J. Math. Soc. Japan
Vol. 30, No. 4, 1978

On the cubics defining abelian varieties

By Tsutomu SEKIGUCHI*

(Received April 12, 1977)
(Revised March 29, 1978)

Introduction.

Let k be an algebraically closed field of characteristic p, X an abelian variety over k of dimension g, and L an ample invertible sheaf on X. For any integer $a\geqq 3$, we denote by $\phi_a: X\rightarrow \boldsymbol{P}(\Gamma(L^a))$ the canonical embedding of X. The purpose of the present paper is to prove, except the case of $p=2$ and 3, the statement:

$\phi_3(X)$ is ideal-theoretically an intersection of cubics.

For generic polarized abelian varieties, the statement is proved by Morikawa [4] for any characteristic, using deformations of polarized abelian varieties. For $a\geqq 4$, Mumford ([5], Theorem 10) proved that for any characteristic, $\phi_a(X)$ is ideal-theoretically an intersection of quadrics. We shall prove our assertion stated above, by reducing it to Mumford's theorem. The essential tool in the reduction process is the normal generation of $\phi_3(X)$, which is discovered by Koizumi [2] for characteristic zero, and later generalized by the author [7], [8] for any characteristic.

Section 1 is devoted to recalling some results concerning the normal generation of abelian varieties. In Section 2, we shall give a slight modification of Mumford's theorem, in order that it will be fit for later use. The proof of our result will be completed in Section 3.

The author would like to thank Mr. R. Sasaki for very useful conversations. In particular, he pointed out the commutativity of such relevance of Segre embedding with diagram as in Lemma 3.1 and the fact that the quadrics considered in Mumford's theorem appear in the equations of Segre embedding.

NOTATION. Throughout the paper, k is an algebraically closed field of characteristic p, and X is an abelian variety over k of dimension g. We denote by $\hat{X}$ the dual abelian variety of X, and by P the Poincaré invertible sheaf on $X\times\hat{X}$. For any $\hat{x}\in\hat{X}$, we put $P_{\hat{x}}=P|_{X\times\{\hat{x}\}}$. For any integer n, we put $X_n=\{x\in X \mid nx=0\}$. For an invertible sheaf L on X, we abbreviate $\Gamma(X, L)$ by $\Gamma(L)$, and we denote

* The author expresses his thanks to the referee for suggesting a simplification of Lemma 1.7.

by $\phi_L : X \to \hat{X}$ the homomorphism defined by $x \mapsto T_x^* L \otimes L^{-1}$. We put $K(L)=\mathrm{Ker}(\phi_L)$, and $\mathcal{G}(L)$=the theta group of L. For an ample invertible sheaf L on X, we denote by $e^L : K(L) \times K(L) \to \boldsymbol{G}_m$ the canonical pairing defined by L. We shall frequently consider any k-valued point z of a scheme Z over k as its B-valued point, compositing z with the structure morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$, for any k-algebra B.

For any finite dimensional k-vector space V, and any non-trivial k-linear map $l : V \to k$, we denote by $[l]$ the k-rational point of the projective space $\boldsymbol{P}(V)$ corresponding to l. For any form in the symmetric algebra $S^*(V)$, we identify it with the hypersurface in $\boldsymbol{P}(V)$ defined by the form. In $S^*(V)$, we denote the symmetric product by $\odot$ following Mumford. For subset W of V, $\langle W \rangle$ means the subspace spanned by W. When a group G acts on V, we denote V^G the subspace of V consisting of G-invariant elements.

§ 1. Normal generation.

We start with Koizumi's theorem generalized in ([8], Theorem 2.4).

THEOREM 1.1. *Let L be any ample invertible sheaf on X, and α, β be two closed points on $\hat{X}$. Then*

$$\Gamma(L^a \otimes P_\alpha) \otimes \Gamma(L^b \otimes P_\beta) \longrightarrow \Gamma(L^{a+b} \otimes P_{\alpha+\beta})$$

is surjective for all integers a, b such that $a \geqq 2$, $b \geqq 3$.

Here we recall a lemma of Mumford in ([5], § 3).

LEMMA 1.2 (Mumford). *Let L and M be invertible sheaves on X such that $\Gamma(L) \neq (0)$, $\Gamma(M) \neq (0)$, and $L \otimes M$ is ample. Then*

$$\sum_{\alpha \in \hat{X}} \Gamma(L \otimes P_\alpha) \otimes \Gamma(M \otimes P_{-\alpha}) \longrightarrow \Gamma(L \otimes M)$$

is surjective.

From this lemma, we can easily deduce

COROLLARY 1.3. *Let L and M be ample invertible sheaves on X. Then, for any non-empty open set U in $\hat{X}$,*

$$\sum_{\alpha \in U} \Gamma(L \otimes P_\alpha) \otimes \Gamma(M \otimes P_{-\alpha}) \longrightarrow \Gamma(L \otimes M)$$

is surjective.

PROOF. Let $p : X \times \hat{X} \to X$ and $q : X \times \hat{X} \to \hat{X}$ be the canonical projections, and we put $\mathcal{L} = q_*(p^* L \otimes P)$ and $\mathcal{M} = q_*(p^* M \otimes P^{-1})$. Since the higher cohomology groups of $L \otimes P_\alpha$ and $M \otimes P_{-\alpha}$ are zero, $\mathcal{L}$ and $\mathcal{M}$ are locally free sheaves on $\hat{X}$ such that

$$\mathcal{L} \otimes \boldsymbol{k}(\alpha) \cong \Gamma(L \otimes P_\alpha)\,;\ \mathcal{M} \otimes \boldsymbol{k}(\alpha) \cong \Gamma(M \otimes P_{-\alpha})\,.$$

Let $\phi : \mathcal{L}\otimes_{\mathcal{O}_{\hat{X}}}\mathcal{M}\to\Gamma(L\otimes M)\otimes_k\mathcal{O}_{\hat{X}}$ be the natural pairing. From Mumford's lemma, there exist a finite number of points $\alpha_1, \cdots, \alpha_N$ in $\hat{X}$ such that

$$\sum_{i=1}^{N}\Gamma(L\otimes P_{\alpha_i})\otimes\Gamma(M\otimes\Gamma_{-\alpha_i}) \longrightarrow \Gamma(L\otimes M)$$

is surjective. Let $p_i : \overbrace{\hat{X}\times\cdots\times\hat{X}}^{N}\to\hat{X}$ be the projection to the i-th component for each $i=1, \cdots, N$, and we put

$$\Phi=\sum_{i=1}^{N}p_i^*(\phi) : \sum_{i=1}^{N}p_i^*(\mathcal{L}\otimes_{\mathcal{O}_{\hat{X}}}\mathcal{M}) \longrightarrow \Gamma(L\otimes M)\otimes_k\mathcal{O}_{\hat{X}^N}.$$

Since $\sum_{i=1}^{N}p_i^*(\mathcal{L}\otimes_{\mathcal{O}_{\hat{X}}}\mathcal{M})$ and $\Gamma(L\otimes M)\otimes_k\mathcal{O}_{\hat{X}^N}$ are locally free sheaves on $\hat{X}^N$, the set

$$V=\{\hat{x}=(\hat{x}_1, \cdots, \hat{x}_N)\in\hat{X}^N \mid \phi\otimes\boldsymbol{k}(\hat{x}) : \sum_i\Gamma(L\otimes P_{\hat{x}_i})\otimes\Gamma(M\otimes P_{-\hat{x}_i}) \longrightarrow \Gamma(L\otimes M) \text{ is surjective}\}$$

is an open set. Since $(\alpha_1, \cdots, \alpha_N)\in V$, V is non-empty, and $(\overbrace{U\times\cdots\times U}^{N})\cap V\neq\emptyset$, which implies our assertion. Q.E.D.

We can see the next lemma, using Mumford's theta structure theorem in the same way as in the proof of Mumford's lemma above.

LEMMA 1.4. *Assume $p\neq3$. Let L be any ample invertible sheaf on X. Then*

$$\sum_{\alpha\in\hat{X}_3}\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L\otimes P_{-\alpha}) \longrightarrow \Gamma(L^3)$$

is surjective.

The following proposition will play an essential role in the proof of our theorem.

PROPOSITION 1.5. *Assume $p\neq2$. Let L be any ample invertible sheaf on X. Then for any α, β in $\hat{X}$, if we take a point γ of $\hat{X}$ in general position,*

$$\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes\Gamma(L^2\otimes P_{\beta-\gamma}) \longrightarrow \Gamma(L^4\otimes P_{\alpha+\beta})$$

is surjective.

REMARK. We can prove the results in the case of $p\neq2$, asserted in Theorem 1.1, more easily using Proposition 1.5 than using the Rank theorem as in the proof given in [2], [7] or [8].

