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Abstract aspects of asymptotic analysis

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Introduction

In the present paper we offer a formal treatment of some simplest classes of the asymptotic methods. Our result is summarized in Theorem 4.5 below. It tells when a given element admits an asymptotic expansion, and also shows the canonical way to derive its expansion.

In many branches of mathematics, various asymptotic methods provide powerful tools, often exhibiting a strong resemblance. This leads one to a suspicion that there be a common structure in these methods of analysis. For instance, in many classes of asymptotic analysis, an asymptotic expansion is just one into homogeneous parts, as a formal series expansion. Thus, for such classes, a speculation may be done that there be an action of the multiplicative group $\boldsymbol{R}_+$ of positive real numbers. We actually observe such $\boldsymbol{R}_+$-actions exist in several standard examples as discussed in §7.

We thus begin by introducing the notion of a differentiable $\boldsymbol{R}_+$-action G in a multiplicatively convex Fréchet algebra A (see §1). However, most formal constructions below will be carried out without referring to the algebra structure of A. The assumption of A being an algebra is mainly to reflect some important cases. The differentiable $\boldsymbol{R}_+$-action in A leads us to define a scale $\{B^{\rho}\,;\,\rho\in\boldsymbol{R}\}$ of Fréchet spaces, and the spaces Γ^{μ}, $\mu\in\boldsymbol{C}$, of G-homogeneous elements (see §2). We then construct the analogues of the spaces of formal series, C^{μ}, from Γ^{μ}'s. We can thus introduce the notions of developable elements and their developments, as generalizations of elements admitting asymptotic expansions and their expansions. The spaces D^{μ} of developable elements are shown to be Fréchet spaces. The mappings α^{μ}, assigning to each element in D^{μ} its development in C^{μ}, are then continuous (see §3). Sufficient conditions on surjectivity of α^{μ} will be discussed in §5. Of course, in such a general situation, α^{μ} are not necessarily surjective (see Example 7.5). The spaces D^{μ} are characterized in terms of the boundary behavior of the differentiable $\boldsymbol{R}_+$-action. This permits us to write down the mappings α^{μ} as a variant of the Taylor expansion (see §4, Theorem 4.5 in particular). We supplement in §6 the cases when A is a Fréchet Montel space.

There are asymptotic classes of practical importance where the groups $\boldsymbol{R}_+^2$ or $\boldsymbol{R}_+^3$ act. Such classes will be discussed elsewhere.

We add a notational remark here. We write $\boldsymbol{C}$ and $\boldsymbol{R}$, respectively, for the fields of complex and real numbers. $\boldsymbol{Z}$ stands for the ring of rational integers. We write $\boldsymbol{R}^+$ and $\boldsymbol{Z}^+$, respectively, for the sets of non-negative reals and non-negative integers. Thus, the superscripted $+$ means the non-negative part. On the other hand, we show by the subscripted $+$ the positive part. In particular, $\boldsymbol{R}_+$ is the multiplicative group of positive reals. Embedding $\boldsymbol{R}_+$ in $\boldsymbol{R}$, we see that the closure of $\boldsymbol{R}_+$ in $\boldsymbol{R}$ coincides with $\boldsymbol{R}^+$. In this sense, we may write $\overline{\boldsymbol{R}_+}=\boldsymbol{R}^+$, $^-$ denoting the closure operation.

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§ 1. A differentiable $\boldsymbol{R}_+$-action.

Let A be a locally multiplicatively convex topological algebra over $\boldsymbol{C}$. Namely, A is topologized by a system of semi-norms Σ such that the (separately continuous) multiplication $\cdot$ in A satisfies the following relation:

$$p(f\cdot g)\leqq p(f)p(g)\,, \qquad f,\, g\in A,\ p\in\Sigma\,. \tag{1.1}$$

We assume for simplicity that A is unitary, $1\in A$. For more details about locally multiplicatively convex topological algebras, we refer to Michael [7].

DEFINITION 1.1. Let X be a locally convex topological vector space over $\boldsymbol{C}$. A family of linear operators $G=\{G_t\,;\ t\in\boldsymbol{R}_+\}$ in X is called a differentiable $\boldsymbol{R}_+$-action in X if the following three conditions are fulfilled:

(1.2) For any compact set K in $\boldsymbol{R}_+$ and any continuous semi-norm p on X there exists a continuous semi-norm q on X such that $p(G_tf)\leqq q(f)$ for any $f\in X$ and $t\in K$.

(1.3) For any $f\in X$, G_tf is strongly differentiable in t.

(1.4) For any t, $s\in\boldsymbol{R}_+$, $G_tG_s=G_sG_t=G_{ts}$ and $G_1=id$.

This definition is supplemented by the following when A is a locally multiplicatively convex topological algebra.

DEFINITION 1.2. A family of linear operators $G=\{G_t\,;\ t\in\boldsymbol{R}_+\}$ in A is called a differentiable $\boldsymbol{R}_+$-action in A if G satisfies (1.2), (1.3), (1.4) (X replaced by A), and

(1.5) For any $t\in\boldsymbol{R}_+$, G_t is multiplicative, that is, $G_t(f\cdot g)=G_tf\cdot G_tg$ for any f, $g\in A$ and $G_t1=1$.

From now on we assume that A is a multiplicatively convex Fréchet algebra, that is, a locally multiplicatively convex topological algebra whose underlying linear space is Fréchet. We fix a system of semi-norms $\{p_n ; n\in \boldsymbol{Z}^+\}$ in A satisfying (1.1) and $p_n(f)\leqq p_{n+1}(f)$ for any $f\in A$. Some of the discussions below are actually done under weaker assumptions on A. In particular, a few results hold good without the algebra structure of A. Note that in Definition 1.2 the requirement (1.2) is in fact a consequence of (1.3) and (1.4) since A is Fréchet (see Kōmura [6], Proposition 1.1). In this respect, we prepare the following

DEFINITION 1.3. A differentiable $\boldsymbol{R}_+$-action G in A is said to be strong if G satisfies, instead of (1.2), the condition:

(1.6) For any $t_0\in \boldsymbol{R}_+$ and $n\in \boldsymbol{Z}^+$ there exist $m\in \boldsymbol{Z}^+$ and $c\in \boldsymbol{R}_+$ such that $p_n(G_t f)\leqq c p_m(f)$ for any $f\in A$ and $t\geqq t_0$.

PROPOSITION 1.4. *Let $t\in \boldsymbol{R}_+$ and set*

$$Ef=G_t^{-1}\Big(t\frac{d}{dt}G_t f\Big), \qquad f\in A. \tag{1.7}$$

Then E is independent of t and a continuous linear operator in A. Furthermore, we have

$$EG_r=G_r E \qquad \text{for all} \quad r\in \boldsymbol{R}_+, \tag{1.8}$$

$$E(f\cdot g)=Ef\cdot g+f\cdot Eg, \qquad f,\ g\in A. \tag{1.9}$$

PROOF. Let $H_s=G_t$ for $t=e^s$, $s\in \boldsymbol{R}$. Then $H=\{H_s ; s\in \boldsymbol{R}\}$ is a locally equi-continuous group of linear operators in A. This is a consequence of (1.2), (1.3) and (1.4). Let E_1 be the infinitesimal generator of H. Then E_1 is a closed linear operator (Kōmura [6], Proposition 1.4.) and defined on all of A, thus is continuous. Since $\frac{d}{ds}H_s=t\frac{d}{dt}G_t$ for $t=e^s$, we have $E=E_1$. (1.8) is then immediate and (1.9) follows from (1.7) and (1.5). Q. E. D.

DEFINITION 1.5. We call E the Euler field of the differentiable $\boldsymbol{R}_+$-action G.

Let $\mu\in \boldsymbol{C}$. $f\in A$ is called G-homogeneous of degree μ if

$$G_t f=t^\mu f \qquad \text{for all} \quad t\in \boldsymbol{R}_+. \tag{1.10}$$

PROPOSITION 1.6. (Euler). *Let $f\in A$. f is G-homogeneous of degree μ if and only if*

$$Ef=\mu f. \tag{1.11}$$

PROOF. (1.7) and (1.10) immediately imply (1.11). On the other hand, we have $EG_t f=\mu G_t f$ from (1.11) via (1.8). Now by (1.7), $t\frac{d}{dt}G_t f=\mu G_t f$, or equivalently, $\frac{d}{dt}(t^{-\mu}G_t f)=0$. Q. E. D.

§ 2. The asymptotic class.

