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Notes on hermitian forms over a ring

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A purpose of this paper is to show fundamental properties on hermitian forms over a ring which are generalizations of ones over a field. First, we shall show that every non degenerate projective and finitely generated hermitian module is characterized by a von Neumann regular and hermitian matrix with respect to a generator of the module. In particular, if the ring is semisimple artinian, then it is derived that every non degenerate hermitian module is finitely generated and isomorphic to a form $\langle a_1\rangle\perp\langle a_2\rangle\perp\cdots\perp\langle a_r\rangle\perp\left\langle\begin{pmatrix}0 & b_1\\ \bar{b}_1 & 0\end{pmatrix}\right\rangle\perp\cdots\perp\left\langle\begin{pmatrix}0 & b_s\\ \bar{b}_s & 0\end{pmatrix}\right\rangle$ for some elements $a_1, a_2, \cdots, a_r, b_1, b_2, \cdots, b_s$ in A with $\bar{a}_i=a_i$, $i=1, 2, \cdots, r$. Next, we discuss a characterization of a metabolic module. Throughout this paper, we assume that a ring A is a non-commutative ring with unit element 1 and has an involution $A\to A$; $a \rightsquigarrow \bar{a}$ satisfying $\overline{(a+b)}=\bar{a}+\bar{b}$, $\overline{ab}=\bar{b}\bar{a}$ and $\bar{\bar{a}}=a$ for every $a, b\in A$. Furthermore, every A-module is unitary.

1. Notations and definitions.

For a ring A with involution, an element a of A is called a von Neumann regular element (simply, regular element) if there is an element $b\in A$ such that $aba=a$. Let I be a finite or infinite set of indices. By $(a_i)_{i\in I}$ (resp. ${}^t(a_i)_{i\in I}$) for $a_i\in A$, we denote an I-row (resp. I-column) vector with i-th component a_i. $(a_i)_{i\in I}$ (resp. ${}^t(a_i)_{i\in I}$) is called I-row (resp. I-column) finite if $a_i=0$ for almost all i in I. We put $A^I=\{(a_i)_{i\in I}\,;\,a_i\in A\}$, ${}^tA^I=\{{}^t(a_i)_{i\in I}\,;\,a_i\in A\}$, $A^{(I)}=\{(a_i)_{i\in I}\in A^I$; I-row finite$\}$ and ${}^tA^{(I)}=\{{}^t(a_i)_{i\in I}\in {}^tA^I$; I-column finite$\}$. As usual, A^I and $A^{(I)}$ are left A-module and ${}^tA^I$ and ${}^tA^{(I)}$ are right A-modules. For any $a_{ij}\in A$ $(i, j\in I)$, $(a_{ij})_{(ij)\in I\times I}$ denotes an $I\times I$-matrix with the (ij)-component a_{ij}. By A_I we denote the set of $I\times I$-matrices with components in A. Then one makes A_I an additive group. Let J and K be another sets of indices. For $I\times J$- and $J\times K$-matrices $(a_{ij})_{(ij)\in I\times J}$ and $(b_{jk})_{(jk)\in J\times K}$, we say that the product $(a_{ij})_{(ij)\in I\times J}\cdot(b_{jk})_{(jk)\in J\times K}$ is defined, if for every $(ik)\in I\times K$, the product $a_{ij}b_{jk}=0$ for almost all j in J. If the product is defined, then $(a_{ij})_{(ij)\in I\times J}\cdot(b_{jk})_{(jk)\in J\times K}=(\sum_{j\in J}a_{ij}b_{jk})_{(ik)\in I\times K}$. We can easily check the following remark;

REMARK 1. Let $X=(x_{hi})_{(hi)\in H\times I}$, $Y=(y_{ij})_{(ij)\in I\times J}$ and $Z=(z_{jk})_{(jk)\in J\times K}$ be matrices over A. If the products XY, $(XY)Z$, YZ and $X(YZ)$ are defined, and if for each $(hk)\in H\times K$, $x_{hi}y_{ij}z_{jk}=0$ for almost all (ij) in $I\times J$, then $(XY)Z=X(YZ)$.

An $I\times J$-matrix $(a_{ij})_{(ij)\in I\times J}$ is called a row-finite (resp. column-finite) matrix, if every its J-row vector $(a_{ij})_{j\in J}$ (resp. I-column vector ${}^t(a_{ij})_{i\in I}$) is in $A^{(J)}$ (resp. ${}^tA^{(I)}$). We denote by $A_{(I}$ (resp. $A_{I)}$) the set of row-finite (resp. column-finite) $I\times I$-matrices over A, and by $A_{(I)}$ the set of row-finite and column-finite matrices over A. Then $A_{(I}$, $A_{I)}$ and $A_{(I)}$ become rings. Let M be a left A-module with a generator $\{m_i\}_{i\in I}$; $M=\sum_{i\in I}Am_i$. For a matrix H in A_I and a subset B of $A^{(I)}$, we put $\mathrm{Ann}(A^{(I)};\{m_i\}_{i\in I})=\{(a_i)_{i\in I}\in A^{(I)};\sum_{i\in I}a_im_i=0\}$, $\mathrm{Ann}(A^{(I)};H)=\{(a_i)_{i\in I}\in A^{(I)};(a_i)_{i\in I}H=0 \text{ in } A^I\}$ and $\mathrm{Ann}({}^tA^I;B)=\{{}^t(a_i)_{i\in I}\in{}^tA^I;(b_i)_{i\in I}\cdot{}^t(a_i)_{i\in I}=\sum_{i\in I}b_ia_i=0 \text{ for all } (b_i)_{i\in I}\in B\}$. For an $I\times J$-matrix $H=(a_{ij})_{(ij)\in I\times J}$ over A, H^* denotes a $J\times I$-matrix $(\bar{a}_{ij})_{(ji)\in J\times I}$. If an $I\times I$-matrix H satisfies $H=H^*$, H is called a hermitian matrix. Let M be a left A-module, and $h:M\times M\to A$; $(x,y)\rightsquigarrow h(x,y)$ a hermitian form, i.e. it satisfies $h(ax,by)=ah(x,y)\bar{b}$, $\overline{h(x,y)}=h(y,x)$ and $h(x+x',y)=h(x,y)+h(x',y)$ for $x,x',y\in M$ and $a,b\in A$. Then (M,h) is called a hermitian left A-module. (M,h) is said non degenerate, if $\theta:M\to\mathrm{Hom}_A(M,A)$; $m\rightsquigarrow h(-,m)$ is bijective. We shall say (M,h) a non degenerate (finitely generated) projective hermitian left A-module, if M is (finitely generated) projective over A and (M,h) is non degenerate. For (M,h) and (M',h'), morphism $f:(M,h)\to(M',h')$ is called isomorphism or isometry, if $f:M\to M'$ is an A-isomorphism satisfying $h(f(x),f(y))=h(x,y)$ for $x,y\in M$. Then we denote by $(M,h)\cong(M',h')$. A hermitian left A-module (M,h) is called metabolic, if there is a hermitian left A-module (N,q) such that $(M,h)\cong(N\oplus N^*,h_q)$ where N^* is a left A-module $\mathrm{Hom}_A(N,A)$ defined by $af(n)=f(n)\bar{a}$ for $f\in\mathrm{Hom}_A(M,A)$, $n\in N$ and $a\in A$, and h_q is a hermitian form $h_q:(N\oplus N^*)\times(N\oplus N^*)\to A$ defined by $h_q(x+f,y+g)=\overline{f(y)}+g(x)+q(x,y)$ for $x,y\in N$, $f,g\in N^*$. The following remark is well known:

REMARK 2. Let (M,h) be a non degenerate hermitian left A-module. (M,h) is metabolic if and only if there is a totally isotropic A-submodule N which is a direct summand of M and satisfies $N^\perp=N$, where $N^\perp=\{m\in M;h(m,n)=0 \text{ for all } n\in N\}$. (The proof is similar to the case of a commutative ring, cf. [2], [3].)

