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# Bounded, periodic and almost periodic classical solutions of some nonlinear wave equations with a dissipative term

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## Introduction.

This paper deals with the questions of the existence and asymptotics of the bounded, periodic and almost periodic classical solutions for the equations

$$\text{(E)} \qquad \frac{\partial^2}{\partial t^2}u+\sum_{|\alpha|,|\beta|\leq m}(-1)^{|\alpha|}D^{\alpha}(a_{\alpha,\beta}(x)D^{\beta}u(x,t))+\nu\frac{\partial}{\partial t}u$$
$$+F(x,t,u)=0 \qquad \text{on} \quad \Omega\times R \ (\text{or } \Omega+R^{+})$$

together with the boundary condition

$$\text{(B)} \qquad D^{\alpha}u|_{\partial\Omega}=0 \qquad \text{for} \quad |\alpha|\leqq m-1,$$

where $\nu$ is a positive constant, $\Omega$ is a bounded domain in $n$-dimensional Euclidean space $R^n$ and $\partial\Omega$ its boundary. We use the following notations

$$\alpha=(\alpha_1,\cdots,\alpha_n), \quad |\alpha|=\sum_{i=1}^{n}|\alpha_i|, \quad D^{\alpha}=\prod_{i=1}^{n}\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i},$$
$$x=(x_1,\cdots,x_n) \quad \text{etc.}$$

The functions to be considered are all real valued and throughout the paper we make the following assumptions:

$$\text{H}_1. \qquad A=\sum_{|\alpha|,|\beta|\leq m}(-1)^{|\alpha|}D^{\alpha}(a_{\alpha,\beta}(x)D^{\beta}u)$$

is formally selfadjoint and coercive on $\overset{\circ}{H}_m$, i.e., there exist some positive constants $c_0$, $c_1$ such that

$$c_1^2\|u\|^2_{\overset{\circ}{H}_m}\geqq\langle Au,u\rangle\geqq c_0^2\|u\|^2_{\overset{\circ}{H}_m} \qquad \text{for} \quad u\in\overset{\circ}{H}_m$$

where we put

$$\langle Au, v\rangle = \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega} a_{\alpha\beta}(x) D^{\beta}u(x) D^{\alpha}v(x)dx, \quad u, v \in \overset{\circ}{H}_m$$

and

$$\|u\|_{\overset{\circ}{H}_m} = \Big(\sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha}u|^2 dx\Big)^{1/2}.$$

$H_2$. $F(x, t, u)$ is of the form

$$F(x, t, u) = g(x, u) + f(x, t)$$

and $g$, $f$ satisfy the conditions:

$$g \in C^{\max([n/2]+1, 3)}(\bar{\Omega} \times R)$$

and
$$\Big|\frac{\partial^j}{\partial u^j} g(x, u)\Big| \leqq K_0 \sum_{i=1,2} |u|^{\max(r_i+1-j, 0)} \qquad (j=0, 1, 2, 3)$$

where $r_i$ ($i=1, 2$) are constants such that

$$0 < r_1 \leqq r_2 < \infty \qquad \text{if} \quad 1 \leqq n \leqq 2m$$

and
$$0 < r_1 \leqq r_2 \leqq \frac{2m}{n-2m} \qquad \text{if} \quad n > 2m,$$

and
$$f(\cdot, t) \in C^3(R; L^2) \cap C^0(R; H_{2m} \cap \overset{\circ}{H}_m)$$

where $C^i(I; V)$ denotes the class of functions from $I \subset R$ to $V$ with uniformly continuous derivatives of order $\leqq i$. Moreover we assume

$$M \equiv \sup_{\substack{t \in R \\ i=0,1,2,3}} \max(\|D_t^i f(t)\|_{L^2}, \|Af(\cdot, t)\|_{L^2}, \|AD_t f(\cdot, t)\|_{L^2}) < +\infty.$$

$H_3$.
$$2m \geqq \Big[\frac{n}{2}\Big] + 1, \quad m \geqq \Big[\frac{n}{3}\Big], \quad a_{\alpha\beta} \in C^{2m+[n/2]+1+|\alpha|}(\bar{\Omega})$$

and
$$\partial\Omega \in C^{2m+[n/2]+1}.$$

Recently, Clements [6] and Biroli [5] have proved the existence of generalized periodic solution to the problem (E)-(B) in the case $F(x, t, u)$ is monotonically increasing in $u$. On the other hand the author [14] has treated a nonmonotonic case, proving the existence of bounded, periodic and almost periodic solutions for the problem (E)-(B) with $m=1$ and with $\nu \frac{\partial}{\partial t} u$ replaced by nonlinear term $\rho\Big(x, \frac{\partial}{\partial t} u\Big)$. But the solutions obtained are also generalized ones.

The object of this paper is to give existence theorems concerning bounded, periodic and almost periodic classical solutions to the problem (E)-(B) and to investigate some asymptotic properties of them.

In this paper we employ a Galerkin's method. In particular techniques are related to those in Sather [**20**], Ebihara-Nakao-Nanbu [7], Amerio-Prouse [3] and Nakao [**14**] [**15**] [**16**]. For other treatments of classical periodic solutions of the nonlinear wave equations with a dissipative term, see Rabinowitz [**19**] and Wahl [**24**], and for quasi-periodic solution see Yamaguchi [**25**].

The existence of a dissipative term is essential for our arguments. Indeed, the problem of the existence and regularity of bounded, periodic and almost periodic solutions for the nonlinear wave equations without dissipative term is extremely difficult and there have been researches only for periodic solutions of essentially one dimensional equations. For this special case see Vejvoda [**23**], Rabinowitz [**18**], Torelli [**22**], Hall [8], Nakao [**11**] [**12**] and the references cited there.

Similar problem for nonlinear parabolic equations has heen treated by Biroli [4], Nakao [**11**] and Nakao-Nanbu [**17**] where differential inequalities concerning several norms of approximate solutions are used extensively, while here we use 'integral inequality' for the estimations of approximate solutions.

## § 1. Preliminaries.

We employ the usual notations for function spaces and for norms associated with them (see, e.g., Lions [**9**]). For $u\in\mathring{H}_m$ we put $\|u\|_A\equiv\langle Au, u\rangle^{1/2}$ and define several functionals on $\mathring{H}_m$ as follows:

$$J_0(u)=\frac{1}{2}\|u\|_A^2+\int_\Omega\int_0^{u(x)}g(x, s)ds\,dx,$$

$$J_1(u)=\|u\|_A^2+\int_\Omega g(x, u(x))u(x)dx,$$

$$\tilde{J}_0(u)=\frac{1}{2}\|u\|_A^2-K_0\sum_{i=1,2}S_{r_i+2}^{r_i+2}/(r_i+2)\cdot\|u\|_A^{r_i+2}$$

and

$$\tilde{J}_1(u)=\|u\|_A^2-K_0\sum_{i=1,2}S_{r_i+2}^{r_i+2}\|u\|_A^{r_i+2},$$

where $S_q$ is the sobolev constant such that

$$\|u\|_{L_q}\leqq S_q\|u\|_A \qquad \text{for} \quad u\in\mathring{H}_m$$

with $0<q\leqq\dfrac{2n}{n-2m}$ if $n>2m$ and $0<q<\infty$ if $1\leqq n\leqq 2m$.