Let G be a differentiable $\boldsymbol{R}_+$-action in a multiplicatively convex Fréchet algebra A. For $\rho\in\boldsymbol{R}$, we denote by B^ρ the totality of $f\in A$ such that for any $a\in\boldsymbol{R}_+$ the set

(2.1) $$\{t^{-\rho}G_t f;\ t\geqq a\} \text{ is bounded in } A.$$

By virtue of (1.2), it is enough to assume (2.1) only for $a=1$.

PROPOSITION 2.1. *Let ρ, σ be any real numbers. Then*

(2.2) *B^ρ is a Fréchet space.*

(2.3) *If $\rho\leqq\sigma$, then $B^\rho\subset B^\sigma$ with the continuous injection.*

(2.4) *The mapping $M_{\rho,\sigma}$: $B^\rho\times B^\sigma\ni(f, g)\rightarrow f\cdot g\in B^{\rho+\sigma}$ is bilinear continuous.*

PROOF. A system of semi-norms in B^ρ is given by

(2.5) $$p_n^\rho(f)=\sup_{t\geq 1}p_n(t^{-\rho}G_t f), \qquad n\in\boldsymbol{Z}^+,\ f\in B^\rho.$$

We prove the completeness B^ρ. Let f_j, $j\in\boldsymbol{Z}^+$, be a Cauchy sequence in B^ρ. Then since $p_n(f_j-f_k)\leqq p_n^\rho(f_j-f_k)$ for any $n\in\boldsymbol{Z}^+$, $\{f_j\}$ is a Cauchy sequence in A. Let f be the limit of $\{f_j\}$ in A. It suffices to show that $f\in B^\rho$, for B^ρ and its eventual completion are both embedded in A. Take any $n\in\boldsymbol{Z}^+$ and set

$$\phi_j(t)=p_n(t^{-\rho}G_t f_j), \qquad j\in\boldsymbol{Z}^+.$$

Since, by the triangle inequality,

$$|\phi_j(t)-\phi_k(t)|\leqq p_n^\rho(f_j-f_k),$$

we see that $\{\phi_j(t)\}$ is a Cauchy sequence in $C_b[1, +\infty)$, the Banach space of uniformly bounded continuous functions on $t\geqq 1$. Thus, there is a $\phi(t)\in C_b[1, +\infty)$ such that $\phi_j(t)\rightarrow\phi(t)$ in $C_b[1, +\infty)$. On the other hand, for each $t\geqq 1$, $t^{-\rho}G_t f_j\rightarrow t^{-\rho}G_t f$ in A. Therefore, $\phi(t)=p_n(t^{-\rho}G_t f)$ and $f\in B^\rho$. Other assertions of the proposition are obvious. Q. E. D.

In view of (2.3), we write $B^{-\infty}=\bigcap_{\rho\in\boldsymbol{R}}B^\rho$.

PROPOSITION 2.2. *$B^{-\infty}$ is a multiplicatively convex Fréchet algebra.*

PROOF. That $B^{-\infty}$ is a Fréchet space is obvious. A system of semi-norms in $B^{-\infty}$ is given by

(2.6) $$p_n^{-m}(f), \qquad n,\ m\in\boldsymbol{Z}^+,\ f\in B^{-\infty},$$

by virtue of (2.3) and (2.5). Then

$$p_n^{-m}(f\cdot g)\leqq p_n^{-m/2}(f)p_n^{-m/2}(g)\leqq p_n^{-m}(f)p_n^{-m}(g)$$

because of (1.5). Q. E. D.

PROPOSITION 2.3. *If the differentiable $\boldsymbol{R}_+$-action is strong, then* $B^\rho=A$ *for* $\rho\geqq 0$.

PROOF. Obvious from (1.6) and (2.1). Q. E. D.

PROPOSITION 2.4. $f\in B^\rho$ *if and only if*

$$p_n\Big(\Big(\frac{d}{dt}\Big)^k G_t f\Big)\leqq C_{n,k}t^{\rho-k}, \qquad t\geqq 1, \tag{2.7}$$

for all n, $k\in\boldsymbol{Z}^+$ *with some positive constants* $C_{n,k}$.

PROOF. (2.7) for $k=0$ is just the requirement (2.1). We observe that the Euler field E preserves B^ρ by virtue of (1.8). By the induction, we have

$$\Big(\frac{d}{dt}\Big)^k G_t f=t^{-k}\sum_{j=1}^{k} a_j^k E^j G_t f, \qquad f\in A,\ k\in\boldsymbol{Z}^+\backslash 0, \tag{2.8}$$

$$a_k^k=1, \quad a_1^k=(-1)^{k-1}(k-1)!, \quad a_j^k=-(k-1)a_j^{k-1}+a_{j-1}^{k-1} \quad \text{for } j=2,\cdots,k-1. \tag{2.9}$$

(2.7) now follows from (2.8). Q. E. D.

Let us denote by Γ^μ the totality of G-homogeneous elements of degree μ. Let $M_{\mu,\nu}$ be the mapping, defined by restricting the multiplication in A, as (2.4).

PROPOSITION 2.5. *Let* μ, ν *be any complex numbers. Then*

(2.10) Γ^μ *is a closed subspace of* A *and of* B^ρ, $\rho>\operatorname{Re}\mu$.

$$\Gamma^\mu\cap\Gamma^\nu=\{0\} \quad \text{if} \quad \mu\neq\nu. \tag{2.11}$$

(2.12) $M_{\mu,\nu}$ *is bilinear continuous from* $\Gamma^\mu\times\Gamma^\nu$ *to* $\Gamma^{\mu+\nu}$.

PROOF. Obvious. Q. E. D.

PROPOSITION 2.6. $\Gamma^0\neq\{0\}$. *If G is strong,* $\Gamma^\mu=\{0\}$ *for* $\operatorname{Re}\mu>0$.

PROOF. $1\in\Gamma^0$ by (1.5). Let $f\in\Gamma^\mu$ for $\operatorname{Re}\mu>0$. Then $G_tf=t^\mu f$, while, by (1.6), $p_n(G_tf)\leqq cp_m(f)$ for $t\geqq 1$ if G is strong. The last assertion also follows from the equi-continuity of G_t, $t\geqq 1$ (see Yosida [**11**], Chapter [IX]).

Q. E. D.

We end this section with a few words on the differentiable $\boldsymbol{R}_+$-action in $B^{-\infty}$. Namely, we have

PROPOSITION 2.7. *The differentiable $\boldsymbol{R}_+$-action G induces in $B^{-\infty}$ a differentiable $\boldsymbol{R}_+$-action which is strong.*

PROOF. That G acts as a differentiable $\boldsymbol{R}_+$-action in $B^{-\infty}$ is obvious. We verify that it is strong. Let $r\in\boldsymbol{R}_+$ and $f\in B^{-\infty}$. Then, for n, $m\in\boldsymbol{Z}^+$,

$$p_n^{-m}(G_r f)=\sup_{t\geqq 1} t^m p_n(G_t G_r f)$$
$$=r^{-m}\sup_{t\geqq 1}(rt)^m p_n(G_{tr} f)\,.$$

Thus,

$$p_n^{-m}(G_r f)\leqq r^{-m} p_n^{-m}(f) \qquad \text{for} \quad r\geqq 1\,. \tag{2.13}$$

Q. E. D.

By the construction of $B^{-\infty}$, it is clear that for any $\mu\in\boldsymbol{C}$, $\mu-E$ is injective in $B^{-\infty}$. We can say a little more.

PROPOSITION 2.8. *For any $\mu\in\boldsymbol{C}$, $\mu-E$ has a continuous inverse in $B^{-\infty}$.*

PROOF. Let $f\in B^{-\infty}$. We solve the equation

$$\mu g-Eg=f \tag{2.14}$$

in $B^{-\infty}$. By (1.7), we then have

$$\mu G_s g-s\frac{d}{ds}G_s g=G_s f\,,$$

or, equivalently,

$$\frac{d}{ds}(s^{-\mu}G_s g)=-s^{-\mu-1}G_s f\,.$$

Hence, (2.14) is solved by

$$g=\int_1^\infty s^{-\mu}G_s f\, ds/s\,. \tag{2.15}$$

Now we verify that $g\in B^{-\infty}$. For any $t\in\boldsymbol{R}_+$,

$$G_t g=\int_1^\infty s^{-\mu}G_{st} f\, ds/s\,.$$

Therefore, for $m\in\boldsymbol{Z}^+$, $m>-\operatorname{Re}\mu$, and $n\in\boldsymbol{Z}^+$,

$$p_n(G_t g)\leqq\int_1^\infty s^{-\rho}p_n(G_{st}f)ds/s\,, \qquad \rho=\operatorname{Re}\mu\,,$$
$$\leqq p_n^{-m}(f)\int_1^\infty s^{-\rho}(st)^{-m}\, ds/s$$
$$\leqq t^{-m}(\rho+m)^{-1}p_n^{-m}(f)\,.$$

That is,

$$p_n^{-m}(g)\leqq(m+\operatorname{Re}\mu)^{-1}p_n^{-m}(f) \tag{2.16}$$

for $m+\operatorname{Re}\mu>0$. Q. E. D.

