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On graded rings, I

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Introduction.

In this paper, we study a Noetherian graded ring R and the category of graded R-modules. We consider injective objects of this category and we define the graded Cousin complex of a graded R-module M. These concepts are essential in this paper (see, (1.2.1), (1.2.4) and (1.3.3)).

We say that R is a graded ring defined over a field k, if R is positively graded, $R_0=k$ and if R is finitely generated over k. We denote by $\mathfrak{m}$ the unique graded maximal ideal of R. $\mathfrak{m}=R_+=\bigoplus_{n>0} R_n$. In the latter part of this paper, we treat graded rings defined over k. If R is a graded ring defined over k, the category of graded R-modules has very simple dualizing functor and dualizing module. The dualizing functor is given by $\underline{\mathrm{Hom}}_k(\ ,\underline{k})$ and the dualizing module is $\underline{\mathrm{Hom}}_k(R, \underline{k})$ (see, (1.2.7) and (1.2.10)).

Also, in this category, the dual of a graded Noetherian (resp. Artinian) R-module is a graded Artinian (resp. Noetherian) R-module. We need not consider the completion of R.

Let R be a graded ring defined over k and let M be a finitely generated graded R-module. We know that several properties of M are determined by its local cohomology groups $\underline{H}^i_{\mathfrak{m}}(M)$ $(i=0, 1, \cdots)$. For example, M is a Macaulay R-module if and only if $\underline{H}^i_{\mathfrak{m}}(M)=0$ for $i<d=\dim M$ and M is a Gorenstein R-module if and only if $\underline{H}^i_{\mathfrak{m}}(M)=0$ for $i<d$ and $\underline{H}^d_{\mathfrak{m}}(M)$ is an injective R-module. So we study several techniques to calculate local cohomology groups for some operations in the category of graded R-modules (see, (2.2.5), (3.1.1) and (4.1.5)).

The theory of the canonical module of a Noetherian local ring was developed in [**15**]. We define the canonical module K_R of a graded ring R defined over k as a graded R-module. $K_R=(\underline{H}^d_{\mathfrak{m}}(R))^*$ ($d=\dim R$ and $(\ )^*$ denotes the dual). If R is a Macaulay ring, R is a Gorenstein ring if and only if $K_R\cong R(a)$ for some integer a. This integer $a=a(R)$ is an important invariant of R and plays an essential role in Chapter 3 and Chapter 4 (see, (3.1.5), (3.2.1) and (4.4.7)).

A graded ring R has a geometric object attached to it——Proj(R). If R_+ is generated by R_1, the relationship of R and Proj(R) is treated in [8]. But the condition "R_+ is generated by R_1" is too strong for us. We must seek better conditions to find more examples of graded rings. We introduce a condition (♯) in Chapter 5 and we will show that this condition is sufficiently strong to relate the geometric languages and ring-theoretic languages. In particular, we see how the canonical module of R and the dualizing module of Proj (R) is related to each other when R is a graded ring defined over k.

In this paper, all rings are assumed to be commutative with identity element. All modules are assumed to be unitary. All homomorphism of rings are assumed to send identity element to identity element.

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Chapter 1. Noetherian graded rings.

In this chapter let $R=\bigoplus_{n\in \boldsymbol{Z}} R_n$ be a Noetherian graded ring.

By definition, a graded R-module is an R-module M with a family $\{M_n\}_{n\in \boldsymbol{Z}}$ of subgroups such that (1) $M=\bigoplus_{n\in \boldsymbol{Z}} M_n$ and (2) $R_nM_m\subset M_{n+m}$ for all $n, m\in \boldsymbol{Z}$.

A homomorphism $f: M \to N$ is, by definition, an R-linear map such that $f(M_n) \subset N_n$ for all $n \in \boldsymbol{Z}$. We denote by $M_H(R)$ the category of all the graded R-modules and their homomorphisms.

Let M, N be graded R-modules and let $n \in \boldsymbol{Z}$. We denote by $N(n)$ the graded R-module which coincides with N as the underlying R-module and whose grading is given by $[N(n)]_m = N_{n+m}$ for all $m \in \boldsymbol{Z}$. Let $\underline{\mathrm{Hom}}_R(M, N)_n$ denote the abelian group of all the homomorphisms from M into $N(n)$. We put $\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{n \in \boldsymbol{Z}} \underline{\mathrm{Hom}}_R(M, N)_n$ and consider it as a graded R-module with $\{\underline{\mathrm{Hom}}_R(M, N)_n\}_{n \in \boldsymbol{Z}}$ as its grading. The derived functors of $\underline{\mathrm{Hom}}_R(\ ,\ )$ will be denoted by $\underline{\mathrm{Ext}}_R^i(\ ,\ )$. Since R is Noetherian, $\underline{\mathrm{Ext}}_R^i(M, N) = \mathrm{Ext}_R^i(M, N)$ as the underlying R-module if M is a finitely generated graded R-module.

Let $(M \underset{R}{\otimes} N)_n$ denote the subgroup of $M \underset{R}{\otimes} N$ generated by the elements of the form $x \otimes y$ where $x \in M_i$ and $y \in N_j$ with $i + j = n$. We consider $M \underset{R}{\otimes} N$ as a graded R-module with $\{(M \underset{R}{\otimes} N)_n\}_{n \in \boldsymbol{Z}}$ as its grading and denote it by $M \underset{\underline{R}}{\otimes} N$.

1. Relation between $\mu_i(\mathfrak{p}, M)$ and $\mu_i(\mathfrak{p}^*, M)$.

R is said to be an H-simple ring, if every non-zero homogeneous element of R is invertible. A graded R-module is called free, if it is isomorphic to a direct sum of graded R-modules of the form $R(n)$ $(n \in \boldsymbol{Z})$.

LEMMA (1.1.1). *The following conditions are equivalent.*

(1) *R is an H-simple ring.*

(2) *$R_0 = k$ is a field, and either $R = k$ or $R = k[T, T^{-1}]$ for some homogeneous invertible element T which is transcendental over k.*

(3) *Every graded R-module is free.*

PROOF. (1)⇒(2) This is essentially proved by [3] (cf. n°8, Section 1, Chap. 5).

(2)⇒(3) We may assume $R \neq k$ and put $d = \deg T$ $(d > 0)$. Let M be a graded R-module. Then every k-basis of $\bigoplus_{i=0}^{d-1} M_i$ will do as an R-free basis of M.

(3)⇒(1) This is obvious.

Let $\mathfrak{m}$ be a graded ideal of R $(\mathfrak{m} \neq R)$. Then $\mathfrak{m}$ is called an H-maximal ideal if $R/\mathfrak{m}$ is an H-simple ring. R is said to be an H-local ring if R has a unique H-maximal ideal.

Let $\mathfrak{p}$ be a prime ideal of R and let S denote the set of all the homogeneous elements of R not contained in $\mathfrak{p}$. Then $S^{-1}R$ (resp. $S^{-1}M$ for a graded R-module M) is again a graded ring (resp. a graded $S^{-1}R$-module) (cf. n°9, Section 2, [3]). $S^{-1}R$ (resp. $S^{-1}M$) is called the homogeneous localization of R (resp. of M) at $\mathfrak{p}$ and is denoted by $R_{(\mathfrak{p})}$ (resp. $M_{(\mathfrak{p})}$). Let $\mathfrak{p}^*$ denote the largest graded ideal of R contained in $\mathfrak{p}$. Then $\mathfrak{p}^*$ is again a prime ideal of R and $(R_{(\mathfrak{p})}, \mathfrak{p}^* R_{(\mathfrak{p})})$ is an

H-local ring.

Let A be an arbitrary Noetherian ring and let M be an A-module. For every prime ideal $\mathfrak{p}$ of A and for every integer $i \geqq 0$, we put

$$\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \mathrm{Ext}^i_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}).$$

(Here $k(\mathfrak{p})$ denotes the field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.) and call it the i-th Bass number of M at $\mathfrak{p}$. Bass [2] showed that, if $0 \to M \to E^0 \to \cdots \to E^i \to \cdots$ is the minimal injective resolution of M, $\mu_i(\mathfrak{p}, M)$ is equal to the number of the injective envelopes $E_A(A/\mathfrak{p})$ of $A/\mathfrak{p}$ which appear in E^i as direct summands.

THEOREM (1.1.2). *Let M be a graded R-module and let $\mathfrak{p}$ be a non-graded prime ideal of R. Then $\mu_0(\mathfrak{p}, M)=0$ and $\mu_{i+1}(\mathfrak{p}, M)=\mu_i(\mathfrak{p}^*, M)$ for every integer $i \geqq 0$.*

PROOF. After homogeneous localization at $\mathfrak{p}$, we may assume that $(R, \mathfrak{p}^*)$ is an H-local ring. Therefore we can express $\mathfrak{p}=fR+\mathfrak{p}^*$ for some $f \in R-\mathfrak{p}^*$, since $R/\mathfrak{p}^*$ is a principal ideal domain by (1.1.1). Applying $\mathrm{Hom}_R(\ , M)$ to the exact sequence $0 \to R/\mathfrak{p}^* \xrightarrow{f} R/\mathfrak{p}^* \to R/\mathfrak{p} \to 0$, we have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(R/\mathfrak{p}, M) \longrightarrow \underline{\mathrm{Hom}}_R(R/\mathfrak{p}^*, M) \xrightarrow{f} \underline{\mathrm{Hom}}_R(R/\mathfrak{p}^*, M) \\ &\longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{p}, M) \longrightarrow \cdots. \end{aligned}$$

This yields a short exact sequence

$$0 \longrightarrow \underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}^*, M) \xrightarrow{f} \underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}^*, M) \longrightarrow \mathrm{Ext}^{i+1}_R(R/\mathfrak{p}, M) \longrightarrow 0$$

and that $\mathrm{Hom}_R(R/\mathfrak{p}, M)=(0)$, since $\underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}^*, M)$ is an $R/\mathfrak{p}^*$-free module by (1.1.1) and since $f \notin \mathfrak{p}^*$. Thus we have $\mu_0(\mathfrak{p}, M)=0$. On the other hand, because $\mathrm{Ext}^{i+1}_R(R/\mathfrak{p}, M)=\underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}^*, M)/f\,\underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}^*, M)$ is also an $R/\mathfrak{p}$-free module, we have

$$\begin{aligned} \mu_{i+1}(\mathfrak{p}, M) &= \dim_{k(\mathfrak{p})} \mathrm{Ext}^{i+1}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \\ &= \mathrm{rank}_{R/\mathfrak{p}} \mathrm{Ext}^{i+1}_R(R/\mathfrak{p}, M) \\ &= \mathrm{rank}_{R/\mathfrak{p}^*} \underline{\mathrm{Ext}}^i_R(R/\mathfrak{p}^*, M) \\ &= \dim_{k(\mathfrak{p}^*)} \mathrm{Ext}^i_{R_{\mathfrak{p}^*}}(k(\mathfrak{p}^*), M_{\mathfrak{p}^*}) \\ &= \mu_i(\mathfrak{p}^*, M). \end{aligned}$$

REMARK. A similar argument is found in [34].

