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Approximation by a sum of polynomials involving primes

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§ 1. Introduction

Based on the Hardy-Littlewood method, Davenport and Heilbronn [3] proved that if $\lambda_1, \cdots, \lambda_s$ are non-zero real numbers, not all of the same sign and not all in rational ratio and if for any given integer $k \geqq 1$, $s \geqq 2^k+1$, then for every $\varepsilon>0$ the inequality

$$|\sum_{j=1}^{s} \lambda_j n_j^k| < \varepsilon$$

has infinitely many solutions in natural numbers n_j. Later, Schwarz [10] showed that if either $s \geqq 2^k+1$ or $s \geqq 2k^2(2\log k+\log\log k+5/2)-1$ (for $k \geqq 12$), then the inequality

$$|\sum_{j=1}^{s} \lambda_j p_j^k| < \varepsilon \tag{1.1}$$

has infinitely many solutions in prime numbers p_j. A. Baker [1] raised a new kind of approximation by proving that when $k=1$, $s=3$ and A is any arbitrary natural number, the ε in (1.1) can be replaced by $(\log \max p_j)^{-A}$. Recently, Ramachandra [9] has obtained this result for arbitrary k and with the p_j^k replaced by arbitrary integer-valued polynomials $f_j(p_j)$ with positive leading coefficients provided that s satisfies the same condition as that required by Schwarz. In 1974, Vaughan [12] made a remarkable progress in this problem by proving that if $k \geqq 4$,

$$\theta=\begin{cases} 2^{1-k} & \text{if } k \leqq 12, \\ (2k^2(2\log k+\log\log k+3))^{-1} & \text{if } k>12, \end{cases} \tag{1.2}$$

$$N=[(-\log 2\theta+\log(1-2/k))/(-\log(1-1/k))] \tag{1.3}$$

and

$$s \geqq 2(k+N)+7, \tag{1.4}$$

then the ε in (1.1) can be replaced by $(\max p_j)^{-\sigma}$, where σ is any positive constant $<(5(k+1)2^{2(k+1)})^{-1}$. In this paper we shall adapt the elegant method of [12] to establish:

THEOREM 1. *Suppose that* $k\geqq 3$, (1.2), (1.3), (1.4) *hold and* $\mathcal{P}_j(x)\,(j=1,\cdots,s)$ *are any polynomials of same degree* k *with integer coefficients and positive leading coefficients. Then for any real number* η *and for any non-zero real numbers* $\lambda_1,\cdots,\lambda_s$ *which are not all of the same sign and not all in rational ratio, there are infinitely many solutions in primes* p_j *of the inequality*

$$|\eta+\sum_{j=1}^{s}\lambda_j\mathcal{P}_j(p_j)|<(\max p_j)^{-\beta}, \tag{1.5}$$

where β *is any constant with*

$$0<\beta<(\sqrt{21}-1)/(15(k+1)2^{2(k+1)}). \tag{1.6}$$

THEOREM 2. *Let* $k\geqq 2$ *and*

$$s\geqq 2^k+1. \tag{1.7}$$

Let η, β, $\mathcal{P}_j(x)$ *and* λ_j $(j=1,\cdots,s)$ *satisfy the same hypotheses of Theorem* 1. *Then* (1.5) *has infinitely many solutions in primes* p_j.

When k is small (1.7) is the same condition required by Schwarz [**10**] and Ramachandra [**9**] which is better than (1.4) only when $k\leqq 4$ (c. f. the table in § 7). So our Theorems 1 and 2 form an improvement of results in [**9**] in both the lower bound of s and the accuracy of the approximation. On the other hand, our Theorems 1, 2, 3 and Lemma 7 extend Theorems 1, 2 and Corollary 2.1 in [**12**] to polynomials of degree $k\geqq 2$ with some improvement in the upper bound of σ When $k=2$, 3, our extension fills up the gap left in [**12**]. By (1.2), (1.3), (1.4) we see that if k is large then the lower bound of $s<Ck\log k$ where the positive constant C can be arbitrarily close to 4.

For positive X, we use $U(X)=U$, with or without a suffix or superfix, to denote a finite set of distinct real numbers such that no element exceeds X in absolute value and each pair of different elements, say u_1, u_2, satisfies $|u_1-u_2|\geqq 1$. $U(X)$ has density ν (<1) if $|U|$, the number of elements of $U(X)$ satisfies

$$|U|>X^{\nu}. \tag{1.8}$$

In § 4 we shall prove

THEOREM 3. *Let* η, β, $\mathcal{P}_j(x)$, λ_j $(j=1,\cdots,s)$ *satisfy the same hypotheses of Theorem* 1. *Suppose that for each integer* $m>0$ *and sufficiently large* X *there exist sets* $U_t(X)$ $(t=1, 2)$ *having density* ν *such that each element in* U_t $(t=1, 2)$ *can be written in the form*

$$\sum_{j=0}^{m-1}\lambda_{M+t+2j}\mathcal{P}_j(p_j) \qquad (t=1,\,2)$$

with $\mathcal{P}_j(p_j)<X$, *where integer* $M=2l+1$ *and*

$$l\geqq k;\quad l>k(1-\nu)/2\theta. \tag{1.9}$$

If $s \geqq M+2m$ *then* (1.5) *has infinitely many solutions in primes* p_j.

As for polynomials we have to make some weaker estimates, especially in our Lemmas 5, 6, it seems that we are not in the position to obtain an even better lower bound for s, when k is small.

The method we use here can also be applied to improve the result in [8] concerning an analogous problem for mixed powers.

§ 2. Notation

Throughout, n and p with or without suffices denote positive integers and primes respectively. The integer k is always $\geqq 2$ except in § 5 where $k \geqq 3$. x is a real variable and $[x]$ is its integral part. We write $e(x)=\exp(2\pi ix)$. P and ε are sufficiently large and small positive numbers respectively such that all the approximations in this paper hold. We put $L=\log P$.