COROLLARY 2.9. *Let $P(E)$ be any non-trivial polynomial in E with complex coefficients. Then $P(E)$ is an isomorphism of $B^{-\infty}$.*

§ 3. The formal asymptotic class.

We keep the assumptions and notations of § 2. Let $\mu\in \boldsymbol{C}$, and set

$$C^{\mu}=\prod_{j\in Z^+}\Gamma^{\mu-j}. \tag{3.1}$$

Since $\Gamma^{\mu-j}$ are Fréchet spaces (see (2.10)), C^{μ} carries a natural Fréchet structure (see, e. g., Treves [**9**], p. 94). A system of semi-norms in C^{μ} is given by

$$q^{\mu}_{n,N}(f_*)=\sum_{j=0}^{N}p_n^{\rho-j}(f_j), \qquad n,\ N\in \boldsymbol{Z}^+,\ p=\mathrm{Re}\,\mu, \tag{3.2}$$

for $f_*=(f_0, \cdots, f_j, \cdots)\in C^{\mu}$. Furthermore, C^{μ} is canonically identified with a closed subspace of $C^{\mu+1}$.

Let $f_*=(f_0, \cdots, f_j, \cdots)\in C^{\mu}$, $g_*=(g_0, \cdots, g_k, \cdots)\in C^{\nu}$, $f_j\in\Gamma^{\mu-j}$, $g_k\in\Gamma^{\nu-k}$, $j,\ k\in \boldsymbol{Z}^+$. We define

$$N_{\mu,\nu}(f_*, g_*)=(h_0, \cdots, h_l, \cdots) \tag{3.3}$$

by

$$h_l=\sum_{j+k=l}f_j\cdot g_k, \qquad l\in \boldsymbol{Z}^+. \tag{3.4}$$

Then $N_{\mu,\nu}$ is a continuous bilinear mapping from $C^{\mu}\times C^{\nu}$ to $C^{\mu+\nu}$. Summarizing, we have shown

PROPOSITION 3.1. *Let μ, ν be any complex numbers. Then*

(3.5) *C^{μ} is a Fréchet space.*

(3.6) *C^{μ} is a closed subspace of $C^{\mu+1}$.*

(3.7) *$N_{\mu,\nu}$ is bilinear continuous from $C^{\mu}\times C^{\nu}$ to $C^{\mu+\nu}$.*

(3.8) *$C^{\mu}\cap C^{\nu}=\{0\}$ if $\mu-\nu\notin \boldsymbol{Z}$.*

PROOF. (3.8) follows from (3.1) and (2.11). Q. E. D.

COROLLARY 3.2. *Let C be the strict inductive limit of C^j, $j\in \boldsymbol{Z}$. Then C is a locally multiplicatively convex topological algebra. $N_{j,k}$, $j,\ k\in \boldsymbol{Z}$, is the restriction to $C^j\times C^k$ of the multiplication of C.*

PROOF. Immediate from the definition of the strict inductive limit (see, e. g., Treves [**9**], Chapter 13). Q. E. D.

DEFINITION 3.3. An element $f\in A$ is said to be developable if there is an element $f_*=(f_0, \cdots, f_j, \cdots)\in C^{\mu}$ such that for any $N\in \boldsymbol{Z}^+$

$$f-\sum_{j<N}f_j\in B^{\rho-N}, \qquad \rho=\mathrm{Re}\,\mu. \tag{3.9}$$

This f_* is called a development of f.

By virtue of (2.11), if f is developable, then its development f_* is uniquely determined.

Let us denote by D^{μ} the totality of developable elements f with their developments $f_*=(f_0, \cdots, f_j, \cdots)\in C^{\mu}$. Let α_N^{μ}, $N\in \boldsymbol{Z}^+$, be the mapping

$$\alpha_N^{\mu}: D^{\mu}\ni f\to (f_0, \cdots, f_{N-1}, f-\sum_{j<N} f_j)\in \prod_{j<N}\Gamma^{\mu-j}\times B^{\rho-N}, \tag{3.10}$$

where $\rho=\mathrm{Re}\,\mu$. We decompose $\alpha_N^{\mu}=\pi_N^{\mu}\times\varepsilon_N^{\mu}$, π_N^{μ} and ε_N^{μ} being respectively the projections to the first product space and to the last factor. We equip D^{μ} with the coarsest topology rendering all α_N^{μ} continuous. Then D^{μ} is a Fréchet space, continuously embedded in B^{ρ}, $\rho=\mathrm{Re}\,\mu$. A system of semi-norms in D^{μ} is given by

$$q_{n,N}^{\mu}(\pi_N^{\mu}(f)) \text{ and } p_n^{\rho-N}(\varepsilon_N^{\mu}(f)), \qquad f\in D^{\mu}, \qquad n, N\in \boldsymbol{Z}^+, \ \rho=\mathrm{Re}\,\mu. \tag{3.11}$$

Furthermore, D^{μ} is a closed subspace of $D^{\mu+1}$. Let $M_{\mu,\nu}$ be the mapping defined by the restriction of the multiplication in A. Then it is immediately seen that $M_{\mu,\nu}$ is continuous from $D^{\mu}\times D^{\nu}$ to $D^{\mu+\nu}$. Thus, we have shown

PROPOSITION 3.4. *Let μ, ν be any complex numbers. Then*

(3.12) *D^{μ} is a Fréchet space.*

(3.13) *D^{μ} is a closed subspace of $D^{\mu+1}$.*

(3.14) *$M_{\mu,\nu}$ is continuous bilinear from $D^{\mu}\times D^{\nu}$ to $D^{\mu+\nu}$.*

(3.15) *$D^{\mu}\cap D^{\nu}=B^{-\infty}$ if $\mu-\nu\notin \boldsymbol{Z}$.*

PROOF. (3.15) follows from (3.8) and (3.9). Q.E.D.

COROLLARY 3.5. *For any $k\in \boldsymbol{Z}^+$,*

$$D^{\mu}=\sum_{j<k}\Gamma^{\mu-j}+D^{\mu-k} \tag{3.16}$$

as a topological direct sum, and $\sum_{j<k}\Gamma^{\mu-j}$ and $D^{\mu-k}$ are closed subspaces of D^{μ}.

PROOF. The mapping ε_k^{μ} is continuous from D^{μ} onto $D^{\mu-k}$ and $(\varepsilon_k^{\mu})^2=\varepsilon_k^{\mu}$. Furthermore, $\sum_{j<k}\Gamma^{\mu-j}=(id-\varepsilon_n^{\mu})D^{\mu}$. Q.E.D.

Since $B^{-\infty}=\bigcap_{k\in Z^+} D^{\mu-k}$, $B^{-\infty}$ is a closed subspace of D^{μ}. However, in general, $B^{-\infty}$ has no topological complement in D^{μ} (see Examples in §7).

Note that if $\Gamma^{\mu}=\{0\}$, then $D^{\mu}=D^{\mu-j}$ for some $j\in \boldsymbol{Z}^+$ if $\Gamma^{\mu-j}\neq\{0\}$ and $\Gamma^{\mu-k}=\{0\}$ for $k<j$. If such j does not exist, then $D^{\mu}=B^{-\infty}$.

DEFINITION 3.6. We denote by α^{μ} the mapping which assigns to each element of D^{μ} its development in C^{μ}.

Then we have the following

PROPOSITION 3.7. *Let μ, ν be any complex numbers. Then*

(3.17) α^{μ} *is continuous linear from* D^{μ} *to* C^{μ}.

(3.18) $$\alpha^{\mu}\circ M_{\mu,\nu}=N_{\mu,\nu}\circ(\alpha^{\mu}\times\alpha^{\nu}).$$

(3.19) $$\ker\alpha^{\mu}=B^{-\infty}.$$

PROOF. Obvious. Q. E. D.

PROPOSITION 3.8. *Let D be the strict inductive limit of D^j, $j\in\boldsymbol{Z}$. Then D is a locally multiplicatively convex topological algebra. $M_{j,k}$, j, $k\in\boldsymbol{Z}$, in* (3.14), *is the restriction to $D^j\times D^k$ of the multiplication of D. Furthermore, $B^{-\infty}$ is a closed ideal of D. There is a continuous algebra homomorphism α from D to C, whose restriction to D^j coincides with α^j, and* $\ker\alpha=B^{-\infty}$.