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and let M be a Macaulay A-module of $\dim_A M=n$. We put $r_A(M)=\dim_k \mathrm{Ext}^n_A(k, M)$ $(=\mu_n(\mathfrak{m}, M))$ and call it the type of M. Various properties of the invariant $r_A(M)$ are discussed by [15]. M is called a Gorenstein A-module if $\dim A=n$ and if M has injective dimension equal to n. The concept of Gorenstein modules was given by Sharp [24] in which we will find useful characterizations of Gorenstein modules. If A is not necessarily a local ring, Gorenstein modules are defined by their local data.

COROLLARY (1.1.3) ([**19**], [**34**] and [**32**]). *Let M be a finitely generated graded R-module and let $\mathfrak{p}$ be a non-graded element of* $\mathrm{Supp}_R M$. *Then* $\mathfrak{p}^* \in \mathrm{Supp}_R M$ *and*

(1) $\dim M_{\mathfrak{p}} = \dim M_{\mathfrak{p}^*} + 1$ *and* $\mathrm{depth}\, M_{\mathfrak{p}} = \mathrm{depth}\, M_{\mathfrak{p}^*} + 1$.

(2) *$M_{\mathfrak{p}}$ is a Macaulay (resp. Gorenstein) $R_{\mathfrak{p}}$-module if and only if $M_{\mathfrak{p}^*}$ is a Macaulay (resp. Gorenstein) $R_{\mathfrak{p}^*}$-module. In this case* $r(M_{\mathfrak{p}}) = r(M_{\mathfrak{p}^*})$.

2. Minimal injective resolutions.

Let M be a graded R-module. We denote by $\underline{E}_R(M)$ the injective envelope of M in $M_H(R)$.

THEOREM (1.2.1). (1) $\mathrm{Ass}_R \underline{E}_R(M) = \mathrm{Ass}_R M$ *for every graded R-module M.*

(2) *Let E be a graded R-module. Then E is an indecomposable injective object of $M_H(R)$ if and only if $E = [\underline{E}_R(R/\mathfrak{p})](n)$ for some graded prime ideal $\mathfrak{p}$ of R and for some integer n. In this case* $\mathrm{Ass}_R E = \{\mathfrak{p}\}$ *and $\mathfrak{p}$ is uniquely determined for E.*

(3) *Every injective object E of $M_H(R)$ can be decomposed into a direct sum of indecomposable injective objects of $M_H(R)$. This decomposition is uniquely determined for E up to isomorphisms.*

The proof follows as in the non-graded case (cf. [**20**]).

LEMMA (1.2.2). *Let E be a graded R-module. Then the following conditions are equivalent.*

(1) *E is an injective object of $M_H(R)$.*

(2) $\underline{\mathrm{Ext}}^1_R(R/\mathfrak{a}, E) = (0)$ *for every graded ideal $\mathfrak{a}$ of R.*

(3) $\underline{\mathrm{Ext}}^i_R(\ \ , E) = (0)$ *for every integer $i > 0$.*

The proof is similar to the non-graded case (cf. Theorem 3.2, [4]). (2) is equivalent to the condition: Let $\mathfrak{a}$ be a graded ideal of R and let $n \in \boldsymbol{Z}$. Then any homomorphism from $\mathfrak{a}(n)$ into E can be extended over $R(n)$.

COROLLARY (1.2.3). *Suppose that R is an H-simple ring. Then every graded R-module is an injective object of $M_H(R)$.*

THEOREM (1.2.4). *Let M be a graded R-module and let*

$$0 \longrightarrow M \longrightarrow \underline{E}^0_R(M) \xrightarrow{d^0} \underline{E}^1_R(M) \longrightarrow \cdots \longrightarrow \underline{E}^i_R(M) \xrightarrow{d^i} \underline{E}^{i+1}_R(M) \longrightarrow \cdots$$

be the minimal injective resolution of M in $M_H(R)$. Then, for every graded prime ideal $\mathfrak{p}$ of R and for every integer $i \geqq 0$, $\mu_i(\mathfrak{p}, M)$ is equal to the number of the graded R-modules of the form $[\underline{E}_R(R/\mathfrak{p})](n)$ ($n \in \boldsymbol{Z}$) which appear in $\underline{E}^i_R(M)$ as direct summands.

PROOF. Since $\underline{E}^i_R(M) = \underline{E}_R(B^i)$ where $B^i = \mathrm{Ker}\, d^i$, it suffices to prove in case $i=0$. Moreover, after homogeneous localization at $\mathfrak{p}$, we may assume that $(R, \mathfrak{p})$

is an H-local ring. Now let us express

$$\underline{E}_R(M)=\bigoplus_{\mathfrak{q}\in V_H(R),n\in Z} a(\mathfrak{q}, n)\quad [\underline{E}_R(R/\mathfrak{q})](n)$$

where $V_H(R)$ is the set of all the graded prime ideals of R and $a(\mathfrak{q}, n)$ denotes the multiplicity of $[\underline{E}_R(R/\mathfrak{q})](n)$. Then, recalling that $\underline{\mathrm{Hom}}_R(R/\mathfrak{a}, \underline{E}_R(N))=\underline{E}_{R/\mathfrak{a}}(\underline{\mathrm{Hom}}_R(R/\mathfrak{a}, N))$ for every graded ideal $\mathfrak{a}$ of R and for every graded R-module N (cf. [2]), we have by (1.2.3)

$$\begin{aligned}\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, M)&=\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, \underline{E}_R(M))\\ &=\bigoplus_{\mathfrak{q}\in V_H(R),n\in Z} a(\mathfrak{q}, n)\,\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, [\underline{E}_R(R/\mathfrak{q})](n))\\ &=\bigoplus_{\mathfrak{q}\in V_H(R),n\in Z} a(\mathfrak{q}, n)\,\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, [R/\mathfrak{q}](n))\\ &=\bigoplus_{n\in Z} a(\mathfrak{p}, n)[R/\mathfrak{p}](n)\,.\end{aligned}$$

Thus we have the assertion: $\mu_0(\mathfrak{p}, M)=\sum_{n\in Z} a(\mathfrak{p}, n)$.

For a graded R-module M, let $\underline{id}_RM$(resp. id_RM) denote the injective dimension of M in $M_H(R)$ (resp. as the underlying R-module).

THEOREM (1.2.5). *Let M be a graded R-module. Then*

(1) $id_RM\leqq \underline{id}_RM+1$.

(2) *Suppose that M is an injective object of $M_H(R)$. Then $id_RM=1$ if and only if $\mathfrak{p}^*\in \mathrm{Ass}_RM$ for some non-graded prime ideal $\mathfrak{p}$ of R.*

PROOF. (1) It suffices to prove in case M is an injective object of $M_H(R)$. Let $\mathfrak{p}$ be a prime ideal of R. If $\mathfrak{p}$ is graded, $\mu_i(\mathfrak{p}, M)=0$ for every $i>0$ since $\underline{\mathrm{Ext}}_R^i(R/\mathfrak{p}, M)=(0)$ by (1.2.2). Suppose that $\mathfrak{p}$ is not a graded ideal. Then, since $\mu_{i+1}(\mathfrak{p}, M)=\mu_i(\mathfrak{p}^*, M)$ by (1.1.2), we have $\mu_{i+1}(\mathfrak{p}, M)=0$ for every $i>0$ by virtue of the result in case $\mathfrak{p}$ is a graded ideal. Thus $\mu_i(\mathfrak{p}, M)=0$ for every prime ideal $\mathfrak{p}$ of R and for every integer $i\geqq 2$. This shows $id_RM\leqq 1$.

(2) By the above discussion, $id_RM=1$ if and only if $\mu_1(\mathfrak{p}_1, M)\neq 0$ for some non-graded prime ideal $\mathfrak{p}$ of R. By (1.1.2), the latter is equivalent to the condition that $\mathfrak{p}^*\in \mathrm{Ass}_RM$.

COROLLARY (1.2.6). *Let $\mathfrak{m}$ be a maximal ideal of R and assume that $\mathfrak{m}$ is graded. Then $\underline{E}_R(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$ as the underlying R-module.*

PROOF. Since $\mathrm{Ass}_R\underline{E}_R(R/\mathfrak{m})=\{\mathfrak{m}\}$ by (1.2.1), $\underline{E}_R(R/\mathfrak{m})$ is an injective R-module. On the other hand $\mu_0(\mathfrak{m}, \underline{E}_R(R/\mathfrak{m}))=1$ by (1.2.4).

For the rest of this section we assume that $R=\bigoplus_{n\geqq 0} R_n$ and that $R_0=k$ is a field. We put $\mathfrak{m}=\bigoplus_{n>0} R_n$ and $\underline{k}=R/\mathfrak{m}$.

Let M be a graded R-module. We define $M^*=\underline{\mathrm{Hom}}_k(M, k)$ and call it the graded k-dual of M. M^* is a graded R-module with $\{\mathrm{Hom}_k(M_{-n}, k)\}_{n\in Z}$ as its grading. Note that $M=M^{**}$ if and only if $[M_n : k]$ is finite for all $n\in \boldsymbol{Z}$. $[\ ]^* : M_H(R) \to M_H(R)$ is a contravariant exact functor.

THEOREM (1.2.7). $R^*=\underline{E}_R(\underline{k})$.

PROOF. R^* is an indecomposable object of $M_H(R)$, since $R=R^{**}$. Moreover R^* contains $\underline{k}=\underline{k}^*$ as a graded R-submodule. Thus it suffices to show that R^* is an injective object of $M_H(R)$.

Let $\mathfrak{a}$ be a graded ideal of R and let $n\in \boldsymbol{Z}$. For every homomorphism $f: \mathfrak{a}(n) \to R^*$, we can find $g: R=R^{**} \to [R(n)]^*$ so that $i^*\circ g=f^*$ where $i: \mathfrak{a}(n) \to R(n)$ denotes the inclusion map. Therefore $g^*\circ i=f$ and hence, by (1.2.2), we have the assertion.

COROLLARY (1.2.8). *R^* is the injective envelope of $\underline{k}$ as the underlying R-module.*

COROLLARY (1.2.9). *Let M be a graded R-module. Then the following conditions are equivalent.*

(1) *M is an Artinian R-module.*

(2) *There is an exact sequence $0 \to M \to \bigoplus_{i=1}^{t} R^*(n_i)$ of graded R-modules.*

The proof follows, by (1.2.8), as in the non-graded case (cf. [**20**]).