For an $I\times I$-matrix H over A, a set $A^{(I)}\cdot H=\{(a_i)_{i\in I}\cdot H\in A^I;(a_i)_{i\in I}\in A^{(I)}\}$ is an A-submodule of A^I, and $H\cdot{}^tA^{(I)}=\{H\cdot{}^t(a_i)_{i\in I}\in{}^tA^I;{}^t(a_i)_{i\in I}\in{}^tA^{(I)}\}$ is an A-submodule of ${}^tA^I$.

2. Non degenerate projective hermitian left A-module.

LEMMA 1. *Let* $M=\sum_{i\in I}Am_i$ *be any left A-module with an arbitrary generator* $\{m_i\}_{i\in I}$. *Then we have the following A-isomorphism as right A-modules;*

$$\mathrm{Hom}_A(M, A) \longrightarrow \mathrm{Ann}({}^tA^I\,;\,\mathrm{Ann}(A^{(I)}\,;\,\{m_i\}_{i\in I}))\,;\, f \rightsquigarrow {}^t(f(m_i))_{i\in I}\,.$$

LEMMA 2. *Let* $M=\sum_{i\in I}Am_i$ *be a left A-module,* $h: M\times M\rightarrow A$ *any hermitian form and* $H=(h(m_i, m_j))_{(ij)\in I\times I}$. *Then* (M, h) *is non degenerate if and only if the following identities are verified*

1) $\mathrm{Ann}(A^{(I)}\,;\,\{m_i\}_{i\in I})=\mathrm{Ann}(A^{(I)}\,;\,H)$,

2) $\mathrm{Ann}({}^tA^I\,;\,\mathrm{Ann}(A^{(I)}\,;\,H))=H\cdot{}^tA^{(I)}$.

PROOF. (M, h) is non degenerate if and only if $\theta: M\rightarrow \mathrm{Hom}_A(M, A)\,;\, m\rightsquigarrow h(-, m)$ is bijective. We have that θ is injective if and only if $\mathrm{Ann}(A^{(I)}\,;\,\{m_i\}_{i\in I})=\mathrm{Ann}(A^{(I)}\,;\,H)$. On the other hand, from Lemma 1, θ is surjective if and only if $\mathrm{Ann}({}^tA^I\,;\,\mathrm{Ann}(A^{(I)}\,;\,\{m_i\}_{i\in I}))\subset H\cdot{}^tA^{(I)}$. Hence, θ is bijective if and only if $\mathrm{Ann}(A^{(I)}\,;\,\{m_i\}_{i\in I})=\mathrm{Ann}(A^{(I)}\,;\,H)$ and $\mathrm{Ann}({}^tA^I\,;\,\mathrm{Ann}(A^{(I)}\,;\,H))\subset H\cdot{}^tA^{(I)}$ are verified. However, in general, $H\cdot{}^tA^{(I)}\subset \mathrm{Ann}({}^tA^I\,;\,\mathrm{Ann}(A^{(I)}\,;\,H))$ is always verified. Therefore, we get this lemma.

LEMMA 3. *Let* $M=\sum_{i\in I}Am_i$ *be a left A-module,* $h: M\times M\rightarrow A$ *a hermitian form and* $H=(h(m_i, m_j))_{(ij)\in I\times I}$. *There is an A-homomorphism* $\xi: M\rightarrow A^{(I)}\cdot H\,;\, \sum_{i\in I}a_im_i \rightsquigarrow (a_i)_{i\in I}\cdot H$ *as left A-modules. If* (M, h) *is non degenerate, then* ξ *is bijective.*

PROOF. Since $\mathrm{Ann}(A^{(I)}\,;\,\{m_i\}_{i\in I})\subset\mathrm{Ann}(A^{(I)}\,;\,H)$ is always verified, there is an A-homomorphism $\xi: M\rightarrow A^{(I)}\cdot H\,;\,\sum_{i\in I}a_im_i\rightsquigarrow (a_i)_{i\in I}\cdot H$. If (M, h) is non degenerate, by Lemma 2, $\mathrm{Ann}(A^{(I)}\,;\,\{m_i\}_{i\in I})=\mathrm{Ann}(A^{(I)}\,;\,H)$, hence ξ is an A-isomorphism.

LEMMA 4. *Let* $M=\sum_{i\in I}Am_i$ *be a left A-module,* $h: M\times M\rightarrow A$ *a hermitian form, and* $H=(h(m_i, m_j))_{(ij)\in I\times I}$. *We assume that* (M, h) *is non degenerate. Then M is a projective A-module if and only if there is a column-finite* $I\times I$*-matrix K* $(\in A_{I)})$ *such that HK is row-finite, i.e.* $HK\in A_{(I}$, *and* $(HK)H=H$.

PROOF. Assume that M is a projective A-module. By Lemma 3, $A^{(I)}H$ is also a projective left A-module. Hence, a surjection $\lambda: A^{(I)}\rightarrow A^{(I)}H\,;\,(a_i)_{i\in I}\rightsquigarrow (a_i)_{i\in I}H$ is split. There is an A-homomorphism $\mu: A^{(I)}H\rightarrow A^{(I)}$ such that $\lambda\circ\mu$ is the identity map on $A^{(I)}H$. For each $j\in I$, let p_j be the j-projection of $A^{(I)}$ to A, i.e. $p_j((a_i)_{i\in I})=a_j$. The composition $p_j\circ\mu\circ\xi$ is in $\mathrm{Hom}_A(M, A)$. By the A-isomorphism $\mathrm{Hom}_A(M, A)\rightarrow\mathrm{Ann}({}^tA^I\,;\,\mathrm{Ann}(A^{(I)}\,;\,H))=H{}^tA^{(I)}: f\rightsquigarrow {}^t(f(m_i))_{i\in I}$ in Lemma 1, there is an element $(a_{ij})_{i\in I}$ in $A^{(I)}$ such that ${}^t(p_j\circ\mu\circ\xi(m_i))_{i\in I}=H{}^t(a_{ij})_{i\in I}$ for each $j\in I$. Put $K=(a_{ij})_{(ij)\in I\times I}$, then K is contained in $A_{I)}$, and we have $(p_j \circ \mu \circ \xi(m_i))_{(ij) \in I \times I} = HK$. Since $(p_j \circ \mu \circ \xi(m_i))_{j \in I} = \mu(\xi(m_i))$ is in $A^{(I)}$ for every $i \in I$, HK is contained in $A_{(I}$. By $\xi(m_i) = (h(m_i, m_j))_{j \in I}$ and $\lambda \circ \mu(\xi(m_i)) = \xi(m_i)$, we get $\xi(m_i) = \lambda \circ \mu(\xi(m_i)) = \lambda((p_j \circ \mu \circ \xi(m_i))_{j \in I}) = \lambda((h(m_i, m_j))_{j \in I} K) = ((h(m_i, m_j))_{j \in I} K) H$, and hence $H = (HK)H$. Conversely, assume that K is a matrix in $A_{I)}$ such that HK is in $A_{(I}$ and $H = (HK)H$. Now, we show that the surjection $\lambda : A^{(I)} \to A^{(I)}H$; $(a_i)_{i \in I} \rightsquigarrow (a_i)_{i \in I} H$ is split. By $K \in A_{I)}$ and $HK \in A_{(I}$ an A-homomorphism $\mu : A^{(I)}H \to A^{(I)}$; $(a_i)_{i \in I} H \rightsquigarrow ((a_i)_{i \in I} H) K = (a_i)_{i \in I}(HK)$ is well defined. Then we get $\lambda \circ \mu((a_i)_{i \in I} H) = \lambda((a_i)_{i \in I}(HK)) = ((a_i)_{i \in I}(HK))H = (a_i)_{i \in I} H$ for every $(a_i)_{i \in I} H \in A^{(I)}H$, that is, $\lambda \circ \mu$ is the identity map on $A^{(I)}H$. Hence $A^{(I)}H$ and also M are projective over A.