PROOF. Obvious from the definitions. Q. E. D.

§ 4. A characterization of the spaces D^{μ}.

We keep the previous assumptions and notations. In particular, recall that $\boldsymbol{R}^+$ is the set of non-negative reals so that $\boldsymbol{R}^+$ is the closure of $\boldsymbol{R}_+$ in $\boldsymbol{R}$.

For $k\in\boldsymbol{Z}^+$, we denote by $\mathcal{E}^k(\boldsymbol{R}^+;A)$ the totality of A-valued functions $u(s)$ defined on $\boldsymbol{R}_+$, strongly continuously differentiable in $\boldsymbol{R}_+$ up to k-times such that $\lim_{s\to 0}\left(\frac{d}{ds}\right)^j u(s)$ exist for $j=0,\cdots,k$. In other words, $u(s)\in\mathcal{E}^k(\boldsymbol{R}^+;A)$ if and only if $u(s)$ is the restriction to $\boldsymbol{R}_+$ of an A-valued function defined in a neighborhood of $\boldsymbol{R}^+$, strongly continuously differentiable up to k-times. We set

$$\mathcal{E}^{\infty}(\boldsymbol{R}^+;A)=\bigcap_{k\in\boldsymbol{Z}^+}\mathcal{E}^k(\boldsymbol{R}^+;A).$$

$\mathcal{E}^{\infty}(\boldsymbol{R}^+;A)$ is naturally a multiplicatively convex Fréchet algebra.

Namely, (1.1) is fulfilled by the system $\{p_{n,k,l};\,n,k,l\in\boldsymbol{Z}^+\}$ of semi-norms in $\mathcal{E}^{\infty}(\boldsymbol{R}^+;A)$. Here we set, for $u\in\mathcal{E}^{\infty}(\boldsymbol{R}^+;A)$,

(4.1) $$p_{n,k,l}(u)=\sup_{0\le s\le l,\,0\le j\le k}2^{k+j}p_n\left(\left(\frac{d}{ds}\right)^j u(s)\right).$$

Note that for any $r\in\boldsymbol{R}_+$ and $u\in\mathcal{E}^{\infty}(\boldsymbol{R}^+;A)$

(4.2) $$(G_r u)(s)=G_r u(s)$$

defines a differentiable $\boldsymbol{R}_+$-action in $\mathcal{E}^{\infty}(\boldsymbol{R}^+;A)$, commuting with the differentiation and multiplication by s.

Let $A_j=A$, $j\in\boldsymbol{Z}^+$, be a countable family of copies of A. We set $A^{\sim}=\prod_{j\in\boldsymbol{Z}^+}A_j$ with the product topology. Then $A^{\sim}$ is a Fréchet space. For

$u\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$, let $\tau(u)\in A^{\sim}$ by

$$\tau(u)=(u_{0}, \cdots, u_{j}, \cdots), \tag{4.3}$$

$$u_{j}=\lim_{s\to 0}\left(\frac{d}{ds}\right)^{j}u(s)/j!, \qquad j\in\boldsymbol{Z}^{+}.$$

PROPOSITION 4.1. *τ is continuous linear from $\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$ onto $A^{\sim}$.*

PROOF. That τ is continuous linear is obvious. We verify the surjectivity of τ. Let $\phi\in C^{\infty}(\boldsymbol{R})$ such that $\phi(s)=1$ for $s<1/2$ and $\phi(s)=0$ for $s>1$. Let us set

$$C_{j,k}=\sup_{0\leqq s\leqq 1}\left|\left(\frac{d}{ds}\right)^{k}(s^{j}\phi(s))\right|, \qquad j, k\in\boldsymbol{Z}^{+}.$$

Let $a_{j}\in A$, $j\in\boldsymbol{Z}^{+}$, be given. We can choose by the diagonal procedure an increasing sequence $\{r_{j}; j\in\boldsymbol{Z}^{+}\}$ of positive numbers such that for each n, $k\in\boldsymbol{Z}^{+}$,

$$\sum_{j\geqq k}(r_{j})^{k-j}C_{j,k}p_{n}(a_{j}) \text{ converges.} \tag{4.4}$$

Let

$$u(s)=\sum_{j\in\boldsymbol{Z}^{+}}\phi(sr_{j})s^{j}a_{j}.$$

Then, for each n, $k\in\boldsymbol{Z}^{+}$,

$$p_{n}\left(\sum_{j\geqq k}\left(\frac{d}{ds}\right)^{k}(\phi(sr_{j})s^{j})a_{j}\right)$$
$$\leqq\sum_{j\geqq k}(r_{j})^{k-j}C_{j,k}p_{n}(a_{j})$$

for all $s\geqq 0$. Thus, by (4.4), we see $u(s)\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$. Furthermore,

$$\left(\frac{d}{ds}\right)^{j}u(s)|_{s=0}=j!\,a_{j}. \qquad \text{Q.E.D.}$$

For each $\mu\in\boldsymbol{C}$, let us denote by F^{μ} the totality of $s^{\mu}G_{s}^{-1}f$, $f\in A$, such that $s^{\mu}G_{s}^{-1}f\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$.

PROPOSITION 4.2. *For each $\mu\in\boldsymbol{C}$, F^{μ} is a closed subspace of $\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$.*

PROOF. Let $u_{j}(s)=s^{\mu}G_{s}^{-1}f_{j}\in F^{\mu}$, $j\in\boldsymbol{Z}^{+}$. Assume $u_{j}(s)$ converge to a $u(s)$ in $\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$. Then since $u_{j}(s)$ is a Cauchy sequence in $\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$, f_{j} is a Cauchy sequence in B^{ρ}, $\rho=\mathrm{Re}\,\mu$, in view of (4.1) and (2.5). Thus, there is an $f\in B^{\rho}$ to which f_{j} converges in B^{ρ} and so in A. In particular, for each $s\in\boldsymbol{R}_{+}$, $u(s)=s^{\mu}G_{s}^{-1}f$ and $s^{\mu}G_{s}^{-1}f\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$. Q.E.D.

Note that $u(s)\in F^{\mu}$ if and only if $u(s)\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$ and

$$G_{r}u(s)=r^{\mu}u(sr^{-1})$$

for all $r\in\boldsymbol{R}_{+}$. This also proves Proposition 4.2.

Now we study behaviors of $G_s^{-1}f$, $f\in A$, in $\mathcal{E}^\infty(\boldsymbol{R}^+;A)$.

LEMMA 4.3. *Set*

$$u(s;f)=G_s^{-1}f, \qquad f\in A. \tag{4.5}$$

Then for $k\in \boldsymbol{Z}^+\backslash 0$, *we have*

$$\left(\frac{d}{ds}\right)^k u(s;f)=\sum_{j=1}^{k} b_j^k s^{-j-k}\left(\frac{d}{dr}\right)^j G_r f|_{r=s^{-1}}, \tag{4.6}$$

$$b_k^k=(-1)^k,\ b_1^k=(-1)^k k!,\ b_j^k=-\{(k+j-1)b_j^{k-1}+b_{j-1}^{k-1}\}, \tag{4.7}$$

$$2\leqq j\leqq k-1.$$

Furthermore,

$$u(s;af+bg)=au(s;f)+bu(s;g), \quad a,b\in\boldsymbol{C}, \quad f,g\in A. \tag{4.8}$$

$$u(s;f\cdot g)=u(s;f)\cdot u(s;g). \tag{4.9}$$

$$G_r u(s;f)=u(s;G_r f)=u(sr^{-1};f), \qquad r\in \boldsymbol{R}_+. \tag{4.10}$$

PROOF. Obvious. Q. E. D.

COROLLARY 4.4. *Let* $f\in B^{\rho-j}$, $j\in\boldsymbol{Z}^+$. *Then*

$$p_n\left(\left(\frac{d}{ds}\right)^k (s^\rho u(s;f)\right)\leqq C_{n,k}s^{j-k} \tag{4.11}$$

for $0<s\leqq 1$, $n,k\in\boldsymbol{Z}^+$, $C_{n,k}$ *being positive constants independent of s.*

PROOF. This follows immediately from (4.6) and (2.7). Q. E. D.