In fact, by Proposition 1.5, there exists an open subset U in $\hat{X}$ such that $\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_\gamma)\to\Gamma(L^4\otimes P_{\alpha+\gamma})$ is surjective for every $\gamma\in U$. Therefore, applying Corollary 1.3 to the diagram

$$\begin{tikzcd}
\Gamma(L^2\otimes P_\alpha)\otimes\{\sum_{\gamma\in U}\Gamma(L\otimes P_{\beta-\gamma})\otimes\Gamma(L^2\otimes P_\gamma)\} \arrow[r] \arrow[d] & \sum_{\gamma\in U}\Gamma(L\otimes P_{\beta-\gamma})\otimes\Gamma(L^4\otimes P_{\alpha+\gamma}) \arrow[d] \\
\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^3\otimes P_\beta) \arrow[r] & \Gamma(L^5\otimes P_{\alpha+\beta}),
\end{tikzcd}$$

we obtain our assertion.

Proposition 1.5 is reduced to the following principal case.

PROPOSITION 1.6. *Assume $p\neq 2$. Let L be a principal invertible sheaf on X, and α, β be two closed points in $\hat{X}$. If we take γ of $\hat{X}$ in general position,*

$$\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes\Gamma(L^2\otimes P_{\beta-\gamma}) \longrightarrow \Gamma(L^4\otimes P_{\alpha+\beta})$$

is surjective.

In the reduction of Proposition 1.5 to the above proposition, we need the following lemma.

LEMMA 1.7. *Let L be a principal invertible sheaf on X, and $(R, \boldsymbol{m})$ be a local ring over k with the residue field k. Let α, β be two R-valued points on $\hat{X}$, and we put $\bar{\alpha}=\alpha\circ\iota$ and $\bar{\beta}=\beta\circ\iota$, where ι : $\mathrm{Spec}\,(R/\boldsymbol{m})\to\mathrm{Spec}\,(R)$ is the canonical morphism. Then, if the map*

$$\Gamma(L^2\otimes P_{\bar{\alpha}})\otimes\Gamma(L^2\otimes P_{\bar{\beta}}) \longrightarrow \Gamma(L^4\otimes P_{\bar{\alpha}+\bar{\beta}})$$

is surjective, the map

$$\Gamma(p_1^*L^2\otimes P_\alpha)\otimes\Gamma(p_1^*L^2\otimes P_\beta) \longrightarrow \Gamma(p_1^*L^4\otimes P_{\alpha+\beta})$$

is surjective, where $p_1: X\times\mathrm{Spec}\,(R)\to X$ is the projection to the first factor.

PROOF. Let δ be any R-valued point of $\hat{X}$, and M be any ample invertible sheaf on X. Since the projection $p_2: X\times\mathrm{Spec}\,(R)\to\mathrm{Spec}\,(R)$ is proper and flat, and $(l_X\times\iota)^*(p_1^*M\otimes P_\delta)\cong M\otimes P_{\bar{\delta}}$ is ample, $p_{2,*}(p_1^*M\otimes P_\delta)=\Gamma(p_1^*M\otimes P_\delta)$ is a free R-module and $p_{2,*}(p_1^*M\otimes P_\delta)\otimes_R(R/\boldsymbol{m})\cong\Gamma(M\otimes P_{\bar{\delta}})$. Therefore, $\Gamma(p_1^*L^2\otimes P_\alpha)$, $\Gamma(p_1^*L^2\otimes P_\beta)$ and $\Gamma(p_1^*L^4\otimes P_{\alpha+\beta})$ are free R-modules and the canonical map $\Gamma(L^2\otimes P_{\bar{\alpha}})\otimes\Gamma(L^2\otimes P_{\bar{\beta}})\to\Gamma(L^4\otimes P_{\bar{\alpha}+\bar{\beta}})$ is obtained by reducting the map $\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_\beta)\to\Gamma(L^4\otimes P_{\alpha+\beta})$ modulo $\boldsymbol{m}$. Hence, by Nakayama's lemma, we obtain our assertion. Q. E. D.

PROOF OF PROPOSITION 1.5 ASSUMING PROPOSITION 1.6. Let H be a maximal isotropic subgroup of $K(L)$, and $\pi: X\to Y=X/H$ be the canonical projection. Then there exists a principal invertible sheaf M on Y such that $\pi^*M\cong L$. We choose closed points α', β' in $\hat{Y}$ such that $\hat{\pi}(\alpha')=\alpha$ and $\hat{\pi}(\beta')=\beta$. By virtue of Proposition 1.6, for almost all γ' in $\hat{Y}$,

$$\Gamma(M^2\otimes P_{\alpha'+\delta+\gamma'})\otimes\Gamma(M^2\otimes P_{\beta'-\gamma'}) \longrightarrow \Gamma(M^4\otimes P_{\alpha'+\beta'+\delta})$$

are surjective for all closed points δ in $\hat{H}$. Here we put $\gamma=\hat{\pi}(\gamma')$ for such a point γ', and we denote by W the image of the map

$$\tau: \Gamma(L^2\otimes P_{\alpha+\gamma})\otimes\Gamma(L^2\otimes P_{\beta-\gamma}) \longrightarrow \Gamma(L^4\otimes P_{\alpha+\beta}).$$

To prove Proposition 1.5, it is sufficient to show that

$$(*)\qquad \begin{cases} \text{for any local ring } (R, \boldsymbol{m}) \text{ over } k \text{ with the residue} \\ \text{field } k \text{ and any } R\text{-valued point } \lambda \text{ of } \mathcal{G}(L^4\otimes P_{\alpha+\beta}), \\ U_\lambda(\pi^*(\Gamma(M^4))\otimes R)\subset W\otimes R. \end{cases}$$

Let $j(\lambda)=u$, where $j:\mathcal{G}(L^4\otimes P_{\alpha+\beta})\to K(L^4\otimes P_{\alpha+\beta})$ is the canonical surjection. Then we have a commutative diagram

$$(1)\qquad \begin{tikzcd} \Gamma(p_1^*L^4\otimes P_{\alpha+\beta})\cong\Gamma(L^4\otimes P_{\alpha+\beta})\otimes R \arrow[r, "T_u^*"] \arrow[r, bend left, "U_\lambda"] & \Gamma(T_u^*p_1^*L^4\otimes P_{\alpha+\beta})\cong\Gamma(L^4\otimes P_{\alpha+\beta})\otimes R \\ \Gamma(p_1^*M^4\otimes P_{\alpha'+\beta'}) \arrow[u, "\pi^*"] \arrow[r, "T_{\pi u}^*"] & \Gamma(T_{\pi u}^*p_1^*M^4\otimes P_{\alpha'+\beta'})\cong\Gamma(p_1^*M^4\otimes P_{\alpha'+\beta'+4\phi_M(\pi u)}). \arrow[u, "\pi^*"'] \end{tikzcd}$$

Here $\pi^*(p_1^*M^2\otimes P_{\alpha'+4\phi_M(\pi u)-\gamma'})\cong p_1^*L^2\otimes P_{\alpha-\gamma}$. So we obtain a commutative diagram

$$(2)\qquad \begin{tikzcd} \{\Gamma(L^2\otimes P_{\alpha+\gamma})\otimes R\}\otimes\{\Gamma(L^2\otimes P_{\beta-\gamma})\otimes R\} \arrow[r, "\tau\otimes R"] & \Gamma(L^4\otimes P_{\alpha+\beta})\otimes R \\ \Gamma(p_1^*M^2\otimes P_{\alpha'+4\phi_M(\pi u)+\gamma'})\otimes\Gamma(p_1^*M^2\otimes P_{\beta'-\gamma'}) \arrow[u, "\pi^*\otimes\pi^*"] \arrow[r, "\tau'"] & \Gamma(p_1^*M^4\otimes P_{\alpha'+\beta'+4\phi_M(\pi u)}). \arrow[u, "\pi^*"'] \end{tikzcd}$$

Since $\hat{\pi}(4\phi_M\pi\bar{u})=4\phi_L(\bar{u})=0$, $4\phi_M(\pi\bar{u})$ is a closed point of $\mathrm{Ker}\,\hat{\pi}=\hat{H}$. Therefore, from the choice of γ',

$$\Gamma(M^2\otimes P_{\alpha'+4\phi_M(\pi\bar{u})+\gamma'})\otimes\Gamma(M^2\otimes P_{\beta'-\gamma'}) \longrightarrow \Gamma(M^4\otimes P_{\alpha'+\beta'+4\phi_M(\pi\bar{u})})$$

is surjective. Hence, by virtue of Lemma 1.7, the map τ' in (2) is surjective. So (*) can be deduced from (1) and (2). Q.E.D.

We need the following three lemmas to prove Proposition 1.6.

LEMMA 1.8. *Let G be an elementary 2-group of order 2^d, and k be a field of characteristic $p\neq 2$. Let $B: G\to \mathbf{M}(n\times n, k)$ be a map, where $\mathbf{M}(n\times n, k)$ is the algebra of $(n\times n)$-matrices with components in k. Then*

$$\det\Big(\chi(a)B(a)\Big)_{(\chi,a)\in\hat{G}\times G}=(-2)^{dn2^{d-1}}\prod_{a\in G}\det B(a).$$

Here $\hat{G}$ is the dual group of G.