From the above lemmas we have

THEOREM 1. *Let* $M = \sum_{i \in I} Am_i$ *be a left A-module with a generator* $\{m_i\}_{i \in I}$, *and* $h : M \times M \to A$ *a hemitian form with an $I \times I$-matrix* $H = (h(m_i, m_j))_{(ij) \in I \times I}$ *with respect to* $\{m_i\}_{i \in I}$. *Then we have the following:*

1) *(M, h) is non degenerate and projective if and only if* $\mathrm{Ann}(A^{(I)}; \{m_i\}_{i \in I}) = \mathrm{Ann}(A^{(I)}; H)$, $\mathrm{Ann}({}^tA^I; \mathrm{Ann}(A^{(I)}; H)) = H\,{}^tA^{(I)}$ *and there is K in $A_{I)}$ such that $HK \in A_{(I}$ and $H = (HK)H$.*

2) *If (M, h) is non degenerate then a map* $\xi : M \to A^{(I)}H$; $\sum_{i \in I} a_i m_i \rightsquigarrow (a_i)_{i \in I} H$ *is an A-isomorphism as left A-modules.*

LEMMA 5. *Let H be an $I \times I$-matrix over A. If there is a K in $A_{I)}$ such that $HK \in A_{(I}$ and $H = (HK)H$, then* $\mathrm{Ann}(A^{(I)}; H) = A^{(I)}(E - HK)$ *and* $\mathrm{Ann}({}^tA^I; \mathrm{Ann}(A^{(I)}; H)) = (HK)\,{}^tA^I$, *where E denotes the unit $I \times I$-matrix.*

PROOF. If $(a_i)_{i \in I}$ is an element in $A^{(I)}$ such that $(a_i)_{i \in I} H = 0$, then $(a_i)_{i \in I} = (a_i)_{i \in I} - ((a_i)_{i \in I} H) K = (a_i)_{i \in I}(E - HK)$, and so $\mathrm{Ann}(A^{(I)}; H) \subset A^{(I)}(E - HK)$. The converse inclusion is obtained from $(E - HK)H = H - (HK)H = 0$. Therefore, we have also $\mathrm{Ann}({}^tA^I; \mathrm{Ann}(A^{(I)}; H)) = \mathrm{Ann}({}^tA^I; A^{(I)}(E - HK))$. If ${}^t(b_i)_{i \in I} \in \mathrm{Ann}({}^tA^I; \mathrm{Ann}(A^{(I)}; H))$, then ${}^t(b_i)_{i \in I} = (HK)\,{}^t(b_i)_{i \in I} + (E - HK)\,{}^t(b_i)_{i \in I} = (HK)\,{}^t(b_i)_{i \in I} \in (HK)\,{}^tA^I$. Conversely, for any $(HK)\,{}^t(b_i)_{i \in I} \in (HK)\,{}^tA^I$ we have $(E - HK)((HK)\,{}^t(b_i)_{i \in I}) = (HK)\,{}^t(b_i)_{i \in I} - (HK)((HK)\,{}^t(b_i)_{i \in I}) = 0$, because of $(HK)((HK)\,{}^t(b_i)_{i \in I}) = ((HK)(HK))\,{}^t(b_i)_{i \in I} = (((HK)H)K)\,{}^t(b_i)_{i \in I} = (HK)\,{}^t(b_i)_{i \in I}$.

COROLLARY 1. *Let* $M = \sum_{i \in I} Am_i$ *be a left A-module,* $h : M \times M \to A$ *a hermitian form and* $H = (h(m_i, m_j))_{(ij) \in I \times I}$. *Then (M, h) is non degenerate and projective if and only if* $\mathrm{Ann}(A^{(I)}; \{m_i\}_{i \in I}) = \mathrm{Ann}(A^{(I)}; H)$ *and there is a K in $A_{I)}$ such that $HK \in A_{(I}$, $H = (HK)H$ and $(HK)\,{}^tA^I = H\,{}^tA^{(I)}$.*

LEMMA 6. *Let H be an $I \times I$-matrix contained in $A_{(I)}$. If there is $K \in A_{I)}$ such that $HK \in A_{(I}$ and $(HK)H = H$, then* $\mathrm{Ann}({}^tA^{(I)}; \mathrm{Ann}(A^{(I)}; H)) = H\,{}^tA^{(I)}$.

PROOF. By $HKH = H$, we have $(HK - E)H = 0$, and so every row of $(HK - E)$ is contained in $\mathrm{Ann}(A^{(I)}; H)$. Hence, ${}^t(a_i)_{i \in I} \in \mathrm{Ann}({}^tA^{(I)}; \mathrm{Ann}(A^{(I)}; H))$ implies $(HK - E)\,{}^t(a_i)_{i \in I} = 0$ and ${}^t(a_i)_{i \in I} = (HK)\,{}^t(a_i)_{i \in I} = H(K\,{}^t(a_i)_{i \in I}) \in H\,{}^tA^{(I)}$. Hence,

$\mathrm{Ann}({}^tA^{(I)}; \mathrm{Ann}(A^{(I)}; H)) \subset H^tA^{(I)}$, and the converse is always verified.