In particular, if $j\geqq 1$, $s^\rho u(s;f)$ vanishes to the $(j-1)$-th order at $s=0$ when $f\in B^{\rho-j}$. Thus, $s^\mu u(s;f)\in\mathcal{E}^{j-1}(\boldsymbol{R}^+;A)$ if $f\in B^{\rho-j}$, $\rho=\mathrm{Re}\,\mu$. On the other hand, if $f\in\Gamma^{\mu-j}$, $j\in\boldsymbol{Z}^+$, then $s^\mu u(s;f)=s^j u(1;f)$, so $s^\mu u(s;f)\in\mathcal{E}^\infty(\boldsymbol{R}^+;A)$.

Now we are ready to state and prove our main result. For each $\mu\in\boldsymbol{C}$, we define a mapping i^μ from D^μ to F^μ by

$$i^\mu(f)=s^\mu G_s^{-1}f, \qquad f\in D^\mu. \tag{4.12}$$

Thus, $i^\mu(f)=s^\mu u(s;f)$ in the above notation. Recall that D^μ is the space of developable elements.

THEOREM 4.5. i^μ *is an isomorphism of* D^μ *onto* F^μ. *Furthermore,*

$$\alpha^\mu=\tau\circ i^\mu. \tag{4.13}$$

PROOF. Let $f\in D^\mu$. Then $\alpha^\mu(f)=(f_0,\cdots,f_j,\cdots)\in C^\mu$, $f_j\in\Gamma^{\mu-j}$, and for any $N\in\boldsymbol{Z}^+$, $f-\sum_{j<N}f_j\in B^{\rho-N}$, $\rho=\mathrm{Re}\,\mu$. But since

$$s^\mu u(s;f)=\sum_{j<N}s^j u(1;f)+s^\mu u(s;f-\sum_{j<N}f_j)$$

and $s^\mu u(s;f-\sum_{j<N}f_j)\in\mathcal{E}^{N-1}(\boldsymbol{R}^+;A)$, we have

$$s^{\mu}u(s\,;\,f)\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}\,;\,A)\,.$$

Thus, i^{μ} is defined on all of D^{μ}. i^{μ} is clearly injective. i^{μ} is continuous in view of (4.1) and (3.11). (4.13) is then immediate. Now let $f\in A$ be such that $s^{\mu}u(s\,;\,f)\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}\,;\,A)$. Let us set

$$f_j=\left(\frac{d}{ds}\right)^j(s^{\mu}u(s\,;\,f))|_{s=0}/j\,!\,,\qquad j\in\boldsymbol{Z}^{+}\,, \tag{4.14}$$

and for any $N\in\boldsymbol{Z}^{+}$

$$v_N(s\,;\,f)=s^{\mu}u(s\,;\,f)-\sum_{j<N}s^jf_j\,.$$

Then, since $s^{\mu}u(s\,;\,f)\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}\,;\,A)$,

$$p_n\left(\left(\frac{d}{ds}\right)^k v_N(s\,;\,f)\right)\leqq C_{n,N,k}\min(1,\,s^{N-k})\,,\qquad 0<s\leqq 1\,, \tag{4.15}$$

for any $n,\,k\in\boldsymbol{Z}^{+}$, $C_{n,N,k}$ some positive constants. On the other hand, since G_r, $r\in\boldsymbol{R}_{+}$, define a differentiable $\boldsymbol{R}_{+}$-action in $\mathcal{E}^{\infty}(\boldsymbol{R}^{+}\,;\,A)$, commuting with the differentiation and multiplication by s, and since $G_rs^{\mu}u(s\,;\,f)=r^{\mu}(sr^{-1})^{\mu}u(sr^{-1}\,;\,f)$, we have

$$\begin{aligned}G_rv_N(s\,;\,f)&=G_rs^{\mu}u(s\,;\,f)-\sum_{j<N}s^jG_rf_j\\&=s^{\mu}u(s\,;\,G_rf)-\sum_{j<N}r^{\mu-j}s^jf_j\,.\end{aligned}$$

Hence, we have $G_rf_j=r^{\mu-j}f_j$, or $f_j\in\Gamma^{\mu-j}$. Furthermore,

$$v_N(s\,;\,f)=s^{\mu}u(s\,;\,f-\sum_{j<N}f_j)\,,$$

and (4.15) now implies

$$p_n(G_t(f-\sum_{j<N}f_j))\leqq C_{n,N}t^{\rho-N}\,,\qquad t\geqq 1,\quad \rho=\operatorname{Re}\mu$$

for $n\in\boldsymbol{Z}^{+}$ with some positive constants $C_{n,N}$. That is,

$$f-\sum_{j<N}f_j\in B^{\rho-N}\,,$$

thus completing the proof. Q. E. D.

The above theorem gives a characterization of developable elements. However, in practice, to check the conditions $s^{\mu}G_s{}^{-1}f\in\mathcal{E}^{\infty}(\boldsymbol{R}^{+}\,;\,A)$ for $f\in A$ is essentially equivalent to give the development of f. Also compare with Wasow [**10**], Chapter III, § 9, p. 39.

§ 5. The surjectivity of the mapping α^{μ}.

We keep the notations and assumptions of the previous sections. We first note the following observation.

PROPOSITION 5.1. *α^{μ} is suriective if and only if $F^{\mu}+\ker\tau$ is a closed subspace of $\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$.*

PROOF. If α^{μ} is surjective, then $\tau(F^{\mu})=C^{\mu}$ by Theorem 4.5. Since C^{μ} is a closed subspace of $A^{\sim}$, $\tau^{-1}(C^{\mu})=F^{\mu}+\ker\tau$ is closed. On the other hand, if $F^{\mu}+\ker\tau$ is closed, then $\complement(F^{\mu}+\ker\tau)=\tau^{-1}(\complement\tau(F^{\mu}))$ is open. Thus, $\complement\tau(F^{\mu})$ is open since τ is an open mapping. The set $\bigcup_{N\in Z^{+}}\prod_{j<N}\Gamma^{\mu-j}$ being dense in C^{μ}, $\tau(F^{\mu})=C^{\mu}$, proving the surjectivity of α^{μ} in view of (4.13). Q.E.D.

The trouble here is that we have few informations on closedness of $F^{\mu}+\ker\tau$. In fact, it happens that even if $F^{\mu}\cap\ker\tau=\{0\}$, $F^{\mu}+\ker\tau$ is not a closed subspace of $\mathcal{E}^{\infty}(\boldsymbol{R}^{+}; A)$.

For practical purposes, it is thus desirable and often more interesting to give conditions assuring a direct proof of the surjectivity of the mapping α^{μ}.

DEFINITION 5.2. An element $e\in A$ is called a convergence factor for the differentiable $\boldsymbol{R}_{+}$-action G if the following two conditions are fulfilled:

(5.1) There is a positive number κ such that

$$p_{n}(G_{t}e)\leqq C_{n}\min(1, t^{\kappa})$$

for all $n\in\boldsymbol{Z}^{+}$ and $t\in\boldsymbol{R}_{+}$ with some constants $C_{n}>0$.

$$p_{n}(1-G_{t}e)\leqq C_{n,N}t^{-N} \tag{5.2}$$

for all N, $n\in\boldsymbol{Z}^{+}$, $t\geqq 1$, with some constants $C_{n,N}>0$.

The requirement (5.1) implies that $G_{r}e\in B^{0}$ for any $r\in\boldsymbol{R}_{+}$. (5.2) means that $1-e\in B^{-\infty}$, whence $1-G_{r}e\in B^{-\infty}$ for any $r\in\boldsymbol{R}_{+}$. Furthermore note that it follows from (5.1)

$$p_{n}(t^{-\kappa}G_{t}e)\leqq C_{n} \tag{5.3}$$

for all $n\in\boldsymbol{Z}^{+}$ and $t\in\boldsymbol{R}_{+}$.

The following proposition shows that there are cases without convergence factors.

PROPOSITION 5.3. *Let G be a strong differentiable $\boldsymbol{R}_{+}$-action. If there is a convergence factor e for G, then $B^{-\infty}$ is dense in A.*

PROOF. Let $f\in A$. Then $f-G_{r^{-1}}e\cdot f\in B^{-\infty}$ for any $r\geqq 1$. By (5.3), $G_{r^{-1}}e\cdot f\to 0$ as $r\to+\infty$. Q.E.D.

PROPOSITION 5.4. *Let $e\in A$ satisfy* (5.3). *Then for any $f\in B^{\rho}$ and $r\geqq 1$, we have*

$$p_{n}^{\rho+\kappa}(r^{\kappa}G_{r^{-1}}e\cdot f)\leqq C_{n}p_{n}^{\rho}(f), \qquad n\in\boldsymbol{Z}^{+}. \tag{5.4}$$

Here p_{n}^{ρ}, $p_{n}^{\rho+\kappa}$ are semi-norms defined by (2.5).

