{n+1}(R^n\times\Omega_S)}.$$

It follows from (4.21), (4.23), and Proposition 2.9 that

$$\|B^{-1}e^{itH_0}E_{0,ac}\varphi\|_{L_2}\leqq Ce^{-St}\|B\varphi\|_{L_2},$$

where $B=b_S(x)(1+|x|^2)^n$. Thus the interpolation theorem shows that for $0\leqq\theta\leqq1$

$$(4.24)\qquad \|B^{-\theta}e^{itH_0}E_{0,ac}\varphi\|_{L_2}\leqq Ce^{-S\theta t}\|B^\theta\varphi\|_{L_2}.$$

Using the homogenity of the function $\min(\lambda_{p+1}(\xi),\lambda_{p+1}(-\xi))$, we obtain that for each $\theta\in(0,1)$ there exists a constant C such that

$$Ca_{S\theta}(x)\leqq(a_S(x))^\theta\leqq C^{-1}a_{S\theta}(x),$$

where $a_S(x)$ is the function defined by (4.17). Thus (2.18) implies

$$(4.25)\qquad C(1+|x|^2)^{-5n\theta/4}\leqq B^\theta/a_{S\theta}(x)\leqq C^{-1}(1+|x|^2)^{(n+1/4)\theta}.$$

For any $\varepsilon>0$, choose S so large as $5nR/S<\varepsilon$. Then (4.25) and the inequality obtained by substituting R/S for θ in (4.24) imply the inequality (4.18) for $\alpha=0$.

For the proof of the inequality (4.18) for $\alpha>0$ we have only to use (4.9) with $s=\alpha/(1-\theta)$.

Next we show (4.18) for $\alpha<0$. We have for any $0\leqq\mu<R$

$$\|Ae^{itH_0}E_{0,ac}A\|_{B(H_0)}\leqq e^{-t\mu}\|a_R(x)^{-1}a_\mu(x)(1+|x|^2)^{-\alpha/2}\|_{L_\infty}^2 . \tag{4.26}$$

But

$$\|a_R(x)^{-1}a_{R(1-1/t)}(x)(1+|x|^2)^{-\alpha/2}\|_{L_\infty}\leqq Ct^{-\alpha} .$$

Hence substitution of $R(1-1/t)$ for μ in (4.26) gives (4.18) for $\alpha<0$.

The inequality (4.19) can be shown in the same way as above. q. e. d.

Finally we give some corollaries of Theorems 4.1 and 4.2 and Propositions 4.4 and 4.5. Consider the system of constant deficit

$$\frac{1}{i}\frac{\partial u}{\partial t}=M(x)P(D)u\equiv M(x)\sum_{j=1}^{n}A_jD_ju .$$

Let H be the self-adjoint operator in $\boldsymbol{H}$ defined by

$$Hu=M(x)P(D)u , \quad u\in D(H)\equiv JD(H_0) .$$

Combining Theorems 4.1 and 4.2 and Proposition 4.4, we have

THEOREM 4.8. *Assume that*

$$|M(x)-I|(1+|x|^2)^{s/2}\in L_\infty(\boldsymbol{R}^n)$$

for some s such that $1<s\leqq n+1$ when n is even, and $s>1$ when n is odd. Then there exists a constant C such that for any $0\leqq\theta\leqq1$ and $t>1$

$$\|(e^{\pm itH}W_\pm-Je^{\pm itH_0}E_{0,ac})(1-\Delta)^{-\theta/2}(1+|x|^2)^{-s\theta/2}\|_{B(H_0,H)}\leqq Ct^{-(s-1)\theta} \tag{4.27}$$

and

$$\|(1+|x|^2)^{-s\theta/2}e^{\pm itH}W_\pm(1-\Delta)^{-\theta/2}(1+|x|^2)^{-s\theta/2}\|_{B(H_0,H)}\leqq Ct^{-(s-1)\theta} . \tag{4.28}$$

In order to give a theorem concerning the exponential decay we assume (A.II′) with $a(x)$ being equal to $a_R(x)(1+|x|^2)^{\alpha/2}$ for some $R>0$ and $\alpha\in\boldsymbol{R}$. That is, we assume

$$|M(x)-I|\,a_R(x)(1+|x|^2)^{\alpha/2}\in L_\infty(\boldsymbol{R}^n) . \tag{4.29}$$

Put

$$\delta=\inf\{r-s\,;\ 0<s<r,\ a_R(x)\{b_R(x)(1+|x|^2)^{r/2}\}^{-1} \text{ and } b_R(x)(1+|x|^2)^{s/2}/a_R(x) \text{ belong to } L_\infty(\boldsymbol{R}^n)\} . \tag{4.30}$$

Then we have by (2.18) that $0\leqq\delta\leqq(n-1)/4$. In particular, $\delta=0$ when $\partial\Omega$ is smooth and its Gaussian curvature never vanishes.