In view of (1.5) we always assume that for each of the above given polynomials $\mathcal{P}_j(x)=\sum_{h=0}^{k} \alpha_{jh} x^h$, the integer coefficients α_{jh} satisfy

$$(\alpha_{jk}, \cdots, \alpha_{j1})=1 \quad \text{and} \quad \alpha_{j0}=0. \tag{2.1}$$

Whenever there is no ambiguity we use $\mathcal{P}(x)=\sum_{n=1}^{k} \alpha_n x^n$ to denote any one of these modified polynomials. Without loss of generality, let $|\lambda_1| \leqq |\lambda_2|$ and suppose that $\lambda_1/\lambda_2<0$ and is irrational. For if λ_1/λ_2 is irrational but >0, then there must be some j with $2<j$ such that $\lambda_j/\lambda_1<0$ and hence $\lambda_j/\lambda_2<0$. But then λ_j/λ_1 and λ_j/λ_2 cannot both be rational. If $X>0$ we use $Y \ll X$ (or $X \gg Y$) to denote $|Y|<AX$, where A is some positive constant which may depend on the given constants k, s, λ_j, α_{jh} and ε only.

§ 3. Proof of Lemma 7

We proceed the proof of our theorems by lemmas.

LEMMA 1. *Suppose that for any given* $x \neq 0$ *there exist integers* a, q *such that* $|\alpha_k x - a/q| \leqq q^{-2}$ *with* $(a, q)=1$ *and* $P^{1-2\varepsilon}<q \leqq P^{k-1+\varepsilon}$. *Then*

$$\sum_{\varepsilon P \leqq n \leqq P} e(x\mathcal{P}(n)) \ll P^{1-\theta+\varepsilon A}, \tag{3.1}$$

where A *is a positive constant depending on* k *only.*

PROOF. When $k \leqq 12$, (3.1) follows from (1.2) and Lemma 3.6 in [**5**, p. 24]. When $k>12$, (3.1) follows from (1.2) and Theorem 9 in [**5**, p. 62]. The slight change of the range of q does not affect the original argument.

For the given β ((1.6)) let

$$\tau = P^{-\beta}, \tag{3.2}$$

$$K_\tau(x)=\begin{cases}\tau^2 & \text{if } x=0,\\ (\sin\pi\tau x)^2/(\pi x)^2 & \text{otherwise}.\end{cases} \tag{3.3}$$

Obviously $K_\tau(x)\ll\tau^2$.

LEMMA 2. *For any real y we have*

$$\int_{-\infty}^{\infty} e(xy)K_\tau(x)dx=\max(0,\ \tau-|y|).$$

PROOF. This follows from Lemma 4 in [3].

For integers a, q let

$$S(a,\ q)=\sum_{n=1}^{q} e(a\,\mathcal{P}(n)/q),$$

$$\mathcal{I}(y)=\int_{\varepsilon P}^{P} e(y\,\mathcal{P}(z))dz.$$

LEMMA 3. *Suppose that for any given x there exist integers a, q such that $|x-a/q|\leqq q^{-1}P^{-k+1-\varepsilon}$ with $(a,\ q)=1$, $1\leqq q\leqq P^{1-\varepsilon}$. Then*

$$\sum_{\varepsilon P\leqq n\leqq P} e(x\,\mathcal{P}(n))-q^{-1}S(a,\ q)\mathcal{I}(y)\ll P^{1-\theta},$$

where $y=x-a/q$.

PROOF. This follows from Lemma 7.11 in [5, p. 87] and (1.2)

LEMMA 4. *If $r\geqq 2$ then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mathcal{I}(y)|^r dy\ll P^{r-k}.$$

PROOF. Using integration by parts we have $\mathcal{I}(y)\ll P^{-k+1}|y|^{-1}$ if $y\neq 0$. Then the lemma follows from this and the partition of the interval $|y|\leqq\frac{1}{2}$ by $\pm P^{-k}$.

LEMMA 5. (a) *If $(q_1,\ q_2)=1$ then*

$$S(a_1q_2+a_2q_1,\ q_1q_2)=S(a_1,\ q_1)S(a_2,\ q_2). \tag{3.4}$$

(b) *Suppose that $(a,\ p)=1$ then for any positive integer h*

$$S(a,\ p^h)\ll p^{h(1-1/k)}. \tag{3.5}$$

PROOF. (3.4) is Theorem 2 in [6, p. 197] which is an easy generalization of a special case of Lemma 1 in [14, p. 46] to polynomials. (3.5) is the Fundamental Lemma in [5, p. 1].

For any positive integer r let

$$A_r(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q} |S(a,\ q)|^r q^{-r}. \tag{3.6}$$

LEMMA 6. *Suppose that* $r \geqq 2k$ *then*

$$\sum_{1 \leqq q \leqq Q} A_r(q) \ll L^C , \tag{3.7}$$

where $Q = P^{1-\varepsilon}$ *and* C *is a large positive constant.*

PROOF. By (3.4), (3.6),

$$A_r(q_1)A_r(q_2) = A_r(q_1 q_2) \tag{3.8}$$

provided that $(q_1, q_2) = 1$. Suppose that

$$\sum_{h=1}^{\infty} A_r(p^h) \ll p^{-1} . \tag{3.9}$$

Then it follows from (3.8), (3.9), $A_r(1)=1$ and $\sum_{p \leqq Q} p^{-1} < 2 \log\log P$ that

$$\sum_{1 \leqq q \leqq Q} A_r(q) \leqq \prod_{p \leqq Q} \Big(1 + \sum_{h=1}^{\infty} A_r(p^h)\Big) \ll \exp\Big(C \sum_{p \leqq Q} p^{-1}\Big) \ll L^C$$

as desired. It remains to prove (3.9). By (3.5), (3.6) and $r \geqq 2k$,

$$\sum_{h=1}^{\infty} A_r(p^h) \ll \sum_{h=1}^{\infty} p^{-h(r/k-1)} \ll p^{1-r/k} \ll p^{-1} .$$

This proves Lemma 6.