PROOF. By (1.1) and (1.5), we have

$$p_n^{\rho+\kappa}(r^\kappa G_r{}^{-1}e\cdot f)=\sup_{t\geq 1} t^{-\rho-\kappa}p_n(G_t(r^\kappa G_r{}^{-1}e\cdot f))$$

$$\leqq \sup_{t\geq 1} t^{-\rho-\kappa}p_n(r^\kappa G_{tr^{-1}}e)p_n(G_t f)$$

$$\leqq \sup_{t\geq 1} p_n(t^{-\kappa}r^\kappa G_{tr^{-1}}e) \sup_{t\geq 1} p_n(t^{-\rho}G_t f)$$

$$\leqq C_n p_n^\rho(f)\,. \qquad \text{Q. E. D.}$$

The following proposition gives a sufficient condition for the surjectivity of α^μ. Its proof is a variant of classical ones (cf. § 7).

PROPOSITION 5.5. *If there is a convergence factor e for the differentiable $\boldsymbol{R}_+$-action G, then, for any $\mu\in\boldsymbol{C}$, the mapping α^μ is surjective.*

PROOF. Let $f_j\in\Gamma^{\mu-j}$, $j\in\boldsymbol{Z}^+$ be given. We show that there is an $f\in D^\mu$ such that for any $N\in\boldsymbol{Z}^+$

$$f-\sum_{j<N} f_j\in B^{\rho-N}\,,\qquad \rho=\mathrm{Re}\,\mu\,. \tag{5.5}$$

In view of Proposition 5.4, we can, by applying the diagonal process, choose an increasing sequence $r_j\geqq 1$, $j\in\boldsymbol{Z}^+$, such that for any $m\in\boldsymbol{Z}^+$ the set

$$\{2^j G_{r_j}{}^{-1}e\cdot f_j\,;\ j\geqq m\} \text{ is bounded in } B^{\rho-m+\kappa}. \tag{5.6}$$

Let $w_j=G_{r_j}{}^{-1}e$, $j\in\boldsymbol{Z}^+$, and $f=\sum_{j\in Z^+} w_j\cdot f_j$. Then since

$$f=\sum_{j<M+1} w_j\cdot f_j+\sum_{j\geq M+1} w_j\cdot f_j\,,$$

and if we take $M=$the integral part of κ, then by (5.6) and (5.1), we see $f\in B^\rho$. Furthermore, for any $N\in\boldsymbol{Z}^+$, since

$$f-\sum_{j<N} f_j=\sum_{j<N}(w_j-1)\cdot f_j+\sum_{j=N}^{N+M} w_j\cdot f_j+\sum_{j\geq N+M+1} w_j\cdot f_j\,,$$

(5.5) holds good in view of (5.6), (5.1) and (5.2). Q. E. D.

Essentially the same proof gives the following

COROLLARY 5.6. *Assume that there be a convergence factor for the differentiable $\boldsymbol{R}_+$-action. Let $f_j\in B^{m_j}$, $j\in\boldsymbol{Z}^+$, be given. Here m_j is a decreasing sequence tending to $-\infty$. Then there is an $f\in B^{m_0}$ such that for any $N\in\boldsymbol{Z}^+$*

$$f-\sum_{j<N} f_j\in B^{m_N}\,.$$

This f is uniquely determined up to the terms in $B^{-\infty}$.

§ 6. The case when A is a Montel space.

In practical cases, A is often a Montel space. We supplement the properties of the spaces B^{ρ}, Γ^{μ}, C^{μ}, D^{μ} under the additional hypothesis that A is a Montel space, keeping the assumptions and notations of the previous section.

The following proposition is then most fundamental.

PROPOSITION 6.1. *Let ρ, σ be any real numbers with $\rho<\sigma$. Let $I_{\sigma,\rho}$: $B^{\rho}\rightarrow B^{\sigma}$ be the inclusion mapping* (2.3). *If A is a Montel space, then $I_{\sigma,\rho}$ maps every bounded set in B^{ρ} to a relatively compact set in B^{σ}.*

PROOF. Let W be a bounded set in B^{ρ}. Thus, there is a sequence C_{n}, $n\in\boldsymbol{Z}^{+}$, of positive constants such that

$$p_{n}(t^{-\rho}G_{t}f)\leqq C_{n}, \qquad f\in W,\ t\geqq 1. \tag{6.1}$$

Fix an $n\in\boldsymbol{Z}^{+}$ and set

$$\phi_{f}(t)=p_{n}(t^{-\sigma}G_{t}f), \qquad f\in W. \tag{6.2}$$

$\phi_{f}(t)$ are continuous functions of $t\geqq 1$, and $\phi_{f}(t)\geqq 0$. Since $\rho<\sigma$, (6.1) implies

$$|\phi_{f}(t)|\leqq C_{n}, \qquad t\geqq 1,\ f\in W. \tag{6.3}$$

Furthermore, for any $\varepsilon>0$, there exists a $t_{0}\geqq 1$ such that

$$|\phi_{f}(t)|<\varepsilon, \qquad t\geqq t_{0},\ f\in W. \tag{6.4}$$

On the other hand, it follows from (2.8) that

$$\frac{d}{dt}(t^{-\sigma}G_{t}f)=-\sigma t^{-\sigma-1}G_{t}f+t^{-\sigma-1}G_{t}Ef,$$

whence

$$p_{n}\Big(\frac{d}{dt}(t^{-\sigma}G_{t}f)\Big)\leqq C_{n}', \qquad t\geqq 1,\ f\in W,$$

with a constant independent of t and f. Thus, for $t\geqq 1$, $t+h\geqq 1$,

$$\begin{aligned}|\phi_{f}(t+h)-\phi_{f}(t)|&\leqq\int_{t}^{t+h}p_{n}\Big(\frac{d}{ds}(s^{-\sigma}G_{s}f)\Big)ds\\&\leqq C_{n}'|h|\end{aligned} \tag{6.5}$$

for all $f\in W$. (6.3), (6.4) and (6.5) imply, by the Ascoli-Arzela theorem, that $\phi_{f}(t)$, $f\in W$, form a relatively compact set in the Banach space $C_{b}[1,+\infty)$ (see the proof of Proposition 2.1). Since A is Montel, and W is bounded in A, there is a sequence $f_{j}\in W$ converging in A to a $g\in A$. We may assume $\phi_{f_{j}}(t)$ converge to a $\phi(t)$ in $C_{b}[1,+\infty)$. Then $\phi(t)=\phi_{g}(t)$ or $g\in B^{\sigma}$. Similarly we see $g\in B^{\sigma'}$ for any $\sigma'>\rho$. Now the same argument applied to $\{f_j-g\}$ implies that $\{f_j-g\}$ converges in B^{σ}. Q.E.D.

COROLLARY 6.2. *Let A be Montel. For any bounded set W in B^{ρ}, $\rho\in\boldsymbol{R}$, the topologies induced on W from all B^{σ}, $\sigma>\rho$, and from A coincide.*

PROOF. Immediate from Proposition 6.1. Q.E.D.

COROLLARY 6.3. *If A is Montel, then so is $B^{-\infty}$.*

PROOF. Immediate from (2.6) and Proposition 6.1. Q.E.D.

COROLLARY 6.4. *If A is a Montel space, then so is Γ^{μ} for each $\mu\in\boldsymbol{C}$.*

PROOF. Since Γ^{μ} is a closed subspace of A, B^{ρ}, $\rho>\operatorname{Re}\mu$, the topologies induced by all B^{ρ}, $\rho>\operatorname{Re}\mu$, to Γ^{μ} coincide. The corollary now follows from Proposition 6.1. Q.E.D.

COROLLARY 6.5. *If A is a Montel space, then so is C^{μ} for each $\mu\in\boldsymbol{C}$.*

PROOF. C^{μ} is the product of Montel spaces $\Gamma^{\mu-j}$. Q.E.D.

COROLLARY 6.6. *If A is a Montel space, then so is D^{μ} for each $\mu\in\boldsymbol{C}$.*

PROOF. This follows from Proposition 6.1, Corollary 6.4 and (3.11). Q.E.D.

COROLLARY 6.7. *If A is a Montel space, then so are the spaces C and D.*

PROOF. C and D are strict inductive limits of Montel spaces. Q.E.D.

§7. Some standard examples.

We illustrate our theory by five standard examples. The mappings α^{μ} are surjective except in the last example. All examples also fall in the situation of §6.