The following theorem is a corollary of Proposition 4.5, Theorems 4.1

and 4.2.

THEOREM 4.9. *Assume* (4.29). *Let* n *be odd. Let* A *be the multiplication operator* $a_R(x)^{-1}(1+|x|^2)^{-(\alpha+\varepsilon)/2}$, *where* $\varepsilon>0$. *Then there exists a constant* C *such that for any* $t>1$ *and* $0\leqq\theta\leqq 1$

$$(4.31)\qquad \|e^{\pm itH}W_\pm - Je^{\pm itH_0}E_{0,ac})(b-\Delta)^{-\theta/2}A^\theta(1+|x|^2)^{-\delta\theta/2}\|_{B(H_0,H)} \leqq Ce^{-R\theta t}t^{-[\alpha+\min(0,\alpha)]\theta}$$

and

$$(4.32)\qquad \|A^\theta e^{\pm itH}W_\pm(b-\Delta)^{-\theta/2}A^\theta(1+|x|^2)^{-\delta\theta/2}\|_{B(H_0,H)} \leqq Ce^{-R\theta t}t^{-[\alpha+\min(0,\alpha)]\theta}.$$

PROOF. We shall show only (4.31) for $\alpha=0$. We have by Theorem 4.1 and Proposition 4.5

$$(4.33)\qquad \|e^{\pm itH}W_\pm - Je^{\pm itH_0}E_{0,ac}\|_{B(X_\theta,H)} \leqq Ce^{-R\theta t},\qquad t>1$$

where $X_\theta=(L_2(\boldsymbol{R}^n), H^1_{1,\rho})_{\theta,2}$ with $\rho(x)=a_R(x)(1+|x|^2)^\varepsilon$. Choose r and s such that $r-s-\varepsilon<\delta$ and

$$(4.34)\qquad C(1+|x|^2)^{-r/2}a_R(x)\leqq b_R(x)\leqq C^{-1}(1+|x|^2)^{-s/2}a_R(x).$$

Proposition 2.8 and the first inequality of (4.34) imply

$$\|f\|_{X_\theta}\leqq C\|b_R(x)^\theta(1+|x|^2)^{(r+2\varepsilon)\theta/2}(b-\Delta)^{\theta/2}f\|_{L_2}.$$

Thus the second inequality of (4.34) gives

$$(4.35)\qquad \|f\|_{X_\theta}\leqq C\|(1+|x|^2)^{(\delta+\varepsilon)\theta/2}a_R(x)^\theta(b-\Delta)^{\theta/2}f\|_{L_2}.$$

Combining (4.33) and (4.35), we obtain (4.31) for $\alpha=0$. q. e. d.

§ 5. Completeness of the wave operators.

In this section we shall study spectral properties of the operator H and the completeness of the wave operators along the line given by [3], [8], [**10**], and [**24**].

To this end we add the following assumptions (A.III′) and (A.IV).

(A.III′) (i) $|Q_j(\xi)(P(\xi)-i)^{-1}|\in L_\infty(\boldsymbol{R}^n)$, $j=1,\cdots,K$;
(ii) The operator H defined by

$$Hu=M(x)(P(D)+\sum_{j=1}^K q_j(x)Q_j(D))u,\qquad u\in D(H)\equiv JD(H_0)$$

is self-adjoint in $\boldsymbol{H}$. (The spectral measure associated with H will be denoted by E.)

(A.IV) (i) There is no root of the equation $\det(\lambda I-P(\xi))=0$ which is identically non-zero constant, and the roots $\lambda_j(\xi)$ $(j=1,\cdots,r)$ which are not identically zero satisfy

$$\lim_{|\xi|\to\infty} |\lambda_j(\xi)| = \infty ;$$

(ii) $$\lim_{|\xi|\to\infty} |Q_j(\xi)(P(\xi)-i)^{-2}| = 0 , \qquad j=1, \cdots, K .$$

REMARK 5.1. If (A.III'. i) is replaced by

(5.1) $$\lim_{|\xi|\to\infty} |Q_j(\xi)(P(\xi)-i)^{-1}| = 0 ,$$

then the operator $\sum_{j=1}^{K} q_j Q_j(D)(P(D)-i)^{-1}$ is a compact operator in $\boldsymbol{H}_0$, and consequently (A.III' ii) is satisfied (see [7]).