LEMMA 7. *Let* $U = U(P^{k-\varepsilon})$ *denote a set with density* ν *and*

$$F(x) = \sum_{u \in U} e(xu) .$$

Let

$$f(x) = \sum_{\varepsilon P \leqq n \leqq P} e(x \mathcal{P}(n)) . \tag{3.10}$$

For the given θ (1.2) *let* r *satisfy*

$$r \geqq 2k \quad \textit{and} \quad r > k(1-\nu)/\theta .$$

Then for each λ_j

$$\int_{-\infty}^{\infty} |f(\lambda_j x)^r F(x)^2| K_\tau(x) dx \ll \tau P^{r-k} L^C |U|^2 ,$$

where C *is the same constant in* (3.7).

PROOF. By Lemmas 1-4, 6 and the same argument as in the proof of Theorem 1 in [**12**, pp. 390-1] we can prove the lemma, so we omit the details here.

4. Proof of Theorem 3.

Let

$$\left.\begin{aligned} g(x) &= \sum_{\varepsilon P \leqq p \leqq P} e(x\,\mathcal{P}(p)), \\ I(x) &= \int_{\varepsilon P}^{P} e(x\,\mathcal{P}(y))/\log y\,dy, \end{aligned}\right\} \tag{4.1}$$

$$D=(\sqrt{21}-1)/10, \qquad \sigma_0=1-D. \tag{4.2}$$

We use $\rho=\sigma+it$ to denote a typical zero of the Riemann zeta function $\zeta(s)$ and $\sum'$ to denote the summation over all those zeros ρ with $|t|\leqq P^D$, $\sigma\geqq\sigma_0$. It is known that ([7, p. 291])

$$\sum{}'1 \ll P^{D3(1-\sigma_0)/(2-\sigma_0)}L^5 \ll P^D. \tag{4.3}$$

Let

$$G(x, \rho)=\sum_{(\varepsilon P)^k \leqq n \leqq P^k} n^{-1+\rho/k} e(x[\mathcal{P}(n^{1/k})])/\log n, \tag{4.4}$$

where $[\mathcal{P}]$ means the integral part of the value of $\mathcal{P}$. Put

$$J(x)=\sum{}' G(x, \rho), \tag{4.5}$$

$$\Delta(x)=g(x)+J(x)-I(x). \tag{4.6}$$

LEMMA 8. (a) *Suppose that* $2\leqq Y\leqq P$. *Then*

$$\sum_{p\leqq Y}\log p+\sum{}' Y^\rho \rho^{-1}-Y \ll P^{\sigma_0}L^2.$$

(b)
$$\sum{}' P^\sigma \ll P\exp(-L^{1/5}).$$

PROOF. Part (a) follows from the same proof as Lemma 3 in [11]. Part (b) can be shown by the same proof as Lemma 8 of [11, p. 379].

LEMMA 9. *We have*

$$\Delta(x) \ll P^{\sigma_0}L^5(1+|x|P^k).$$

PROOF. As one may regard the integral in (4.1) as the difference of two integrals with different upper limits εP, P respectively but with the same lower limit 2, in the proof we replace εP in (4.1) simply by 2. By the same reason we replace $(\varepsilon P)^k$ in (4.4) by 2. Let

$$a_n=\begin{cases} \log n+\sum' n^{-1+\rho/k} & \text{if } n=p^k \text{ for some } p\leqq P, \\ \sum' n^{-1+\rho/k} & \text{otherwise}. \end{cases} \tag{4.7}$$

$$b_n=e(x[\mathcal{P}(n^{1/k})])/\log n \quad\text{and}\quad b'_n=e(x\mathcal{P}(n^{1/k}))/\log n.$$

Then we have

$$g(x)+J(x)=\sum_{2\leqq n\leqq P^k}\{a_n(b_n-b'_n)+a_n b'_n\}=S_1+S_2, \text{ say.} \tag{4.8}$$

As for any real y

$$e(x[y])-e(xy)\ll|x|$$

and $\mathcal{P}(n)$ is integral valued we have

$$S_1=\sum{}'\sum_{2\leq n\leq P^k}n^{-1+\rho/k}(b_n-b_n')$$

$$\ll|x|\sum{}'P^\sigma\ll|x|P\exp(-L^{1/5}). \tag{4.9}$$

The last inequality follows from Lemma 8 (b).

We come now to consider S_2. By a similar argument as the first part of the proof of Lemma 5 in [**11**] we have

$$\sum_{n\leq z}n^{(\rho/k)-1}-z^{\rho/k}(k/\rho)\ll P^D \tag{4.10}$$

if $z\leqq P^k$, $\sigma_0\leqq\sigma<1$, $|t|\leqq P^D$. It follows from (4.7), (4.10), (4.3), (4.2) and Lemma 8 (a) that for any $z\leqq P^k$

$$\sum_{n\leq z}a_nk^{-1}-z^{1/k}=\sum_{p\leq z^{1/k}}\log p+\sum{}'z^{\rho/k}\rho^{-1}-z^{1/k}+O(P^D)\sum{}'1\ll P^{\sigma_0}L^5. \tag{4.11}$$

Let $A(z)=\sum_{n\leq z}a_n$, $P_1=([P^k]+1)^{1/k}$ and $\mathcal{D}(x, z)=\dfrac{d}{dz}\{e(x\mathcal{P}(z^{1/k}))/\log z\}$. We see that

$$b_{n+1}'-b_n'=\int_n^{n+1}\mathcal{D}(x, z)dz.$$

By Abel's partial summation we have

$$S_2=A(P^k)e(x\mathcal{P}(P_1))/(k\log P_1)-a_1b_2'-\sum_{2\leq n\leq P^k}A(n)\int_n^{n+1}\mathcal{D}(x, z)dz$$