EXAMPLE 7.1. (The classical Taylor expansion). Let $A=\mathcal{E}(\boldsymbol{R}^n)$, the ring of $\boldsymbol{C}$-valued C^{∞} functions on the n-dimensional Euclid space $\boldsymbol{R}^n$. We equip $\mathcal{E}(\boldsymbol{R}^n)$ with its standard multiplicatively convex Fréchet algebra structure (see Michael [7], Proposition 2.4, h), p. 48). Namely, (1.1) is satisfied by the following system of semi-norms in $\mathcal{E}(\boldsymbol{R}^n)$:

$$p_{j,k}(f)=\sup_{|x|\leq j,|\alpha|\leq k}2^{|\alpha|+k}|\partial_x^{\alpha}f(x)| \tag{7.1}$$

for $f\in\mathcal{E}(\boldsymbol{R}^n)$, $j,k\in\boldsymbol{Z}^+$. Here $\alpha=(\alpha_1,\cdots,\alpha_n)\in(\boldsymbol{Z}^+)^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\partial_x^{\alpha}=\partial^{|\alpha|}/(\partial x_1)^{\alpha_1}\cdots(\partial x_n)^{\alpha_n}$ and $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$ for $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$.

The multiplicative group $\boldsymbol{R}_+$ acts on $\boldsymbol{R}^n$ by

$$g_t:\boldsymbol{R}^n\ni x\rightarrow t^{-1}x\in\boldsymbol{R}^n, \qquad t\in\boldsymbol{R}_+.$$

For $f\in\mathcal{E}(\boldsymbol{R}^n)$, we set

$$(G_tf)(x)=f(g_tx), \qquad t\in\boldsymbol{R}_+.$$

Then $G=\{G_t;t\in\boldsymbol{R}_+\}$ is immediately seen to be a differentiable $\boldsymbol{R}_+$-action in $\mathcal{E}(\boldsymbol{R}^n)$. Furthermore, G is strong. Thus, $B^{\rho}=\mathcal{E}(\boldsymbol{R}^n)$ for $\rho\geqq 0$. For $\rho<0$, we have $f\in B^{\rho}$ if and only if

$$|(\partial_x^\alpha f)(x)|\leqq C\min(1,\ |x|^{-\rho-|\alpha|})$$

for $|x|\leqq l$, $|\alpha|\leqq m$, l, $m\in \boldsymbol{Z}^+$, C being some positive constant depending only on l, m. Therefore,

$$B^{-j}=\{f\in\mathcal{E}(\boldsymbol{R}^n)\,;\,(\partial_x^\alpha f)(0)=0 \quad \text{for} \quad |\alpha|\leqq j-1\}$$

for $j\in\boldsymbol{Z}^+$ and $B^{-j+\theta}=B^{-j}$ for $0\leqq\theta<1$. $B^{-\infty}$ is thus the set of flat functions at $x=0$.

Furthermore, $\Gamma^{\mu}=\{0\}$ for non-real μ and also for $\mu>0$. If $j\in\boldsymbol{Z}^+$, then

$$\Gamma^{-j}=\text{the totality of homogeneous polynomials of degree } j,$$

and $\Gamma^{-j+\theta}=\{0\}$ when $0<\theta<1$. For $j\in\boldsymbol{Z}^+$,

$$C^{-j}=\left\{\sum_{k=j}^{\infty}\sum_{|\alpha|=k}a_\alpha x^\alpha\,;\ a_\alpha\in\boldsymbol{C}\right\}$$

with the topology of simple convergence of the coefficients (see Treves [**9**], Example III, p. 91.) Here $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for $\alpha=(\alpha_1,\cdots,\alpha_n)\in(\boldsymbol{Z}^+)^n$ and $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$. If $\theta\in\boldsymbol{C}$, $0<\mathrm{Re}\,\theta<1$ or $\mathrm{Im}\,\theta\neq0$, $C^{-j-\theta}=\{0\}$.

Every element of $\mathcal{E}(\boldsymbol{R}^n)$ is developable, and thus, $D^{-j}=B^{-j}$, $j\in\boldsymbol{Z}^+$. Other D^{μ} spaces reduce to $B^{-\infty}$. Developments of elements in D^{-j} are nothing but their classical Taylor expansions at the origin. The mappings α^{-j} are surjective. There is no convergence factor for the differentiable $\boldsymbol{R}_+$-action (in the sense of Definition 5.2). The surjectivity of α^{-j} can be proved in an analogous way to the proof of Proposition 4.1. For another proof, see Treves ([**9**], Theorem 38.1). A proof is also given at the end of the following Example 7.2. The space $B^{-\infty}$ has no topological complement in D^{-j} (see Glaeser [**3**], IV. Prolongement de Whitney et prolongateur, p. 130).

EXAMPLE 7.2. Let $A^{\wedge}=\mathcal{E}(\boldsymbol{R}^n\setminus\{0\})$, the ring of $\boldsymbol{C}$-valued C^∞ functions on $\boldsymbol{R}^n\setminus\{0\}$ with the standard multiplicatively convex Fréchet algebra structure. Namely, (1.1) is satisfied by the following system of semi-norms in $\mathcal{E}(\boldsymbol{R}^n\setminus\{0\})$:

$$\hat{p}_{j,k}(f)=\sup_{j^{-1}\leq|x|\leq j,\,|\alpha|\leq k}2^{|\alpha|+k}|\partial_x^\alpha f(x)|$$

for $f\in\mathcal{E}(\boldsymbol{R}^n\setminus\{0\})$, j, $k\in\boldsymbol{Z}^+$, $j>0$. The multiplicative group $\boldsymbol{R}_+$ acts on $\boldsymbol{R}^n\setminus\{0\}$ by

$$\hat{g}_t:\ \boldsymbol{R}^n\setminus\{0\}\ni x\to t^{-1}x\in\boldsymbol{R}^n\setminus\{0\},\qquad t\in\boldsymbol{R}_+.$$

For $f\in\mathcal{E}(\boldsymbol{R}^n\setminus\{0\})$, we set

$$(\hat{G}_t f)(x)=f(\hat{g}_t x),\qquad t\in\boldsymbol{R}_+.$$

Then $\hat{G}=\{\hat{G}_t\,;\,t\in\boldsymbol{R}_+\}$ is a differentiable $\boldsymbol{R}_+$-action in $\mathcal{E}(\boldsymbol{R}^n\setminus\{0\})$. In the present example, we write $\hat{B}^\rho$, $\hat{\Gamma}^\mu$, $\hat{C}^\mu$, $\hat{D}^\mu$ etc. instead of B^ρ, Γ^μ, C^μ, D^μ etc. to avoid any possible confusion with the previous example.

For $\rho\in\boldsymbol{R}$, $f\in\hat{B}^\rho$ if and only if

$$|(\partial_x^\alpha f)(x)|\leqq C|x|^{-\rho-|\alpha|}$$

for $0<|x|\leqq l$, $|\alpha|\leqq m$, $\alpha\in(\boldsymbol{Z}^+)^m$, l, $m\in\boldsymbol{Z}^+$, C being a positive constant depending only on l, m. In particular, for $\rho<0$, $f\in\hat{B}^\rho$ vanishes to the order $p-1$ at $x=0$ if p is the integral part of $-\rho$. Hence, $\hat{B}^{-\infty}$ coincides with the set of C^∞ functions on $\boldsymbol{R}^n$ vanishing to the infinite order at $x=0$ (compare with Example 7.1).

Note the diffeomorphim $\boldsymbol{R}^n\backslash\{0\}\cong\boldsymbol{S}^{n-1}\times\boldsymbol{R}_+$. Then $\hat{\Gamma}^0$ coincides with $\mathcal{E}(\boldsymbol{S}^{n-1})$, the space of C^∞ functions on $\boldsymbol{S}^{n-1}$. For each $\mu\in\boldsymbol{C}$, $\hat{\Gamma}^\mu$ is isomorphic to $\hat{\Gamma}^0$ and $f\in\hat{\Gamma}^\mu$ if and only if $|x|^\mu f\in\hat{\Gamma}^0$. Hence, for $\mu\in\boldsymbol{C}$,

$$\hat{C}^\mu=\{|x|^{-\mu}\sum_{j=0}^{\infty}a_j(x)|x|^j\,;\ a_j\in\hat{\Gamma}^0\}$$

with the product topology. There is a convergence factor for the differentiable $\boldsymbol{R}_+$-action $\hat{G}$, namely, $e(x)\in C_0^\infty(\boldsymbol{R}^n)$ such that $e(x)=1$ in a neighborhood of $x=0$. Now let us compare with the previous example. Then $B^{-\infty}=\hat{B}^{-\infty}$ and for $j\in\boldsymbol{Z}^+$, $B^{-j}\subset\hat{B}^{-j}$ and Γ^{-j} is a closed (in fact finite dimensional) subspace of $\hat{\Gamma}^{-j}$. Thus, C^{-j} is a closed subspaces of $\hat{C}^{-j}$. Furthermore, if $f_j\in B^{-j}$, $j\in\boldsymbol{Z}^+$, then there is an $f\in\hat{B}^0$ such that $f-\sum_{j<k}f_j\in\hat{B}^{-k}$ (Corollary 5.6). This f belongs to $\mathcal{E}(\boldsymbol{R}^n)$ since the elements in $\hat{B}^{-k}$ are differentiable at $x=0$ up to $(k-1)$-times. Then by the same reason $f-\sum_{j<k}f_j\in B^{-k}$. Therefore we give a proof of the surjectivity of α in Example 7.1 by using a convergence factor in the present example.