Let $N_H(R)$ (resp. $A_H(R)$) denote the full subcategory of $M_H(R)$ consisting of all the Noetherian (resp. Artinian) graded R-modules. By (1.2.9), we obtain

THEOREM (1.2.10). *$[\ ]^* : N_H(R) \to A_H(R)$ establishes an equivalence of categories.*

THE INVERSE SYSTEM OF MACAULAY (1.2.11). Let k be a field, $R=k[X_1, X_2, \cdots, X_r]$ be a polynomial ring and put $R^{\#}=k[X_1^{-1}, X_2^{-1}, \cdots, X_r^{-1}]$. For $f\in R$ and $\varphi\in R^{\#}$ we define

$$f\cdot\varphi=\text{the non-positive part of the product } f\varphi$$

and call it the f-deviate of φ (cf. Section 60, [**18**]). We consider $R^{\#}$ as a graded R-module by this action and call it the inverse system of Macaulay. Now let us identify $R^{\#}$ with R^* by regarding $\{X_1^{-1}, X_2^{-1}, \cdots, X_r^{-1}\}$ as the k-dual basis of $\{X_1, X_2, \cdots, X_r\}$. We define, for every $\mathfrak{m}$-primary graded ideal $\mathfrak{q}$ of R and for every finitely generated graded R-submodule M of R^*,

$$\mathfrak{q}^{-1}=\{\varphi\in R^* \mid f\cdot\varphi=0 \quad \text{for all } f\in\mathfrak{q}\},$$

$$M^{-1}=\{f\in R \mid f\cdot\varphi=0 \quad \text{for all } \varphi\in M\}.$$

$\mathfrak{q}^{-1}=(R/\mathfrak{q})^*$ is a finitely generated graded R-submodule of R^* and M^{-1} is an $\mathfrak{m}$-primary graded ideal of R. (Note $(\mathfrak{q}^{-1})^{-1}=\mathfrak{q}$ and $(M^{-1})^{-1}=M$.) Later we will show that $R/\mathfrak{q}$ is a Gorenstein ring if and only if $\mathfrak{q}^{-1}$ is principal (cf. (2.1.3). See [**18**] and [**29**]).

3. Cousin complexes and local cohomology modules.

The Cousin complexes were given by Hartshorne [12] in terms of geometry and in this section we will reconstruct them in terms of algebra——namely of graded modules. The method is the same as that of Sharp [23] and so, though he considered no sort of grading, we may refer the detail to [23].

Let M be a graded R-module and let $V_H(M)$ denote the set of all the graded prime ideals of R contained in $\mathrm{Supp}_R M$. We put $U_H^i(M)=\{\mathfrak{p}\in V_H(M)/\dim M_\mathfrak{p}\geqq i\}$ for every integer $i\geqq 0$.

LEMMA (1.3.1) ([23]). *Let U and U' be subsets of $V_H(R)$ such that $U'\subset U$ and suppose that every element of $U-U'$ is minimal in U. Let M be a graded R-module and assume that $V_H(M)\subset U$. Then*

$$\varphi: M \longrightarrow \bigoplus_{\mathfrak{p}\in U-U'} M_{(\mathfrak{p})}$$

$$x \longmapsto \{x/1\}$$

is a well-defined homomorphism of graded R-modules and $V_H(\mathrm{Coker}\,\varphi)\subset U'$.

Construction of $\underline{C}_R(M)$.

Let M be a graded R-module. We put $M^{-2}=(0)$, $M^{-1}=M$ and $d^{-2}=0$. Let $i\geqq 0$ be an integer and assume that there exists a complex of graded R-modules

$$M^{-2}\xrightarrow{d^{-2}} M^{-1}\xrightarrow{d^{-1}} M^0\longrightarrow\cdots\longrightarrow M^{i-2}\xrightarrow{d^{i-2}} M^{i-1}$$

such that $V_H(\mathrm{Coker}\,d^{i-2})\subset U_H^i(M)$. Of course this assumption is satisfied for $i=0$. We put $M^i=\bigoplus_{\mathfrak{p}\in U_H^i(M)-U_H^{i+1}(M)}[\mathrm{Coker}\,d^{i-2}]_{(\mathfrak{p})}$ and define $d^{i-1}=\varphi\circ\varepsilon$ where $\varepsilon: M^{i-1}\to\mathrm{Coker}\,d^{i-2}$ is the canonical epimorphism and $\varphi:\mathrm{Coker}\,d^{i-2}\to M^i$ denotes the homomorphism induced by (1.3.1). Then $d^{i-1}\circ d^{i-2}=0$, and $V_H(\mathrm{Coker}\,d^{i-1})\subset U_H^{i+1}(M)$ by (1.3.1). Thus inductively we obtain a complex $\underline{C}_R(M)$ of graded R-modules

$$0\longrightarrow M=M^{-1}\xrightarrow{d^{-1}} M^0\longrightarrow\cdots\longrightarrow M^i\xrightarrow{d^i} M^{i+1}\longrightarrow\cdots$$

which we call the Cousin complex of M. The i-th cohomology module of $\underline{C}_R(M)$ will be denoted by $H^i(M)$.

LEMMA (1.3.2) ([23]). *Let M be a graded R-module and let $n>0$ be an integer. Suppose that $H^i(M)=(0)$ for $i\leqq n-2$. Then*

$$\underline{\mathrm{Ext}}_R^i(L, M)=\underline{\mathrm{Ext}}_R^{i-n}(L, \mathrm{Coker}\,d^{n-2})$$

for every integer i and for every finitely generated graded R-module L with

$$V_H(L)\cap[V_H(M)-U_H^n(M)]=\emptyset\,.$$

Recalling (1.1.3), the following theorem can be proved similarly as in the non-graded case (cf. [23] and [24]).

THEOREM (1.3.3). *Let M be a non-zero finitely generated graded R-module. Then M is a Macaulay (resp. Gorenstein) R-module if and only if $\underline{C}_R(M)$ is exact (resp. $\underline{C}_R(M)$ provides the minimal injective resolution of M in $M_H(R)$).*

In the following we assume that $(R, \mathfrak{m})$ is an H-local ring. For every integer $i \geqq 0$, we put

$$\underline{H}^i_{\mathfrak{m}}(\ ) = \varinjlim_{t} \underline{\mathrm{Ext}}^i_R(R/\mathfrak{m}^t,\ \ )$$

and call it the i-th local cohomology functor (cf. [11]). $\underline{H}^0_{\mathfrak{m}}(\ )$ is left exact and $\{\underline{H}^i_{\mathfrak{m}}(\ )\}_{i \geqq 0}$ will do as its derived functors.

THEOREM (1.3.4). *Let M be a Macaulay graded R-module of* $\dim M_{\mathfrak{m}} = n$. *Then*

(1) $\underline{\mathrm{Ext}}^n_R(N, M) = \underline{\mathrm{Hom}}_R(N, \underline{H}^n_{\mathfrak{m}}(M))$ *for every finitely generated graded R-module N such that $V_H(N) \subset \{\mathfrak{m}\}$.*

(2) $M^n = \underline{H}^n_{\mathfrak{m}}(M)$.

(3) *M is a Gorenstein R-module if and only if $\underline{H}^n_{\mathfrak{m}}(M)$ is an injective object of $M_H(R)$.*

PROOF. $\underline{C}_R(M)$ is exact by (1.3.3), and we know that $M^i = (0)$ for every $i > n$ and that $V_H(M^n) = \{\mathfrak{m}\}$ by the construction of $\underline{C}_R(M)$. Thus we have $\underline{\mathrm{Ext}}^n_R(N, M) = \underline{\mathrm{Hom}}_R(N, M^n)$ by (1.3.2), since $M^n = \mathrm{Coker}\, d^{n-2}$. Moreover, if we take $N = R/\mathfrak{m}^t$ $(t > 0)$, $\underline{\mathrm{Ext}}^n_R(R/\mathfrak{m}^t, M) = \underline{\mathrm{Hom}}_R(R/\mathfrak{m}^t, M^n)$ and this implies $\underline{H}^n_{\mathfrak{m}}(M) = M^n$ as $V_H(M^n) = \{\mathfrak{m}\}$. Hence (1) and (2) are proved.

Now consider (3). The necessity follows from (1.3.3). For the sufficiency, we note $\underline{\mathrm{Ext}}^{i+n}_R(R/\mathfrak{m}, M) \cong \underline{\mathrm{Ext}}^i_R(R/\mathfrak{m}, M^n)$ for every integer $i > 0$, as $\underline{C}_R(M)$ is exact. Thus $\underline{\mathrm{Ext}}^{i+n}_R(R/\mathfrak{m}, M) = (0)$ for every $i > 0$, as $M^n = \underline{H}^n_{\mathfrak{m}}(M)$ is an injective object of $M_H(R)$, and consequently $\mu_i(\mathfrak{m}, M) = 0$ for every integer $i > n$. This implies that M is a Gorenstein R-module (cf. (1.1.3) and [24]).

Added in proof. The theorems (1.2.1), (1.2.4) and (1.2.5) are given independently in [35] and [36].

Chapter 2. The canonical module of a graded ring defined over a field.

Let k be a field. We say that a graded ring R is defined over k, if

(i) $R = \bigoplus_{n \geqq 0} R_n$

(ii) R is finitely generated over k

(iii) $R_0 = k$.

If R is a graded ring defined over k, then R is H-local with the maximal ideal $\mathfrak{m} = R_+$. We consider k itself as a graded ring defined over k.

As in Section 2 of Chapter 1, we put

$\underline{k} = R/\mathfrak{m}$.

$E_R = R^* = \underline{\mathrm{Hom}}_k(R, \underline{k})$.

$M^* = \underline{\mathrm{Hom}}_k(M, \underline{k}) \cong \underline{\mathrm{Hom}}_R(M, E_R)$ (M is a graded R-module; cf. (2.1.1)).

If M and N are graded R-modules, $M \cong N$ means that M and N are isomorphic as graded R-modules.

In this chapter, all rings are graded rings defined over k and all modules are graded modules. All homomorphisms are graded of degree 0 and k-linear.

1. Definition of the canonical module and duality.

LEMMA (2.1.1). *Let R, S be graded rings and $S \to R$ be a homomorphism of graded rings. If P is a graded S-module and M, N are graded R-modules, then*

$$\underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_S(N, P)) \cong \underline{\mathrm{Hom}}_S(M \otimes_R N, P).$$

PROOF. See Cartan-Eilenberg [**4**], Proposition 5.2 of Chapter II.