On the other hand, by (4.1), integration by parts and $y=z^{1/k}$ we have

$$I(x)=\int_2^{P_1}\frac{e(x\mathcal{P}(y))}{\log y}dy+O(1/L)=O(1+|x|)+e(x\mathcal{P}(P_1))P/\log P_1$$

$$-\int_2^{P_1^k}kz^{1/k}\mathcal{D}(x, z)dz$$

since $\int_2^{2^k}kz^{1/k}\mathcal{D}(x, z)dz\ll|x|$. So in view of (4.11) we have

$$S_2-I(x)=-a_1b_2'+O(1+|x|)+\frac{e(x\mathcal{P}(P_1))}{\log P_1}(A(P^k)k^{-1}-P)$$

$$+\int_2^{P_1^k}ke(x\mathcal{P}(z^{1/k}))\{2\pi ixz\log z\frac{d}{dz}\mathcal{P}(z^{1/k})-1\}(z(\log z)^2)^{-1}(z^{1/k}-A(z)k^{-1})dz$$

$$\ll P^{\sigma_0}L^5(1+|x|P^k) \tag{4.12}$$

since by (4.7), (4.3) we have $a_1 b_2' \ll \sum' 1 \ll P^D$. Lemma 9 follows from (4.8), (4.9) and (4.12).

LEMMA 10. *Suppose that β_1 is a positive constant $<2D/3$ ((4.2)). Let*

$$\gamma = P^{-k+\beta_1}. \tag{4.13}$$

We have

$$I(x) \ll P \min(1, |x|^{-1} P^{-k}), \tag{4.14}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |J(x)|^2 dx \ll P^{2-k} \exp(-2L^{1/5}), \tag{4.15}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |I(x)|^2 dx \ll P^{2-k}, \tag{4.16}$$

$$\int_{-\gamma}^{\gamma} |\Delta(x)|^2 dx \ll P^{2-k} \exp(-2L^{1/5}), \tag{4.17}$$

$$\int_{-\gamma}^{\gamma} |g(x)|^2 dx \ll P^{2-k}. \tag{4.18}$$

PROOF. (4.14) follows from (4.1) by partial integration.

By (4.5) and Schwarz's inequality,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |J(x)|^2 dx \leqq \Big(\sum' \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x, \rho)|^2 dx\Big)^{1/2}\Big)^2. \tag{4.19}$$

Note that for any large integers m, n with $|m-n| \geqq 2$,

$$[\mathcal{P}(m^{1/k})] \neq [\mathcal{P}(n^{1/k})].$$

Let $H(n) = n^{-1+\sigma/k}(\log n)^{-1}$. Then by (4.4), Parseval's identity and $\sigma < 1$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |G(x, \rho)|^2 dx \ll \sum_{(\varepsilon P)^k \leqq n \leqq P^k} (H(n)^2 + H(n)H(n-1) + H(n)H(n+1)) \ll P^{-k+2\sigma} L^{-2}. \tag{4.20}$$

Then (4.15) follows from (4.19), (4.20) and Lemma 8(b).

(4.16) follows from (4.14) and the partition of the interval $|x| \leqq \frac{1}{2}$ by $\pm P^{-k}$.

By Lemma 9 and (4.13),

$$\int_{-\gamma}^{\gamma} |\Delta(x)|^2 dx \ll P^{2\sigma_0} L^{10} P^{-k+3\beta_1}.$$

(4.17) follows from this and (4.2), $\beta_1 < 2D/3$.

(4.18) follows from (4.6), (4.15), (4.16), (4.17) easily. This proves Lemma 10.

In what follows the suffix j in any function $h_j(x)$ depending on a polynomial $\mathcal{P}$ indicates that this polynomial is now specified to be the given polynomial

$\mathcal{P}_j$. For simplicity, sometimes we shall denote $h_j(\lambda_j x)$ by h_j. Let

$$\Psi(x)=\prod_{j=1}^{M} g_j(\lambda_j x), \qquad \Psi^*(x)=\prod_{j=1}^{M} I_j(\lambda_j x). \tag{4.21}$$

For the given β ((1.6)) let β_1 be any constant satisfying

$$\alpha^{-1}\beta<\beta_1<2D/3, \tag{4.22}$$

where

$$\alpha=(2^{2(k+1)}(k+1))^{-1} \tag{4.23}$$

and $k\geqq 2$.

Partition the real line into E_1, E_2, E_3 as

$$E_1=\{x \mid |x|\leqq P^{-k+\beta_1}\}, \qquad E_2=\{x \mid P^{-k+\beta_1}<|x|\leqq P^{\beta_1}\},$$
$$E_3=\{x \mid |x|>P^{\beta_1}\}. \tag{4.24}$$

LEMMA 11. *We have*

$$\int_{E_1} |\Psi(x)-\Psi^*(x)| K_\tau(x)dx \ll \tau^2 P^{M-k} \exp(-L^{1/5}).$$

PROOF. By (3.3), (4.21), (4.6),

$$\int_{E_1} |\Psi(x)-\Psi^*(x)| K_\tau(x)dx \ll \tau^2 \int_{E_1} \Big| \sum_{j=1}^{M} (\Delta_j - J_j) \prod_{1}^{j-1} g_n \prod_{j+1}^{M} I_n \Big| dx,$$

where $\prod_{1}^{0} g_n=\prod_{M+1}^{M} I_n=1$. Note that $|g_n|$, $|I_n|<P$ and then for each j

$$\prod_{1}^{j-1} g_n \prod_{j+1}^{M} I_n \ll (|g_1|+|I_M|)P^{M-2}.$$

By Schwarz's inequality and Lemma 10, for each $j=1, \cdots, M$ we have

$$\int_{E_1} |\Delta_j - J_j| (|g_1|+|I_M|)dx \ll P^{2-k} \exp(-L^{1/5}).$$

Hence Lemma 11 follows.