EXAMPLE 7.3. (The algebra of symbols). Let X be a paracompact C^∞ manifold of dimension n, and U a principal $\boldsymbol{R}_+$-bundle over X. U is thus a cone bundle over X (see Boutet de Monvel [1], Hörmander [5]). Since $\boldsymbol{R}_+$ is contractible, U is trivial, $U=X\times\boldsymbol{R}_+$. Let $A=S(U)$, the ring of $\boldsymbol{C}$-valued C^∞ functions on U with the standard (multiplicatively convex) Fréchet (algebra) structure as defined in a similar way to (7.1). We denote by g_t the $\boldsymbol{R}_+$-action on U, that is,

$$g_t:\ U\ni(x,\ r)\rightarrow(x,\ tr)\in U\,,\qquad t\in\boldsymbol{R}_+\,.$$

For $p\in S(U)$, we set

$$(G_tp)(x,\ r)=p(g_t(x,\ r))\,,\qquad t\in\boldsymbol{R}_+\,.$$

Then $G=\{G_t\,;\ t\in\boldsymbol{R}_+\}$ is a differentiable $\boldsymbol{R}_+$-action in $S(U)$. In this case, we write S^ρ, $\rho\in\boldsymbol{R}$, instead of B^ρ. Then $S^\rho\ni p$ if and only if

$$|\Lambda\partial_r^k p(x, r)| \leqq Cr^{\rho-k}, \qquad \partial_r = \partial/\partial r,$$

for $x\in K$, $r\geqq r_0$, $k\in \boldsymbol{Z}^+$, Λ any differential operator on X, r_0 any positive number, K any compact subset of X, and C a positive constant depending only on K, r_0, k and Λ. $p\in\Gamma^\mu$ if and only if

$$p(x, tr)=t^\mu p(x, r), \qquad t\in \boldsymbol{R}_+,\ \mu\in \boldsymbol{C}.$$

Therefore, Γ^0 is isomorphic to $\mathcal{E}(X)$ and $p\in\Gamma^\mu$ if and only if $r^{-\mu}p\in\Gamma^0$. C^μ is the totality of formal sums $\sum_{j=0}^{\infty} p_j$ with p_j homogeneous of degree $\mu-j$. If p is developable, its development is just the usual asymptotic expansion (see Hörmander [4], [5], for instance). Thus, developable elements are essentially classical symbols. A convergence factor $e(x, r)$ for the differentiable $\boldsymbol{R}_+$-action is given by $e(x, r)=e(r)\in S(U)$ such that $e(r)=1$ for $r>1$ and $e(r)=0$ for $r<1/2$.

Let $V=X\times\boldsymbol{R}^+$ and consider $\mathcal{E}(V)$, the space of C^∞ functions on V. Recall that $\boldsymbol{R}^+$ is the set of non-negative reals. Thus, $f\in\mathcal{E}(V)$ if and only if f is a restriction to $X\times\boldsymbol{R}^+$ of a C^∞ function defined on a neighborhood of $X\times\boldsymbol{R}^+$ in $X\times\boldsymbol{R}$. In the customary notations, $S^\rho\cap\mathcal{E}(V)$ is written as $S^\rho_{1,0}(U)$ (see Hörmander [4]). In applications to the theory of pseudo-differential operators, $p\in S^\rho_{1,0}(U)$ is usually understood as vanishing near the zero section. It is well-known that many important symbols admit asymptotic expansions (classical symbols), that is, developable in our terminology.

EXAMPLE 7.4. (The classical asymptotic expansion in an open sector). Let Σ be an open sector in $\boldsymbol{C}\setminus\{0\}$, given by

$$z\in\boldsymbol{C}, \qquad z\neq 0, \qquad |\arg z|<p$$

for some positive p. Let $A=\mathcal{O}(\Sigma)$, the ring of holomorphic functions on Σ with the standard (multiplicatively convex) Fréchet (algebra) structure. Namely, for any compact subset K of Σ,

$$p_K(f)=\sup_{z\in K}|f(z)|, \qquad f\in\mathcal{O}(\Sigma),$$

is a semi-norm satisfying (1.1). The multiplicative group $\boldsymbol{R}_+$ acts on Σ by

$$g_t : \Sigma\ni z\to tz\in\Sigma, \qquad t\in\boldsymbol{R}_+.$$

For $f\in\mathcal{O}(\Sigma)$, we set

$$(G_t f)(z)=f(g_t z), \qquad t\in\boldsymbol{R}_+.$$

Then $G=\{G_t;\ t\in\boldsymbol{R}_+\}$ is a differentiable $\boldsymbol{R}_+$-action in $\mathcal{O}(\Sigma)$. For $\rho\in\boldsymbol{R}$, we write O^ρ instead of B^ρ. Then $f\in O^\rho$ if and only if

$$|f(z)|\leqq C|z|^\rho$$

for all $z \in \Sigma$ such that $|\arg z| \leqq p_0$, $|z| \geqq q_0$, p_0, q_0 any positive numbers and $p_0 < p$. C is a positive constant depending only on p_0 and q_0. Thus, in the classical notation, $O^\rho = O(z^\rho)$ (see, e. g., Erdelyi [2], Olver [8], Wasow [10]).

$$\Gamma^\mu = \{az^\mu;\ a \in \boldsymbol{C}\}, \qquad \mu \in \boldsymbol{C},$$

where $z^\mu = \exp(\mu \log z)$. C^μ is the totality of the formal series

$$z^\mu \sum_{j=0}^{\infty} a_j z^{-j}, \qquad a_j \in \boldsymbol{C},$$

with the topology of simple convergence of the coefficients (see Treves [9], Example III, p. 91). If f is developable, then its development is the classical asymptotic expansion of f (see, e. g., Erdelyi [2], Olver [8], Wasow [10]).

A convergence factor for the differentiable $\boldsymbol{R}_+$-action is given by

$$e(z) = 1 - e^{-z^\lambda}$$

with $0 < \lambda < \pi/2p$ (compare with Olver [8], p. 22, Wasow [10], p. 42). Note that the above construction is also valid for holomorphic functions on Σ with values in any multiplicatively convex Fréchet algebra.

EXAMPLE 7.5. Let $A = \mathcal{O}(\boldsymbol{C}^n)$, the ring of entire analytic functions on $\boldsymbol{C}^n$, equipped with the standard multiplicatively convex Fréchet algebra structure. The group $\boldsymbol{R}_+$ acts in $\boldsymbol{C}^n$ by

$$g_t : \boldsymbol{C}^n \ni z \to t^{-1}z \in \boldsymbol{C}^n, \qquad t \in \boldsymbol{R}_+.$$

Then

$$(G_t f)(z) = f(g_t z), \qquad f \in \mathcal{O}(\boldsymbol{C}^n), \quad t \in \boldsymbol{R}_+,$$

determines a strong differentiable $\boldsymbol{R}_+$-action in $\mathcal{O}(\boldsymbol{C}^n)$. For $\rho \geqq 0$, $B^\rho = \mathcal{O}(\boldsymbol{C}^n)$. $f \in B^{-j}$, $j \in \boldsymbol{Z}^+$, if and only if f vanishes to the j-th order at $z=0$.

$B^{-j+\theta} = B^{-j+1}$ for $0 < \theta \leqq 1$. $B^{-\infty} = \{0\}$. For $j \in \boldsymbol{Z}^+$, Γ^{-j} is the totality of homogeneous polynomials of degree j. Other Γ^μ's reduce to $\{0\}$. Thus $D^{-j} = B^{-j}$ for $j \in \boldsymbol{Z}^+$ and other D^μ spaces reduce to $\{0\}$. The mappings α^{-j} are nothing but the Taylor expansion at $z=0$. Clearly α^{-j} are not surjective.

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