The following Definition is due to Kunz-Herzog [**15**].

DEFINITION (2.1.2). If R is a graded ring and $\dim R = n$, we put

$$K_R = (\underline{H}^n_{\mathfrak{m}}(R))^* \cong \underline{\mathrm{Hom}}_R(\underline{H}^n_{\mathfrak{m}}(R), E_R)$$

the dual of the n-th local cohomology group of R. As $\underline{H}^n_{\mathfrak{m}}(R)$ is an Artinian graded R-module, K_R is a finitely generated graded R-module by (1.2.10). We call K_R the canonical module of R.

PROPOSITION (2.1.3). *If R is a Macaulay ring, then R is a Gorenstein ring if and only if $K_R \cong R(d)$ for some $d \in \boldsymbol{Z}$.*

PROOF. If

$$0 \longrightarrow R \longrightarrow R^0 \longrightarrow \cdots \longrightarrow R^n \longrightarrow 0$$

is the graded Cousin complex of R, then by (1.3.4), $R^n \cong \underline{H}^n_{\mathfrak{m}}(R)$ and R is Gorenstein if and only if $\underline{H}^n_{\mathfrak{m}}(R)$ is injective. As $E_R(\underline{k}) = R^*$, R is Gorenstein if and only if $\underline{H}^n_{\mathfrak{m}}(R) \cong R^*(-d)$ for some integer d. Taking duals of both sides, we get the proposition.

REMARK. If R is a Gorenstein ring and $K_R \cong R(d)$, this integer d is uniquely determined by R and is an important invariant of R.

EXAMPLE (2.1.4). If $R = k[X]$, then the graded Cousin complex of R is

$$0 \longrightarrow R \longrightarrow R^0 \cong k[X, X^{-1}] \longrightarrow R^1 \longrightarrow 0.$$

As R is a Macaulay ring, the graded Cousin complex is exact and $\underline{H}^1_{\mathfrak{m}}(R) \cong R^1$ by (1.3.4). Thus we have

$$\underline{H}^1_{\mathfrak{m}}(R) \cong k[X, X^{-1}]/k[X] \cong X^{-1}k[X^{-1}] \cong R^*(d) \qquad (d = \deg X)$$

and

$$K_R = (\underline{H}^1_{\mathfrak{m}}(R))^* \cong X \cdot k[X] \cong R(-d).$$

PROPOSITION (2.1.5). *If M is a finitely generated graded R-module, there is a natural isomorphism*

$$(\underline{H}_{\mathfrak{m}}^{n}(M))^{*}\cong\underline{\mathrm{Hom}}_{R}(M, K_{R}) \qquad (n=\dim R).$$

PROOF. We put $T^{0}(M)=(\underline{H}_{\mathfrak{m}}^{n}(M))^{*}$ for a finitely generated graded R-module M. Then T^{0} is a covariant left-exact functor. As $T^{0}(R)=K_{R}$, the proof is the same as that of [11], Proposition 4.2.

PROPOSITION (2.1.6). *The following conditions are equivalent for a graded ring R of dimension n.*

(i) *R is a Macaulay ring.*

(ii) *For every finitely generated graded R-module M and for every integer j, there is a natural isomorphism*

$$(\underline{H}_{\mathfrak{m}}^{n-j}(M))^{*}\cong\underline{\mathrm{Ext}}_{R}^{j}(M, K_{R}).$$

PROOF. We put $T^{j}(M)=(\underline{H}_{\mathfrak{m}}^{n-j}(M))^{*}$ for a finitely generated graded R-module M. As $T^{0}(M)=\underline{\mathrm{Hom}}_{R}(M, K_{R})$ and $\underline{\mathrm{Ext}}_{R}^{j}(*, K_{R})$ $(j>0)$ are derived functors of $\underline{\mathrm{Hom}}_{R}(*, K_{R})$, it suffices to show that T^{j} $(j>0)$ are derived functors of T^{0}. To show this, it suffices to show

(a) if $0\rightarrow M'\rightarrow M\rightarrow M''\rightarrow 0$ is an exact sequence of finitely generated graded R-modules, then there exists a long exact sequence

$$0\longrightarrow T^{0}(M'')\longrightarrow T^{0}(M)\longrightarrow T^{0}(M')\longrightarrow T^{1}(M'')\longrightarrow T^{1}(M)\longrightarrow\cdots,$$

(b) $T^{j}(R(d))=0$ for $j>0$ and for every integer d.

The statement (a) follows from the exact sequence of local cohomology modules and the exactness of the functor ( )* considering that $\underline{H}_{\mathfrak{m}}^{j}(M)=0$ for $j>n$ and for every R-module M. The statement (b) is equivalent to say that R is a Macaulay ring.

LEMMA (2.1.7). *Let F, M be finitely generated graded R-modules and $f: F\rightarrow M$ be a surjective homomorphism of graded R-modules. Then f is minimal* ($\mathrm{Ker}(f)\subset\mathfrak{m}F$) *if and only if $f^{*}: M^{*}\rightarrow F^{*}$ is essential.*

PROOF. f is minimal

$\Leftrightarrow$ for every proper graded R-submodule F' of F, the composition map $F'\hookrightarrow F\xrightarrow{f} M$ is not surjective

$\Leftrightarrow$ for every proper graded quotient module F'' of F^{*}, the composition map $M^{*}\xrightarrow{f^{*}} F^{*}\rightarrow F''$ (the right arrow is the canonical surjection) is not injective

$\Leftrightarrow$ f^{*} is essential.

NOTATION. For a finitely generated graded R-module M, we write $\nu(M)=[M/\mathfrak{m}M: k]$.

PROPOSITION (2.1.8). *If R is a Macaulay ring, then*

(i) *K_{R} is a Macaulay R-module*

(ii) *K_{R} has a finite injective dimension as an R-module*

(iii) $r_R(K_R)=1$

(iv) $r(R)=\nu(K_R)$.

PROOF. By (2.1.6),

$$\underline{\mathrm{Ext}}_R^j(\underline{k}, K_R)\cong(\underline{H}_{\mathfrak{m}}^{n-j}(\underline{k}))^*=\begin{cases}\underline{k} & (j=n)\\ 0 & (j\neq n)\end{cases}\quad (n=\dim R).$$

The statements (i), (ii) and (iii) follow from this isomorphism. If $f: F\to K_R$ is a minimal free resolution of K_R, $f^*: (K_R)^*=\underline{H}_{\mathfrak{m}}^n(R)\to F^*$ is an essential extention by (2.1.7). If $\nu(K_R)=r$ and $F\cong\bigoplus_{i=1}^r R(d_i)$, $F^*\cong\bigoplus_{i=1}^r R^*(-d_i)$ and $\underline{\mathrm{Ext}}_R^n(\underline{k}, R)\cong \underline{\mathrm{Hom}}_R(\underline{k}, \underline{H}_{\mathfrak{m}}^n(R))\cong\bigoplus_{i=1}^r \underline{k}(-d_i)$ by (1.3.4). Thus $r(R)=r$.

EXAMPLE (2.1.9). Let H be a numerical semigroup. That is, H is an additive subsemigroup of $\boldsymbol{N}$ (the set of natural numbers), $0\in H$ and H contains all but finite natural numbers. We say that H is symmetric if there exists an integer d such that for every integer n, $n\in H$ if and only if $d-n\notin H$.

We put $R=k[H]=k[T^h\mid h\in H]\subset k[T]$. We put $\deg(T)=1$ and consider R as a graded ring defined over k. As R is a one-dimensional domain, R is a Macaulay ring. It was proved by Herzog-Kunz [**14**] that R is a Gorenstein ring if and only if H is symmetric. We will put a new proof of this fact using the graded Cousin complex of R and we will compute K_R for general H.

The graded Cousin complex of R is

$$0\longrightarrow R\longrightarrow R^0=k[T, T^{-1}]\longrightarrow \underline{H}_{\mathfrak{m}}^1(R)\longrightarrow 0.$$

As this sequence is exact, $\underline{H}_{\mathfrak{m}}^1(R)\cong k[T, T^{-1}]/R$. Thus $\underline{H}_{\mathfrak{m}}^1(R)$ is generated by $\{T^n\mid n\in\boldsymbol{Z}, n\notin H\}$ as k-vector space and $K_R=(\underline{H}_{\mathfrak{m}}^1(R))^*$ is the fractional ideal of R generated by $\{T^{-n}\mid n\in\boldsymbol{Z}, n\notin H\}$ as a k-vector space and as an R-module. By (2.1.3), R is Gorenstein if and only if $K_R\cong R(d)$ for some integer d. It is easy to see that this condition is equivalent to say that H is symmetric.

2. Calculation of local cohomology groups and canonical modules.

LEMMA (2.2.1). *If a graded R-module M satisfies the condition*

(*) *For every element $x\neq 0$ of M, $\mathrm{Ann}_R(x)$ is an $\mathfrak{m}$-primary ideal,*

then $\underline{H}_{\mathfrak{m}}^0(M)=M$ and $\underline{H}_{\mathfrak{m}}^q(M)=0$ for $q\neq 0$.

PROOF. Let $(I^\cdot)$ be the minimal injective resolution of M in the category of graded R-modules. As $\mathrm{Ass}_R(M)=\{\mathfrak{m}\}$, $\mathrm{Ass}_R(I^j)=\{\mathfrak{m}\}$ for every j and $\underline{H}_{\mathfrak{m}}^q(M)=H^q(\underline{H}_{\mathfrak{m}}^0(I^\cdot))=H^q(I^\cdot)$.

LEMMA (2.2.2). *If a graded R-module M satisfies the condition*

(**) *There exists* $f\in R_d$, $d>0$, *such that the multiplication map* f_M *is bijective, then* $\underline{H}^q_{\mathfrak{m}}(M)=0$ *for every integer* q.

PROOF. The multiplication map f_M induces the multiplication map of f on $\underline{H}^q_{\mathfrak{m}}(M)$ and the latter map must be bijective. But this is impossible unless $\underline{H}^q_{\mathfrak{m}}(M)=0$.

LEMMA (2.2.3). *If* $E=E_R(R/\mathfrak{p})$ *where* $\mathfrak{p}$ *is a homogeneous prime ideal of* R *and* $\mathfrak{p}\neq\mathfrak{m}$, E *satisfies the condition* (**).

PROOF. We consider $M=R/\mathfrak{p}$ as a submodule of E. We denote by f_M (resp. f_E) the multiplication map of f on M (resp. on E). If $f\in R_d$ and if $f\notin\mathfrak{p}$, f_M is injective. As E is an essential extension of M, $f_E: E\to E(d)$ is injective, too. As E is an injective module, f_E must split. But as $E(d)$ is indecomposable, f_E is bijective.