Notice that $J_0(u)\geqq\tilde{J}_0(u)$ and $J_1(u)\geqq\tilde{J}_1(u)$ for $u\in\mathring{H}_m$. For convenience we put, for $x\geqq 0$,

$$J_0(x)=\frac{1}{2}x^2-K_0\sum_{i=1,2}S_{r_i+2}^{r_i+2}/(r_i+2)\cdot x^{r_i+2}$$

and

$$J_1(x)=x_2-K_0\sum_{i=1,2}S_{r_i+2}^{r_i+2}\cdot x^{r_i+2}.$$

Let $D_1$, $D_0$ and $x_0$ be positive numbers such that

$$D_1=\max_{x\geqq 0}\tilde{J}_1(x)=\tilde{J}_1(x_0) \quad \text{and} \quad D_0=\tilde{J}_0(x_0)$$

Then it is easy to see that

$$(1.1) \qquad D_0=\max_{0\leqq x\leqq x_0}\tilde{J}_0(x)<\min(D_1, \max_{x\geqq 0}\tilde{J}_0(x)).$$

Now, let $\{\phi_j\}_{j=1}^{\infty}$ be the set of eigen functions of the operator $A$ in $L^2$ with the Dirichlet boundary condition:

$$A\phi_j=\mu_j\phi_j \quad \text{in } \Omega \text{ and } \quad D^\alpha\phi_j=0 \quad \text{on } \partial\Omega \text{ for } \quad |\alpha|\leqq m-1.$$

As is well known, $\phi_j\in C^{4m+[n/2]+1}(\bar{\Omega})$ and $\{\phi_j\}$ is dense both in $L_2$ and $\mathring{H}_m$ (cf. Agmon [1]).

## §2. Approximate solutions.

We employ the Galerkin's method for our purpose. Let $\{\phi_j\}$ be the basis of $\mathring{H}_m$ consisting of the eigen functions of $A$, and consider the system of ordinary differential equations

$$(2.1) \qquad (u_r''(t), \phi_j)+(Au_r(t), \phi_j)+\nu(u_r'(t), \phi_j)+(F(\cdot, t, u_r), \phi_j)=0,$$

$$j=1, 2, \cdots, r,$$

with the initial condition

$$(2.2) \qquad u_r'(0)=u_r(0)=0,$$

where $u_r(t)=\sum_{j=1}^{r}\lambda_j^r(t)\phi_j$.

The equation (2.1) is equivalent to

$$(2.1)' \qquad u_r''(t)+Au_r(t)+\nu u_r'(t)+P_rF(x, t, u_r)=0,$$

where $P_r$ is the projection of the space $L_2$ onto the $r$-dimensional subspace spanned by $\{\phi_j\}_{j=1}^{r}$.

Since $F(x, t, u)$ is locally Lipschitz continuous in $u$, $u_r(t)$ exists uniquely on some interval, say $[0, T_r)$. In what follows, we shall estimate $u_r(t)$ in various norms. First we show

LEMMA 2.1. *There exists a positive constant $M_0$ such that if $M<M_0$, then $u_r(t)$ exist on $[0, \infty)$ and the estimates*

$$\|u_r(t)\|_A \leqq x_0(M) < x_0 \quad \text{and} \quad \|u_r'(t)\|_{L_2} \leqq \sqrt{2k_0(M)} < \sqrt{2D_0} \tag{2.3}$$

*hold for all $r$, where $x_0(M)$ and $k_0(M)$ are certain constants tending to* 0 *as* $M \to 0$.

PROOF. Let $k_0(M)$ be a positive constant ($<D_0$) determined later, and $x_0(M)$ ($<x_0$) be the solution of the numerical equation in $x$:

$$\tilde{J}_0(x) = k_0(M), \quad x \geqq 0. \tag{2.4}$$

In order to prove the lemma it suffices to show

$$\frac{1}{2}\|u_r'(t)\|_{L_2}^2 + J_0(u_r(t)) \leqq k_0(M) \quad \text{for} \quad \forall t \in [0, T_r). \tag{2.5}$$

The proof of (2.5) can be carried out essentially in the same manner as the proof of Lemma 2.1 in [**14**], and we give it briefly. Suppose that the inequality (2.5) were false. There would then exist a time $\bar{t} \in [0, T_r)$ such that

$$\frac{1}{2}\|u_r'(t)\|_{L_2}^2 + J_0(u_r(t)) \leqq k_0(M) \qquad \text{for} \quad t \leqq \bar{t} \tag{2.6}$$

and

$$\frac{1}{2}\|u_r'(t_\varepsilon)\|_{L_2}^2 + J_0(u_r(t_\varepsilon)) > k_0(M) \qquad \text{for some} \quad t_\varepsilon > \bar{t}, \tag{2.7}$$

where $t_\varepsilon$ can be chosen as close to $\bar{t}$ as wanted. For any $t_1, t_2 \in [0, T_r)$, we obtain by (2.1)

$$\begin{aligned} &\frac{1}{2}\|u_r'(t_2)\|_{L_2}^2 + J_0(u_r(t_2)) + \nu\int_{t_1}^{t_2}\|u_r'(t)\|_{L_2}^2\,dt \\ &= \frac{1}{2}\|u_r'(t_1)\|_{L_2}^2 + J_0(u_r(t_1)) - \int_{t_1}^{t_2}(f(t), u_r'(t))\,dt. \end{aligned} \tag{2.8}$$