The assumptions (A.II) and (A.IV) ensure that for each ζ with $\mathrm{Im}\,\zeta \neq 0$ the operator

$$(1+|x|^2)^{s/4}(HJ-JH_0)(H_0-\zeta)^{-1}(H_0-i)^{-1}$$

is a compact operator from $\boldsymbol{H}_0$ to $\boldsymbol{H}$. Thus in order to apply the results of [**3**] and [**10**] we have only add the assumption which ensure the existence of a "good" spectral representation of the operator H_0. Here we shall give two such conditions: the first one asserts that the set Λ_e of all exceptional values of $P(\xi)$ is discrete (for definition of Λ_e, see (2.5)), and the second one asserts that $P(\xi)$ is homogeneous.

In order to state the first one we prepare some notations and a proposition. We put

(5.2) $$\Lambda = \Lambda_e \cup \Lambda_c ,$$

where Λ_c is the set defined by (2.14). Note that Λ_c is a finite set. Put

(5.3) $$\mathfrak{R} = \{\lambda \in \boldsymbol{R} ;\ \det(\lambda - P(\xi)) = 0 \text{ for some } \xi \in \boldsymbol{R}^n\} .$$

Let I be an open interval contained in $\mathfrak{R} \setminus \Lambda$. For each $j=1, \cdots, r$, put

$$\Omega_j = \{\xi ;\ \lambda_j(\xi) \in I\} , \quad S_{j,\lambda} = \{\xi ;\ \lambda_j(\xi) = \lambda\} , \quad \lambda \in I .$$

Fixing $c \in I$, we write $S_j = S_{j,c}$. Let $d\sigma_j$ be the surface element of S_j, and $L_2(S_j) = L_2(S_j, d\sigma_j ; \boldsymbol{C}^m)$. Then we can obtain the following proposition, (c. f. [**10**, II, Propositions 2.2 and 2.3]).

PROPOSITION 5.2. *Assume* (A.IV. i). *Then, for each j there exist a homeomorphism ϕ_j from $I \times S_j$ onto Ω_j and a positive measurable function p_j on $I \times S_j$ satisfying the following properties:* (i) *for each $\lambda \in I$, ϕ_j maps $\{\lambda\} \times S_j$ on $S_{j,\lambda}$;* (ii) *for any $f \in L_2(\Omega_j)$*

$$\| p_j(\lambda, \omega) F_j(\phi_j(\lambda, \omega)) f(\phi_j(\lambda, \omega)) \|_{L_2(I;\, L_2(S_j))} = \| F_j(\xi) f(\xi) \|_{L_2(\Omega_j)} .$$

Furthermore, for each $\lambda \in I$ there exists a bounded linear operator $\Gamma_j(\lambda)$ from $H^{s/2}(\boldsymbol{R}^n)$ $(s>1)$ to $L_2(S_j)$ satisfying the following properties: (iii) *for any*

$f\in H^{s/2}(\boldsymbol{R}^n)$

(5.4) $$(\Gamma_j(\lambda)f)(\omega)=p_j(\lambda, \omega)F_j(\phi_j(\lambda, \omega))f(\phi_j(\lambda, \omega));$$

(iv) *$\Gamma_j(\lambda)$ depends locally Hölder continuously on λ in operator norm.*

This proposition shows the existence of a good spectral representation of H_0 restricted to $E_0(\mathfrak{R}\backslash\Lambda)$. Thus the completeness of the wave operators will follow if we add the following assumption.

(A.V) The set Λ_e is discrete in $\boldsymbol{R}$.

Under the assumptions (A.I), (A.II), (A.III′), (A.IV), and (A.V) we obtain the following three theorems. (The proof is omitted since they can be shown along the line given in [3], [8], [**10**], and [**24**].)