LEMMA 12. *Suppose that for any given x there are some integers a, q with* $(a, q)=1$, $1\leqq q$ *such that* $|x-a/q|\leqq q^{-2}$. *If*

$$\log V>2^{6k-2}(2k+1) \log\log P, \tag{4.25}$$

where $V=\min(P^{1/3}, q, P^k q^{-1})$. *Then*

$$\sum_{p\leqq P} e(x\mathcal{P}(p)) \ll PV^{-\alpha},$$

where α is defined in (4.23).

PROOF. This lemma follows easily from the theorem in [**13**, p. 5].

LEMMA 13. *Suppose that $j=1$, 2 and $x\in E_2$. If there are integers a_j, q_j with* $(a_j, q_j)=1$ *and* $q_j\geqq 1$ *such that*

$$|\lambda_j x - a_j/q_j| \leqq q_j^{-1} P^{-k+\beta_1 \varepsilon} \tag{4.26}$$

then

$$\max(q_1, q_2) \geqq P^{\beta_1}.$$

PROOF. The argument is based on the method used in the proof of Lemma 13 in [4]. We see that

$$a_1 a_2 \neq 0. \tag{4.27}$$

For if $a_1=0$ then by (4.26), $|x| \leqq P^{-k+\beta_1}\varepsilon|\lambda_1|^{-1}$ which contradicts $x \in E_2$ ((4.24)) as ε would be very small. Suppose that

$$\max(q_1, q_2) < P^{\beta_1}. \tag{4.28}$$

By (4.26), (4.28), and $x \in E_2$ we have

$$\left|\frac{a_2}{q_2}\right|\left|\frac{1}{\lambda_2 x}\right|\left|\lambda_1 x - \frac{a_1}{q_1}\right| q_1 q_2 \leqq (|\lambda_2 x| + \varepsilon q_2^{-1} P^{-k+\beta_1})|\lambda_2 x|^{-1} q_2 \varepsilon P^{-k+\beta_1}$$
$$\leqq (q_2 + \varepsilon|\lambda_2|^{-1})\varepsilon P^{-k+\beta_1} \leqq 2\varepsilon P^{-k+2\beta_1}. \tag{4.29}$$

Similarly since $|\lambda_1| \leqq |\lambda_2|$ we have

$$\left|\frac{a_1}{q_1}\frac{1}{\lambda_2 x} q_1 q_2 \left(\lambda_2 x - \frac{a_2}{q_2}\right)\right| \leqq 2\varepsilon P^{-k+2\beta_1}. \tag{4.30}$$

It follows from (4.29), (4.30) that

$$\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| \leqq 4\varepsilon P^{-k+2\beta_1}. \tag{4.31}$$

On the other hand, since λ_1/λ_2 is irrational it is known that there are infinitely many convergents a/q with $(a, q)=1$, $1 \leqq q$ such that

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a}{q}\right| < \frac{1}{2q^2}.$$

So for any integers a', q' with $1 \leqq q' < q$ we have

$$\left|q'\frac{\lambda_1}{\lambda_2} - a'\right| \geqq q'\left(\frac{|aq'-a'q|}{qq'} - \left|\frac{\lambda_1}{\lambda_2} - \frac{a}{q}\right|\right) > \frac{1}{q} - \frac{q'}{2q^2} > \frac{1}{2q}. \tag{4.32}$$

If $P = q^{(1/(k-2\beta_1))}$ and $\varepsilon < 1/8$ then in view of (4.27), (4.31) and (4.32) we must have

$$|a_2 q_1| \geqq q = P^{k-2\beta_1}. \tag{4.33}$$

But by (4.26), (4.28) and $x \in E_2$ we have

$$\left|\frac{a_2}{q_2}\right| q_1 q_2 \leqq (|\lambda_2 x| + \varepsilon q_2^{-1} P^{-k+\beta_1}) P^{2\beta_1} \leqq 2|\lambda_2| P^{3\beta_1}. \tag{4.34}$$

Thus (4.34) contradicts (4.33) since by (4.22) $\beta_1 < 2D/3 < k/5$. This proves Lemma 13.

LEMMA 14. *We have, for any constant $B>0$*

$$\sup_{x\in E_2}\min(|g_1(\lambda_1 x)|,\ |g_2(\lambda_2 x)|)\ll\tau PL^{-B}.$$

PROOF. For any $x\in E_2$ and $j=1, 2$ by Dirichlet's theorem there are integers a_j, q_j with $(a_j, q_j)=1$ and $1\leqq q_j\leqq P^{k-\beta_1}\varepsilon^{-1}$ such that

$$|\lambda_j x-a_j/q_j|\leqq q_j^{-1}P^{-k+\beta_1}\varepsilon.$$

Then in view of Lemma 13, we may let $q_1\geqq P^{\beta_1}$. We see that now (4.25) is satisfied with $V=\min(P^{1/3}, q, P^k q^{-1})\geqq\varepsilon P^{\beta_1}$. By Lemma 12, (4.22), (3.2)

$$g_1(\lambda_1 x)\ll P^{1-\alpha\beta_1}\ll\tau PL^{-B}.$$

This proves Lemma 14.

In what follows we always assume that $U_t=U_t(P^{k-\varepsilon})$ $(t=1, 2)$ satisfy (1.8) and

$$F_t(x)=\sum_{u\in U_t}e(xu).$$

LEMMA 15. *Let M in* (4.21) *be an odd integer and $l=(M-1)/2$ satisfying* (1.9). *Then for any constant $B>0$*

$$\int_{E_2}|\Psi(x)F_1(x)F_2(x)|K_\tau(x)dx\ll\tau^2 P^{M-k}|U_1||U_2|L^{-B+C},$$

where C is the same constant in (3.7).