Taking $t_2 = \bar{t}$ and $t_1 = 0$ in the above we have

$$\begin{aligned} &\frac{1}{2}\|u_r'(\bar{t})\|_{L_2}^2 + J_0(u_r(\bar{t})) + \nu\int_0^{\bar{t}}\|u_r'(t)\|_{L_2}^2\,dt \\ &\leqq \nu\int_0^{\bar{t}}\|u_r'(t)\|_{L_2}^2 + \frac{1}{4\nu}\int_0^{\bar{t}}\|f(t)\|_{L_2}^2\,dt, \end{aligned} \tag{2.9}$$

and hence

$$4\nu k_0(M) \leqq \int_0^{\bar{t}} \|f(t)\|_{L_2}^2 dt,$$

which implies $\bar{t}>1$ if we choose $k_0(M)$ satisfying

$$k_0(M) > \frac{1}{4\nu} M^2. \tag{2.10}$$

Under the assumption (2.10), we take $t_2 = t_\varepsilon$ and $t_1 = t_\varepsilon - 1$ ($<\bar{t}$) in (2.8) to obtain

$$\int_{t_\varepsilon - 1}^{t_\varepsilon} \|u_r'(t)\|_{L_2}^2 dt \leqq (\nu^{-1}M)^2, \tag{2.11}$$

and hence there exist points $\bar{t}_1 \in \left[t_\varepsilon - 1,\ t_\varepsilon - \frac{3}{4}\right]$ and $\bar{t}_2 \in \left[t_\varepsilon - \frac{1}{4},\ t_\varepsilon\right]$ such that

$$\|u_r'(\bar{t}_i)\|_{L_2} \leqq 2\nu^{-1}M \quad (i=1,\ 2). \tag{2.12}$$

Therefore, multiplication (2.1) by $\lambda_j^r(t)$, summation over $j$ and integration by parts yield

$$\int_{\bar{t}_1}^{\bar{t}_2} J_1(u_r(t))\,dt \leqq |(u_r'(\bar{t}_1),\ u_r(\bar{t}_1))| + |(u_r'(\bar{t}_2),\ u_r(\bar{t}_2))|$$

$$+ \int_{\bar{t}_1}^{\bar{t}_2} \{\|u_r'(t)\|_{L_2}^2 + \nu|(u_r',\ u_r)| + |(f(t),\ u_r(t)|\}\,dt$$

$$\leqq (\nu^{-1}M)^2 + 4\nu^{-1}c_0^{-1}Mx_0 + 2Mc_0^{-1}\left(\int_{\bar{t}_1}^{\bar{t}_2} \|u_r(s)\|_A^2 ds\right)^{1/2}. \tag{2.13}$$

On the other hand we know $\|u_r(t)\|_A \leqq x_0$ if $t \leqq t_\varepsilon$, and hence

$$\int_{\bar{t}_1}^{\bar{t}_2} J_1(u_r(t))\,dt \geqq \int_{\bar{t}_1}^{\bar{t}_2} \tilde{J}_1(u_r(t))\,dt \geqq c_2 \int_{\bar{t}_1}^{\bar{t}_2} \|u_r(t)\|_A^2 dt, \tag{2.14}$$

where $c_2 = 1 - k_0 \sum_{i=1,2} S_{r_i+2}^{r_i+2} x_0^{r_i} > 0$.

From (2.13) and (2.14) we obtain

$$\int_{\bar{t}_1}^{\bar{t}_2} \|u_r(t)\|_A^2 dt \leqq c_2^{-1}\{(2(\nu^{-1}M)^2 + 8\nu^{-1}c_0^{-1}Mx_0 + 4M^2c_0^{-2}c_2^{-1}\}. \tag{2.15}$$

By (2.11) and (2.15) we see that there exists a point $t^* \in [\bar{t}_1,\ \bar{t}_2]$ such that

$$\frac{1}{2}(\|u_r'(t^*)\|_{L_2}^2 + \|u_r(t^*)\|_A^2) \leqq k_0'(M), \tag{2.16}$$

where

$$k_0'(M)\equiv(1+2c_2^{-1})(\nu^{-1}M)^2+4c_2^{-1}M(2\nu^{-1}c_0^{-1}x_0+Mc_0^{-2}c_0^{-1}).$$

Therefore, by (2.8) with $t_2=t_\varepsilon$ and $t_1=t^*$, we obtain

$$(2.17)\qquad \frac{1}{2}\|u_r'(t_\varepsilon)\|_{L_2}^2+J_0(u_r(t_\varepsilon))\leqq\frac{1}{2}\|u_r'(t^*)\|_{L_2}^2+J_0(u_r(t^*))+\frac{1}{4\nu}M^2$$
$$\leqq k_0(M),$$

where we define

$$k_0(M)=k_0'(M)+\frac{1}{4\nu}M^2+k_0\sum_{i=1,2}\frac{1}{r_i+2}S_{r_i+2}^{r_i+2}(2k_0'(M))^{(r_i+2)/2}.$$

Then, (2.10) is automatically valid, and if we choose $M_0$ so that $k_0(M_0)=D_0$, the inequality (2.7) contradicts (2.17) for $M<M_0$. Q.E.D.

REMARK. As is easily seen from the proof, Lemma 1.1 is valid in fact even if we replace $M$ by $\sup_t\left(\int_t^{t+1}\|f(s)\|_{L_2}^2 ds\right)^{1/2}$.

We proceed to further estimation of the approximate solutions. To do so it is convenient to prepare an estimate of the approximate solutions $\{U_r(t)\}$ of the linear equation (E) with $F(x, t, u)=F(x, t)$.

LEMMA 2.2. *Let $\{U_r(t)\}$ be the solution of the equation (2.1)-(2.2) with $F(x, t, u_r)=F(x, t)$. Then we have*

$$(2.18)\qquad \max_{s\in[t,t+1]}\|U_r(s)\|_E^2\leqq c_3(\|U_r(t)\|_E^2-\|U_r(t+1)\|_E^2)+c_4N(t)^2$$
$$\text{for}\quad t\in[0,\infty),$$

*where*

$$N(t)=\left(\int_t^{t+1}\|F(\cdot, s)\|_{L_2}^2 ds\right)^{1/2},\quad \|U\|_E^2=\|U'(t)\|_{L_2}^2+\|U(t)\|_A^2,$$

$$c_3=2\nu^{-1}\{2\nu+5+4(4+\nu)^2c_0^{-2}\}\quad and\quad c_4=c_3\nu^{-1}+2+8c_0^{-2}.$$

PROOF. By (2.1) we have, for $0\leqq t_1<t_2$,

$$(2.19)\qquad \nu\int_{t_1}^{t_2}\|U_r'(s)\|_{L_2}^2 ds=\frac{1}{2}(\|U_r(t_1)\|_E^2-\|U_r(t_2)\|_E^2)$$
$$-\int_{t_1}^{t_2}(F(\cdot, s), U_r'(s))ds,$$

and hence, taking $t_2=t+1$, $t_1=t$,

$$(2.20)\qquad \int_t^{t+1}\|U_r'(s)\|_{L_2}^2 ds\leqq\nu^{-1}(\|U_r(t)\|_E^2-\|U_r(t+1)\|_E^2)+(\nu^{-1}N)^2$$
$$\equiv D(t)^2.$$

It follows that there exist two points $t_1\in\left[t,\ t+\frac{1}{4}\right]$ and $t_2\in\left[t+\frac{3}{4},\ t+1\right]$ such that

$$\|U_r'(t_i)\|_{L_2}\leqq 2D(t)\qquad (i=1,\ 2),$$

and hence by (2.1) we have, as in (2.13),

$$\int_{t_1}^{t_2}\|U_r(s)\|_A^2 ds\leqq D(t)^2+((4+\nu)D(t)+N)c_0^{-1}\max_{s\in[t,\ t+1]}\|U_r(s)\|_A. \tag{2.21}$$