The following is a standard technique of homological algebra.

LEMMA (2.2.4). *If* $0\to M\to K^0\to K^1\to\cdots$ *is a resolution of a graded* R-*module* M *by graded* R-*modules and if* $\underline{H}^q_{\mathfrak{m}}(K^i)=0$ *for every* i *and every* $q\neq 0$, *then* $\underline{H}^q_{\mathfrak{m}}(M)=H^q(\underline{H}^0_{\mathfrak{m}}(K^\cdot))$ *for every* $q\geqq 0$.

THEOREM (2.2.5). *Let* R, S *be graded rings defined over* k *and* $\mathfrak{m}=R_+$, $\mathfrak{n}=S_+$ *be their H-maximal ideals. We put* $T=R\underline{\otimes}_k S$ *and* $\mathfrak{M}=T_+$. *If* A (*resp.* B) *is a graded* R- (*resp.* S-) *module, we have*

$$\underline{H}^q_{\mathfrak{M}}(A\underline{\otimes}_k B)=\bigoplus_{i+j=q}(\underline{H}^i_{\mathfrak{m}}(A)\underline{\otimes}_k\underline{H}^j_{\mathfrak{n}}(B)).$$

Before proving this theorem, we need some notations.

NOTATION (2.2.6) If M is a graded R-module and if $(I^\cdot)$ is the minimal injective resolution of M in the category of graded R-modules, we put $I^j={}'I^j\oplus{}''I^j$ for every j, where $\mathrm{Ass}_R({}'I^j)=\{\mathfrak{m}\}$ and $\mathfrak{m}\notin\mathrm{Ass}_R({}''I^j)$. Note that $({}'I^\cdot)$ is a subcomplex of $(I^\cdot)$ and $H^q({}'I^\cdot)=\underline{H}^q_{\mathfrak{m}}(M)$. We denote by $({}''I^\cdot)$ the quotient complex $(I^\cdot/{}'I^\cdot)$. This decomposition depends on (1.2.1).

PROOF OF (2.2.5). We put $C=A\underline{\otimes}_k B$. Let $(I^\cdot)$ (resp. $(J^\cdot)$) be the minimal injective resolution of A (resp. of B) in the category of graded R- (resp. S-) modules. We define the complex $(E^\cdot)$ by putting

$$E^q=\bigoplus_{i+j=q}(I^i\underline{\otimes}_k J^j).$$

Then by the Künneth formula of tensor products of complexes over a field (cf. [37], Chapter V, (10.1)), $(E^\cdot)$ is a resolution of C. We put

$${}'E^q=\bigoplus_{i-j=q}({}'I^i\underline{\otimes}_k{}'J^j)\quad\text{and}\quad {}''E^q=\bigoplus_{i+j=q}[({}''I^i\underline{\otimes}_k{}'J^j)\oplus({}'I^i\underline{\otimes}_k{}''J^j)\oplus({}''I^i\underline{\otimes}_k{}''J^j)]$$

where ${}'I^i$, ${}''I^i$, ${}'J^j$ and ${}''J^j$ are defined as in (2.2.6). Then it is easy to see that ${}'E^q$ satisfies the condition (*) and ${}''E^q$ is a direct sum of modules which satisfy the condition (**) for every q. By (2.2.4), we have $\underline{H}^q_{\mathfrak{M}}(C)=H^q('E^\cdot)$ and as $'E^\cdot='I^\cdot\otimes_k 'J^j$, we have (2.2.5) again by the Künneth formula.

COROLLARY (2.2.7). *If R, S and T be as in* (2.2.5). *Then* $K_T=K_R\underline{\otimes}_k K_S$.

COROLLARY (2.2.8). *If* $R=k[X_1, \cdots, X_n]$ *where* $\deg(X_i)=d_i$, *then*

$$K_R=X_1X_2\cdots X_nR=R(-d) \qquad (d=d_1+\cdots+d_n).$$

PROOF. This follows from (2.1.4) and (2.2.7).

PROPOSITION (2.2.9). *Let $f\colon R\to S$ be a homomorphism of graded rings defined over k. We assume that R is a Macaulay ring and that S is a finite R-module. (We do not assume that f is injective.) If we put* $t=\dim R-\dim S$, *then*

$$K_S=\underline{\mathrm{Ext}}^t_R(S, K_R).$$

PROOF. We put $r=\dim R$, $s=\dim S$, $\mathfrak{m}=R_+$ and $\mathfrak{n}=S_+$. As S is finite over R, $\underline{H}^j_{\mathfrak{m}}(M)=\underline{H}^j_{\mathfrak{n}}(M)$ for every graded S-module M and every j. By (2.1.6), $\underline{\mathrm{Ext}}^t_R(S, K_R)\cong(\underline{H}^{r-t}_{\mathfrak{m}}(S))^*=(\underline{H}^s_{\mathfrak{n}}(S))^*=K_S$.

PROPOSITION (2.2.10). *If $S=R/(x_1, \cdots, x_t)$, where $(x_1, \cdots, x_t)$ is an R-regular sequence and if R is a Macaulay ring, then*

$$K_S\cong(K_R/(x_1, \cdots, x_t)K_R)(d) \qquad (d=d_1+\cdots+d_t).$$

PROOF. It suffices to treat the case $t=1$. (We omit the subscript 1.) By the exact sequence

$$0\longrightarrow R(-d)\xrightarrow{\;x\;}R\longrightarrow S\longrightarrow 0,$$

we have $K_S\cong\underline{\mathrm{Ext}}^1_R(S, K_R)\cong(K_R/xK_R)(d)$.

REMARK. If R is not a Macaulay ring, (2.2.10) is not true in general.

PROPOSITION (2.2.11). *If R is a one-dimensional graded integral domain defined over k and if k is algebraically closed, R is isomorphic to a semigroup ring. Moreover, if R_+ is generated by R_1, R is a polynomial ring over k.*

PROOF. Let $0\to R\to R^0=Q\to R^1=\underline{H}^1_{\mathfrak{m}}(R)\to 0$ be the graded Cousin complex of R. Q is the graded total quotient ring of R. As R is an integral domain, Q is H-simple and by (1.1.1), $Q\cong K[T, T^{-1}]$ where $K=Q_0$ is a field. But as $R_0=k$ and $(\underline{H}^1_{\mathfrak{m}}(R))_0$ is a finite-dimensional k-vector space ($\underline{H}^1_{\mathfrak{m}}(R)$ is an Artinian R-module), K is a finite extension of k. As k is algebraically closed, we have $K=k$ and R is a graded subring of $k[T]$. Then it is clear that R is a semigroup ring.

REMARK (2.2.12). Let R be a graded ring defined over k and let k' be an extension field of k. If we put $R'=R\otimes_k k'$ and $\mathfrak{m}'=(R')_+$, then R' is a graded ring defined over k'. If M is a graded R-module and if we put $M'=M\underline{\otimes}_k k'\cong M\underline{\otimes}_R R'$, there is a natural isomorphism

$$\underline{H}^p_{\mathfrak{m}'}(M')\cong\underline{H}^p_{\mathfrak{m}}(M)\otimes_k k'$$

for every integer p.

PROOF. Let $(I^{\cdot})$ be an injective resolution of M in the category of graded R-modules. As in (2.2.6), we write $I^j={}'I^j\oplus{}''I^j$ for every j. Then, ${}'I^j\otimes_k k'$ satisfies the condition (*) of (2.2.1) and ${}''I^j\otimes_k k'$ is a direct sum of modules which satisfy the condition (**) of (2.2.2). Thus

$$\underline{H}^p_{\mathfrak{m}'}(M')\cong H^p(\underline{H}^0_{\mathfrak{m}'}(I^{\cdot}\otimes_k k'))\cong H^p({}'I^{\cdot}\otimes_k k')\cong\underline{H}^p_{\mathfrak{m}}(M)\otimes_k k'.$$

REMARK. If we discuss some properties of a graded R-module M using the local cohomology groups of M, it frequently occurs that we may consider M' instead of M by the aid of (2.2.12). In these cases, we may assume that k is an infinite field.

Chapter 3. The Veronesean subrings of a graded ring.

Let R be a graded ring. For a positive integer d, we define

$$R^{(d)}=\bigoplus_{n\in Z}R_{nd}$$

and call it the Veronesean subring of R of order d. We consider $R^{(d)}$ as a graded ring by $(R^{(d)})_n=R_{nd}$. In this Chapter, we continue the study of [7] and investigate the condition for $R^{(d)}$ to be a Gorenstein ring when R is a graded ring defined over a field k and R is a Gorenstein ring.

As $R^{(d)}$ is a direct summand of R as an $R^{(d)}$-module, $R^{(d)}$ is a pure subring of R (cf. [17], Section 6), and R is integral over $R^{(d)}$. So, if R is a Macaulay ring, so is $R^{(d)}$ and if R is an integrally closed domain, so is $R^{(d)}$.

In this Chapter, R is a graded ring defined over k (cf. Chapter 2). We use the notations of Chapter 2. We fix a positive integer d and we put

$$R'=R^{(d)}$$

$$\mathfrak{m}'=(R')_+=\mathfrak{m}^{(d)}.$$

If M is a graded R-module, we put

$$M^{(d)}=\bigoplus_{n\in Z}M_{nd}.$$

$M^{(d)}$ is a graded R'-module in a natural way and the functor $(\ )^{(d)}$ is an exact functor.

1. Calculation of local cohomology groups and the canonical module.

THEOREM (3.1.1). *If M is a graded R-module, we have*

$$\underline{H}^p_{\mathfrak{m}'}(M^{(d)})\cong(\underline{H}^p_{\mathfrak{m}}(M))^{(d)}$$

for every integer p.

To prove this, we need a lemma.

LEMMA (3.1.2). *Let M be a graded R-module. If M satisfies the condition* (*) *of* (2.2.1), (*resp. the condition* (**) *of* (2.2.2),) *so does $M^{(d)}$.*

PROOF OF (3.1.1). Let

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be the minimal injective resolution of M. If we apply the functor $(\ )^{(d)}$ to this sequence,

$$0 \longrightarrow M^{(d)} \longrightarrow (I^0)^{(d)} \longrightarrow (I^1)^{(d)} \longrightarrow \cdots$$

is a resolution of $M^{(d)}$ and for every j, $(I^j)^{(d)}$ is a direct sum of modules which satisfy condition (*) or condition (**) by (3.1.2) and (2.2.3). By (2.2.4),

$$\underline{H}^p_{\mathfrak{m}'}(M^{(d)}) \cong H^p(\underline{H}^0_{\mathfrak{m}'}((I^\cdot)^{(d)})) \cong H^p((\underline{H}^0_{\mathfrak{m}}(I^\cdot))^{(d)}) \cong (H^p(\underline{H}^0_{\mathfrak{m}}(I^\cdot))^{(d)} \cong (\underline{H}^p_{\mathfrak{m}}(M))^{(d)}.$$

COROLLARY (3.1.3). $K_{R'}=(K_R)^{(d)}$.