THEOREM 5.3. (i) *The spectrum of H in $\boldsymbol{R}\backslash\mathfrak{R}$ consists of eigenvalues. The set $\sigma_p(H)\cap(\boldsymbol{R}\backslash\mathfrak{R})$ is bounded.* (ii) *$\sigma_p(H)$ has no points of accumulation in $\boldsymbol{R}\backslash\Lambda$. Each eigenvalue $\lambda\notin\Lambda$ is of finite multiplicity.* (iii) *H restricted to $E(\mathfrak{R}\backslash(\sigma_p(H)\cup\Lambda))\boldsymbol{H}$ is absolutely continuous.*

THEOREM 5.4. *Let $I=\boldsymbol{R}\backslash(\sigma_p(H)\cup\Lambda)$. Then the function $R(\zeta)=(H-\zeta)^{-1}$ on $\Pi^{\pm}=\{\zeta\in\boldsymbol{C};\ \pm\mathrm{Im}\,\zeta>0\}$ can be extended to $\Pi^{\pm}\cup I$ as a $\boldsymbol{B}(L_2^{s/2}, L_2^{-s/2})$-valued locally Hölder continuous function. In particular, for any $\lambda\in I$ and $f\in L_2^{s/2}$ the limit $R(\lambda\pm i0)=\lim_{\varepsilon\downarrow 0}R(\lambda+i\varepsilon)f$ exists in $L_2^{-s/2}$. $R(\lambda\pm i0)f$ satisfies $(H-\lambda)R(\lambda\pm i0)f=f$ in the sense of distribution.*

THEOREM 5.5. *$R(W_+)=R(W_-)=E(\mathfrak{R}\backslash(\sigma_p(H)\cup\Lambda))\boldsymbol{H}$.*

Next we treat the homogeneous case. That is, we assume

(A.VI) $P(\xi)$ is a homogeneous polynomial of order N.

Then the assumption (A.IV. i) is equivalent to the condition

(5.5) the rank of the matrix $P(\xi)$ is constant for all $\xi\in\boldsymbol{R}^n\backslash\{0\}$.

Under the assumption (A.VI) and (5.5) we have

PROPOSITION 5.6. *Let $I=\boldsymbol{R}^{\pm}$ and $c=\pm 1$. Then the conclusion of Proposition 5.2 is valid.*

PROOF. We shall treat only the case that $I=\boldsymbol{R}^+$, $c=1$, and $S_j=S_{j,1}$. We see that S_j is a compact hypersurface without boundary which is smooth except possibly on a real analytic set with codimension more than 1 in $\boldsymbol{R}^n$. Thus the surface element $d\sigma_j$ of S_j is well-defined. Put

$$\phi_j(\lambda, \omega)=\lambda^{1/N}\omega, \quad (\lambda, \omega)\in I\times S_j.$$

Then we see that (i) ϕ_j is a homeomorphism from $I\times S_j$ onto $\{\xi\in\boldsymbol{R}^n;\ \lambda_j(\xi)>0\}$; (ii) for each $\lambda\in I$, ϕ_j maps $\{\lambda\}\times S_j$ on $S_{j,\lambda}$; and (iii) there exists a constant a such that $d\xi=a^2\lambda^{n/N-1}d\lambda d\sigma_j$. Thus, setting

$$p_j=a\lambda^{n/2N-1/2},$$

we obtain the first half of the conclusion of Proposition 5.2. It remains to show the latter half. Choose an open interval J such that $\bar{J}\subset I$. Since $F_j(\lambda^{1/N}\omega)=F_j(\omega)$, we have

$$\left\|\frac{\partial}{\partial\lambda}(p_j(\lambda, \omega)F_j(\lambda^{1/N}\omega)f(\lambda^{1/N}\omega)\right\|_{L_2(J;\, L_2(S_j))}\leqq C\|f\|_{H^1(R^n)}.$$

Hence the interpolation theorem shows

$$\|p_j(\lambda, \omega)F_j(\lambda^{1/N}\omega)f(\lambda^{1/N}\omega)\|_{B^{s/2}_{2,2}(J;\, L_2(S_j))}\leqq C\|f\|_{H^{s/2}(R^n)}, \qquad 1<s<2.$$

Applying the imbedding theorem, we obtain that the operator $\Gamma_j(\lambda)$ defined by (5.4) belongs to $\boldsymbol{B}(H^{s/2}(\boldsymbol{R}^n), L_2(S_j))$ for each $\lambda\in J$, and it depends Hölder continuously on λ in operator norm, the exponent being $(s-1)/2$. q. e. d.

Using this proposition we can obtain the following theorem.

THEOREM 5.7. *Assume* (A.I), (A.II), (A.III′), (A.IV), *and* (A.VI). *Then each conclusion of Theorems* 5.3, 5.4, *and* 5.5 *holds with Λ being replaced by* $\{0\}$.