From (2.20) and (2.21) we see that there exists a point $t^*\in[t_1,\ t_2]$ such that

$$\|U_r(t^*)\|_E^2\leqq 4D(t)^2+2((4+\nu)D(t)+N)c_0^{-1}\max_{s\in[t,\ t+1]}\|U_r(s)\|_A. \tag{2.22}$$

Using the estimate (2.22) we have from (2.19) with $t_1=t^*$ (or $t_2=t^*$)

$$\begin{aligned}\max_{s\in[t,\ t+1]}\|U_r(s)\|_E^2&\leqq 2\nu\int_t^{t+1}\|U_r'(s)\|_{L_2}^2 ds\\ &\quad+\|U_r(t^*)\|_E^2+2\int_t^{t+1}\|F(s)\|_{L_2}\|U_r'(s)\|_{L_2}ds\\ &\leqq 2\nu D(t)^2+4D(t)^2+2((4+\nu)D(t)+N)c_0^{-1}\\ &\quad\times\max_{s\in[t,\ t+1]}\|U_r(s)\|_E+2ND(t)\\ &\leqq\{5+2\nu+4(4+\nu)^2c_0^{-2}\}D(t)^2+(1+4c_0^{-2})N^2\\ &\quad+\frac{1}{2}\max_{s\in[t,\ t+1]}\|U_r(s)\|_E^2,\end{aligned}$$

which proves the lemma. Q. E. D.

We obtain further the following.

LEMMA 2.3. *Let $U_r(t)$ be as in Lemma 2.2. Then*

$$\begin{aligned}\|U_r(t)\|_E^2&\leqq\max(\max_{s\in[0,\ 1]}\|U_r(s)\|_E^2,\ c_4\bar{N}^2)\\ &\leqq\|U_r(0)\|_E^2+c_4\bar{N}^2,\end{aligned} \tag{2.23}$$

*where* $\bar{N}=\sup_{t\in R}N(t)$.

PROOF. The former inequality in (2.23) is an immediate consequence of (2.18). For $t\in[0,\ 1]$ we have

$$\|U_r(t)\|_E^2 \leqq \|U_r(0)\|_E^2 + \frac{1}{2\nu}\int_0^t \|F(s)\|_{L_2}^2 ds$$

$$\leqq \|U_r(0)\|_E^2 + \frac{1}{2\nu}\bar{N}^2,$$

which gives (2.23), because $c_4 > \dfrac{1}{2\nu}$. Q. E. D.

Now we return to the estimation of $\{u_r(t)\}$.

LEMMA 2.4. *Let $M_1'$ be the positive number such that*

$$c_5(M_1') \equiv 1 - 4c_4 \sum_{i=1,2} K_0^2 S_{2r_i+2}^{2r_i+2} x_0 (M_1')^{2r_i} = 0.$$

*Then for $M < M_1 \equiv \min(M_0, M_1')$ we have*

$$\|D_t u_r(t)\|_E \leqq k_1(M) \qquad \text{for} \quad t \in [0, \infty), \tag{2.24}$$

*where $k_1(M) = (1+2c_4)^{1/2} M c_5(M)^{-1/2}$.*

PROOF. By Lemma 2.3 with $U_r(t) = D_t u_r(t)$ and $F(x, t) = D_t F(x, t, u_r)$, we have

$$\max_{t\in R^+} \|D_t u_r(t)\|_E^2 \leqq \|D_t u_r(0)\|_E^2 + c_4 \max_{t\in R^+} \|D_t\{g(\cdot, u_r(\cdot, t)) + f(t)\}\|_{L_2}^2. \tag{2.25}$$

$$\|D_t\{g(\cdot, u_r(\cdot, t)) + f(t)\}\|_{L_2}$$

$$\leqq M + K_0\Big(\int_\Omega \Big(\sum_{i=1,2} |u_r|^{r_i} |D_t u_r(x, t)|\Big)^2 dx\Big)^{1/2}$$

$$\leqq M + \sqrt{2}\, K_0 \Big(\sum_{i=1,2} S_{2r_i+2}^{2r_i+2} x_0 (M)^{2r_i}\Big)^{1/2} \|D_t u_r(t)\|_E, \tag{2.26}$$

and also

$$\|D_t u_r(0)\|_E^2 = \|D_t^2 u_r(0)\|_{L_2}^2 + \|D_t u_r(0)\|_A^2$$

$$= \|-\nu u_r'(0) - A u_r(0) + P_r(g(\cdot, u_r(0)) + f(0))\|_{L_2}^2$$

$$\leqq M^2. \tag{2.27}$$

The inequalities (2.25)-(2.27) yield (2.24). Q. E. D.

LEMMA 2.5. *Let $M_1$ be the number given in Lemma 2.4. Then if $M < M_1$, we have*

$$\|D_t^2 u_r(t)\|_E \leqq k_2(M) \qquad \text{for} \quad t \in [0, \infty), \tag{2.28}$$

*where*

$$k_2(M)=\left\{(2\nu^2+3)M^2+4c_4M^2+8c_4K_2^2\sum_{i=1,2}S_{4+2\tilde{r}_i}^{4+2\tilde{r}_i}x_0(M)^{2\tilde{r}_i}k_1(M)^4\right\}^{1/2}$$

$$\times c_5(M)^{-1/2}\quad(\tilde{r}_i=\max(r_i-1,0)).$$

PROOF.

$$\|D_t^2F(\cdot,t,u_r(t))\|_{L_2}^2$$

$$=\int_\Omega\left|D_t^2f(x,t)+\frac{\partial}{\partial u}g(x,u_r(t))D_t^2u_r+\left(\frac{\partial}{\partial u}\right)^2g(x,u_r(t))\right.$$

$$\left.\times(D_tu_r(x,t))^2\right|^2dx$$

$$\leqq 4M^2+4K_0^2\sum_{i=1,2}S_{2r_i+2}^{2r_i+2}\|u_r(t)\|_A^{2r_i}\|D_t^2u_r(t)\|_A^2$$

$$+8K_0^2\sum_{i=1,2}S_{4+2\tilde{r}_i}^{4+2\tilde{r}_i}\|u_r(t)\|_A^{2\tilde{r}_i}\|D_tu_r(t)\|_A^4$$

$$\leqq 4M^2+4K_0^2\sum_{i=1,2}S_{2r_i+2}^{2r_i+2}x_0(M)^{2r_i}\|D_t^2u_r(t)\|_E^2$$

$$+8K_0^2\sum_{i=1,2}S_{4+2\tilde{r}_i}^{4+2\tilde{r}_i}x_0(M)^{2\tilde{r}_i}k_1(M)^4.$$