PROOF. As $\dim R'=\dim R$ and the functor $(\ )^*$ commutes with the functor $(\ )^{(d)}$, this is a direct consequence of (3.1.1).

DEFINITION (3.1.4). We put

$$a(R)=-\min\{m \mid (K_R)_m \neq 0\}=\max\{m \mid (\underline{H}^n_{\mathfrak{m}}(R))_m \neq 0\} \quad (n=\dim R).$$

If R is a Gorenstein ring, $a(R)$ is defined so that $K_R=R(a(R))$. If R is an Artinian ring, we define

$$t(R)=\max\{m \mid R_m \neq 0\}.$$

If R is an Artinian ring, $K_R=R^*$ and we have $a(R)=t(R)$.

COROLLARY (3.1.5). *If R is a Gorenstein ring and if $a(R)\equiv 0 \pmod d$, $R^{(d)}$ is a Gorenstein ring.*

PROOF. If we put $a(R)=b\cdot d$, $K_{R'}\cong(R(bd))^{(d)}\cong R'(b)$ by (3.1.3). As R' is a Macaulay ring, R' is a Gorenstein ring by (2.1.3).

REMARK (3.1.6). If R is a Macaulay ring and if $(f_1, \cdots, f_t)$ is a homogeneous R-regular sequence, then

$$a(R/(f_1, \cdots, f_t))=a(R)+\sum_{i=1}^{t} \deg f_i.$$

This is an immediate consequence of (2.2.10).

EXAMPLE (3.1.7). We put $R=k[X, Y, Z]/(X^p+Y^q+Z^r)$. We can consider R as a graded ring by putting $\deg X=qr$, $\deg Y=rp$ and $\deg Z=pq$. In this case, $a(R)=pqr-pq-qr-rp$. $a(R)<0$ if and only if R is a "rational singularity" and $a(R)=0$ if and only if R is a "simple elliptic singularity" (cf. [**31**]).

2. Veronesean subrings of R which satisfies the condition $R=k[R_1]$.

In this section, we assume the following conditions for R.

(i) $R=k[R_1]$.

(ii) R is a Macaulay ring.

The following gives the converse to (3.1.5).

THEOREM (3.2.1). *If $R^{(d)}$ is a Gorenstein ring and if* $\dim R\geqq 2$, *R is a Gorenstein ring and* $a(R)\equiv 0 \pmod d$.

REMARK. The two conditions in (3.2.1) for R is necessary. Counterexamples will be shown in Section 3.

PROOF OF THE THEOREM. As $R'=R^{(d)}$ is a Gorenstein ring, we have $K_{R'}\cong(K_R)^{(d)}\cong R'(p)$ for some integer p. We will show that $K_R\cong R(pd)$. Let us take $f\in(K_R)_{-pd}$ which generates the R'-module $[K_R]^{(d)}$. First we will show that $\mathrm{ann}_R(f)=0$. Let x be a homogeneous element of R such that $xf=0$. If y is any homogeneous element of R such that $xy\in R'$, $xy=0$ since $\mathrm{ann}_{R'}(f)=0$. As $\mathfrak{m}=R_+$ is generated by R_1, $\mathrm{ann}_R(x)$ is an m-primary ideal. As R is a Macaulay ring and $\dim R\geqq 1$, $x=0$. Thus we have an exact sequence

$$0 \longrightarrow Rf \cong R(pd) \longrightarrow K_R \longrightarrow K_R/Rf \longrightarrow 0\,.$$

As $[K_R]^{(d)}=R'f=[Rf]^{(d)}$, $[K_R/Rf]^{(d)}=0$. So, depth $(K_R/Rf)=0$. But, on the other hand, as R is a Macaulay ring and $\dim R\geqq 2$, depth $Rf=$depth $K_R=\dim R\geqq 2$. So, if $K_R/Rf\neq 0$, depth $K_R/Rf>0$. Thus we have $K_R=Rf=R(pd)$. By (2.1.3), R is a Gorenstein ring.

NOTATION (3.2.2). We put $H(n, R)=[R_n: k]$.

LEMMA (3.2.3). *If* $\dim R\geqq 1$, $H(n+1, R)\geqq H(n, R)$ *for every integer* n.

PROOF. Considering $R\otimes_k\bar{k}$ instead of R ($\bar{k}$ is the algebraic closure of k), we may assume that k is an infinite field. Then we can take an R-regular element $x\in R_1$. Then we have

$$H(n+1, R)-H(n, R)=H(n+1, R/xR)\geqq 0\,.$$

LEMMA (3.2.4). *$[R(a)]^{(d)}$ is generated by $R(a)_n=R_{a+n}$ over R', where n is the smallest integer such that $n\equiv 0 \bmod d$ and $n+a\geqq 0$.*

PROOF. As $R=k[R_1]$, $R_{n+1}=R_1R_n$ and $R_{n+d}=R_dR_n$ for every integer $n\geqq 0$.

THEOREM (3.2.5). *If R is a Gorenstein ring and if $R=k[R_1]$,*

$$r(R')=H(n+a(R), R)$$

where n is the smallest integer such that $n\equiv 0 \bmod d$ and $n+a(R)\geqq 0$.

PROOF. By (2.1.8), $r(R')=\nu(K_{R'})=\nu([K_R]^{(d)})$. But, as $K_R\cong R(a(R))$, the result follows from (3.2.4).

The following was proved by Matsuoka [**21**].

COROLLARY (3.2.6). $r(k[X_1, \cdots, X_s]^{(d)}) = \binom{n-1}{s-1}$ *where n is the smallest integer such that* $n \geqq s$ *and* $n \equiv 0 \bmod d$. (*We put* $\deg(X_i)=1$ *for every* i.)

PROOF. If $R=k[X_1, \cdots, X_s]$, R is a Gorenstein ring and $a(R)=-s$.

COROLLARY (3.2.7). *If R is a Gorenstein ring,* $\dim R \geqq 1$ *and if* $H(1, R) \geqq 2$, *then* $R^{(d)}$ *is a Gorenstein ring if and only if* $a(R) \equiv 0 \bmod d$.

PROOF. By (3.2.5), $r(R')=H(n+a(R), R)$. But by the assumption and (3.2.3), $H(n+a(R), R)=1$ if and only if $n+a(R)=0$. The result follows from the definition of n in (3.2.5).

REMARK. If $H(1, R)=1$, R' is Gorenstein for every d.

When R is an Artinian ring, the following was proved in [7].

THEOREM (3.2.8). *If R is an Artinian Gorenstein ring,* $R=k[R_1]$ *and if* $H(1, R) \geqq 2$, *then* $R^{(d)}$ *is a Gorenstein ring if and only if* $a(R) \equiv 0 \bmod d$ *or* $d > a(R)$.

3. Examples.

(3.3.1). "$R^{(d)}$ is a Gorenstein ring" does not imply "R is a Macaulay ring".

We put $S=k[x, y, z]=k[X, Y, Z]/(F)$ where F is a homogeneous polynomial of degree 3. We assume that $xyz \neq 0$ in S. If we put

$$R=k[x^3, x^2y, xy^2, y^3, y^2z, yz^2, z^3, z^2x, zx^2] \subset S^{(3)},$$

R is not a Macaulay ring. But for every $d \geqq 2$, $R^{(d)}=S^{(3d)}$ is a Gorenstein ring since $a(S)=0$.

(3.3.2). Examples of Gorenstein rings whose Veronesean subrings of all order are Gorenstein rings.

(a) If R is a Gorenstein ring and if $a(R)=0$, $R^{(d)}$ is a Gorenstein ring for every d. For example, if $R=k[X, Y, Z]/(X^2+Y^3+Z^6)$, $\deg(X)=3$, $\deg(Y)=2$, $\deg(Z)=1$, $a(R)=0$ and $R^{(d)}$ is Gorenstein for every d. If we write the images of X, Y and Z in R by x, y and z, it is easy to see that $R^{(2)}=k[z^2, y, xz] \cong k[U, V, W]/(W^2+UV^3+U^4)$, $R^{(3)}=k[z^3, yz, x] \cong k[U, V, W]/(U^3+V^3+UW^2)$ and $R^{(4)}=k[z^4, z^2y, y^2, xz] \cong k[U, V, W, T]/(V^2-UW, T^2+VW+U^2)$. (Cf. [26] and Section 2 of Chapter 5.)

(b) If $R=k[T^2, T^a]$ (a is a positive odd integer) or $R=k[T^3, T^4]$, $R^{(d)}$ is Gorenstein for every d but $a(R) \neq 0$. If $R=k[T^3, T^4, T^5]$ or $R=k[T^4, T^6, T^7, T^9]$, R is not Gorenstein but $R^{(d)}$ is Gorenstein for every $d \geqq 2$.

(c) If R is a Gorenstein ring, $a(R) \neq 0$ and if the set $\{d>0 \mid R^{(d)}$ is a Gorenstein ring$\}$ is an infinite set, then we can show that $\dim R \leqq 1$ by the aid of (3.1.3). The detailed proof is omitted.

(3.3.3) We put $R=k[X, Y]$ with $\deg X=p$ and $\deg Y=q$. We assume that p and q are relatively prime. Then $R^{(d)}$ is Gorenstein if and only if $p'q+pq' \equiv 0 \bmod d$ where $p'=(p, d)$ and $q'=(q, d)$. We can further say that $R^{(d)}$ is a polynomial ring if and only if $p'q \equiv 0 \bmod d$. If, for example, $p=2$ and $q=3$, $R^{(d)}$ is Gorenstein if and only if $d=1, 2, 3, 4, 5, 6, 8, 9$ or 12 and $R^{(d)}$ is a polynomial ring if and only if $d=1, 2, 3$ or 6.

(3.3.4) We put $R=k[X, Y, Z]/(Z^2+Y^3+X^5)$ with $\deg X=6$, $\deg Y=10$ and $\deg Z=15$. Then R is a Gorenstein ring and $a(R)=-1$. In this case, $R^{(d)}$ is a Gorenstein ring if and only if $d=1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 18, 20, 21, 25, 30, 36, 40, 45$ or 60. $R^{(d)}$ is a polynomial ring if and only if $d=2, 3, 5, 6, 10, 15$ or 30. Thus if R is not generated by R_1 over k, it is rather complicated to determine whether $R^{(d)}$ is a Gorenstein ring or not.