In the special case that $q_j(x)\equiv 0$ $(j=1, \cdots, K)$ we can get more information about the structure of $\sigma_p(H)$. (As for the other case, we refer the reader to [8].) That is, we have

THEOREM 5.8. *Assume* (A.I), (A.II), (A.IV.i), (A.VI), *and* $q_j\equiv 0$ $(j=1, \cdots, K)$. *Then zero is not an accumulation point of* $\sigma_p(H)$.

PROOF. For the sake of simplicity we shall treat only the case that $N=1$. We use the same notation as in the latter half of §4: (i) $\lambda_1(\xi)\leqq\cdots\leqq\lambda_p(\xi)<0<\lambda_{p+1}(\xi)\leqq\cdots\leqq\lambda_{2p}(\xi)$, $\xi\neq 0$; (ii) $F_j(\xi)$ $(j=1, \cdots, 2p)$ and $F_{2p+1}(\xi)$ are the orthogonal projection associated with $\lambda_j(\xi)$ $(j=1, \cdots, 2p)$ and 0, respectively.

We first claim that for any $\varepsilon>0$ there exists a constant $R>0$ such that for each $u\in\boldsymbol{H}$ and λ $(0<|\lambda|<1)$ with $Hu=\lambda u$

$$\|u\|_{L_2(|x|>R)}\leqq\varepsilon\|u\|_{\boldsymbol{H}}. \tag{5.6}$$

Let us show (5.6) for $0<\lambda<1$. Putting $v(\xi)=\mathcal{F}[(M(x)^{-1}-I)u(x)](\xi)$, we have

$$(P(\xi)-\lambda)\hat{u}(\xi)=\lambda v(\xi).$$

Thus

$$(\lambda_j(\xi)-\lambda)F_j(\xi)\hat{u}(\xi)=\lambda F_j(\xi)v(\xi), \qquad j=1, \cdots, 2p+1. \tag{5.7}$$

Since $v\in H^s(\boldsymbol{R}^n)$ by (A.II), (5.7) implies

$$F_j(\xi)v(\xi)|_{\lambda_j(\xi)=\lambda}=0, \qquad j=p+1, \cdots, 2p.$$

Thus we obtain for $j=p+1, \cdots, 2p$

$$\frac{F_j(\xi)v(\xi)}{\lambda_j(\xi)-\lambda}=F_j(\omega)\sum_{k=1}^{n}\int_0^1\frac{\partial v}{\partial\xi_k}\{(t(\lambda_j-\lambda)+\lambda)\omega\}\,\omega_k dt.$$

Hence

$$\|(\lambda_j(\xi)-\lambda)^{-1}F_j(\xi)v(\xi)\|_{L_2(R^n)}\leqq C\|v\|_{H^s(R^n)}, \qquad j=p+1, \cdots, 2p. \tag{5.8}$$

Since $H^{s-1}(\boldsymbol{R}^n)\subset L_q(\boldsymbol{R}^n)$ for some $q>2$, we obtain similarly

$$\|(\lambda_j(\xi)-\lambda)^{-1}F_j(\xi)v(\xi)\|_{L_q(\boldsymbol{R}^n)}\leqq C\|v\|_{H^s(\boldsymbol{R}^n)}, \qquad j=p+1, \cdots, 2p. \tag{5.9}$$

For any $\delta>0$ and $1\leqq j\leqq 2p$, we put $f_j^\delta(\xi)=\exp\{-(\lambda_j(\xi)-\lambda)^2/\delta^2\}$. Putting

$$V_j=\lambda(\lambda_j(\xi)-\lambda)^{-1}f_j^\delta F_j v, \quad U=\sum_{j=1}^{2p}\lambda(\lambda_j(\xi)-\lambda)^{-1}(1-f_j^\delta)F_j v,$$

we have by (5.7)

$$\hat{u}(\xi)=\sum_{j=1}^{2p}V_j+U-F_{2p+1}v. \tag{5.10}$$

Choosing a sufficiently small number a, we obtain by (5.9) and Hölder's inequality

$$\sum_{j=p+1}^{2p}\|V_j\|_{L_2(|\lambda_j(\xi)-\lambda|<a)}\leqq\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}.$$

Choose δ so small as $\sup\{f_j^\delta(\xi);\ |\lambda_j(\xi)-\lambda|\geqq a\}\leqq\varepsilon/p$. Then (5.8) implies

$$\sum_{j=p+1}^{2p}\|V_j\|_{L_2(|\lambda_j(\xi)-\lambda|\geqq a)}\leqq\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}.$$

Hence

$$\sum_{j=p+1}^{2p}\|V_j\|_{L_2(\boldsymbol{R}^n)}\leqq 2\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}. \tag{5.11}$$