PROOF. We prove our assertion by induction on d. In the case of $d=1$, $G=\boldsymbol{Z}/2\boldsymbol{Z}$ and $\hat{G}=(\boldsymbol{Z}/2\boldsymbol{Z})^{\wedge}$. Therefore

$$\det\Big(\chi(a)B(a)\Big)_{(\chi,a)\in\hat{G}\times G}=\begin{vmatrix} B(0) & B(1) \\ B(0) & -B(1)\end{vmatrix}=\begin{vmatrix} B(0) & B(1) \\ 0 & -2B(1)\end{vmatrix}$$

$$=(-2)^n \det B(0) \det B(1)\,.$$

We assume our assertion for $d\geqq 1$. Then for $d+1$, we put $G=G_1\times H$ and $\hat{G}=\hat{G}_1\times\hat{H}$ with $|G_1|=2^d$ and $H=\boldsymbol{Z}/2\boldsymbol{Z}$. In this case,

$$\det\Big(\chi(a)B(a)\Big)_{(\chi,a)\in\hat{G}\times G}=\det\Big(\mu(h)\Big(\chi_1(a)B(a+h)\Big)_{(\chi_1,a)\in\hat{G}_1\times G_1}\Big)_{(\mu,h)\in\hat{H}\times H}$$

$\Big($defining $B': H\longrightarrow \mathbf{M}((n2^d)\times(n2^d), k)$ by $B'(h)=\Big(\chi_1(a)B(a+h)\Big)_{(\chi_1,a)\in\hat{G}_1\times G_1}\Big)$

$$=\det\Big(\mu(h)B'(h)\Big)_{(\mu,h)\in\hat{H}\times H}$$

$$=(-2)^{n2^d}\det B'(0)\det B'(1)$$

(using the inductive hypothesis,)

$$=(-2)^{(d+1)n2^d}\prod_{a\in G}\det B(a)\,.$$

Q. E. D.

LEMMA 1.9. *Let G be an additive group of finite order n, and let k be a field of characteristic p with $p\nmid n$. Let $\{T(\lambda)\mid\lambda\in G\}$ be a set of independent variables over k, bijectively corresponding to G. If we define an $(n\times n)$-matrix M by*

$$M=(T(\lambda-\mu))_{(\lambda,\mu)\in G\times G}\,,$$

then we have

$$\det M=\prod_{\chi\in\hat{G}}\Big(\sum_{\lambda\in G}\chi(\lambda)T(\lambda)\Big)$$

(cf. [3], § 2, Lemma 2.1).

Hereafter, let $p\neq 2$ and L be a principal symmetric invertible sheaf on X. Let α, β be two fixed closed points in $\hat{X}$. Let $\xi: X\times X\to X\times X$ be a homomorphism defined by $(x, y)\mapsto(x-y, x+y)$. Then easily we have an isomorphism

$$\phi:\xi^*(p_1^*(L^2\otimes P_\alpha)\otimes p_2^*(L^2\otimes P_\beta))\xrightarrow{\ \sim\ } p_1^*(L^4\otimes P_{\alpha+\beta})\otimes p_2^*(L^4\otimes P_{\beta-\alpha})$$

(cf. [7], § 1, Proposition 1.2). That is, we obtain an inclusion

$$(3)\qquad \xi^*:\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_\beta)\hookrightarrow\Gamma(L^4\otimes P_{\alpha+\beta})\otimes\Gamma(L^4\otimes P_{\beta-\alpha})\,.$$

Let $K(L^4\otimes P_{\alpha+\beta})=K(L^4\otimes P_{\beta-\alpha})=H(4)_1\oplus H(4)_2$ be a Göpel decomposition, and put

$2H(4)_i=H(2)_i$ for $i=1, 2$. Then $K(L^2\otimes P_\alpha)=K(L^2\otimes P_\beta)=H(2)_1\oplus H(2)_2$ is a Göpel decomposition. The isomorphism ϕ defines a lifting of the group $K=\mathrm{Ker}\,\xi$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{G}_m & \longrightarrow & \mathcal{G}(p_1^*(L^4\otimes P_{\alpha+\beta})\otimes p_2^*(L^4\otimes P_{\beta-\alpha})) & \longrightarrow & X_4\times X_4 & \longrightarrow & 0\,. \\ & & & & \cup & & \cup & & \\ & & & & K^* & \xrightarrow{\ \sim\ } & K & & \end{array}$$

If we denote by $\mathcal{G}^*$ the centralizer of K^*, then we have a canonical isomorphism

$$\bar{\xi}:\ \mathcal{G}^*/K^* \xrightarrow{\ \sim\ } \mathcal{G}(p_1^*(L^2\otimes P_\alpha)\otimes p_2^*(L^2\otimes P_\beta))\,.$$

We put $H(4)_1^\Delta=\{(x, -x)\,|\,x\in H(4)_1\}$ and $H(4)_2^\Delta=\{(x, x)\,|\,x\in H(4)_2\}$. Since $H(4)_i^\Delta$ ($i=1, 2$) are isotropic subgroups in $K(p_1^*(L^4\otimes P_{\alpha+\beta})\otimes p_2^*(L^4\otimes P_{\beta-\alpha}))$, there exist level subgroups $H(4)_i^{\Delta*}$ of $H(4)_i^\Delta$ for $i=1, 2$. Let $\pi: \mathcal{G}(L^4\otimes P_{\alpha+\beta})\times\mathcal{G}(L^4\otimes P_{\beta-\alpha})\rightarrow \mathcal{G}(p_1^*(L^4\otimes P_{\alpha+\beta})\otimes p_2^*(L^4\otimes P_{\beta-\alpha}))$ be the canonical map and we take a level subgroup $H(4)_i^*$ in $\mathcal{G}(L^4\otimes P_{\beta-\alpha})$ of $H(4)_i$ for each $i=1, 2$. Then there exists a level subgroup $H'(4)_i^*$ in $\mathcal{G}(L^4\otimes P_{\alpha+\beta})$ and the subgroup $H(4)_i^{\Delta*}$ defines an isomorphism $':H(4)_i^*\rightarrow H'(4)_i^*$ by the relation $\pi(\lambda', \lambda)\in H(4)_i^{\Delta*}$ for each $i=1, 2$. Moreover, obviously $H(4)_i^{\Delta*}\subset G^*$. Therefore we can consider the images $\bar{\xi}(H(4)_i^{\Delta*})$ for $i=1, 2$. Since the diagrams

$$\begin{tikzcd} X\times X \arrow[r, "\xi"] \arrow[d, "T_y\times T_y"'] & X\times X \arrow[d, "1_X\times T_{2y}"] \\ X\times X \arrow[r, "\xi"] & X\times X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X\times X \arrow[r, "\xi"] \arrow[d, "T_y\times T_{-y}"'] & X\times X \arrow[d, "T_{2y}\times 1_X"] \\ X\times X \arrow[r, "\xi"] & X\times X \end{tikzcd}$$

commute for any $y\in X$, $\bar{\xi}(H(4)_1^{\Delta*})$ and $\bar{\xi}(H(4)_2^{\Delta*})$ are level subgroups of $H(2)_1\times\{0\}$ and $\{0\}\times H(2)_2$ in $\mathcal{G}(p_1^*(L^2\otimes P_\alpha)\otimes p_2^*(L^2\otimes P_\beta))$, respectively. So we can identify these subgroups with the level subgroups $H(2)_1^*$ in $\mathcal{G}(L^2\otimes P_\alpha)$ and $H(2)_2^*$ in $\mathcal{G}(L^2\otimes P_\beta)$, respectively. Under these notations we have

LEMMA 1.10. *Let u, v and θ be non-zero sections in $\Gamma(L^2\otimes P_\alpha)^{H(2)_1^*}$, $\Gamma(L^2\otimes P_\beta)^{H(2)_2^*}$ and $\Gamma(L^4\otimes P_{\beta-\alpha})^{H(4)_2^*}$, respectively. Then for a non-zero section t in $\Gamma(L^4\otimes P_{\alpha+\beta})^{H'(4)_2^*}$, we have*

$$\xi^*(u\otimes v)=\sum_{\lambda\in H(4)_1^*} U_{\lambda'}t\otimes U_\lambda\theta\,. \tag{4}$$

PROOF. From the choice of θ, $\{U_\lambda\theta\}_{\lambda\in H(4)_1^*}$ is a basis of $\Gamma(L^4\otimes P_{\beta-\alpha})$. Hence $\xi^*(u\otimes v)$ can be written in a form

$$\xi^*(u\otimes v)=\sum_{\lambda\in H(4)_1^*} t_\lambda\otimes U_\lambda\theta \qquad \text{with } t_\lambda\in\Gamma(L^4\otimes P_{\alpha+\beta})\,.$$