PROOF. By Lemmas 2, 7, for $j=1, 2, \cdots, M$; $t=1, 2$,

$$\int_{-\infty}^{\infty}|g_j(\lambda_j x)^l F_t(x)|^2 K_\tau(x)dx\ll\int_{-\infty}^{\infty}|f_j(\lambda_j x)^l F_t(x)|^2 K_\tau(x)dx$$
$$\ll\tau P^{2l-k}|U_t|^2 L^C. \tag{4.35}$$

By Hölder's inequality we have

$$\mathcal{S}_1=\int_{E_2}\prod_{j=2}^{M}|g_j F_1(x)F_2(x)|K_\tau(x)dx$$
$$\leqq\Big(\int_{-\infty}^{\infty}\prod_{j=2}^{l+1}|g_j F_1(x)|^2 K_\tau(x)dx\Big)^{\frac12}\Big(\int_{-\infty}^{\infty}\prod_{j=l+2}^{M}|g_j F_2(x)|^2 K_\tau(x)dx\Big)^{\frac12}$$
$$\leqq\Big(\prod_{j=2}^{l+1}\Big(\int_{-\infty}^{\infty}|g_j^l F_1(x)|^2 K_\tau(x)dx\Big)^{1/l}\Big)^{\frac12}\Big(\prod_{j=l+2}^{M}\Big(\int_{-\infty}^{\infty}|g_j^l F_2(x)|^2 K_\tau(x)dx\Big)^{1/l}\Big)^{\frac12}.$$

The same result holds for $\mathcal{S}_2=\int_{E_2}\prod_{j\neq 2}|g_j F_1(x)F_2(x)|K_\tau(x)dx$. So by Lemma 14 and (4.35)

$$\int_{E_2}|\Psi(x)F_1(x)F_2(x)|K_\tau(x)dx\ll\tau PL^{-B}(\mathcal{S}_1+\mathcal{S}_2)$$
$$\ll\tau PL^{-B}(\tau P^{2l-k}|U_1||U_2|L^C).$$

This proves Lemma 15.

LEMMA 16. *Let* $\Omega(x)=\sum e(x\,\omega(y_1, \cdots, y_n))$, *where* ω *is any real valued function and the summation is over any finite set of values of* $y_1, \cdots, y_n$. *Then for any* $X>4/\tau$ *we have*

$$\int_{|x|>X}|\Omega(x)|^2K_\tau(x)dx\ll(X\tau)^{-1}\int_{-\infty}^{\infty}|\Omega(x)|^2K_\tau(x)dx\,.$$

PROOF. By Lemma 2 in [4], for any $A>4$

$$\int_{|z|>A}|\Omega(z)|^2K_1(z)dz\ll(1/A)\int_{-\infty}^{\infty}|\Omega(z)|^2K_1(z)dz\,,$$

where $K_1(z)$ means $K_\tau(z)$ with τ replaced by 1. Then our Lemma 16 follows from simple substitutions.

LEMMA 17. *With the same hypotheses as Lemma* 15 *on* M, *we have for any constant* $B>0$,

$$\int_{E_3}|\Psi(x)F_1(x)F_2(x)|K_\tau(x)dx\ll\tau^2P^{M-k}|U_1||U_2|L^{-B+C}\,,$$

where C *is the same constant in* (3.7).

PROOF. By Lemmas 16, 7 and (3.2), (4.23), for any $j=1, 2, \cdots, M$; $t=1, 2$,

$$\int_{E_3}|g_j(\lambda_jx)^lF_t(x)|^2K_\tau(x)dx\ll P^{-\beta_1}P^{2l-k}|U_t|^2L^C$$
$$\ll\tau^2P^{2l-k}|U_t|^2L^{-B+C}\,. \tag{4.36}$$

Then Lemma 17 follows from Hölder's inequality, (4.36) and a similar argument as that in Lemma 15.

LEMMA 18. *For any constant* $B>0$, *we have*

$$\int_{x\in E_1}|\Psi^*(x)|K_\tau(x)dx\ll\tau^2P^{M-k}L^{-B}\,, \tag{4.37}$$

$$\int_{-\infty}^{\infty}\Psi^*(x)F_1(x)F_2(x)e(\eta x)K_\tau(x)dx\gg\tau^2P^{M-k}|U_1||U_2|L^{-M}\,. \tag{4.38}$$

PROOF. By (4.14), for $|x|>P^{-k+\beta_1}$,

$$I_j(\lambda_jx)\ll P^{1-k}|x|^{-1}\,. \tag{4.39}$$

(4.37) follows from (4.39), (3.3).

We come now to prove (4.38). For each pair (u_1, u_2) with $u_t\in U_t$ $(t=1, 2)$ define a set $\mathscr{B}^*$ in the M-dimensional space by the following (4.40), (4.41), (4.42).

$$\varepsilon^2P^k\leqq z_j\leqq2\varepsilon^2P^k\qquad(j=3, 4, \cdots, M)\,, \tag{4.40}$$

$$\varepsilon|\lambda_1/\lambda_2|P^k\leqq z_2\leqq2\varepsilon|\lambda_1/\lambda_2|P^k\,. \tag{4.41}$$

For real y with $|y|\leqq\dfrac{\tau}{2}$, let $z_1>0$ satisfy

$$\mathcal{P}_1(z_1^{1/k})=-\mathcal{P}_2(z_2^{1/k})\lambda_2/\lambda_1+y/\lambda_1-\xi/\lambda_1, \tag{4.42}$$

where

$$\xi=\eta+u_1+u_2+\sum_3^M \lambda_j\,\mathcal{P}_j(z_j^{1/k}).$$

Such z_1 is uniquely defined if $\mathcal{P}_1(z_1^{1/k})$ is large. We shall prove that

$$(\varepsilon P)^k<z_1<P^k.$$

Hence if $\mathcal{B}$ denotes the cartesian product of the intervals $(\varepsilon P)^k\leqq z_j\leqq P^k$ $(j=1, 2, \cdots, M)$ then

$$\mathcal{B}^*\subset\mathcal{B}. \tag{4.43}$$

Note that for large z_j,

$$\frac{1}{2}\alpha_{jk}z_j<\mathcal{P}_j(z_j^{1/k})<2\alpha_{jk}z_j. \tag{4.44}$$