Since $|\lambda(\lambda_j(\xi)-\lambda)^{-1}|\leqq 1$ for $j=1, \cdots, p$, we have by similar computations

$$\sum_{j=1}^{p}\|V_j\|_{L_2(\boldsymbol{R}^n)}\leqq 2\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}. \tag{5.12}$$

Choose a C^∞-function φ such that $\varphi=1$ in a neighborhood of zero and $\operatorname{Supp}\varphi$ is contained in a sufficiently small ball. Since $F_{2p+1}\in C^\infty(\boldsymbol{R}^n\setminus\{0\})$ by (5.5), we obtain

$$\|\varphi F_{2p+1}v\|_{L_2(\boldsymbol{R}^n)}\leqq\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}, \quad \|(1-\varphi)F_{2p+1}v\|_{H^s(\boldsymbol{R}^n)}\leqq C_\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}, \tag{5.13}$$

where C_ε is a constant depending only on ε. Choose d so large as $|\lambda_j(\xi)-\lambda|^{-1}\leqq\varepsilon/4p$ for $|\xi|\geqq d$ and $j=1, \cdots, 2p$, and choose a C^∞-function ψ such that $\psi(\xi)=1$ on $\{|\xi|\leqq d\}$ and $\operatorname{Supp}\psi\subset\{|\xi|<d+1\}$. Then we have

$$\|(1-\psi)U\|_{L_2(\boldsymbol{R}^n)}\leqq\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}. \tag{5.14}$$

On the other hand,

$$\psi U=-\frac{1}{2\pi i}\int_{|z|=b}\lambda(z-\lambda)^{-1}[1-e^{-(z-\lambda)^2/\delta^2}]\psi(P(\xi)-z)^{-1}(1-F_{2p+1})v\,dz,$$

where $b=\sup\{|\lambda_j(\xi)|+2;\ |\xi|\leqq d+1,\ j=1, \cdots, 2p\}$. Thus we obtain

$$\|\varphi\psi U\|_{L_2(\boldsymbol{R}^n)}\leqq\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}, \quad \|(1-\varphi)\psi U\|_{H^s(\boldsymbol{R}^n)}\leqq C_\varepsilon\|v\|_{H^s(\boldsymbol{R}^n)}. \tag{5.15}$$

Combining (5.10)~(5.15), we have

$$\|u\|_{L_2(|x|>R)}\leqq(7\varepsilon+2C_\varepsilon R^{-s})\|v\|_{H^s(R^n)}\leqq C(\varepsilon+C_\varepsilon R^{-s})\|u\|_H .$$

This shows (5.6) for $0<\lambda<1$. The inequality (5.6) for $-1<\lambda<0$ is shown similarly.

Suppose that there exist infinite eigenvalues of H such that $0<|\lambda_j|<1$ and $\lim_{j\to\infty}\lambda_j=0$. Let u_j be the normalized eigenfunctions associated with λ_j. Since u_j is orthogonal to each eigenfunctions associated with zero and

$$\|u_j\|_H+\|Hu_j\|_H\leqq 2 ,$$

it follows from Theorem 2.1 in [**19**] that $\{u_j\}_{j=1}^\infty$ is precompact in $L_{2,\mathrm{loc}}(\boldsymbol{R}^n)$. But (5.6) implies that for any $\varepsilon>0$ there exists $R>0$ such that for all $j=1, 2, \cdots$

$$\|u_j\|_{L_2(|x|>R)}\leqq\varepsilon .$$

Hence $\{u_j\}_{j=1}^\infty$ is precompact in $\boldsymbol{H}$. Since it is an orthonormal system, this leads to a contradiction. q. e. d.

Appendix.

In this appendix we shall show a precise version of Proposition 2.8 in the case $\rho(x)=(1+|x|^2)^{1/2}$. That is, we shall show

PROPOSITION. *For* $\sigma_0\leqq\sigma_1$, $s_0\leqq s_1$, *and* $0<\theta<1$, *one has*

$$(H_{s_0}^{\sigma_0}, H_{s_1}^{\sigma_1})_{\theta,2}=[H_{s_0}^{\sigma_0}, H_{s_1}^{\sigma_1}]_\theta=H_{s_0(1-\theta)+s_1\theta}^{\sigma_0(1-\theta)+\sigma_1\theta} .$$