LEMMA 7. *Let A be a semisimple artinian ring and (M, h) a non degenerate hermitian left A-module. Then there exists a generator $\{x_i\}_{i\in I}$ of M as left A-module such that the $I\times I$-matrix $H=(h(x_i, x_j))_{(ij)\in I\times I}$ is rowfinite and columnfinite, i.e. $H\in A_{(I)}$. For such a generator $\{x_i\}_{i\in I}$, every f in $\mathrm{Hom}_A(M, A)$ has $f(x_i)=0$ for almost all $i\in I$.*

PROOF. Since A is semisimple artinian, M is projective over A, hence there are $\{f_i\}_{i\in I}$ in $\mathrm{Hom}_A(M, A)$ and $\{y_i\}_{i\in I}$ in M such that for each $x\in M$, $f_i(x)=0$ for almost all $i\in I$ and $x=\sum_{i\in I} f_i(x)y_i$. Since (M, h) is non degenerate, there is a system of elements $\{x_i\}_{i\in I}$ in M such that $f_i=h(-, x_i)$ for $i\in I$. Then $H=(h(x_i, x_j))_{(ij)\in I\times I}$ is a row-finite and column-finite $I\times I$-matrix. Put $N=\sum_{i\in I} Ax_i$, then we have $N^{\perp}=0$. Since A is semisimple artinian, N is a direct summand of M, hence $N=(N^{\perp})^{\perp}=M$.

THEOREM 2.

1) *Let (M, h) be a hermitian left A-module. Assume that $M=\sum_{i\in I} Am_i$ with $m_i\neq 0$ for $i\in I$ and $H=(h(m_i, m_j))_{(ij)\in I\times I}$ is contained in $A_{(I)}$. (M, h) is non degenerate and projective if and only if I is a finite set, in consequence M is finitely generated, $\mathrm{Ann}(A^I; \{m_i\}_{i\in I})=\mathrm{Ann}(A^I; H)$ and there is K in A_I such that $HKH=H$.*

2) *If A is a semisimple artinian ring and (P, h) a non degenerate hermitian left A-module, then P is a finitely generated A-module.*

3) *Let $H=(h_{ij})$ be a hermitian $n\times n$-matrix in A_n. Then a hermitian left A-module (A^nH, h) is defined by $h: A^nH\times A^nH\to A$; $((a_i)_{i\in I}H, (b_i)_{i\in I}H) \rightsquigarrow \sum_{i,j\in I} a_ih_{ij}\bar{b}_j$, where $I=\{1, 2, \cdots, n\}$. Then (A^nH, h) is non degenerate and projective if and only if H is a von Neumann regular element in A_n.*

PROOF. 1): Suppose that (M, h) is non degenerate and projective. From Corollary 1, there is K in $A_{I)}$ such that $HK\in A_{(I}$, $(HK)H=H$ and $HK^tA^I=H^tA^{(I)}$. Since H is in $A_{(I)}$, $(HK)^tA^I=H^tA^{(I)}$ is contained in $A^{(I)}$, and so HK is in $A_{(I)}$. Hence $(HK)^tA^I\subset A^{(I)}$ implies that almost all row-vectors of HK are a zero vector. Namely, $H=(HK)H$ has only a finite number of non-zero row-vectors. Since (M, h) is non degenerate and $m_i\neq 0$ for $i\in I$, I is a finite set. The other conditions are obtained from Theorem 1. By Lemma 6 and Theorem 1, the converse is obvious. 2): By Lemma 7, (P, h) has a generator $\{x_i\}_{i\in I}$ such that $H=(h(x_i, x_j))_{(ij)\in I\times I}$ is in $A_{(I)}$. Hence from 1), P is a finitely generated A-module. 3): Let $H=(h_{ij})$ be a hermitian matrix in A_n, and put $I=\{1, 2, \cdots, n\}$. A hermitian form $h: A^IH\times A^IH\to A$; $((a_i)_{i\in I}H, (b_i)_{i\in I}H) \rightsquigarrow \sum_{i,j\in I} a_ih_{ij}\bar{b}_j$ is well defined. Because, if $(a_i)_{i\in I}H=0$ or $(b_i)_{i\in I}H=0$ then $\sum_{i,j\in I} a_ih_{ij}\bar{b}_j =((a_i)_{i\in I}H)^t(\bar{b}_i)_{i\in I}=(a_i)_{i\in I}((b_i)_{i\in I}H)^*=0$. A^IH has a generator $\{(h_{ij})_{j\in I}; i=1, 2,$

$\cdots, n\}$, and the matrix of $(A^I H, h)$ with respect to this generator is $(h((h_{ik})_{kI}, (h_{jk})_{k\in I}))_{(ij)\in I\times I}=(h_{ij})_{(ij)\in I\times I}=H$. By 1), $(A^I H, h)$ is non degenerate and projective if and only if there is $K\in A_n$ such that $HKH=H$.

DEFINITION. Let H be an $n\times n$-matrix in A_n. We call H a regular hermitian matrix, if H is a hermitian matrix and von Neumann regular element in A_n. For a regular hermitian matrix H, by $\langle H\rangle$ we denote the hermitian left A-module $(A^n H, h)$ defined in 3) of Theorem 2.

As is well known, we have

LEMMA 8. *Let (M, h) be a non degenerate hermitian left A-module. If N is an A-submodule of M such that $(N, h|N)$ is non degenerate, N is an orthogonal direct summand of M, i. e. $M=N\oplus N^{\perp}$.* (The proof of this lemma is similar to the case over a commutative ring, cf. [3].)

COROLLARY 2. *Every non degenerate and finitely generated projective hermitian left A-module is isomorphic to $\langle H\rangle$ defined by a regular hemitian matrix. If (M, h) is a non degenerate hermitian left A-module and $N=\sum_{i=1}^{r} An_i$ is A-submodule of M such that the matrix $H=(h(n_i, n_j))$ is regular hermitian matrix of degree r, then there is an A-submodule $N'=\sum_{i=1}^{r} An'_i$ of N such that $h(n_i, n_j)=h(n'_i, n'_j)$; $i, j=1, 2, \cdots, r$, and $(N', h|N')$ is non degenerate and projective. Furthermore, it satisfies $N=N'\oplus(N\cap N'^{\perp})$, $(N', h|N')\cong\langle H\rangle$, $M=N'\oplus N'^{\perp}$, and $(N'^{\perp}, h|N'^{\perp})$ is also non degenerate.*

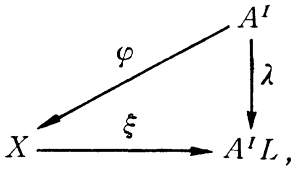
THEOREM 3. *Let H and H' be regular hermitian matrices of degree n and n' respectively. Then $\langle H\rangle\cong\langle H'\rangle$ if and only if there exist matrices L and L' such that $H=LH'L^*$, $LL'H=H$ and $L'LH'=H'$.*

PROOF. This is easily obtained by the definitions of $\langle H\rangle$ and $\langle H'\rangle$.