So, for any $\mu\in H(4)_2^*$,

$$
\begin{aligned}
\xi^*(u\otimes v)&=\xi^*(U_{\bar{\xi}\pi(\mu',\mu)}(u\otimes v))\\
&=\sum_{\lambda\in H(4)_1^*} U_{\mu'}t\otimes U_{\mu}U_{\lambda}\theta\\
&=\sum_{\lambda\in H(4)_1^*} U_{\mu'}t_{\lambda}\otimes e(\mu,\lambda)U_{\lambda}\theta\,.
\end{aligned}
$$

Therefore we have

$$U_{\mu'}t_{\lambda}=e(\lambda,\mu)t_{\lambda}\,.$$

In particular, if we put $t=t_0$, $t\in\Gamma(L^4\otimes P_{\alpha+\beta})^{H'(4)_2^*}$. Moreover, for any $\nu\in H(4)_1^*$,

$$
\begin{aligned}
\xi^*(u\otimes v)&=\xi^*(U_{\bar{\xi}\pi(\nu',\nu)}(u\otimes v))\\
&=\sum_{\lambda\in H(4)_1^*} U_{\nu'}t_{\lambda}\otimes U_{\nu}U_{\lambda}\theta\\
&=\sum_{\lambda\in H(4)_1^*} U_{\nu'}t_{\lambda}\otimes U_{\lambda+\nu}\theta\,.
\end{aligned}
$$

Hence we have

$$U_{\nu'}t_{\lambda}=t_{\lambda+\nu}\,.$$

In particular, $U_{\nu'}t=t_{\nu}$ for any $\nu\in H(4)_1^*$. Since $\xi^*(u\otimes v)\neq 0$, $t\neq 0$. Therefore we obtain our assertion. Q. E. D.

PROOF OF PROPOSITION 1.6. If necessary, slightly modifying α and β, we may assume that L is symmetric. Since $H(2)_i\subset H(4)_i$, $H(2)_i$ is automatically lifted up to a subgroup $H(2)_i^*$ in $H(4)_i^*$ for each $i=1, 2$. Obviously, $\pi(\{1\}\times H(2)_i^*)\subset\mathcal{G}^*$. Moreover, for any $y\in X$, the diagram

$$
\begin{tikzcd}
X\times X \arrow[r, "\xi"] \arrow[d, "1_X\times T_y"'] & X\times X \arrow[d, "T_{-y}\times T_y"] \\
X\times X \arrow[r, "\xi"] & X\times X
\end{tikzcd}
$$

commutes. Therefore $\bar{\xi}\pi(\{1\}\times H(2)_i^*)$ is a level subgroup of the group $\{(y, y)\,|\,y\in H(2)_i\}$ for each $i=1, 2$. So, by the equation (4), for any $\mu\in H(2)_1^*$,

$$\xi^*(u\otimes U_{\mu}v)=\xi^*(U_{\mu}u\otimes U_{\mu}v)=\sum_{\lambda\in H(4)_1^*} U_{\lambda'}t\otimes U_{\lambda+\mu}\theta\,,$$

and for any $\nu\in H(2)_2^*$,

$$
\begin{aligned}
e^{L^2}(\nu,\mu)\xi^*(U_{\nu}u\otimes U_{\mu}v)&=\xi^*(U_{\nu}u\otimes U_{\nu}U_{\mu}v)\\
&=\sum_{\lambda\in H(4)_1^*} U_{\lambda'}t\otimes U_{\nu}U_{\lambda+\mu}\theta\\
&=\sum_{\lambda\in H(4)_1^*} U_{\lambda'}t\otimes e^{L^4}(\nu,\lambda)U_{\lambda+\mu}\theta\,.
\end{aligned}
$$

Therefore, to prove Proposition 1.6, we have only to show that if we put $A=(e(\nu,\lambda)U_{\lambda+\mu}\theta)_{(\lambda,(\mu,\nu))\in H(4)_1^*\times X_2^*}$, $\det A\neq 0$. Here we put $B=(U_{\lambda+\mu}\theta)_{(\lambda,\mu)\in H(2)_1^*\times H(2)_1^*}$. Then, denoting a complete system of representatives of $H(4)_1^*/H(2)_1^*$ by $\mathrm{Rep}\,(H(4)_1^*/H(2)_1^*)$,

$$A=(e(\nu,\lambda)U_{\lambda+\mu}\theta)_{(\lambda,(\mu,\nu))\in H(4)_1^*\times X_2^*}$$

$$=(e(\nu,\bar{\lambda})U_{\bar{\lambda}}B)_{(\bar{\lambda},\nu)\in \mathrm{Rep}(H(4)_1^*/H(2)_1^*)\times H(2)_2^*}.$$

Moreover, since $(H(4)_1^*/H(2)_1^*)^\wedge=\{e(\nu,\bar{\lambda})\,|\,\nu\in H(2)_2^*\}$, by Lemma 1.8,

$$\det A=(-2)^{g2^{g-1}\cdot 2^g}\prod_{\bar{\lambda}\in\mathrm{Rep}(H(4)_1^*/H(2)_1^*)}\det(U_{\bar{\lambda}}B)$$

$$=2^{g\cdot 2^{2g-1}}\prod_{\bar{\lambda}\in\mathrm{Rep}(H(4)_1^*/H(2)_1^*)}U_{\bar{\lambda}}\det B\,.$$

On the other hand, by Lemma 1.9,

$$\det B=\prod_{\chi\in\hat{H}(2)^*}\Big(\sum_{\lambda\in H(2)_1^*}\chi(\lambda)U_\lambda\theta\Big)\neq 0\,.$$

This completes the proof of Proposition 1.6. Q. E. D.

§ 2. Quadrics defining abelian varieties.

Let L be an ample invertible sheaf on X. Let p, q be integers such that $p\geqq 2$, $q\geqq 2$; and we put $n=p+q$. For any $\alpha\in\hat{X}$ and any $s_1, s_2\in\Gamma(L^p\otimes P_\alpha)$, $t_1, t_2\in\Gamma(L^q\otimes P_{-\alpha})$, we put

$$Q^{(\alpha)}_{s_1,t_1,s_2,t_2}=s_1t_1\circledast s_2t_2-s_1t_2\circledast s_2t_1\in S^2(\Gamma(L^n))\,.$$

Let $\psi_n: X\hookrightarrow \boldsymbol{P}(\Gamma(L^n))$ be the canonical embedding. Then under these notations, Mumford shows

THEOREM 2.1. *Ideal-theoretically,*

$$\psi_n(X)=\bigcap_{\alpha\in\hat{X}}\ \bigcap_{\substack{s_1,s_2\in\Gamma(L^p\otimes P_\alpha)\\ t_1,t_2\in\Gamma(L^q\otimes P_{-\alpha})}}Q^{(\alpha)}_{s_1,t_1,s_2,t_2}$$

(cf. [5], § 4, Theorem 10).

We soup this theorem up into the following style.

COROLLARY 2.2. *Let U be any non-empty open subset of $\hat{X}$. Then, ideal-theoretically*

$$\psi_n(X)=\bigcap_{\alpha\in U}\ \bigcap_{\substack{s_1,s_2\in\Gamma(L^p\otimes P_\alpha)\\ t_1,t_2\in\Gamma(L^q\otimes P_{-\alpha})}}Q^{(\alpha)}_{s_1,t_1,s_2,t_2}\,.$$

PROOF. Without loss of generality, we can assume that L is symmetric. Let $\xi: X\times X\to X\times X$ be the homomorphism defined by $(x,y)\mapsto(x-qy,\ x+py)$. Then we have

$$\xi^*(p_1^*L^p\otimes p_2^*L^q)\cong p_1^*L^n\otimes p_2^*L^{pqn}\,,$$

i. e.,

$$\Gamma(L^p)\otimes\Gamma(L^q)\xrightarrow{\xi^*}\Gamma(L^n)\otimes\Gamma(L^{pqn})\,. \tag{5}$$

Let $\{u_i\}_{i=1,\cdots,k}$, $\{v_i\}_{i=1,\cdots,l}$ and $\{w_i\}_{i=1,\cdots,m}$ be bases of $\Gamma(L^p)$, $\Gamma(L^q)$ and $\Gamma(L^n)$, respectively. Then from (5),

$$\xi^*(u_i\otimes v_j)=\sum_{\mu=1}^{m}w_\mu\otimes\theta_\mu^{(i,j)}\qquad\text{with }\theta_\mu^{(i,j)}\in\Gamma(L^{pqn})\,,$$

i. e.,

$$u_i(x-qy)v_j(x+py)=\sum_{\mu=1}^{m}w_\mu(x)\theta_\mu^{(i,j)}(y)\,,$$

or

$$(T^*_{-qy}u_i)(x)(T^*_{py}v_j)(x)=\sum_{\mu=1}^{m}w_\mu(x)\theta_\mu^{(i,j)}(y)\,. \tag{6}$$