Also we have

$$\|D_t^2u_r(\cdot,0)\|_E^2=\|-\nu D_t^2u_r(0)-AD_tu_r(0)+P_rD_tF(\cdot,t,u_r(0))\|_{L_2}^2$$

$$+\|-\nu D_tu_r(0)-Au_r(0)+P_rF(\cdot,t,u_r(0))\|_A^2$$

$$=\|-\nu(-\nu D_tu_r(0)-Au_r(0)+P_rF(\cdot,t,u_r(0)))$$

$$+P_rD_tf(\cdot,0)\|_{L_2}^2+\|P_rf(\cdot,0)\|_A^2$$

$$\leqq 2(\nu^2+1)M^2+M\|AP_rf(\cdot,0)\|_{L_2}$$

$$\leqq(2\nu^2+3)M^2.$$

Here we have used the fact that $A$ and $P_r$ commute on $H_{2m}\cap\mathring{H}_m$. Thus by Lemma 2.3 with $U_r(t)=D_t^2u_r(t)$ and with $F(x,t)=D_t^2F(x,t,u_r(t))$, we obtain (2.28). Q.E.D.

LEMMA 2.6. *Let* $M<M_1$. *Then*

(2.29) $$\|D_t^3u_r(t)\|_E\leqq k_3(M)\quad\text{for}\quad t\in[0,\infty),$$

*where*

$$k_3(M)=\Big[\{(\nu^2+\nu+2)^2+(\nu+1)^2\}M^2+6c_4\{M^2+2K_0^2\sum_{i=1,2}S_{6+2\tilde{\tilde{r}}_i}^{6+2\tilde{\tilde{r}}_i}x_0(M)^{2\tilde{\tilde{r}}_i}k_1(M)^6$$

$$+6K_0^2\sum_{i=1,2}S_{4+2\tilde{r}_i}^{4+2\tilde{r}_i}x_0(M)^{2\tilde{r}_i}k_1(M)^2k_2(M)^2\}\Big]^{1/2}c_5(M)^{-1/2}.$$

$$(\check{\tilde{r}}_i=\max(r_i-2,\ 0).)$$

PROOF. Easy calculations give

$$\|D_t^3F(\cdot,\ t,\ u_r(t)\|_{L_2}^2\leqq 6\Big(M^2+2K_0^2\sum_{i=1,2}S_{6+2\tilde{\tilde{r}}_i}^{6+2\tilde{\tilde{r}}_i}\|u_r(t)\|_A^{2\tilde{\tilde{r}}_i}\|D_tu_r(t)\|_A^6$$

$$+18K_0^2\sum_{i=1,2}S_{4+2\tilde{r}_i}^{4+2\tilde{r}_i}\|u_r(t)\|_A^{2\tilde{r}_i}\|D_tu_r(t)\|_A^2$$

$$\times\|D_t^2u_r(t)\|_A^2\Big)$$

$$+4K_0^2\sum_{i=1,2}S_{2+2r_i}^{2+2r_i}\|u_r(t)\|_A^{2r_i}\|D_t^3u_r(t)\|_A^2$$

$$\leqq 6\Big(M^2+2K_0^2\sum_{i=1,2}S_{6+2\tilde{\tilde{r}}_i}^{6+2\tilde{\tilde{r}}_i}x_0(M)^{2\tilde{\tilde{r}}_i}k_1(M)^6$$

$$+18K_0^2\sum_{i=1,2}S_{4+2\tilde{r}_i}^{4+2\tilde{r}_i}x_0(M)^{2\tilde{r}_i}k_1(M)^2k_2(M)^2\Big)$$

$$+4K_0^2\sum_{i=1,2}S_{2+2r_i}^{2+2r_i}x_0(M)^{2r_i}\|D_t^3u_r(t)\|_A^2.$$

Also we have

$$\|D_t^3u_r(0)\|_E^2=\|\{D_t(\nu D_tu_r(t)+Au_r(t)+P_rF(\cdot,\ t,\ u_r(t))\}_{t=0}\|_E^2$$

$$=\|\nu\{-\nu D_tu_r(t)-Au_r(t)-P_rF(\cdot,\ t,\ u_r(t))\}_{t=0}$$

$$+\{AD_tu_r(t)+P_rD_tF(\cdot,\ t,\ u_r(t))\}_{t=0}\|_E^2$$

$$=\|\nu^2D_t^2u_r(0)+\nu P_rf(\cdot,\ 0)+A\{-\nu D_tu_r(t)-Au_r(t)$$

$$-P_rF(\cdot,\ t,\ u_r(t))\}_{t=0}+P_rD_t^2f(\cdot,\ 0)\|_{L_2}^2$$

$$+\|\nu P_rf(\cdot,\ 0)+P_rD_tf(\cdot,\ 0)\|_A^2$$

$$\leqq \|\nu^2\{-\nu D_t u_r(0) - Au_r(0) - P_r F(\cdot, 0, u_r(0))\}$$

$$+\nu P_r f(\cdot, 0) - AP_r f(0) + P_r D_t^2 f(\cdot, 0)\|_{L_2}^2$$

$$+\|\nu P_r f(\cdot, 0) + P_r D_t f(\cdot, 0)\|_{L_2} \|A(\nu P_r f(\cdot, 0) + P_r D_t f(\cdot, 0))\|_{L_2}$$

$$\leqq (\nu^2+\nu+2)^2 M^2 + (\nu+1)^2 M^2.$$

(Note that $D_t f \in H_{2m} \cap \overset{\circ}{H}_m$.)

Thus, by Lemma 2.3, the proof is completed. Q. E. D.

Now we have finished the estimation of the approximate solutions.

## § 3. Passage to the limit.

In this section we shall study the convergency properties of the approximate solutions $u_r(t)$ as $r \to \infty$. Always we assume $M < M_1$. Then by the estimates given in the previous section we see first that there exists a subsequence of $\{u_r(t)\}$, which will be denoted by the same notation for simplicity, and a function $u(x, t)$ such that

(3.1) $D_t^i u_r(t) \longrightarrow D_t^i u(t)$ weakly* in $L^\infty(R^+; \overset{\circ}{H}_m)$, a.e. on $\Omega \times R^+$, and uniformly in $L_2$ on each compact set of $R^+$ $(i=0, 1, 2, 3)$,

and

(3.2) $D_t^4 u_r(t) \longrightarrow D_t^4 u(t)$ weakly* in $L^\infty(R^+; L_2)$.