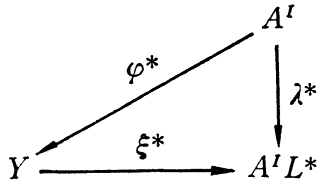
COROLLARY 3. 1) *Let a and b be von Neumann regular elements in A such that $\bar{a}=a$ and $\bar{b}=b$. $\langle a\rangle\cong\langle b\rangle$ if and only if there are $c, d\in A$ such that $cda=a$, $dcb=b$ and $a=cb\bar{c}$.* 2) *Let (Am, h) be a hermitian cyclic left A-module with $a=h(m, m)\in A$. (Am, h) is non degenerate and projective if and only if a is a von Neumann regular element of A satisfying that for any $c\in A$, $ca=0$ implies $cm=0$. If (Am, h) is non degenerate then $(Am, h)\cong\langle a\rangle$.* 3) *If H_1 and H_2 are regular hermitian matrices, then $\langle H_1\rangle\perp\langle H_2\rangle\cong\left\langle\begin{pmatrix}H_1 & 0\\ 0 & H_2\end{pmatrix}\right\rangle$.*

THEOREM 4. *Let (M, h) be a hermitian left A-module. Assume that $X=\sum_{i=1}^{n} Ax_i$ is a totally isotropic A-submodule of (M, h) such that there is an A-submodule $Y=\sum_{i=1}^{n} Ay_i$ of M with a von Neumann regular matrix $L=(h(x_i, y_j))$ in A_n. Then there are A-submodules $X'=\sum_{i=1}^{n} Ax'_i$ and $Y'=\sum_{i=1}^{n} Ay'_i$ of M such that X' and Y' are direct summands of X and Y, respectively, $(X'+Y', h|X'+Y')$ is* *a non degenerate projective and metabolic submodule and* $h(x_i, y_j)=h(x'_i, y'_j)$ *for* $i, j=1, 2, \cdots, n$. *Furthermore,* X *is expressed as* $X=X'\oplus((X'+Y')^{\perp}\cap X)$.

PROOF. Put $I=\{1, 2, \cdots, n\}$. From the fact that $\mathrm{Ann}(A^I ; \{x_i\}_{i\in I})$ is contained in $\mathrm{Ann}(A^I ; L)$, an A-homomorphism $\xi ; X=\sum_{i=1}^{n} Ax_i \rightarrow A^I L : \sum_{i=1}^{n} a_i x_i \rightsquigarrow (a_i)_{i\in I}L$ is well defined, and makes the following diagram commute;



where $\varphi : A^I \rightarrow X ; (a_i)_{i\in I} \rightsquigarrow \sum_{i=1}^{n} a_i x_i$ and $\lambda : A^I \rightarrow A^I L ; (a_i)_{i\in I} \rightsquigarrow (a_i)_{i\in I}L$ are natural A-homomorphisms. Since L is a regular element in A_n, there is a matrix K in A_n such that $LKL=L$. For an A-homomorphism $\mu : A^I L \rightarrow A^I$; $(a_i)_{i\in I}L \rightsquigarrow (a_i)_{i\in I}LK$, the composition $\lambda\circ\mu$ is the identity map on $A^I L$. Put $\phi=\varphi\circ\mu$, $X'=\mathrm{Im}\,\phi$ and $X''=\mathrm{Ker}\,\xi$. Then we get $\xi\circ\phi=\xi\circ\varphi\circ\mu=\lambda\circ\mu$, and so $X=X'\oplus X''$. If we put $x'_i=\phi\circ\xi(x_i)$ in X' and $x''_i=x_i-x'_i$ for $i=1, 2, \cdots, n$, then we have $X'=\sum_{i=1}^{n} Ax'_i$, $X''=\sum_{i=1}^{n} Ax''_i$ and $h(x_i, y_j)=h(x'_i, y_j)$ for $i, j=1, 2, \cdots, n$. Therefore, $X''\subset(X+Y)^{\perp}\cap X$ and $L=(h(x'_i, y_j))$. On the other hand, since we have $L^*K^*L^*=L^*$ and $\mathrm{Ann}(A^I ; \{y_i\}_{i\in I})\subset\mathrm{Ann}(A^I ; L^*)$ for $L^*=(h(y_i, x'_j))$, we can define an A-homomorphism $\xi^* : Y=\sum_{i=1}^{n} Ay_i \rightarrow A^I L^* ; \sum_{i=1}^{n} a_i y_i \rightsquigarrow (a_i)_{i\in I}L^*$ making the following diagram commute;



where $\varphi^* : A^I \rightarrow Y ; (a_i)_{i\in I} \rightsquigarrow \sum_{i=1}^{n} a_i y_i$ and $\lambda^* : Y \rightarrow A^I L^* ; (a_i)_{i\in I} \rightsquigarrow (a_i)_{i\in I}L^*$ are natural A-homomorphisms. As well as the above argument, for an A-homomorphism $\mu^* : A^I L^* \rightarrow A^I$; $(a_i)_{i\in I}L^* \rightsquigarrow (a_i)_{i\in I}L^*K^*$, the composition $\lambda^*\circ\mu^*$ is the identity map on $A^I L^*$. Put $\phi^*=\varphi^*\circ\mu^*$, $Y'=\mathrm{Im}\,\phi^*$, $Y''=\mathrm{Ker}\,\xi^*$, and $y'_i=\phi^*\circ\xi^*(y_i)$, $y''_i=y_i-y'_i$ for $i=1, 2, \cdots, n$. Then we have $Y=Y'\oplus Y''$, $Y'=\sum_{i=1}^{n} Ay'_i$ and $L=(h(x'_i, y'_j))$. Furthermore, the restriction $\xi^*|Y'$ of ξ^* within Y' is an A-isomorphism with the inverse map ϕ^*. Put $B=(h(y'_i, y'_j))$ in A_n and $H=\begin{pmatrix} 0 & L \\ L^* & B \end{pmatrix}$

in A_{2n}. Then we can easily check $L^*K^*B=B$, because of $h(y'_i, y'_j)=h(\psi^*\circ\xi^*(y_i), y'_j)=h(\varphi^*((\delta_{ik})_{k\in I}L^*K^*), y'_j)=(\delta_{ik})_{k\in I}L^*K^{*t}(h(y'_k, y'_j))_{k\in I}$, ($\delta_{ij}$ denotes the Kronecker's delta). Put $F=\begin{pmatrix}-K^*BK & K^*\\ K & 0\end{pmatrix}$. Then we have $HFH=H$, that is, H is a regular hermitian matrix in A_{2n}. Now we shall show $\mathrm{Ann}(A^{2n}; \{x'_1, \cdots, x'_n, y'_1, \cdots, y'_n\})=\mathrm{Ann}(A^{2n}; H)$. Suppose $((a_i)_{i\in I}, (b_i)_{i\in I})\in\mathrm{Ann}(A^{2n}; H)$. By $0=((a_i)_{i\in I}, (b_i)_{i\in I})H=((b_i)_{i\in I}L^*, (a_i)_{i\in I}L+(b_i)_{i\in I}B)$, we have $(b_i)_{i\in I}L^*=0$ and $(a_i)_{i\in I}L=-(b_i)_{i\in I}B$. Hence we have $\sum_{i=1}^{n}a_ix'_i+\sum_{i=1}^{n}b_iy'_i=\psi((a_i)_{i\in I})+\psi^*((b_i)_{i\in I}L^*)=\psi(-(b_i)_{i\in I}B)=\psi(-((b_i)_{i\in I}L^*)K^*B)=0$. The converse is always satisfied. Accordingly, by Corollary 2, $(X'+Y', h|X'+Y')$ is non degenerate and projective. From $X=X'\oplus Y'$ and $X''\subset(X'+Y')^\perp$, we get $X=X'\oplus(X'+Y')^\perp\cap X$. Now we shall show that $(X'+Y', h|X'+Y')$ is metabolic. Put $Q=X'+Y'$. By Remark 2, if $X'^\perp\cap Q=X'$ and $Q=X'\oplus Y'$, then $(Q, h|Q)$ is metabolic. In order to show $X'^\perp\cap Q=X'$ and $Q=X'\oplus Y'$ it suffices to prove $X'^\perp\subset Y'=0$, because of $X'\subset X'^\perp$. Suppose that $x=\sum_{i=1}^{n}a_iy'_i$ is any element of $X'^\perp\cap Y'$. Then we have $x=\sum_{i=1}^{n}a_iy'_i=\psi^*((a_i)_{i\in I}L^*)=\psi^*((\sum_{i=1}^{n}a_ih(y'_i, x_j))_{j\in I})=\psi^*((h(x, x'_j))_{j\in I})=0$. The proof is concluded.