Here $\{T^*_{-qy}u_i\}$ and $\{T^*_{py}v_j\}$ are bases of $\Gamma(L^p\otimes P_{-pq\phi_L(y)})$ and $\Gamma(L^q\otimes P_{pq\phi_L(y)})$, respectively. We put $I=\langle\{Q^{(\alpha)}_{s_1,t_1,s_2,t_2}\mid\alpha\in\hat{X},\ s_1,\ s_2\in\Gamma(L^p\otimes P_\alpha),\ t_1,\ t_2\in\Gamma(L^q\otimes P_{-\alpha})\}\rangle$ in $S^2(\Gamma(L^n))$. Then there exist a finite number of points $\alpha_1,\cdots,\alpha_N$ in $\hat{X}$ such that

$$I=\langle\{Q^{(\alpha_i)}_{s_1,t_1,s_2,t_2}\mid i=1,\cdots,N;\ s_1,\ s_2\in\Gamma(L^p\otimes P_{\alpha_i}),\ t_1,\ t_2\in\Gamma(L^q\otimes P_{-\alpha_i})\}\rangle\,. \tag{7}$$

We choose points $y_1^{(0)},\cdots,y_N^{(0)}$ in X so that $pq\phi_L(y_i^{(0)})=-\alpha_i$ for $i=1,\cdots,N$. From the equation (6), for any point $(y_1,\cdots,y_N)$ in X^N, we have

$$\begin{aligned}
&Q^{(\alpha_i)}_{T^*_{-qy_i}u_{i_1},\,T^*_{py_i}v_{j_1},\,T^*_{-qy_i}u_{i_2},\,T^*_{py_i}v_{j_2}}\\
&=T^*_{-qy_i}u_{i_1}T^*_{py_i}v_{j_1}\circledast T^*_{-qy_i}u_{i_2}T^*_{py_i}v_{j_2}-T^*_{-qy_i}u_{i_1}T^*_{py_i}v_{j_2}\circledast T^*_{-qy_i}u_{j_2}T^*_{py_i}v_{j_1}\\
&=\Big(\sum_{\mu=1}^{m}w_\mu\theta_\mu^{(i_1,j_1)}(y_i)\Big)\circledast\Big(\sum_{\nu=1}^{m}w_\nu\theta_\nu^{(i_2,j_2)}(y_i)\Big)\\
&\quad-\Big(\sum_{\mu=1}^{m}w_\mu\theta_\mu^{(i_1,j_2)}(y_i)\Big)\circledast\Big(\sum_{\nu=1}^{m}w_\nu\theta_\nu^{(i_2,j_1)}(y_i)\Big)\\
&\underset{\text{put}}{=}\sum_{1\leqq\mu\leqq\nu\leqq m}\Theta^{(i_1,j_1,i_2,j_2)}_{(\mu,\nu)}(y_i)w_\mu\circledast w_\nu\,.
\end{aligned}$$

Here we put $\{(\mu,\nu)\mid 1\leqq\mu\leqq\nu\leqq m\}=\{q_1,\cdots,q_M\}$ and $\{(i_1,j_1,i_2,j_2)\}=\{p_1,\cdots,p_K\}$. Then (7) implies that

$$\operatorname{rank}\big(\Theta^{p_i}_{q_j}(y_k^{(0)})\big)_{((i,k),j)\in([1,K]\times[1,N])\times[1,M]}=\dim I\,. \tag{8}$$

On the other hand, the map

$$(y_1,\cdots,y_N)\longmapsto\operatorname{rank}\big(\Theta^{p_i}_{q_j}(y_k)\big)_{((i,k),j)}$$

is lower semi-continuous. Therefore the set V of points $(y_1, \cdots, y_N)$ at which the equation (8) is satisfied is a non-empty open set in X^N. Since X^N is irreducible, if we put $(-pq\phi_L)^{-1}(U)=U'$,

$$\underbrace{(u'\times \cdots \times U')}_{N}\cap V\neq\emptyset\,.$$

Hence we obtain our assertion using Theorem 2.1. Q. E. D.

§ 3. Cubics defining abelian varieties.

Let L be an ample invertible sheaf on X. For any integer a with $a\geqq 3$, we denote by $\psi_a: X\to \boldsymbol{P}(\Gamma(L^a))$ the canonical embedding. Let $\mathfrak{R}$ be the subspace in $S^3(\Gamma(L^3))$ defined by the kernel of the canonical map $S^3(\Gamma(L^3))\to\Gamma(L^9)$. The purpose of this section is to prove

MAIN THEOREM. *Assume that the characteristic $p\neq 2, 3$. Then, ideal-theoretically,*

$$\psi_3(X)=\bigcap_{F\in\mathfrak{R}} F\,.$$

We start with the following lemma which will be used in the last part of the proof of our theorem.

LEMMA 3.1. *The diagram*

$$(*)\qquad \begin{tikzcd}[column sep=large] & \boldsymbol{P}(\Gamma(L^3))\times\boldsymbol{P}(\Gamma(L^4)) \arrow[dr, "\zeta"] & \\ X \arrow[ur, "{(\psi_3,\, \psi_4)}"] \arrow[dr, "\psi_7"'] & & \boldsymbol{P}(\Gamma(L^3)\otimes\Gamma(L^4)) \\ & \boldsymbol{P}(\Gamma(L^7)) \arrow[ur, "\boldsymbol{P}(\tau)"'] & \end{tikzcd}$$

is commutative, where ζ is the Segre embedding and τ is the canonical surjection: $\Gamma(L^3)\otimes\Gamma(L^4)\to\Gamma(L^7)$. Moreover, we have the equality

$$\zeta^{-1}(\operatorname{Im}(\boldsymbol{P}(\tau)))\cap\{\boldsymbol{P}(\Gamma(L^3))\times\psi_4(X)\}=\operatorname{Im}(\psi_3, \psi_4)\,.$$

PROOF. The commutativity of the diagram (*) is obvious. Therefore, automatically we have the inclusion

$$\boldsymbol{P}(\tau)^{-1}(\zeta(\boldsymbol{P}(\Gamma(L^3))\times\psi_4(X)))\supset\psi_7(X)\,.$$

Moreover, for any point $\alpha\in\hat{X}$ and for any sections $u, v\in\Gamma(L^3)$; $u^{(\alpha)}, v^{(\alpha)}\in\Gamma(L^2\otimes P_\alpha)$; and $u^{(-\alpha)}, v^{(-\alpha)}\in\Gamma(L^2\otimes P_{-\alpha})$, we put

$$\begin{aligned} F_{u,v,u^{(\alpha)},\, u^{(-\alpha)},\, v^{(\alpha)},\, v^{(-\alpha)}}&=(u\otimes u^{(\alpha)}u^{(-\alpha)})\circledcirc(v\otimes v^{(\alpha)}v^{(-\alpha)})\\ &\quad-(u\otimes u^{(\alpha)}v^{(-\alpha)})\circledcirc(v\otimes v^{(\alpha)}u^{(-\alpha)})\\ &\quad\in S^2(\Gamma(L^3)\otimes\Gamma(L^4))\,.\end{aligned}$$

Then, obviously

$$\zeta(\boldsymbol{P}(\Gamma(L^3))\times\psi_4(X))\subset\bigcap_{\alpha\in\hat{X}}\ \bigcap_{\substack{u,v\in\Gamma(L^3)\\ u^{(\alpha)},v^{(\alpha)}\in\Gamma(L^2\otimes P_\alpha)\\ u^{(-\alpha)},v^{(-\alpha)}\in\Gamma(L^2\otimes P_{-\alpha})}} F_{u,v,u^{(\alpha)},u^{(-\alpha)},v^{(\alpha)},v^{(-\alpha)}},$$

and

$$\boldsymbol{P}(\tau)^{-1}(\zeta(\boldsymbol{P}(\Gamma(L^3))\times\psi_4(X)))\subset\bigcap_{\alpha\in\hat{X}}\ \bigcap_{\substack{u,v\\ u^{(\alpha)},v^{(\alpha)}\\ u^{(-\alpha)},v^{(-\alpha)}}} S^2(\tau)(F_{u,v,u^{(\alpha)},u^{(-\alpha)},v^{(\alpha)},v^{(-\alpha)}}).\tag{9}$$

On the other hand,

$$\begin{aligned}S^2(\tau)(F_{u,v,u^{(\alpha)},u^{(-\alpha)},v^{(\alpha)},v^{(-\alpha)}})&=\tau(u\otimes u^{(\alpha)}u^{(-\alpha)})\circledcirc\tau(v\otimes v^{(\alpha)}v^{(-\alpha)})\\&\quad-\tau(u\otimes u^{(\alpha)}v^{(-\alpha)})\circledcirc\tau(v\otimes v^{(\alpha)}u^{(-\alpha)})\\&=Q^{(\alpha)}_{uu^{(\alpha)},u^{(-\alpha)},vv^{(\alpha)},v^{(-\alpha)}},\end{aligned}$$

where $Q^{(\alpha)}_{uu^{(\alpha)},u^{(-\alpha)},vv^{(\alpha)},v^{(-\alpha)}}$ is Mumford's quadric stated in Section 2. Moreover, by virtue of Theorem 1.1, the canonical map $\Gamma(L^3)\otimes\Gamma(L^2\otimes P_\alpha)\to\Gamma(L^5\otimes P_\alpha)$ is surjective. So, in $S^2(\Gamma(L^7))$,

$$\begin{aligned}&\langle\{Q^{(\alpha)}_{uu^{(\alpha)},u^{(-\alpha)},vv^{(\alpha)},v^{(-\alpha)}}\mid u,v\in\Gamma(L^3),u^{(\alpha)},v^{(\alpha)}\in\Gamma(L^3\otimes P_\alpha),u^{(-\alpha)},v^{(-\alpha)}\in\Gamma(L^2\otimes P_{-\alpha})\}\rangle\\&\quad=\langle\{Q^{(\alpha)}_{s_1,t_1,s_2,t_2}\mid s_1,\ s_2\in\Gamma(L^5\otimes P_\alpha),\ t_1,\ t_2\in\Gamma(L^2\otimes P_{-\alpha})\}\rangle.\end{aligned}$$

Hence, by virtue of Theorem 2.1 and the inclusion (9), we obtain the converse relation

$$\boldsymbol{P}(\tau)^{-1}(\zeta(\boldsymbol{P}(\Gamma(L^3)\times\psi_4(X)))\subset\psi_7(X).$$

This implies the required equality. Q. E. D.