(3.3.5) We put $S=k[X, Y]/(X^5+2Y^5)=k[x, y]$

and
$$R=S[y^2/x]=k[x, y, y^2/x],$$
where $\deg X=\deg Y=1$, $ch(k)\neq 2$. R is a Macaulay ring, $\dim R=1$ and $r(R)=2$. But $R^{(2)}$ is a Gorenstein ring. Thus (3.2.1) is not true if $\dim R=1$.

Chapter 4. Segre product of two graded rings defined over a field.

Let us consider the ring $R=k[X_{ij} \mid 1\leqq i\leqq r, 1\leqq j\leqq s]/\mathfrak{a}$ where X_{ij} are indeterminates over a field k and $\mathfrak{a}$ is the ideal generated by 2×2 minors of the matrix (X_{ij}). Then, as is well known, R is isomorphic to the subring $k[S_iT_j \mid 1\leqq i\leqq r, 1\leqq j\leqq s]$ of the polynomial ring $k[S_1, \cdots, S_r, T_1, \cdots, T_s]\cong k[S_1, \cdots, S_r]\otimes_k k[T_1, \cdots, T_s]$. The ring R is the homogeneous coordinate ring of the Segre embedding of $\boldsymbol{P}^{r-1}\times \boldsymbol{P}^{s-1}$ in $\boldsymbol{P}^{rs-1}$. This concept was generalized by Chow [**5**] to the concept of the Segre products of two graded rings defined over the same field k. We also define Segre products of two graded modules. We compute the canonical modules and the local cohomology modules of the Segre products.

In this Chapter, we use the following

NOTATIONS (4.0.1). R and S are graded rings defined over k with H-maximal ideals $\mathfrak{m}=R_+$ and $\mathfrak{n}=S_+$ respectively. We put $r=\dim R$ and $s=\dim S$.
$$T=R\,\#\,S=\bigoplus_{n\geqq 0} R_n\otimes_k S_n .$$
We consider T as a graded ring by $T_n=R_n\otimes_k S_n$.
$$T'=R\,\underline{\otimes}_k\, S .$$
For a graded R-module M and a graded S-module N, we put
$$M\,\#\,N=\bigoplus_{n\in \boldsymbol{Z}} M_n\otimes_k N_n .$$
We consider $M\# N$ as a graded T-module by $(M\# N)_n=M_n\otimes_k N_n$. If $x\in M_n$ and $y\in N_n$, we denote the image of $x\otimes y$ in $M\# N$ by $x\# y$.
$$\mathfrak{P}=T_+=R_+\,\#\,S_+ .$$

PROPOSITION (4.0.2). (i) *T is a Noetherian ring.* (*So, T is also a graded* *ring defined over k.)*

(ii) *If M and N are finitely generated modules, $M\# N$ is a finitely generated T-module.*

PROOF. (i) As T is a direct summand of T' as a T-module, T is a pure subring of T'. As T' is Noetherian, T is Noetherian by Proposition 6.15 of [**17**].

(ii) If Q is a T-submodule of $M\# N$, $Q'=T'\cdot Q$ is a T'-submodule of $M\underline{\otimes}_k N$ and $Q=Q'\cap(M\# N)$. If $(Q_n)_{n\geq 0}$ is an ascending chain of T-submodules of $M\# N$, this ascending chain terminates because $M\underline{\otimes}_k N$ is a finitely generated T'-module.

REMARK (4.0.3). (i) The functors $M\#\cdot$ and $\cdot\# N$ are exact functors and commute with direct sums.

(ii) If M_n and N_n are finite-dimensional k-vector spaces for every integer n, there is a natural isomorphism

$$(M\# N)^*\cong M^*\# N^*.$$

(iii) If M (resp. N) is an Artinian R- (resp. S-) module, $M\# N$ is an Artinian T-module.

(iv) If T' is a normal domain, T is a normal domain.

(v) If k is algebraically closed and if R and S are normal domains, T is a normal domain.

PROOF. (i) and (ii) are direct consequences of definitions. (iii) follows from the duality between Noetherian modules and Artinian modules and (4.0.2).

(iv) As T is a pure subring of T', this follows from Proposition 6.15 of [**17**].

(v) In this case, T' is an integrally closed domain by [**10**], (6.5.4).

1. Calculation of local cohomology groups and the canonical module.

In this section, M is a graded R-module and N is a graded S-module. We write the minimal injective resolution of M (resp. N) in the category of graded R- (resp. S-) modules by $(E^\cdot)$ (resp. by $(I^\cdot)$). For each i, we write

$$E^i={}'E^i\oplus{}''E^i \quad (\text{resp. } I^i={}'I^i\oplus{}''I^i)$$

as in (2.2.6).

LEMMA (4.1.1). (i) *If M or N satisfies the condition* (*) *of* (2.2.1), *$M\# N$ satisfies the condition* (*) *as a T-module.*

(ii) *If M and N satisfy the condition* (**) *of* (2.2.2), *$M\# N$ satisfies the condition* (**) *as a T-module.*

PROOF. If $f\in R_d$ and $g\in S_e$ are such that f_M and g_N are bijective, the multiplication map of $f^e\# g^d$ on $M\# N$ is bijective.

LEMMA (4.1.2). *If N satisfies the condition* (**),

$$\underline{H}^q_{\mathfrak{P}}(M \# N) \cong \underline{H}^q_{\mathfrak{m}}(M) \# N$$

for every integer q.

PROOF. By (2.2.4) and (4.1.1), $\underline{H}^q_{\mathfrak{P}}(M \# N) \cong H^q(\underline{H}^0_{\mathfrak{P}}(E^{\cdot} \# N)) = H^q('E^{\cdot} \# N) = H^q('E^{\cdot}) \# N = \underline{H}^q_{\mathfrak{m}}(M) \# N$, for $\cdot \# N$ is an exact functor.

REMARK. If $S = k[Y]$, $\deg Y = d$ and $N = k[Y, Y^{-1}]$, $R \# S \cong R^{(d)}$ and $M \# N \cong M^{(d)}$. So (3.1.1) is a corollary of (4.1.2).

REMARK (4.1.3). As $0 \to 'E^{\cdot} \to E^{\cdot} \to ''E^{\cdot} \to 0$ is an exact sequence of complexes, we have the long exact sequence of cohomology groups of these complexes. But as $H^0(E^{\cdot}) = M$ and $H^q(E^{\cdot}) = 0$ for $q \geqq 1$, we have

$$H^q(''E^{\cdot}) \cong H^{q+1}('E^{\cdot}) = \underline{H}^{q+1}_{\mathfrak{m}}(M) \qquad (q \geqq 1)$$

and we have an exact sequence

$$0 \longrightarrow \underline{H}^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow H^0(''E^{\cdot}) \longrightarrow \underline{H}^1_{\mathfrak{m}}(M) \longrightarrow 0\,.$$

LEMMA (4.1.4) *Let (A^i, d'_i) (resp. (B^j, d''_j)) be a complex of graded R- (resp. S-) modules. If we define a complex (C^q, d_q) of graded T-modules by*

$$C^q = \bigoplus_{i+j=q} (A^i \# B^j) \quad \text{and} \quad d_q = \sum_{i+j=q} (d'_i \# 1 + (-1)^i (1 \# d''_j))\,,$$

then we have

$$H^q(C^{\cdot}) \cong \bigoplus_{i+j=q} (H^i(A^{\cdot}) \# H^j(B^{\cdot}))\,.$$

PROOF. As the Segre product is the direct sum of tensor products over k of all degrees, we can use the usual Künneth formula of tensor products of complexes over a field. (Cf. [37], Chapter V, (10.1).)

THEOREM (4.1.5). *We assume that $\underline{H}^q_{\mathfrak{m}}(M)$* (resp. $\underline{H}^q_{\mathfrak{n}}(N)$) *vanishes for $q = 0$, 1. Then*

$$\underline{H}^q_{\mathfrak{m}}(M \# N) \cong (M \# \underline{H}^q_{\mathfrak{n}}(N)) \oplus (\underline{H}^q_{\mathfrak{m}}(M) \# N) \oplus (\bigoplus_{i+j=q+1} (\underline{H}^i_{\mathfrak{m}}(M) \# \underline{H}^j_{\mathfrak{n}}(N)))$$

for every $q \geqq 0$.

PROOF. If we put $F^q = \bigoplus_{i+j=q} E^i \# I^j$ and make a complex $(F^{\cdot})$ as in (4.1.4), $(F^{\cdot})$ is a resolution of $M \# N$ by (4.1.4). If we put

$$'F^q = \bigoplus_{i+j=q} (('E^i \# 'I^j) \oplus ('E^i \# ''I^j) \oplus (''E^i \# 'I^j))$$

and

$$''F^q = \bigoplus_{i+j=q} (''E^i \# ''I^j)\,,$$

$'F^q$ satisfies the condition (*) and $''F^q$ satisfies the condition (**) for every q. Moreover, $('F^{\cdot})$ is a subcomplex of $(F^{\cdot})$ and by the assumption, $'F^q = 0$ for $q = 0, 1$. As usual, we denote by $(''F^{\cdot})$ the quotient complex $(F^{\cdot}/'F^{\cdot})$. By (2.2.4), $\underline{H}^q_{\mathfrak{P}}(M \# N) \cong H^q(\underline{H}^0_{\mathfrak{P}}(F^{\cdot})) = H^q('F^{\cdot})$. Using the same argument as in (4.1.3), $H^q('F^{\cdot}) \cong H^{q-1}(''F^{\cdot})$ for $q \geqq 2$ and $H^q('F^{\cdot}) = 0$ for $q = 0$, 1 by the assumption. We use

(4.1.4) again for $({}''F^{\cdot})$ and we get

$$\underline{H}^q_{\mathfrak{P}}(M \# N) \cong H^{q-1}({}''F^{\cdot}) \cong \bigoplus_{i+j=q-1} (H^i({}''E^{\cdot}) \# H^j({}''I^{\cdot})).$$

By (4.1.3) and by the assumption, $H^0({}''E^{\cdot}) \cong M$, $H^0({}''I^{\cdot}) \cong N$ and $H^i({}''E^{\cdot}) \cong \underline{H}^{i+1}_{\mathfrak{m}}(M)$, $H^i({}''I^{\cdot}) = \underline{H}^{i+1}_{\mathfrak{n}}(N)$ for $i \geqq 1$. The assertion follows from these facts.