COROLLARY 4.

1) *Let A be a von Neumann regular ring, and (M, h) a non degenerate hermitian left module. If M is finitely generated as A-module, then M is projective. If $X=\sum_{i=1}^{n}Ax_i$ is a totally isotropic submodule of (M, h), then for any elements $y_1, y_2, \cdots, y_n$, there is a non degenerate projective and metabolic submodule $\sum_{i=1}^{n}Ax'_i+\sum_{i=1}^{n}Ay'_i$ of $X+\sum_{i=1}^{n}Ay_i$ with $h(x_i, y_j)=h(x'_i, y'_j)$ for $i, j=1, 2, \cdots, n$. If $X=\sum_{i=1}^{n}Ax_i$ is any submodule of M with the matrix $L=(h(x_i, x_j))$, then there is a non degenerate and projective hermitian submodule $X'=\sum_{i=1}^{n}Ax'_i$ of (M, h) such that $L=(h(x'_i, x'_j))$ and $X'\subset X$.*

2) *Let A be a semisimple artinian ring. Then any non degenerate hermitian left A-module is projective and finitely generated, and has a form $\langle a_1\rangle\perp\langle a_2\rangle\perp\cdots\perp\langle a_r\rangle\perp\left\langle\begin{pmatrix}0 & b_1\\ \bar{b}_1 & 0\end{pmatrix}\right\rangle\perp\cdots\perp\left\langle\begin{pmatrix}0 & b_t\\ \bar{b}_t & 0\end{pmatrix}\right\rangle$.*

THEOREM 5. *Let H be a regular hermitian matrix in A_n. Then $\langle H\rangle$ is metabolic if and only if there is a matrix L in A_n such that $L^2=L$, $LHL^*=0$ and $A^nH\cap A^nH(E-L^*)=A^nLH$. Furthermore, $\langle H\rangle$ is hyperbolic if and only if there is a matrix L in A_n such that $L^2=L$, $LHL^*=0$ and $LH=H(E-L^*)$. (E denotes the unit matrix in A_n.)*

PROOF. Suppose $\langle H\rangle$ is metabolic (resp. hyperbolic). Then there is a projective left A-module M such that $(M\oplus M^*, h')\xrightarrow[\cong]{\pi}\langle H\rangle=(A^nH, h)$, where

$M^*=\mathrm{Hom}_A(M, A)$ and $(M\oplus M^*, h')$ is a hermitian module with $h'(M^*, M^*)=0$ and $h'(m, f)=f(m)$ for $f\in M^*$, $m\in M$. Put $A^nH=\sum_{i=1}^{n} Au_i$ and $H=(h(u_i, u_j))$. Since A^nH is projective over A, we can choose $\varphi_1, \varphi_2, \cdots, \varphi_n$ in $\mathrm{Hom}_A(A^nH, A)$ such that every element $x\in A^nH$ is expressed by $x=\sum_{i=1}^{n}\varphi_i(x)u_i$. Put $\pi^{-1}(u_i)=m_i+f_i$ by $m_i\in M$ and $f_i\in M^*$ for $i=1, 2, \cdots, n$. Then $m_1, m_2, \cdots, m_n$ and $f_1, f_2, \cdots, f_n$ generate M and M^* as A-modules, respectively. Since a map $M^*\to A$; $f \rightsquigarrow \varphi_i(\pi(f))$ is an A-homomorphism, there is an element m_i' in M determined by an A-isomorphism $M\to\mathrm{Hom}_A(M^*, A)$; $m \rightsquigarrow [f \rightsquigarrow \overline{f(m)}]$, hence $\varphi_i(\pi(f))=\overline{f(m_i')}$ for $i=1, 2, \cdots, n$. Then we have $\pi(f)=\sum_{i=1}^{n}\varphi_i(\pi(f))u_i=\sum_{i=1}^{n}\overline{f(m_i')}u_i$ and so $f=\sum_{i=1}^{n}\overline{f(m_i')}f_i$ for all $f\in M^*$. Put $L=(\overline{f_i(m_j')})$ in A_n. Then we get $L^2=L$, $\pi(M^*)=A^nLH$ and $\pi(M)=A^n(E-L)H$. Because, by $\overline{f_i(m_j')}=\overline{(\sum_{k=1}^{n}\overline{f_i(m_k')}f_k)(m_j')}=\sum_{k=1}^{n}\overline{f_k(m_j')f_i(m_k')}=\sum_{k=1}^{n}\overline{f_i(m_k')}\,\overline{f_k(m_j')}$, we have $L^2=L$. From $\pi(f_i)=\sum_{j=1}^{n}f_i(m_j')u_j$ and $\pi(m_i')=u_i-\pi(f_i)$, it follows that $\pi(M^*)=\sum_{i=1}^{n}A\pi(f_i)=A^nLH$ and $\pi(M^*)=\sum_{i=1}^{n}A(m_i)=A^n(E-L)H$. Since M^* is totally isotropic, we get $LHL^*=0$. Now, we make mention of the following observation: Under the assumption $L^2=L$ and $LHL^*=0$, $(A^nLH)^{\perp}=A^nLH$ in $\langle H\rangle$ and $A^nH=A^nLH\oplus A^n(E-L)H$ are verified if and only if $A^nH\cap A^n(E-L)H=A^nLH$. Because, $LHL^*=0$ implies that A^nLH is totally isotropic. Hence, $(A^nLH)^{\perp}=A^nLH$ and $A^nH=A^nLH\oplus A^n(E-L)H$ are verified if and only if $(A^nLH)^{\perp}\cap A^n(E-L)H=0$. However, $(A^nLH)^{\perp}\cap A^n(E-L)H=0$ if and only if $\mathrm{Ann}(A^n; HL^*)$ is contained in $\mathrm{Ann}(A^n; (E-L)H)$, that is, $A^nH\cap A^n(E-L^*)\subset A^nLH$. Since by $LHL^*=0$, $A^nLH\subset A^nH\cap A^n(E-L^*)$, we get the observation. Hence we conclude that $\langle H\rangle$ is metabolic if and only if there is $L\in A_n$ such that $L^2=L$, $LHL^*=0$ and $A^nH\cap A^n(E-L^*)=A^nLH$. Furthermore, under the assumption $L^2=L$ and $LHL^*=0$, we have that $(A^nLH)^{\perp}=A^nLH$, $(A^n(E-L)H)^{\perp}=A^n(E-L)H$ and $A^nH=A^nLH\oplus A^n(E-L)H$ are verified if and only if $LH=H(E-L^*)$. Because, we have easily that $LH=H(E-L^*)$ implies $A^nH\cap A^n(E-L^*)=A^nLH$, and $A^n(E-L)H$ is totally isotropic if and only if $(E-L)H(E-L^*)=0$. By $(E-L)H(E-L^*)=H(E-L^*)-LH$, we get that $\langle H\rangle$ is hyperbolic if and only if there is $L\in A_n$ such that $L^2=L$, $LHL^*=0$ and $LH=H(E-L^*)$.