Here we abbreviated $H_{s,\rho}^\sigma$ to H_s^σ. The proof is decomposed into the following four lemmas. Throughout the appendix we use the notations:

$$\rho(x)=(1+|x|^2)^{1/2},\quad U=\rho(x)\cdot,\quad V=(1-\Delta)^{1/2}.$$

LEMMA 1. *For each* $\sigma, \tau, \gamma, s\in\boldsymbol{R}$ *the operator* $V^{\sigma+\tau i}$ *is an isomorphism from* H_s^γ *to* $H_s^{\gamma-\sigma}$. *Furthermore, there exists a constant C depending only on s such that*

$$\|V^{\sigma+\tau i}f\|_{H_s^{\gamma-\sigma}}\leqq C\rho(\tau)^{|s|}\|f\|_{H_s^\gamma} . \tag{1}$$

PROOF. Since $\|\rho^{\tau i}\|_{W^k(R^n)}\leqq C\rho(\tau)^k$ for a non-negative integer k, we have

$$\|V^{\tau i}\|_{B(L_2^s)}=\|U^{\tau i}\|_{B(H^s)}\leqq C\rho(\tau)^{|s|} .$$

Hence

$$\|V^{\sigma+\tau i}f\|_{H_s^{\gamma-\sigma}}=\|V^{\tau i}V^\gamma f\|_{L_2^s}\leqq C\rho(\tau)^{|s|}\|f\|_{H_s^\gamma} . \tag{2}$$

Since $(V^{\sigma+\tau i})^{-1}=V^{-(\sigma+\tau i)}$, (2) implies the lemma.

LEMMA 2. $[H_{s_0}^{\sigma_0}, H_{s_1}^{\sigma_1}]_\theta=H_{s_0(1-\theta)+s_1\theta}^{\sigma_0(1-\theta)+\sigma_1\theta}$ *for* $\sigma_0\leqq\sigma_1$, $s_0\leqq s_1$, *and* $0<\theta<1$.

PROOF. For simplicity of notations we shall treat only the case that $\sigma_0=s_0=0$. (The proof below can be applied also to the general case.) Put $\sigma=\sigma_1$, $s=s_1$, $T=\{z\in \boldsymbol{C};\ 0<\mathrm{Re}\, z<1\}$, and $\bar{T}=\{z\in \boldsymbol{C};\ 0\leqq \mathrm{Re}\, z\leqq 1\}$. Let $\mathcal{A}$ be the Banach space of all L_2-valued continuous functions F on T such that F is holomorphic in T, bounded in $\bar{T}$, and

$$\|F\|_{\mathcal{A}}=\max(\sup_{\tau\in \boldsymbol{R}}\|F(i\tau)\|_{L_2},\ \sup_{\tau\in \boldsymbol{R}}\|F(1+i\tau)\|_{H_s^\sigma})<\infty\,.$$

Then we have by the definition of the complex interpolation space that $[L_2, H_s^\sigma]_\theta$ is equal to the set of those $f\in L_2$ such that $f=F(\theta)$ for some $F\in\mathcal{A}$, and

$$\|f\|_{[L_2, H_s^\sigma]_\theta}=\inf\{\|F\|_{\mathcal{A}};\ F\in\mathcal{A},\ f=F(\theta)\}\,.$$

First we show the inclusion $H_{s\theta}^{\sigma\theta}\subset[L_2, H_s^\sigma]_\theta$. For each $f\in H_{s\theta}^{\sigma\theta}$ and $z\in\bar{T}$, set

$$F(z)=e^{(z-\theta)^2}V^{-\sigma z}U^{-sz}g,\quad g=U^{s\theta}V^{\sigma\theta}f\,.$$

It suffices to show that F belongs to the space $\mathcal{A}$ and there exists a constant C independent of f such that

$$\|F\|_{\mathcal{A}}\leqq C\|f\|_{H_{s\theta}^{\sigma\theta}}\,. \tag{3}$$

We obtain by (1)

$$\|F(t+\tau i)\|_{H_{st}^{\sigma t}}=\|e^{(t+\tau i-\theta)^2}U^{st}V^{-\sigma\tau i}U^{-s(t+\tau i)}g\|_{L_2}\leqq Ce^{-\tau^2}\rho(\sigma\tau)^{|s|t}\|g\|_{L_2}\leqq C\|f\|_{H_{s\theta}^{\sigma\theta}}\,.$$

This proves (3) and the boundedness of F in $\bar{T}$. We have

$$F'(z)=e^{(z-\theta)^2}V^{-\sigma z}[2(z-\theta)-\sigma\log(1-\Delta)^{1/2}-s\log\rho(x)]U^{-sz}g\,.$$

This implies that for any $\varepsilon>0$

$$\sup_{\varepsilon\leqq t\leqq 1}\|F'(z)\|_{L_2}\leqq C\|f\|_{H_{s\theta}^{\sigma\theta}}\,,$$

from which it follows the holomorphy of $F(z)$ in T. The continuity is an easy corollary of Lebesgue's dominated convergence theorem.