For any point $\alpha\in\hat{X}$, we put

$$\mathfrak{Q}_\alpha=\{Q^{(\alpha)}_{s_1,t_1,s_2,t_2}\mid s_1,\ s_2\in\Gamma(L^2\otimes P_\alpha),\ t_1,\ t_2\in\Gamma(L\otimes P_{-\alpha})\}\subset S^2(\Gamma(L^3)).\tag{10}$$

Note that if L is principal, $\mathfrak{Q}_\alpha=\{0\}$. The next lemma can be seen by easy calculation.

LEMMA 3.2. *For a point* $[l]$ *in* $\boldsymbol{P}(\Gamma(L^3))$, *the following two conditions are equivalent:*

($\mathfrak{Q}_\alpha$) $[l]$ *is a common zero of all forms in* $\mathfrak{Q}_\alpha$.

($\mathfrak{Q}'_\alpha$) *There exist linear maps* $m_\alpha:\Gamma(L^2\otimes P_\alpha)\to k$ *and* $n_{-\alpha}:\Gamma(L\otimes P_{-\alpha})\to k$ *such that the diagram*

$$\begin{tikzcd}\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L\otimes P_{-\alpha}) \arrow[rr] \arrow[dr, "m_\alpha\otimes n_{-\alpha}"'] & & \Gamma(L^3) \arrow[dl, "l"] \\ & k & \end{tikzcd}$$

commutes.

Moreover, if α satisfies the condition

($\mathrm{C1}_l$) *l is non-trivial on the image of $\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L\otimes P_{-\alpha})$ in $\Gamma(L^3)$, then m_α and $n_{-\alpha}$ are uniquely determined by l up to constant multiples.*

REMARK. For any positive integer n and any point $[l]$ in $\boldsymbol{P}(\Gamma(L^n))$, also true is the same type of assertion as in the above lemma.

Note that if a point $[l]\in\boldsymbol{P}(\Gamma(L^3))$ satisfies the condition

($\mathfrak{K}$) $[l]$ is a common zero of all forms in $\mathfrak{K}$,

then obviously it satisfies the conditions ($\mathfrak{Q}_\alpha$) for any $\alpha\in\hat{X}$.

LEMMA 3.3. *Let $[l]$ be a point in $\boldsymbol{P}(\Gamma(L^3))$ satisfying the condition ($\mathfrak{K}$) and α be a point in $\hat{X}$ satisfying the condition*

(C2) $$\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha})\longrightarrow\Gamma(L^4) \text{ is surjective.}$$

Moreover, we assume that α and $-\alpha$ satisfy the condition ($\mathrm{C1}_l$). Then l defines, uniquely up to constant multiples, non-trivial linear maps $z_\alpha:\Gamma(L^4)\to k$ and $y_\alpha:\Gamma(L^7)\to k$, by the commutative diagrams

(11)
$$\begin{tikzcd}
\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha}) \arrow[rr] \arrow[dr, "m_\alpha\otimes m_{-\alpha}"'] & & \Gamma(L^4) \arrow[dl, "z_\alpha"] \\
& k &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Gamma(L^3)\otimes\Gamma(L^4) \arrow[rr] \arrow[dr, "l\otimes z_\alpha"'] & & \Gamma(L^7), \arrow[dl, "y_\alpha"] \\
& k &
\end{tikzcd}$$

where m_α and $m_{-\alpha}$ are linear maps given in Lemma 3.2.

PROOF. Since $[l]$ satisfies the condition ($\mathfrak{K}$), there exists a linear map $x:\Gamma(L^9)\to k$ such that the diagram

$$\begin{tikzcd}
\Gamma(L^3)\otimes\Gamma(L^3)\otimes\Gamma(L^3) \arrow[rr] \arrow[dr, "l\otimes l\otimes l"'] & & \Gamma(L^9) \arrow[dl, "x"] \\
& k &
\end{tikzcd}$$

commutes. Hence we obtain a commutative diagram

$$\begin{tikzcd}
& \Gamma(L\otimes P_{-\alpha})\otimes\Gamma(L\otimes P_\alpha)\otimes\Gamma(L^7) \arrow[dr] & \\
\Gamma(L^3)\otimes\Gamma(L\otimes P_{-\alpha})\otimes\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha}) \arrow[r] \arrow[ur] \arrow[dr, "l\otimes n_{-\alpha}\otimes m_\alpha\otimes n_\alpha\otimes m_{-\alpha}"'] & \Gamma(L^3)\otimes\Gamma(L^3)\otimes\Gamma(L^3) \arrow[r] \arrow[d, "l\otimes l\otimes l"] & \Gamma(L^9). \arrow[dl, "x"] \\
& k &
\end{tikzcd}$$

Therefore there exists a linear map $y'_\alpha:\Gamma(L\otimes P_{-\alpha})\otimes\Gamma(L\otimes P_\alpha)\otimes\Gamma(L^7)\to k$ which makes the diagram

$$\begin{tikzcd}
\Gamma(L\otimes P_{-\alpha})\otimes\Gamma(L\otimes P_{\alpha})\otimes\Gamma(L^3)\otimes\Gamma(L^2\otimes P_{\alpha})\otimes\Gamma(L^2\otimes P_{-\alpha}) \arrow[rr] \arrow[dr, "n_{-\alpha}\otimes n_{\alpha}\otimes l\otimes m_{\alpha}\otimes m_{-\alpha}"'] & & \Gamma(L\otimes P_{-\alpha})\otimes\Gamma(L\otimes P_{\alpha})\otimes\Gamma(L^7) \arrow[dl, "y'_{\alpha}"] \\
& k &
\end{tikzcd}$$

commute. From our assumption, there exist elements $\theta^{(-\alpha)}\in\Gamma(L\otimes P_{-\alpha})$ and $\theta^{(\alpha)}\in\Gamma(L\otimes P_{\alpha})$ such that $n_{-\alpha}(\theta^{(-\alpha)})n_{\alpha}(\theta^{(\alpha)})\neq 0$. Here we can define a linear map $y_{\alpha}:\Gamma(L^7)\to k$ by

$$y_{\alpha}(v)=\frac{y'_{\alpha}(\theta^{(-\alpha)}\otimes\theta^{(\alpha)}\otimes v)}{n_{-\alpha}(\theta^{(-\alpha)})n_{\alpha}(\theta^{(\alpha)})}\,.$$

Then, since $n_{-\alpha}(\theta^{(-\alpha)})n_{\alpha}(\theta^{(\alpha)})l(u)m_{\alpha}(u^{(\alpha)})m_{-\alpha}(u^{(-\alpha)})=y'_{\alpha}(\theta^{(-\alpha)}\otimes\theta^{(\alpha)}\otimes uu^{(\alpha)}u^{(-\alpha)})$ for every $u\otimes u^{(\alpha)}\otimes u^{(-\alpha)}\in\Gamma(L^3)\otimes\Gamma(L^2\otimes P_{\alpha})\otimes\Gamma(L^2\otimes P_{-\alpha})$, we obtain the equality

$$y_{\alpha}(uu^{(\alpha)}u^{(-\alpha)})=l(u)m_{\alpha}(u^{(\alpha)})m_{-\alpha}(u^{(-\alpha)})\,,$$

i.e., the diagram

$$\begin{tikzcd}
\Gamma(L^3)\otimes\Gamma(L^2\otimes P_{\alpha})\otimes\Gamma(L^2\otimes P_{-\alpha}) \arrow[rr] \arrow[dr, "l\otimes m_{\alpha}\otimes m_{-\alpha}"'] & & \Gamma(L^7) \arrow[dl, "y_{\alpha}"] \\
& k &
\end{tikzcd}$$

commutes. Moreover, obviously there exists a linear map $z'_{\alpha}:\Gamma(L^3)\otimes\Gamma(L^4)\to k$ which makes the diagram