Moreover it is clear that the limit function $u(t) = u(x, t)$ is in $B^2(R^+; \overset{\circ}{H}_m) \cap B^3(R^+; L_2)$ and satisfies the estimates (2.3), (2.24), (2.28) and (2.29) for $u_r = u$, where we put, for a Banach space $V$ and an interval $I \subset R$,

$$B^i(I; V) = \{u \in C^i(I; V) \mid \sup_{\substack{t \in I \\ 0 \leq j \leq i}} \|D^j_t u(t)\|_V < +\infty\}.$$

Next, to show the uniformly continuity of $D_t^3 u(t)$ in $E$, we shall prove the uniformly equi-continuity of $\{D_t^3 u_r(t)\}$ with respect to the norm $\|\cdot\|_E$. For each $t \in R^+$, $u_r(t)$ is the solution of the elliptic equation

$$Au_r(t) = -u_r''(t) - \nu u_r'(t) + P_r F(x, t, u_r(t))$$

$$\equiv h_r(x, t) \in B(R^+; L_2)$$

and by the theory of elliptic boundary value problem (cf. Agmon-Douglis-Nirenberg [2]) we have for $\delta>0$

$$|u_r(x, t+\delta)-u_r(x, t)|\leqq \text{const.}\|u_r(\cdot, t+\delta)-u_r(\cdot, t)\|_{H_{2m}}$$

$$\leqq \text{const.}\|h_r(\cdot, t+\delta)-h_r(\cdot, t)\|_{L_2}$$

$$\leqq\phi_1(\delta)\longrightarrow 0 \quad (\text{as } \delta\to 0), \tag{3.3}$$

where $\phi_i(\delta)$ $(i=1, 2, \cdots)$ denotes a function of $\delta$ independent of $r$, $t$, tending to 0 as $\delta\to 0$. Using the estimates (2.24), (2.28), (2.29), (3.3) and the hypothesis that $D_t^2 f$ is uniformly continuous in $L_2$, we can show in a straight foward manner

$$\|D_t^2 F(\cdot, t+\delta, u_r(t+\delta)-D_t^2 F(\cdot, t, u_r(t))\|_{L_2}^2\leqq\phi_2(\delta) \tag{3.4}$$

and

$$\|D_t^2 u_r(\delta)-D_t^2 u_r(0)\|_E^2\leqq\phi_3(\delta). \tag{3.5}$$

Therefore, applying Lemma 2.3 we have

$$\max_{t\in R^+}\|D_t^2 u_r(t+\delta)-D_t^2 u_r(t)\|_E^2\leqq\phi_3(\delta)+c_4\phi_2(\delta), \tag{3.6}$$

which implies that $\{D_t^2 u_r(t)\}$ is uniformly equicontinuous in the norm $\|\cdot\|_E$, and we can conclude by the resonance theorem

$$D_t^2 u(t)\in B^0(R^+; E). \tag{3.7}$$

Now, the limit functions $D_t^i u(t)$ $(0\leqq i\leqq 2)$ satisfy the following elliptic equations in a generalized sense

$$AD_t^i u(t)=-(D_t^{2+i}u(t)+\nu D_t^{i+1}u(t)+D_t^i F(\cdot, t, u(t))) \tag{3.8}$$

$$(\text{a.e. } t\in R^+).$$

Since $D_t u$, $D_t^2 u$, $F(\cdot, t, u(t))\in B^0(R^+; L_2)$, the equation (3.8) with $i=0$ implies

$$u\in B^0(R^+; H_{2m}\cap \mathring{H}_m)\subset C^0(\Omega\times R^+)\cap L^\infty(\Omega\times R^+) \tag{3.9}$$

and hence

$$D_t F(\cdot, t, u(t))\in B^0(R^+; L^2(\Omega)).$$

Therefore (3.8) with $i=1$ implies

$$D_t u\in B^0(R^+; H_{2m}\cap \mathring{H}_m)\subset C^0(\bar{\Omega}\times R^+)\cap L^\infty(\Omega\times R^+). \tag{3.10}$$

By (3.1), (3.7) and (3.10) it is easy to see that the right hand side of (3.8) with $i=2$ belongs to $B^0(R^+; L_2)$, and we have

$$D_t^2 u \in B^0(R^+; H_{2m} \cap \overset{\circ}{H}_m) \subset C^0(\bar{\Omega} \times R^+) \cap L^\infty(\Omega \times R^+). \tag{3.11}$$

Finally we note that $g(\cdot, u) \in B^0(R^+; H_{[n/2]+1})$ because $u(t) \in B^0(R^+; H_{2m}) \subset B^0(R^+; H_{[n/2]+1})$ (cf. Mizohata [Theorem 7.1; **10**]). Thus by (3.10) and (3.11) the right hand side of the equation (3.8) with $i=0$ belongs to $B^0(R^+; H_{[n/2]+1})$ and by the theory of elliptic equation we have

$$u(t) \in B^0(R^+; H_{2m+[n/2]+1} \cap \overset{\circ}{H}_m) \subset B^0(R^+; C^{2m}(\bar{\Omega})). \tag{3.12}$$

We summarize above results in the following

THEOREM 3.1. *Suppose $H_1$-$H_3$ and let $M<M_1$. Then the initial-boundary value problem* (E)-(B) *on $\Omega \times R^+$ with initial data $u(x, 0) \equiv u'(x, 0) \equiv 0$ admits a unique bounded classical solution $u$ satisfying* (2.3), (2.24), (2.28), (2.29) *and*

$$\|u(t)\|_Q \equiv \max_{x \in \bar{\Omega}} \Big( \sum_{|\alpha| \leqq 2m} |D^\alpha u(x, t)| + \sum_{i=0}^{2} \Big| \frac{\partial^i}{\partial t^i} u(x, t) \Big| \Big) \tag{3.13}$$

$$\leqq k_4(M) \qquad for \quad t \in [0, \infty),$$

*where $k_4(M)$ is a certain constant tending to 0 as $M \to 0$.*

The proof of the uniqueness in Theorem 3.1 is trivial and omitted. Also it is easy from the procedure deriving the estimates (3.10)-(3.12) to see that the estimate (3.13) holds.

## §4. Bounded, periodic and almost periodic solution.

Let us make the assumptions as in Theorem 3.1. Then the problem

$$\frac{\partial^2}{\partial t^2} u + \sum_{|\alpha|, |\beta| \leqq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha,\beta}(x) D^\beta u(x, t)) + \nu \frac{\partial}{\partial t} u + F(x, t, u) = 0$$

$$\text{on} \quad \Omega \times (-r, \infty), \quad u(x, -r) = \frac{\partial}{\partial t} u(x, -r) = 0 \tag{4.11}$$

$$\text{and} \quad D^\alpha u|_{\partial\Omega} = 0 \qquad \text{for} \quad |\alpha| \leqq m-1 \quad (r = 1, 2, 3, \cdots)$$

admits a unique bounded solution $u_r(x, t)$ satisfying (2.3), (2.24), (2.28), (2.29) and (3.11)-(3.13) for $t \geqq -r$. Also it is easily seen from the proof of (3.13) that for each $T>0$ the family $\{u_r(t)\}$ $(r \geqq T)$ is uniformly equicontinuous in the norm $\|\cdot\|_Q$ on $[-T, T]$. Since the imbedding maps from $H_{2m+[n/2]+1}$ and $H_{2m}$ into $C^{2m}(\bar{\Omega})$ and $C(\bar{\Omega})$, respectively, are both compact, the vectorial

Ascoli-Arzela's Lemma implies that $u_r(t)$ is convergent to a function $u(x, t)$ on $\bar{\Omega}\times R$ in the following sense; for each $T>0$ and for $r>T$,

$$u_r(x, t) \longrightarrow u(x, t) \quad \text{in} \quad C^{2m}(\bar{\Omega}) \qquad \text{uniformly on} \quad [-T, T]$$

and

$$\frac{\partial^i}{\partial t^i} u_r \longrightarrow \frac{\partial^i}{\partial t^i} u \quad \text{in} \quad C(\bar{\Omega}) \qquad \text{uniformly on} \quad [-T, T]$$

($i=1, 2$).