It follows from $\lambda_1/\lambda_2<0$, (4.42), (4.44), (4.41), (4.40) and $|u_t|\leqq P^{k-\varepsilon}$ that

$$\mathcal{P}_1(z_1^{1/k})\geqq\frac{1}{2}\alpha_{2k}\varepsilon P^k-(2|\lambda_1|)^{-1}$$

$$-(|\eta|+2P^{k-\varepsilon}+4\varepsilon^2P^k\sum_3^M|\lambda_j|\alpha_{jk})/|\lambda_1|>\frac{1}{3}\alpha_{2k}\varepsilon P^k$$

and then $z_1>(\varepsilon P)^k$. Similarly we have $z_1<P^k$. This proves (4.43). By Lemma 2, (4.42), (4.43)

$$\int_{-\infty}^{\infty}\Psi^*(x)F_1(x)F_2(x)e(x\eta)K_\tau(x)dx$$

$$=\sum_{u_1,u_2}\int_{\mathfrak{B}}\prod_1^M z_j^{-1+1/k}(\log z_j)^{-1}\max(0,\ \tau-|\xi+\sum_1^2\lambda_j\mathcal{P}_j(z_j^{1/k})|)dz_1\cdots dz_M$$

$$\geqq\frac{1}{2}\tau\sum_{u_1,u_2}\int_{\mathfrak{B}^*}\prod_1^M z_j^{-1+1/k}(\log z_j)^{-1}dz_1\cdots dz_M$$

$$\gg\tau P^{(1-k)M}L^{-M}|U_1||U_2|\tau P^{(M-1)k}.$$

This proves (4.38).

We come now to the proof of Theorem 3. Let $X=P^{k-\varepsilon}$. It suffices to consider $s=M+2m$. By Lemma 2,

$$\int_{-\infty}^{\infty}e(x\eta)\Psi(x)F_1(x)F_2(x)K_\tau(x)dx$$

$$=\sum\max(0,\ \tau-|\eta+\sum_1^s\lambda_j\mathcal{P}_j(p_j)|),$$

where $\sum$ is over $u_t\in U_t$ $(t=1, 2)$; $\varepsilon P<p_j<P$ $(j=1, 2, \cdots, M)$. Obviously Theorem 3

follows if the above integral tends to infinity as $P\rightarrow\infty$. Let the constants B in Lemmas 14, 15, 17, 18 satisfy

$$B>M+C, \tag{4.45}$$

where C is the same constant in (3.7). By (4.38), (4.37), Lemma 11 and (4.45),

$$\int_{E_1}\Psi(x)F_1(x)F_2(x)e(x\eta)K_\tau(x)dx\gg\tau^2P^{M-k}|U_1||U_2|L^{-M}. \tag{4.46}$$

Then by Lemma 15, 17, (4.46), (4.45), the first integral in the proof $\gg\tau^2P^{M-k}|U_1||U_2|L^{-M}$ as desired. This proves Theorem 3 as $M>k+2\beta$ ((1.9), (1.6)).

§ 5. Proof of Theorem 1.

LEMMA 19. *For each large X let $U(X)$ be a set with density $\nu-\varepsilon$, where $1/k<\nu<1$ and let*

$$\mu=\mu(\nu)=k^{-1}(1+(k-1)\nu)+(\nu/k)\max_{n\leq k-2}(n+1-\nu(k-1))/(2^n-1+\nu). \tag{5.1}$$

Suppose further that λ is a non-zeroreal number and

$$\phi(\nu)=(\mu k-1)/\nu k. \tag{5.2}$$

Then, for each sufficiently large Y, there is a set $U^*(Y)$ with density $\mu-\varepsilon$ such that every element in U^* can be written in the form $\lambda\mathcal{P}(p)+u$ with $\mathcal{P}(p)<Y$ and $u\in U(Y^\phi)$.

PROOF. Following the same argument as Lemma 16 in [12] we can prove Lemma 19 without difficulty, since the main tool, Theorems 1, 2 in [2] can be easily generalized to polynomials with integral coefficients.

We come now to the proof of Theorem 1. In view of Theorem 3 it suffices to show that for each integer $m>0$ there exist $U_1(P^{k-\varepsilon})$, $U_2(P^{k-\varepsilon})$ satisfying all hypotheses in Theorem 3. We shall construct U_1 only and U_2 can be obtained in exactly the same way.

Let

$$\nu_0=1/k+\varepsilon/2,$$

$$\nu_j=\begin{cases}2/k & \text{if } j=1,\\ k^{-1}(1+(k-1)\nu_{j-1}) & \text{if } 2\leqq j\leqq m-1.\end{cases} \tag{5.3}$$

Whenever there is no ambiguity we shall drop j from ν_j. Let (cf. (5.1), (5.2))

$$\mu_{j+1}=\mu(\nu_j),\qquad \phi_{j+1}=\phi(\nu_j),$$

$$\kappa(n)=(n+1-\nu(k-1))/(2^n-1+\nu).$$

Obviously, for $j\geqq1$

$$\mu_{j+1}=\nu_{j+1}+(\nu_j/k)\max_n \kappa(n)\,, \tag{5.4}$$

$$\phi_{j+1}=(1-1/k)+k^{-1}\max_n \kappa(n)\,, \tag{5.5}$$

where $n\leqq k-2$. By (5.3), for $j\geqq 1$

$$\nu_j=1-(1-2/k)(1-1/k)^{j-1} \tag{5.6}$$

and so

$$1/k<\nu_j<1\,. \tag{5.7}$$

Also we have

$$\nu_j<\mu_j\,. \tag{5.8}$$

For if $j\geqq 2$, (5.8) follows from (5.4) and $\max_n \kappa(n)\geqq\kappa(k-2)>0$. For $j=1$, by (5.1),