$$\begin{tikzcd}
\Gamma(L^3)\otimes\Gamma(L^2\otimes P_{\alpha})\otimes(L^2\otimes P_{-\alpha}) \arrow[rr] \arrow[dr] \arrow[ddr, "l\otimes m_{\alpha}\otimes m_{-\alpha}"'] & & \Gamma(L^7) \arrow[ddl, "y_{\alpha}"] \\
& \Gamma(L^3)\otimes\Gamma(L^4) \arrow[ur] \arrow[d, "z'_{\alpha}"] & \\
& k &
\end{tikzcd}$$

commute. Since l is non-trivial, there exists an element $u_0\in\Gamma(L^3)$ such that $l(u_0)\neq 0$. We define $z_{\alpha}:\Gamma(L^4)\to k$ by

$$z_{\alpha}(w)=\frac{z'_{\alpha}(u_0\otimes w)}{l(u_0)}\,.$$

Then, since $l(u_0)m_{\alpha}(u^{(\alpha)})m_{-\alpha}(u^{(-\alpha)})=z'_{\alpha}(u_0\otimes u^{(\alpha)}u^{(-\alpha)})$ for every $u^{(\alpha)}\otimes u^{(-\alpha)}\in\Gamma(L^2\otimes P_{\alpha})\otimes\Gamma(L^2\otimes P_{-\alpha})$, we have the equality

$$z_{\alpha}(u^{(\alpha)}u^{(-\alpha)})=m_{\alpha}(u^{(\alpha)})m_{-\alpha}(u^{(-\alpha)})\,,$$

i. e., the diagram

$$\begin{tikzcd}
\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha}) \arrow[rr] \arrow[dr, "m_\alpha\otimes m_{-\alpha}"'] & & \Gamma(L^4) \arrow[dl, "z_\alpha"] \\
& k &
\end{tikzcd}$$

commutes. Moreover, since α satisfies (C2), these y_α and z_α are those which we searched for. Q. E. D.

Here, for a point $\hat{x}$ in $\hat{X}$, we consider a condition

$$(\mathrm{C3}_l)\qquad \begin{cases} l \text{ does not vanish on the image of } \Gamma(L)\otimes\Gamma(L\otimes P_{\hat{x}})\otimes\Gamma(L\otimes P_{-\hat{x}}) \\ \text{in } \Gamma(L^3)\,. \end{cases}$$

Since the diagram

$$(12)\qquad \begin{tikzcd}
& \Gamma(L^2\otimes P_{\hat{x}})\otimes\Gamma(L\otimes P_{-\hat{x}}) \arrow[dr] & \\
\Gamma(L)\otimes\Gamma(L\otimes P_{\hat{x}})\otimes\Gamma(L\otimes P_{-\hat{x}}) \arrow[ur] \arrow[r] \arrow[dr] & \Gamma(L)\otimes\Gamma(L^2) \arrow[r] & \Gamma(L^3) \\
& \Gamma(L\otimes P_{\hat{x}})\otimes\Gamma(L^2\otimes P_{-\hat{x}}) \arrow[ur] &
\end{tikzcd}$$

commutes, $\hat{x}$ and $-\hat{x}$ satisfy the condition $(\mathrm{C1}_l)$, providing that $\hat{x}$ satisfies $(\mathrm{C3}_l)$. Therefore, in Lemma 3.3, the condition on α can be replaced by $(\mathrm{C3}_l)$.

LEMMA 3.4. *If a point* $[l]$ *in* $\boldsymbol{P}(\Gamma(L^3))$ *satisfies* $(\mathfrak{K})$, *and* α, β *in* $\hat{X}$ *satisfy* (C2) *and* $(\mathrm{C3}_l)$, *then the* z_α *and* z_β *(a fortiori,* y_α *and* y_β*) in Lemma* 3.3 *differ possibly only by a scalar.*

PROOF. By the condition $(\mathrm{C3}_l)$ on α, β, there exists a section $\theta\in\Gamma(L)$ such that $n_0(\theta)\neq 0$, where $m_0:\Gamma(L^2)\to k$ and $n_0:\Gamma(L)\to k$ are linear maps given in Lemma 3.2. Moreover, in view of the diagram (12) for $\hat{x}=\alpha$ and β, m_0 is non-trivial on the images of $\Gamma(L\otimes P_\alpha)\otimes\Gamma(L\otimes P_{-\alpha})$ and of $\Gamma(L\otimes P_\beta)\otimes\Gamma(L\otimes P_{-\beta})$ in $\Gamma(L^2)$. Therefore we can choose elements $\theta^{(\alpha)}\in\Gamma(L\otimes P_\alpha)$, $\theta^{(-\alpha)}\in\Gamma(L\otimes P_{-\alpha})$, $\theta^{(\beta)}\in\Gamma(L\otimes P_\beta)$, and $\theta^{(-\beta)}\in\Gamma(L\otimes P_{-\beta})$ such that

$$\begin{aligned}
l(\theta\theta^{(\alpha)}\theta^{(-\alpha)}) &= m_\alpha(\theta\theta^{(\alpha)})n_{-\alpha}(\theta^{(-\alpha)}) \\
&= m_{-\alpha}(\theta\theta^{(-\alpha)})n_\alpha(\theta^{(\alpha)}) \\
&= n_0(\theta)m_0(\theta^{(\alpha)}\theta^{(-\alpha)})\neq 0
\end{aligned}$$

and

$$\begin{aligned} l(\theta\theta^{(\beta)}\theta^{(-\beta)}) &= m_\beta(\theta\theta^{(\beta)})n_{-\beta}(\theta^{(-\beta)}) \\ &= m_{-\beta}(\theta\theta^{(-\beta)})n_\beta(\theta^{(\beta)}) \\ &= n_0(\theta)m_0(\theta^{(\beta)}\theta^{(-\beta)}) \neq 0 . \end{aligned}$$

Let w be any element of $\Gamma(L^4)$. Since α and β satisfy the condition (C2), there exist elements $\sum_i u_i^{(\alpha)}\otimes u_i^{(-\alpha)} \in \Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha})$ and $\sum_j v_j^{(\beta)}\otimes v_j^{(-\beta)} \in \Gamma(L^2\otimes P_\beta)$ $\otimes\Gamma(L^2\otimes P_{-\beta})$, whose images in $\Gamma(L^4)$ are w. So, the cubic

$$\begin{aligned} &\theta\theta^{(\beta)}\theta^{(-\beta)} \circledcirc (\sum_j \theta^{(-\alpha)}u_i^{(\alpha)} \circledcirc \theta^{(\alpha)}u_i^{(-\alpha)}) \\ &\qquad -\theta\theta^{(\alpha)}\theta^{(-\alpha)} \circledcirc (\sum_j \theta^{(-\beta)}v_j^{(\beta)} \circledcirc \theta^{(\beta)}v_j^{(-\beta)}) \end{aligned}$$

is contained in $\mathfrak{R}$. Hence we have

$$\begin{aligned} &l(\theta\theta^{(\beta)}\theta^{(-\beta)})\{\sum_i l(\theta^{(-\alpha)}u_i^{(\alpha)})l(\theta^{(\alpha)}u_i^{(-\alpha)})\} \\ &\qquad - l(\theta\theta^{(\alpha)}\theta^{(-\alpha)})\{\sum_j l(\theta^{(-\beta)}v_j^{(\beta)})l(\theta^{(\beta)}v_j^{(-\beta)})\} \\ &= l(\theta\theta^{(\beta)}\theta^{(-\beta)})n_{-\alpha}(\theta^{(-\alpha)})n_\alpha(\theta^{(\alpha)})z_\alpha(w) \\ &\qquad - l(\theta\theta^{(\alpha)}\theta^{(-\alpha)})n_{-\beta}(\theta^{(-\beta)})n_\beta(\theta^{(\beta)})z_\beta(w) = 0 . \end{aligned}$$

This completes the proof. Q.E.D.

Of course, there exists $[l]$ in $\boldsymbol{P}(\Gamma(L^3))$ having no points satisfying the condition (C3$_l$). So, in our proof, we must soonsider the isomorphisms $U_z : \Gamma(L^3) \xrightarrow[\sim]{T_z^*} \Gamma(T_z^* L^3) \xrightarrow{\sim} \Gamma(L^3)$ for any closed points $z \in K(L^3)$. These U_z induce canonically automorphisms $S^*(U_z)$ of the symmetric algebra $S^*(\Gamma(L^3))$. Obviously, $\mathfrak{R} = S^3(U_z)(\mathfrak{R})$. Therefore, if a point $[l]$ in $\boldsymbol{P}(\Gamma(L^3))$ satisfies the condition ($\mathfrak{R}$), so does the point $[l\circ U_z]$.