Clearly $u(x, t)$ is a classical solution of the equation (1) on $\Omega\times R$, satisfying the same bounded properties as $u_r(x, t)$. Thus we have

THEOREM 4.1. *Under the same assumptions in Theorem* 3.1 *the problem* (E)-(B) *admits a unique bounded classical solution* $u$ *satisfying the estimates* (2.3), (2.24), (2.28), (2.29) *and* (3.13) *for* $t\in R$.

PROOF. It remains to show the uniqueness of bounded solution satisfying such estimates. In fact this follows from the estimates (2.3) and (2.24) only. Indeed, let $u$ and $v$ be two bounded solutions satisfying (2.3) and (2.24) for $t\in R$. Then $w=u-v$ is a bounded solution of the equation

$$w''+Aw+\nu w'=g(\cdot, u)-g(\cdot, v). \tag{4.2}$$

Now, using the mean value theorem, we can show easily

$$\|g(\cdot, u)-g(\cdot, v)\|_{L_2}^2\leqq 2K_0^2 \sum_{i=1,2} S_{2r_i+2}^{2r_i+2} x_0(M)^{2r_i}\| w \|_A^2,$$

and hence by Lemma 2.2

$$\max_{s\in[t, t+1]} \| w(s)\|_E^2\leqq c_3(\| w(t)\|_E^2-\| w(t+1)\|_E^2)$$

$$+2c_4 K_0^2 \sum_{i=1,2} S_{2r_i+2}^{2r_i+2} x_0(M)^{2r_i}\| w(t)\|_E^2.$$

Thus, recalling the way of choice of the number $M_1$, we have

$$\max_{s\in[t, t+1]} \| w(s)\|_E^2\leqq c_3 c_5^{-1}(M)(\| w(t)\|_E^2-\| w(t+1)\|_E^2)$$

for $t\in R$, which together with the boundedness of $\|w(t)\|_E$ implies easily $w(t)\equiv 0$ (see [14]). Q.E.D.

The following is an immediate consequence of the above theorem.

COROLLARY 4.1. *In addition to the assumption of Theorem* 4.1, *suppose that* $f(\cdot, t)$ *is* $\omega$*-periodic in* $t$. *Then the bounded solution* $u(x, t)$ *in Theorem* 4.1 *is also* $\omega$*-periodic.*

Now we shall study the almost periodicity of the bounded solution.

THEOREM 4.2. *In addition to the assumption of Theorem* 4.1, *suppose that* $f(\cdot, t)$ *is almost periodic as a function from* $R$ *to* $C(\bar{\Omega})$. *Then the bounded solution* $u(x, t)$ *is almost periodic with respect to the norm* $\|\cdot\|_Q$.

PROOF. Let $\{t_r\}_{r=1}^{\infty}$ be any real sequence. By Bochner's criterion (see [3]) it suffices to show that $\{u(t+t_r)\}$ contains a subsequence which is convergent in the norm $\|\cdot\|_Q$ uniformly on $R$. Suppose that this were false. Then there would exist a positive constant $\varepsilon_0$, two subsequences $\{t_{r_i}\}$ $(i=1, 2)$ of $\{t_r\}$ and a real sequences $s_r$ such that

$$\|u(s_r+t_{r_1})-u(s_r+t_{r_2})\|_Q \geqq \varepsilon_0. \tag{4.3}$$

Consider the sequences of functions $\{u(t+t_{r_i}+s_r)\}$ $(i=1, 2)$. They satisfy the estimate (2.3), (2.24), (2.28), (2.29) for $t\in R$, and by the same reason in the proof of Theorem 4.1 we may assume $u(t+t_{r_i}+s_r)\to U_i(t)$ in $\|\cdot\|_Q$ uniformly on each compact set in $R$ as $r\to\infty$. $U_i(t)$ $(i=1, 2)$ have, of course, the same boundedness properties as $u(x, t)$. Now, since $f(t)$ is almost periodic in $C(\bar{\Omega})$, we may assume $f(t+t_{r_i}+s_r)$ converge to a function $F(x, t)$ in the norm $C(\bar{\Omega})$ uniformly on $R$, and hence $U_i(t)$ $(i=1, 2)$ are both bounded classical solutions of the problem (E)-(B) with $f(x, t)$ replaced by $F(x, t)$. Therefore the local uniqueness concerning bounded solution implies $U_1(t)=U_2(t)$. However by (4.3) we have

$$\|U_1(0)-U_2(0)\|_Q \geqq \varepsilon_0>0$$

which is a contradiction. Q. E. D.

## § 5. Asymptotic properties of bounded solutions.

In this section we shall investigate some asymptotic properties of the bounded solutions of the problem (E)-(B). For this purpose we let $u_i(t)$ $(i=1,2)$ be solutions on $\Omega\times(-r_0,\infty)$ of the equation (1) with $f=f_i$ $(i=1,2)$, respectively. Let us suppose that $u_i(t)$ fulfill the following boundedness conditions:

$$\|u_i(t)\|_A \leqq a_0 \ (<x_0(M_1)), \quad \|u_i'(t)\|_{L_2} \leqq b_0 \ (<k_0(M_1)) \tag{5.1}$$

$$\|D^j_t u_i(t)\|_E \leqq a_j \ (<k_j(M_1)) \qquad \text{for} \quad j=1,2,3, \quad t\in[-r_0,\infty)$$

with positive constants $a_0$, $b_0$, $a_j$.

From (5.1), we note, the same arguments as in the proof of Theorem 3.1 give

$$\|u_i(t)\|_Q \leqq a_4 \qquad \text{for} \quad j=1,2, \quad t\in[-r_0,\infty) \tag{5.2}$$

with some positive constant $a_4$.

Assume that $f_i(t)$, $i=1,2$, satisfy $H_2$ and we put

$$\delta_j(t)=\left(\int_t^{t+1}\|D_t^j(f_1(s)-f_2(s))\|_{L_2}^2 ds\right)^{1/2}$$

$(j=0,1,2,3)$.