3. The orthogonal group of $\langle H\rangle$.

Let H be a regular hermitian matrix in A_n. In this section we consider the orthogonal group of $\langle H\rangle$. By $O(\langle H\rangle)$ we denote the orthogonal group of $\langle H\rangle$. Let K be a matrix in A_n such that $HKH=H$. Put $O(H)=\{L\in A_nHK;\ LHL^*=H$ and there exists $L'\in A_nHK$ such that $LL'=L'L=HK\}$.

Then we have

THEOREM 6. *Let H, K and $O(H)$ be as above. Then $O(H)$ is a multiplicative group, and $O(\langle H\rangle)\rightarrow O(H)$; $f \leadsto L_f-E$ is a group isomorphism, where $L_g=(a_{ij})$ is a matrix in A_nHK determined by $g((\delta_{ij})_{j\in I}H)=(a_{ij})_{j\in I}$ for $i\in I=\{1,2,\cdots,n\}$, $g\in O(\langle H\rangle)$, (δ_{ij} is the Kronecker's delta).*

PROOF. Let $f:\langle H\rangle\rightarrow\langle H\rangle$ be an isometry. A matrix L_f in A_nHK is uniquely determined by $L_f=(a_{ij})_{(ij)\in I\times I}$ and $f((\delta_{ij})_{j\in I}H)=(a_{ij})_{j\in I}H$ for $i\in I=\{1,\cdots,n\}$, where δ_{ij} is Kronecker's delta. Let us denote it by $f(H)=L_fH$. If $f(H)=L_fH=L'_fH$ and L_f, $L'_f\in A_nHK$, then we have $L_f=L_fHK=L'_fHK=L'_f$. Hence the map $O(\langle H\rangle)\rightarrow O(H)$; $f \leadsto L_f-E$ is well defined and is a bijection because of Theorem 4. For f and g in $O(\langle H\rangle)$, we have $f\cdot g(H)=f(L_gH)=L_gf(H)=L_gL_fH$, hence $L_{(fg)}-E=(L_f-E)(L_g-E)$. Therefore, the map $O(\langle H\rangle)\rightarrow O(H)$ is a group isomorphism.

Now, we will characterize the group $O(\langle H\rangle)$ as a subgroup of the unit group of a ring. Let Λ be a ring with an involution $\Lambda\rightarrow\Lambda$; $a \leadsto a^*$. An element h in Λ will be called a hermitian element if it satisfies $h=h^*$. Let h be a hermitian element which is a regular element in Λ, i.e. $h=h^*$, there is $k\in\Lambda$ such that $hkh=h$. Then we can choose k which satisfies $k^*=k$ and $khk=k$. Because, the h and k satisfy $hk^*h=h$, $hkhk^*h=h$ and $khk^*hkhk^*=khk^*$, hence we may take khk^* as k. Put $e=hk$, then e is an idempotent and satisfies $eh=h=he^*$, $e^*k=k=ke$ and $e^*=kh$. Put $O(h)=\{a\in\Lambda e\,;\ aha^*=h,\ ^{\exists}a'\in\Lambda e\,;\ a'a=aa'=e\}$.

COROLLARY 5.

1) *$O(h)$ is a group with the unit element e.*

2) *For any element $a\in O(h)$, the inverse element is given by ha^*k.*

3) *$O(h)=\{a\in U(e\Lambda e)\,;\ aha^*=h\}=\{a\in e\Lambda e\,;\ aha^*=h$ and $a^*ka=k\}=\{a\in U(e\Lambda e)\,;\ a(ha^*k)=e\}$, where $U(e\Lambda e)$ denotes the unit group of ring $e\Lambda e$ with identity element e.*

PROOF. 1) is obtained from the definition of $O(h)$. 2) If a is in $e\Lambda e$ and satisfies $aha^*=h$, then $a(ha^*k)=(aha^*)k=hk=e$. 3) is obvious from 1) and 2).

Appendix.

We consider the following statement;

(A): If (M,h) is a non degenerate projective hermitian left A-module, then M is finitely generated over A.

Before this paper is revised, Dr. Naomasa Maruyama pointed out that there is an error in the author's proof of this statement (A). We prove here the statement for some ring A, but, in general, the author can not prove. The author wishes to thank Dr. N. Maruyama for reading this paper and useful advice.

LEMMA B. *Let A be any ring, and J the Jacobson radical of A. Assume that M is a projective left A-module such that M/JM is finitely generated over*

A. If A is either a non commutative left noetherian ring or a commutative ring, then M is finitely generated over A.

PROOF. If M is a free A-module, it is easy that if M/JM is finitely generated over A then so is M. First, suppose that A is a commutative ring and M/JM is finitely generated over A, i. e. there are $x_1, x_2, \cdots, x_n$ in M such that $M=\sum_{i=1}^{n} Ax_i+JM$. Since $M=\sum_{i=1}^{n} Ax_i+\mathfrak{m}M$ for any maximal ideal $\mathfrak{m}$ of A, the localization $M_{\mathfrak{m}}$ is finitely generated over $A_{\mathfrak{m}}$. By Nakayama's Lemma, $M_{\mathfrak{m}}=\sum_{i=1}^{n} A_{\mathfrak{m}}x_i\otimes 1+\mathfrak{m}A_{\mathfrak{m}}M_{\mathfrak{m}}$ implies $M_{\mathfrak{m}}=\sum_{i=1}^{n} A_{\mathfrak{m}}x_i\otimes 1$. Therefore, we get $M=\sum_{i=1}^{n} Ax_i$. Next, suppose that A is a non commutative left noetherian ring. By using the method of proof of Proposition 2.7 in [**4**], we have the following. Let $M\oplus N=F=\sum_{i\in I}\oplus Av_i$ be a free A-module with a free basis $\{v_i\,;\,i\in I\}$. Let q be the projection of M to N, and put $v_i=m_i+n_i$ with $m_i\in M$, $n_i\in N$ for $i\in I$. Since M/JM is finitely generated, there are $v_1, v_2, \cdots, v_t$ such that, for every $i\in I$, m_i is expressed as $m_i=\sum_{j=1}^{t} a_{ij}v_j+\sum_{j\in I'} b_{ij}v_j$ with $a_{ij}\in A$ and $b_{ij}\in J$, where $I'=I-\{1, 2, \cdots, t\}$. For any finite number of indices $j_1, j_2, \cdots, j_n\in I'$, $n_{j_1}, n_{j_2}, \cdots, n_{j_n}$ are linearly independent over A, because an $n\times n$-matrix $E-(b_{j_ij_k})$ is invertible in the $n\times n$-matrix ring A_n. Therefore $\sum_{j\in I'} An_j$ is an A-free module with a free basis $\{n_j\,;\,j\in I'\}$. Since $(\sum_{i=1}^{t} An_i)\cap(\sum_{j\in I'} An_j)$ is finitely generated over A, there are $t+1$, $t+2, \cdots, s$ in I' such that $(\sum_{i=1}^{t} An_i)\cap(\sum_{j\in I'} An_j)\subset\sum_{k=t+1}^{s} An_k$. For any $x\in M$, x is expressed as $x=\sum_{i=1}^{t} a_iv_i+\sum_{j\in I'} a_jv_j$ by the free basis $\{v_i\,;\,i\in I\}$, and $0=q(x)=\sum_{i=1}^{t} a_in_i+\sum_{j\in I'} a_jn_j$, hence $\sum_{j\in I'} a_jn_j=-\sum_{i=1}^{t} a_in_i$ is contained in $\sum_{k=t+1}^{s} An_k$. By the linearly independency of $\{n_j\,;\,j\in I'\}$ we get $x=\sum_{i=1}^{s} a_iv_i\in\sum_{i=1}^{s} Av_i$. Therefore, $M\subset\sum_{i=1}^{s} Av_i$ and so M is finitely generated over A.