LEMMA 3.5. *For any point $[l] \in \boldsymbol{P}(\Gamma(L^3))$, there exists a closed point z in $K(L^3)$ such that $l\circ U_z$ does not vanish identically on the image of $\Gamma(L)\otimes\Gamma(L^2)$ in $\Gamma(L^3)$.*

PROOF. By virtue of Lemma 1.4,

$$\sum_{\alpha\in\hat{X}_3} \Gamma(L\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha}) \longrightarrow \Gamma(L^3)$$

is surjective. Therefore there exists a point $\alpha \in \hat{X}_3$ such that l does not vanish identically on the image of $\Gamma(L\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha})$ in $\Gamma(L^3)$. Here we take a closed point $z \in K(L^3)$ with $\phi_L(z) = \alpha$. Then, since the diagram

$$\begin{tikzcd} \Gamma(L)\otimes\Gamma(L^2) \arrow[r] \arrow[d, "T_z^*\otimes T_z^*"'] & \Gamma(L^3) \arrow[d, "T_z^*"] \arrow[dd, bend left, "U_z"] \\ \Gamma(T_z^*L)\otimes\Gamma(T_z^*L^2) \arrow[r] \arrow[d, "\wr"'] & \Gamma(T_z^*L^3) \arrow[d, "\wr"'] \\ \Gamma(L\otimes P_\alpha)\otimes\Gamma(L^2\otimes P_{-\alpha}) \arrow[r] & \Gamma(L^3) \end{tikzcd}$$

commutes, $l\circ U_z$ is non-trivial on the image of $\Gamma(L)\otimes\Gamma(L^2)$ in $\Gamma(L^3)$. Q.E.D.

For completing the proof of our theorem, we need the following Lemma whose proof is a modification of Mumford's proof in [**5**, p. 83 and p. 88].

LEMMA 3.6. *Let p, q be positive integers and $n=p+q$. Then for a non-trivial linear map $l:\Gamma(L^n)\to k$, there exists an open subset V in $\hat{X}$ such that l does not vanish identically on the image of $\Gamma(L^p\otimes P_\alpha)\otimes\Gamma(L^q\otimes P_{-\alpha})$ in $\Gamma(L^n)$ for every point $\alpha\in V$.*

PROOF. In the same way as in the proof of Corollary 1.3, we have locally free sheaves $\mathcal{L}_1=q_*(p^*L^p\otimes P)$ and $\mathcal{L}_2=q_*(p^*L^q\otimes P^{-1})$ on $\hat{X}$ such that $\mathcal{L}_1\otimes \boldsymbol{k}(\alpha)\cong\Gamma(L^p\otimes P_\alpha)$ and $\mathcal{L}_2\otimes\boldsymbol{k}(\alpha)\cong\Gamma(L^q\otimes P_{-\alpha})$, and the canonical pairing

$$\phi:\ \mathcal{L}_1\otimes_{\mathcal{O}_{\hat{X}}}\mathcal{L}_2\longrightarrow q_*(p^*L^n)\cong\Gamma(L^n)\otimes_k\mathcal{O}_{\hat{X}}\,.$$

Let $D=\operatorname{Ker}(l)\subset\Gamma(L^n)$, which is a proper subspace. We put $\bar{\phi}$ the composite homomorphism

$$\bar{\phi}:\ \mathcal{L}_1\otimes_{\mathcal{O}_{\hat{X}}}\mathcal{L}_2\xrightarrow{\ \phi\ }\Gamma(L^n)\otimes_k\mathcal{O}_{\hat{X}}\longrightarrow(\Gamma(L^n)/D)\otimes_k\mathcal{O}_{\hat{X}}\,.$$

Then $\bar{\phi}$ is a non-zero section of the locally free sheaf $\mathcal{H}om_{\mathcal{O}_{\hat{X}}}(\mathcal{L}_1\otimes_{\mathcal{O}_{\hat{X}}}\mathcal{L}_2,$ $(\Gamma(L^n)/D)\otimes_k\mathcal{O}_{\hat{X}})$ on $\hat{X}$. Therefore the set

$$\begin{aligned} V&=\{\alpha\in\hat{X}\mid\bar{\phi}(\alpha)\neq 0\}\\ &=\left\{\alpha\in\hat{X}\,\middle|\,\begin{array}{l} l \text{ does not vanish identically on the image}\\ \text{of } \Gamma(L^p\otimes P_\alpha)\otimes\Gamma(L^q\otimes P_{-\alpha}) \text{ in } \Gamma(L^n)\end{array}\right\}\end{aligned}$$

is a non-empty open set and we are done. Q.E.D.

Under these preliminaries, we now prove our main theorem. In the main theorem, the inclusion relation $\subset$ is obvious. Conversely, let $[l]$ be a point of $\bigcap_{F\in K}F$.

The diagram

$$\begin{tikzcd} X \arrow[r, "\phi_3"] \arrow[d, "T_z"'] & \boldsymbol{P}(\Gamma(L^3)) \arrow[d, "\boldsymbol{P}(U_z)"] \\ X \arrow[r, "\phi_3"] & \boldsymbol{P}(\Gamma(L^3)) \end{tikzcd}$$

is commutative for any closed point z in $K(L^3)$. Therefore, if we prove that $[l\circ U_z]$ is contained in $\phi_3(X)$ for some $z\in K(L^3)$, we see that $[l]$ is also contained in $\phi_3(X)$. Hence, to prove our assertion, by virtue of Lemma 3.5, we may assume that l does not vanish identically on the image of $\Gamma(L)\otimes\Gamma(L^2)$ in $\Gamma(L^3)$. Since $[l]$ satisfies $(\mathfrak{K})$, a fortiori $(\mathfrak{Q}_\alpha)$ for any $\alpha\in\hat{X}$, there exist linear maps m_α, $n_{-\alpha}$ such that the diagram

$$\begin{tikzcd} \Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L\otimes P_{-\alpha}) \arrow[rr] \arrow[dr, "m_\alpha\otimes n_{-\alpha}"'] & & \Gamma(L^3) \arrow[dl, "l"] \\ & k & \end{tikzcd}$$

commutes for any $\alpha\in\hat{X}$. In particular, there exist non-trivial linear maps, m, n which make the diagram:

$$\begin{tikzcd} \Gamma(L^2)\otimes\Gamma(L) \arrow[rr] \arrow[dr, "m\otimes n"'] & & \Gamma(L^3) \arrow[dl, "l"] \\ & k & \end{tikzcd}$$

commute. Applying Lemma 3.6 to m, there exists an open subset U_1 in $\hat{X}$ such that m does not vanish identically on the image of $\Gamma(L\otimes P_\alpha)\otimes\Gamma(L\otimes P_{-\alpha})$ in $\Gamma(L^2)$ for every point α in U_1. Moreover, by virtue of Proposition 1.5, there exists an open set U_2 in $\hat{X}$ every point of which satisfies the condition (C2). Here we put $U=U_1\cap U_2$ which is a non-empty subset of $\hat{X}$. Then every point of U satisfies the conditions (C2) and $(C3_l)$. Therefore, in view of Lemma 3.4, z_α (resp. y_α) for all $\alpha\in U$, in Lemma 3.3, define a same point in $\boldsymbol{P}(\Gamma(L^4))$ (resp. in $\boldsymbol{P}(\Gamma(L^7))$), which is independent of α and is denoted by $[z]$ (resp. $[y]$). Moreover, Lemma 3.3 implies that $([l], [z])=\boldsymbol{P}(\tau)([y])$. On the other hand, in view of the remark to Lemma 3.2 and Corollary 2.2, we see that $[z]$ is contained in $\phi_4(X)$. Hence, by virtue of Lemma 3.1, $([l], [z])\in\mathrm{Im}(\phi_3, \phi_4)$, i.e., $[l]$ is contained in $\phi_3(X)$. These arguments are true for $(k[X]/(X^2))$-valued points instead of k-valued points. So we complete the proof of our main theorem. Q.E.D.

References

[1] S. Koizumi, On the structure of the graded $\boldsymbol{C}$-algebra of theta functions, Proc. Japan Acad., **51** (1975), 1-5.

[2] S. Koizumi, Theta relations and projective normality of abelian varieties, Amer. J. Math., **98** (1976), 865-889.

[3] S. Koizumi, The rank theorem on matrices of theta functions, J. Fac. Sci. Univ. Tokyo Sect. IA, **24** (1977), 115-122.

[4] H. Morikawa, On the defining equations of abelian varieties, Nagoya Math. J., **30** (1967), 143-162.

[5] D. Mumford, Varieties defined by quadratic equations, Questioni sulle varieta algebriche, Corsi dal I. L. M. E., Edizioni Cremonese, Roma, 1969.

[6] D. Mumford, Abelian varieties, Tata Inst. Studies in Math., Oxford Univ. Press, London and New York, 1970.

[7] T. Sekiguchi, On projective normality of abelian varieties, J. Math Soc. Japan, **28** (1976), 307-322.

[8] T. Sekiguchi, On projective normality of abelian varieties II, J. Math. Soc. Japan, **29** (1977), 709-727.

Tsutomu SEKIGUCHI
Department of Mathematics
Faculty of Science and Engineering
Chuo University
Kasuga, Bunkyo-ku
Tokyo, Japan