Then similar arguments as in the proofs of Lemmas 2.2-2.6 yield the following inequalities

$$\max_{s\in[t,t+1]}\|w(s)\|_E^2\leqq c_3 c_5'(a_0)^{-1}(\|w(t)\|_E^2-\|w(t+1)\|_E^2) + 2c_4 c_5'(a_0)^{-1}\delta_0(t)^2, \tag{5.3}$$

$$\max_{s\in[t,t+1]}\|D_t w(s)\|_E^2\leqq c_3 c_5'(a_0)^{-1}(\|D_t w(t)\|_E^2-\|D_t w(t+1)\|_E^2) + 4c_4 c_5'(a_0)^{-1}\Big(2K_0^2\sum_{i=1,2} S_{4+2\bar{r}_i}^{4+2\bar{r}_i} a_0^{2\bar{r}_i} a_1^2\|w(t)\|_E^2+\delta_1(t)^2\Big) \tag{5.4}$$

and

$$\begin{aligned}\max_{s\in[t,t+1]}&\|D_t^2 w(s)\|_E^2\\ &\leqq c_3 c_5'(a_0)^{-1}(\|D_t^2 w(t)\|_E^2-\|D_t^2 w(t+1)\|_E^2)\\ &\quad+4c_4 c_5'(a_0)^{-1}\Big\{2K_0^2\sum_{i=1,2}\Big[S_{6+2\bar{r}_i}^{6+2\bar{r}_i} a_0^{2\bar{r}_i} a_1^4\|w(t)\|_E^2\\ &\quad+S_{4+2\bar{r}_i}^{4+2\bar{r}_i} a_0^{2\bar{r}_i}(4a_1^2\|D_t w\|_E^2+a_2^2\|w(t)\|_E^2)\Big]+\delta_2(t)^2\Big\}\end{aligned} \tag{5.5}$$

where $w(t)=u_1(t)-u_2(t)$ and

$$c_5'(a_0)=1-4c_4K_0^2\sum_{i=1,2} S_{2r_i+2}^{2r_i+2} a_0^{2r_i} \quad (>0).$$

Here we make an additional assumption;

$H_2'$. $g(x,u)$ is 4-times continuously differentiable in $u$.

Then we can easily show

$$\begin{aligned}\int_\Omega &|D_t^3 g(x,u_1(t))-D_t^3 g(x,u_2(t))+D_t^3(f_1-f_2)|^2 dx\\ &\leqq c_6(a_4)\Big(\sum_{i=0}^{2}\|D_t^i w(t)\|_E^2+\delta_3(t)^2\Big)+4K_2^0\sum_{i=1,2} S_{2r_i+2}^{2r_i+2} a_0^{2r_i}\|D_t^3 w(t)\|_E^2\end{aligned}$$

with a constant $c_6(a_4)$ depending on $a_4$, and hence we have by Lemma 2.2

(5.6)
$$\max_{s\in[t,t+1]}\|D_t^3 w(s)\|_E^2 \leqq c_3 c_5'(a_0)^{-1}(\|D_t^3 w(t)\|_E^2-\|D_t^3 w(t+1)\|_E^2) + 2c_4 c_5'(a_0)^{-1} c_6(a_4)\Big(\delta_3(t)^2+\sum_{i=0}^{2}\|D_t^i w(t)\|_E^2\Big).$$

The following elementary lemma is useful.

LEMMA 5.1. *Let $\phi(t)$ be a bounded nonnegative function on $[-r_0, \infty)$ satisfying*

$$\max_{s\in[t,t+1]}\phi(s)\leqq c_7(\phi(t)-\phi(t+1))+k(t)$$

*with $c_7>0$ and $k(t)\leqq c_8 e^{-\theta(t-r_0)}$ $(c_7, \theta>0)$.*

*Then we have*

$$\phi(t)\leqq c_9 e^{-\eta(t-r_0)} \quad (t\geqq r_0)$$

*where $c_9$ and $\eta$ are positive constants depending on $c_7$, $c_8$, and $\theta$.*

The above lemma is a special case of Lemma 2.1 in [**16**] and the proof is omitted (see also [**15**]).

Combining Lemma 5.1 with the estimates (5.3), (5.4), (5.5) and (5.6) we obtain that if $\delta_j(t)\leqq Ke^{-\theta(t-r_0)}$ $(j=0,1,2,3)$ with $\theta>0$, then

(5.7)
$$\sum_{j=0}^{3}\|D^j{}_t w(t)\|_E\leqq K'e^{-\eta(t-r_0)} \qquad \text{for} \quad t\geqq r_0$$

with some positive constants $K'$, $\eta>0$.

Thus combining (5.7) and the standard theory of elliptic equations as was used in the proof of Theorem 3.1 we can conclude

THEOREM 5.1. *Suppose* $H_1$, $H_2$, $H_2'$, $H_3$ *and let*

$$\sum_{j=0}^{3}\delta_j(t)+\Big(\int_t^{t+1}\|f_1(s)-f_2(s)\|_{H_{2m}}^2 ds\Big)^{1/2}\leqq K_1 e^{-\theta(t-r_0)} \quad (\theta>0).$$

*Then for any solutions $u_i(t)$ $(i=1, 2)$ on $[-r_0, \infty)$ of the problem* (E)-(B) *with $f(t)$ replaced by $f_i(t)$, which satisfy* (5.1), *we have*

(5.8)
$$\|u_1(t)-u_2(t)\|_Q\leqq K_2 e^{-\eta(t-r_0)} \qquad \text{for} \quad t\geqq r_0$$

*with some positive constants $K_2$, $\eta$.*

The following is immediate from above.

COROLLARY 5.1. *The bounded solution in Theorem* 4.1 *is exponentially stable with respect to the norm $\|\cdot\|_Q$ in the class of solutions satisfying* (5.1). *And also if $\sum_{j=0}^{3}\|D^j{}_t f(t)\|_{L_2}+\|f(t)\|_{H_{2m}}$ decays exponentially as $t\to\infty$, then the bounded solution decays exponentially in the norm $\|\cdot\|_Q$.*

The former part of the Corollary can be regarded as an extension in part of the result of Sattinger [21], where the existence of global classical solution and the asymptotic stability of the stationary solution have been investigated for the equation

$$u_{tt}-\sum_{i,j}\frac{\partial}{\partial x_j}\Big(a^{ij}\frac{\partial}{\partial x_i}u(x,t)\Big)+2\alpha u_t=P(u), \quad \alpha>0,$$

with linear homogeneous boundary condition $\mathfrak{B}u=0$ on $\partial\Omega$, when $n=1, 2, 3$. In [21], the method of developing the solutions in power series is employed and quitely different from ours. Quite recently, A. Matsumura [26] has extended the result of [21] to the more general nonlinear equations of second order. The method of [26], however, is also different from ours.

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