LEMMA C. *Let A a ring with an involution $A\to A$; $a \rightsquigarrow \bar{a}$, and $\mathfrak{a}$ an ideal of A such that $\bar{\mathfrak{a}}=\mathfrak{a}$. If (M, h) is a non degenerate projective hermitian left A-module, then $(M/\mathfrak{a}M, h)$ is also a non degenerate projective hermitian left $A/\mathfrak{a}$-module, where $h: M/\mathfrak{a}M\times M/\mathfrak{a}M\to A/\mathfrak{a}$ is defined by $h([m], [n])=[h(m, n)]$ for $[m], [n]\in M/\mathfrak{a}M$, ([ ] denotes a residue class).*

PROOF. $M/\mathfrak{a}M$ is obviously projective over $A/\mathfrak{a}$. We show that $\tilde{\theta}: M/\mathfrak{a}M\to\mathrm{Hom}_{A/\mathfrak{a}}(M/\mathfrak{a}M, A/\mathfrak{a})$; $[m]\rightsquigarrow h(-, [m])$ is a bijection. Since (M, h) is non-degenerate projective, there are $\{x_i, y_i\,;\,i\in I\}$ in M such that any $x\in M$ is expressed as $x=\sum_{i\in I} h(x, y_i)x_i$, where $h(x, y_i)=0$ for almost all $i\in I$. If $[m]$ is in $\mathrm{Ker}\,\tilde{\theta}$, then, for any $x\in M$, $h(x, m)$ is contained in $\mathfrak{a}$, hence $m=\sum_{i\in I} h(m, y_i)x_i=\sum_{i\in I}\overline{h(y_i, m)}x_i$ is contained in $\bar{\mathfrak{a}}M=\mathfrak{a}M$, i.e. $[m]=0$. Therefore, $\tilde{\theta}$ is injective. For any $f\in\mathrm{Hom}_{A/\mathfrak{a}}(M/\mathfrak{a}M, A/\mathfrak{a})$, there is an A-homomorphism $f: M\to A$ such that

$f([x])=[f(x)]$ for every $x\in M$, because M is projective over A. Hence, there is $y\in M$ such that $f=h(-,y)$, and $h([x],[y])=[h(x,y)]=[f(x)]=f([x])$ for any $x\in M$, i.e. $\tilde{\theta}([y])=f$. Therefore, $\tilde{\theta}$ is surjective.

THEOREM D. *If A is either a commutative ring or a non commutative left noetherian ring such that A/J is a semisimple artinian ring, then the statement (A) is true.*

PROOF. First, suppose that A is a commutative ring. Let (M,h) be a non degenerate projective hermitian left A-module. Then there are $x_i\in M$ and $f_i\in M^*=\mathrm{Hom}_A(M,A)$ with $i\in I$ such that, for each $x\in M$, $f_i(x)=0$ for almost all $i\in I$ and $x=\sum_{i\in I}f_i(x)x_i$, we shall say this system $\{x_i,f_i\,;i\in I\}$ "a projective system of M over A". Put $L=\sum_{i\in I}Af_i$. For any maximal ideal $\mathfrak{m}$ of A, $\mathfrak{n}=\mathfrak{m}\cap\bar{\mathfrak{m}}$ satisfies $\mathfrak{n}=\bar{\mathfrak{n}}$ and $A/\mathfrak{n}$ is a semisimple artinian ring. By Lemma C and Theorem 2, $M/\mathfrak{n}M$ is finitely generated over $A/\mathfrak{n}$. Hence $M/\mathfrak{m}M$ is also finitely generated over $A/\mathfrak{m}$. By $M/\mathfrak{m}M=M\otimes_A A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}=M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}}$, $M_{\mathfrak{m}}$ is finitely generated over $A_{\mathfrak{m}}$. We have an isomorphism $\Phi:(M^*)_{\mathfrak{m}}=M^*\otimes_A A_{\mathfrak{m}}\to(M_{\mathfrak{m}})^*=\mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},A_{\mathfrak{m}})$; $f\otimes1\rightsquigarrow[x\otimes1\rightsquigarrow f(x)\otimes1]$. Because, the system $\{x_i\otimes1,f_i\otimes1\,;i\in I\}$ also becomes a projective system of $M_{\mathfrak{m}}$ over $A_{\mathfrak{m}}$, and $\Phi(f_i\otimes1)=0$ for almost all $i\in I$ since $M_{\mathfrak{m}}$ is finitely generated over $A_{\mathfrak{m}}$. Hence the finite set $\{\Phi(f_i\otimes1)\neq0\,;i=1,2,\cdots,t\}=\{\Phi(f_i\otimes1)\,;i\in I\}$ generates $\mathrm{Hom}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}},A_{\mathfrak{m}})$, that is, Φ is surjective. From the bijection $M\to M^*$; $x\rightsquigarrow h(-,x)$, we have a bijection $M_{\mathfrak{m}}\to(M^*)_{\mathfrak{m}}$, and the composition $M_{\mathfrak{m}}\to(M^*)_{\mathfrak{m}}\xrightarrow{\Phi}(M_{\mathfrak{m}})^*$ is surjective. Since $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module with finite rank $[M_{\mathfrak{m}}:A_{\mathfrak{m}}]=[(M_{\mathfrak{m}})^*:A_{\mathfrak{m}}]$, Φ must be bijective. We may regard the injection $L_{\mathfrak{m}}\to(M^*)_{\mathfrak{m}}$ as an inclusion, then we have $\Phi(L_{\mathfrak{m}})=(M_{\mathfrak{m}})^*$ and so $L_{\mathfrak{m}}=(M^*)_{\mathfrak{m}}$. Hence we get $M^*=L=\sum_{i\in I}Af_i$. Let y_i be an element of M determined by $h(-,y_i)=f_i$ for each $i\in I$. By $M^*=\sum_{i\in I}Af_i$, we have $M=\sum_{i\in I}Ay_i$, and the $I\times I$-matrix $H=(h(y_i,y_j))_{(i,j)\in I\times I}$ is a row-finite and column-finite matrix. By Theorem 2, M is finitely generated over A. Next, suppose that A is a non commutative noetherian ring such that A/J is a semisimple artinian ring. Since $\bar{J}=J$, by Lemma C and Theorem 2, M/JM is finitely generated over A/J. By Lemma B, M is finitely generated over A.

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