Next we show the opposite inclusion. For each $f\in[L_2, H_s^\sigma]_\theta$ and $\varepsilon>0$, there exists $F\in\mathcal{A}$ such that $F(\theta)=f$ and

$$\|F\|_{\mathcal{A}}\leqq\|f\|_{[L_2, H_s^\sigma]_\theta}+\varepsilon\,. \tag{4}$$

Set

$$G(z)=e^{(z-\theta)^2}U^{sz}V^{\sigma z}F(z)\,.$$

Then $G(z)$ is, an $\mathcal{S}'$-valued function, holomorphic in T and continuous on $\bar{T}$. Moreover there exists for any $\varphi\in\mathcal{S}$ a constant C such that $|(G(t+i\tau), \varphi)|\leqq Ce^{-\tau^2}$. Hence we have in $\mathcal{S}'$

(5) $$U^{s\theta}V^{\sigma\theta}f=G(\theta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(1+i\tau)}{1+i\tau-\theta}d\tau-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(i\tau)}{i\tau-\theta}d\tau .$$

On the other hand

(6) $$\|G(i\tau)\|_{L_2}+\|G(1+i\tau)\|_{L_2}\leqq Ce^{-|\tau|}(\|F(i\tau)\|_{L_2}+\|F(1+i\tau)\|_{H_s^\sigma}) .$$

Combining (4), (5) and (6) we get

$$\|f\|_{H_{s\theta}^{\sigma\theta}}\leqq C(\|f\|_{[L_2, H_s^\sigma]_\theta}+\varepsilon) .$$

This implies that $\|f\|_{H_{s\theta}^{\sigma\theta}}\leqq C\|f\|_{[L_2, H_s^\sigma]_\theta}$. q. e. d.

LEMMA 3. *For $s, t, \sigma, r\in \boldsymbol{R}$, the operator U^{s+ti} is an isomorphism from H_r^σ to H_{r-s}^σ.*

PROOF. For a non-negative integer k, we have

$$\|U^{s+ti}f\|_{H_{r-s}^{2k}}=\|\rho(x)^{r-s}(1-\Delta)^k\rho(x)^{s+ti}f\|_{L_2}\leqq C\|f\|_{H_r^{2k}} .$$

This and Lemma 2 imply

(7) $$\|U^{s+ti}f\|_{H_{r-s}^\sigma}\leqq C\|f\|_{H_r^\sigma}, \qquad \sigma\geqq 0 .$$

Since $(U^{s+ti})^{-1}=U^{-(s+ti)}$, (7) proves the lemma in the case $\sigma\geqq 0$. If $\sigma<0$, we have only to use the facts: $H_r^\sigma=(H_{-r}^{-\sigma})'$ and $U^{s+ti}=(U^{s-ti})^*$. q. e. d.

LEMMA 4. *$(L_2, H_s^\sigma)_{\theta,2}=[L_2, H_s^\sigma]_\theta$ for $\sigma\geqq 0$, $s\geqq 0$, and $0<\theta<1$.*

PROOF. Let B be the operator in L_2 defined by

$$B=U^{s/2}V^\sigma U^{s/2}, \quad D(B)=H_s^\sigma .$$

Then we obtain by Lemmas 1 and 3 that B^{-1} is a bounded operator in L_2. Thus B is a positive self-adjoint operator. Hence we have $D(B^\theta)=(L_2, D(B))_{\theta,2}$ $=[L_2, D(B)]_\theta$ (c.f. [**9**] and [**13**]). This proves the lemma.

Finally we remark that Lemmas 1 and 3 imply the following proposition.

PROPOSITION 5. (i) *For $\sigma, s\in\boldsymbol{R}$, there exist positive constants C and C' such that*

(8) $$C\|f\|_{H_s^\sigma}\leqq\|(1-\Delta)^{\sigma/2}(1+|x|^2)^{s/2}f\|_{L_2}\leqq C'\|f\|_{H_s^\sigma}, \qquad f\in H_s^\sigma .$$

(ii) *$\mathscr{F}(H_s^\sigma)=H_\sigma^s$ for $\sigma, s\in\boldsymbol{R}$.*

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