$$\mu(\nu_0)\geqq k^{-1}(1+(k-1)\nu_0+\nu_0\kappa(1))\geqq k^{-1}\Big(2+\frac{1}{2}\varepsilon(k-2)\Big)\,,$$

as

$$\nu_0\kappa(1)>1/k-\varepsilon/2\,.$$

Next, if $\kappa(n)$ attains its maximum at $h>1$ then

$$(h-(k-1)\nu)/(2^{h-1}-1+\nu)\leqq(h+1-(k-1)\nu)/(2^h-1+\nu)$$

or

$$\kappa(h)\leqq 2^{1-h}\,.$$

Whence

$$\max_n \kappa(n)<1 \tag{5.9}$$

as for the case $h=1$, (5.7) can be applied. It follows from (5.5), (5.9) that

$$\phi_j<1\,. \tag{5.10}$$

Let η_j be m non-zero real numbers and $\Pi_j(x)$ be m polynomials satisfying (2.1). For any large real number X there is a set $U^{(0)}(X)$ with density $\nu_0-\varepsilon$ such that every element in $U^{(0)}$ is of the form $\eta_0\Pi_0(p_0)$ with $\Pi_0(p_0)<X$. For by the prime number theorem the above requirements produce the set $U^{(0)}(X)$ immediately. By (5.7), (5.8) we are now able to apply Lemma 19 iteratively and finally for large Y we obtain a set $U^{(m-1)}(Y)$ with density $\nu_{m-1}-\varepsilon$ such that every element in $U^{(m-1)}$ is of the form

$$\sum_{j=0}^{m-1}\eta_j\Pi_j(p_j)$$

with $\Pi_j(p_j)<Y^{\phi'_j}$, where $\phi'_j=\prod_{j+1}^{m-1}\phi_n$ if $j<m-1$ and $\phi'_{m-1}=1$. By (5.10), all $\Pi_j(p_j)<Y$.

On the other hand, if we set $m=N+3$, then by (1.3)

$$m>2+(\log(1-2/k)-\log 2\theta)/(-\log(1-1/k)).$$

Hence by (5.6)

$$1-2\theta<\nu_{m-1}.$$

If we take $l(=(M-1)/2)=k$, we have $l>k(1-\nu_{m-1})/2\theta$. So $M=2k+1$ satisfies (1.9). Putting $Y=P^{k-\varepsilon}$, $\eta_j=\lambda_{M+2j+1}$, $\Pi_j=\mathcal{P}_{M+2j+1}$ and $U^{(m-1)}(Y)=U_1$ we see that this $U_1(P^{k-\varepsilon})$ satisfies all hypotheses in Theorem 3. This proves Theorem 1.

§ 6. Proof of Theorem 2.

In Lemmas 20, 21, 22, C' will denote the same positive constant depending on k only.

LEMMA 20. *We have*

$$\int_0^1 |\sum_{n\leq P} e(x\mathcal{P}(n))|^{2k} dx \ll P^{2k-k}L^{C'}.$$

PROOF. This is Theorem 4 in [5, p. 19].

LEMMA 21. *Let $r=2^k$. For each λ_j we have*

$$\int_{-\infty}^{\infty} |f(\lambda_j x)|^r K_\tau(x)dx \ll \tau P^{r-k}L^{C'},$$

where $f(\lambda_j x)$ is defined in (3.10).

PROOF. Let $t=r/2$. It follows from Lemma 2 that

$$\begin{aligned}\int_{-\infty}^{\infty} |f(\lambda_j x)|^r K_\tau(x)dx &= \sum\nolimits_1 \int_{-\infty}^{\infty} e(x\lambda_j \sum_{l=1}^{t} \mathcal{P}(n_l)-\mathcal{P}(m_l))K_\tau(x)dx \\ &= \sum\nolimits_1 \max(0, \tau-|\lambda_j \sum_{l=1}^{t} \mathcal{P}(n_l)-\mathcal{P}(m_l)|) \\ &\leqq \tau \sum\nolimits_2 1, \end{aligned} \tag{6.1}$$

where the summation $\sum_1$ is over all integers $\varepsilon P\leqq n_l$, $m_l\leqq P$ $(l=1, \cdots, t)$ and the summation $\sum_2$ is over all positive integers n_l, $m_l\leqq P$ $(l=1, \cdots, t)$ satisfying $\sum_{l=1}^{t} \mathcal{P}(n_l)-\mathcal{P}(m_l)=0$. Since

$$\sum\nolimits_2 1=\int_0^1 |\sum_{n\leq P} e(x\mathcal{P}(n))|^{2t}dx$$

Lemma 21 follows from (6.1) and Lemma 20.

LEMMA 22. *Let M in* (4.21) *be defined by*

$$M=2^k+1.$$

Then for any constant $B>0$ *we have*

$$\int_{E_2}|\Psi(x)|K_\tau(x)dx \ll \tau^2 P^{M-k} L^{-B+C'},$$

$$\int_{E_3}|\Psi(x)|K_\tau(x)dx \ll \tau^2 P^{M-k} L^{-B+C'},$$

$$\int_{-\infty}^{\infty}\Psi^*(x)e(\eta x)K_\tau(x)dx \gg \tau^2 P^{M-k} L^{-M}.$$

PROOF. The proof follows from the same argument as that of Lemmas 15, 17, 18 except now we should apply Lemma 21 instead of Lemma 7.

With the help of Lemma 2, 11, 22 and (4.37) we can prove Theorem 2 by a similar argument as that of Theorem 3.

§ 7. Remark.

By (1.4), (1.7), we obtain here the numerical lower bound of s in our theorems when $k \leqq 12$.

k	2	3	4	5	6	7	8	9	10	11	12
$2(k+N)+7$		11	19	31	43	61	79	103	127	155	185
2^k+1	5	9	17	33	65	129	257	513	1025	2049	4097

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