REMARK (4.1.6). If we do not assume that $\underline{H}^q_{\mathfrak{m}}(M)$ (resp. $\underline{H}^q_{\mathfrak{n}}(N)$) vanishes for $q=0, 1$, (4.1.5) must be modified a little. We put $M'' = H^0({}''E^{\cdot})$ and $N'' = H^0({}''I^{\cdot})$. By (4.1.3), there are exact sequences

$$0 \longrightarrow A \longrightarrow M \longrightarrow M'' \longrightarrow B \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow N \longrightarrow N'' \longrightarrow D \longrightarrow 0$$

where A, B, C and D are modules which satisfy the condition (*). By the aid of local cohomology long exact sequences, we have $\underline{H}^q_{\mathfrak{m}}(M) \cong \underline{H}^q_{\mathfrak{m}}(M'')$ and $\underline{H}^q_{\mathfrak{n}}(N) \cong \underline{H}^q_{\mathfrak{n}}(N'')$ for $q \geqq 2$ and for $q=0, 1$, $\underline{H}^q_{\mathfrak{m}}(M'')=0$ and $\underline{H}^q_{\mathfrak{n}}(N'')=0$. On the other hand, we have $\underline{H}^q_{\mathfrak{P}}(M \# N) = \underline{H}^q_{\mathfrak{P}}(M'' \# N'')$ for $q \geqq 2$. Thus we have

$$\underline{H}^q_{\mathfrak{P}}(M \# N) = (M'' \# \underline{H}^q_{\mathfrak{n}}(N)) \oplus (\underline{H}^q_{\mathfrak{m}}(M) \# N'') \oplus (\bigoplus_{i+j=q+1} (\underline{H}^i_{\mathfrak{m}}(M'') \# \underline{H}^j_{\mathfrak{n}}(N'')))$$

for $q \geqq 2$.

REMARK (4.1.7). Let M be a Macaulay R-module of $\dim M=1$. Then the Cousin complex

$$0 \longrightarrow M \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0$$

of M is exact, M^0 satisfies the condition (**) and $M^1 \cong \underline{H}^1_{\mathfrak{m}}(M)$ satisfies the condition (*). Applying the functor $\cdot \# N$ to this exact sequence, we get the exact sequence

$$0 \longrightarrow M \# N \longrightarrow M^0 \# N \longrightarrow M^1 \# N \longrightarrow 0.$$

In this exact sequence, $M^1 \# N$ satisfies the condition (*). So, by the local cohomology long exact sequence and by (4.1.2), we have an exact sequence

$$0 \longrightarrow \underline{H}^0_{\mathfrak{P}}(M \# N) \longrightarrow M^0 \# \underline{H}^0_{\mathfrak{n}}(N) \longrightarrow \underline{H}^1_{\mathfrak{m}}(M) \# N$$
$$\longrightarrow \underline{H}^1_{\mathfrak{P}}(M \# N) \longrightarrow M^0 \# \underline{H}^1_{\mathfrak{n}}(N) \longrightarrow 0$$

and isomorphisms $\underline{H}^q_{\mathfrak{P}}(M \# N) \cong M^0 \# \underline{H}^q_{\mathfrak{n}}(N)$ for $q \geqq 2$. If $M \# \underline{H}^q_{\mathfrak{n}}(N)=0$ for some integer $q \geqq 2$, we have

$$\underline{H}^q_{\mathfrak{p}}(M \# N) \cong \underline{H}^1_{\mathfrak{m}}(M) \# \underline{H}^q_{\mathfrak{n}}(N).$$

2. Dimension and depth of the Segre product.

(4.2.1) We first recall two fundamental facts. If M is a finitely generated graded R-module,

$$\dim M = \max\{q \mid \underline{H}^q_{\mathfrak{m}}(M) \neq 0\}$$

and

$$\operatorname{depth} M = \min\{q \mid \underline{H}^q_{\mathfrak{m}}(M) \neq 0\}.$$

PROPOSITION (4.2.2). *If $R=k[X_1, \cdots, X_r]$ and $S=k[Y_1, \cdots, Y_s]$ are polynomial rings with $\deg(X_i)=\deg(Y_j)=1$ for every i, j and if $M=R(a)$ and $N=S(b)$ for some integers a and b, then*

(i) $\dim M \# N = r+s-1$ *if $r \geqq 1$ and $s \geqq 1$.*

(ii) *If $r \geqq 2$ and $s \geqq 2$,*

(a) *$M \# N$ is a Macaulay T-module if and only if $s > a-b > -r$,*

(b) *if $a-b \geqq s$, depth $M \# N = s$,*

(c) *if $a-b \leqq -r$, depth $M \# N = r$.*

(iii) *If $r=1$ and $s \geqq 2$,*

(a) *if $a \geqq b$, $M \# N \cong S(b)$ and $M \# N$ is a Macaulay T-module,*

(b) *if $a < b$,* depth $M \# N = 1$.

(iv) *If $r=s=1$, $M \# N$ is a Macaulay T-module for every a and b.*

PROOF. By (2.2.8) and (2.1.5), we have

$$\underline{H}^q_{\mathfrak{m}}(M) \cong \begin{cases} R^*(a+r) & (q=r) \\ 0 & (q \neq r) \end{cases}$$

$$\underline{H}^q_{\mathfrak{n}}(N) = \begin{cases} S^*(b+s) & (q=s) \\ 0 & (q \neq s). \end{cases}$$

If $r \geqq 2$ and $s \geqq 2$, then by (4.1.5),

$$\underline{H}^q_{\mathfrak{P}}(M \# N) = \begin{cases} M \# \underline{H}^s_{\mathfrak{n}}(N) = R(a) \# S^*(b+s) & (q=s) \\ \underline{H}^r_{\mathfrak{m}}(M) \# N = R^*(a+r) \# S(b) & (q=r) \\ \underline{H}^r_{\mathfrak{m}}(M) \# \underline{H}^s_{\mathfrak{n}}(N) = R^*(a+r) \# S^*(b+s) & (q=r+s-1) \\ 0 \ \text{(otherwise)}. \end{cases}$$

The assertions (i) and (ii) follow from the above isomorphisms by the aid of (4.2.1). The assertions (iii) and (iv) follow from (4.1.7).

THEOREM (4.2.3). *Let R and S be graded rings defined over k. We put* $\dim R = r$ *and* $\dim S = s$.

(i) *If $r \geqq 1$ and $s \geqq 1$,* $\dim T = r+s-1$.

(ii) *If $r \geqq 2$, $s \geqq 2$ and if R and S are Macaulay rings, T is a Macaulay ring if and only if $R \# \underline{H}^s_{\mathfrak{n}}(S) = 0$ and $\underline{H}^r_{\mathfrak{m}}(R) \# S = 0$.*

(iii) *If $r=1$, $s \geqq 2$ and if R and S are Macaulay rings, T is a Macaulay ring if and only if $\underline{H}^1_{\mathfrak{m}}(R) \# S = 0$.*

(iv) *If $r=s=1$ and if R and S are Macaulay rings, T is a Macaulay ring.*

PROOF. (i) We can take a parameter system $(x_1, \cdots, x_r)$ of R (resp.

$(y_1, \cdots, y_s)$ of S) such that $x_i \in R_d$ for every i (resp. $y_j \in S_d$ for every j). If we put $R'=k[x_1, \cdots, x_r]$, $S'=k[y_1, \cdots, y_s]$ and $T'=R' \# S'$, $\dim T'=r+s-1$ by (4.2.2). As T is a finite module over T' and as $T' \subset T$, $\dim T=\dim T'=r+s-1$.

(ii) follows from (4.1.5). (iii) and (iv) follow from (4.1.7).

PROPOSITION (4.2.4). *If M (resp. N) is a Macaulay R-(resp. S-) module of dimension $r \geqq 1$ (resp. $s \geqq 1$), $\dim M \# N=r+s-1$ or $M \# N=0$.*

PROOF. Let $(x_1, \cdots, x_r)$ and $(y_1, \cdots, y_s)$ be parameter systems of M and N respectively. We assume $x_i \in R_d$ and $y_j \in S_d$ for every i and j. If we put $R'=k[x_1, \cdots, x_r]$ and $S'=k[y_1, \cdots, y_s]$, R' and S' are polynomial rings over k and M and N are finitely generated free modules over R' and S' respectively. Thus $M \# N$ is a direct sum of modules of the form $R'(a) \# S'(b)$. If $a \not\equiv b \bmod d$, $R'(a) \# S'(b)=0$. If $a=b+nd$ for some integer n, as $R'(a) \# S'(b)=(R'(nd) \# S')(b)$, $\dim R'(a) \# S'(b)=r+s-1$.

REMARK. If we put $R=S=k[X_1, \cdots, X_r]$ where $\deg X_i=d>1$ for every i, $M=R(1)$ and $N=S$, then $M \# N=0$. If we put $M=R(1) \oplus R/(X_1, \cdots, X_i)$ and $N=S$, $\dim M=\dim N=r$ and $\dim M \# N=2r-1-i$. Thus the assumption "M and N are Macaulay" in (4.2.4) is necessary.

PROPOSITION (4.2.5). *Let M (resp. N) be a Macaulay R- (resp. S-) module of dimension r (resp. s). We assume that $M \# N \neq 0$.*

(i) *If $r \geqq 2$ and $s \geqq 2$, $M \# N$ is a Macaulay T-module if and only if $M \# \underline{H}^s_{\mathfrak{n}}(N)=0$ and $\underline{H}^r_{\mathfrak{m}}(M) \# N=0$.*

(ii) *If $r=1$ and $s \geqq 2$, $M \# N$ is a Macaulay T-module if and only if $\underline{H}^1_{\mathfrak{m}}(M) \# N=0$.*

(iii) *If $r=s=1$, $M \# N$ is a Macaulay T-module.*

PROOF. These results follow from (4.1.5), (4.1.7) and (4.2.4).

3. The canonical module of the Segre product.

In this section, we put $\dim R=r$ and $\dim S=s$. We assume $r \geqq 1$ and $s \geqq 1$.

THEOREM (4.3.1). *If $r \geqq 2$ and $s \geqq 2$, $K_T \cong K_R \# K_S$.*

PROOF. By (4.2.3), $\dim T=r+s-1$ and by (4.1.5) and (4.1.6), $\underline{H}^{r+s-1}_{\mathfrak{P}}(T) \cong \underline{H}^r_{\mathfrak{m}}(R) \# \underline{H}^s_{\mathfrak{n}}(S)$. By (4.0.3), $K_T=(\underline{H}^{r+s-1}_{\mathfrak{P}}(T))^* \cong (\underline{H}^r_{\mathfrak{m}}(R))^* \# (\underline{H}^s_{\mathfrak{n}}(S))^*=K_R \# K_S$.

COROLLORY (4.3.2). *If $r \geqq 2$, $s \geqq 2$ and if T is a Macaulay ring, T is a Gorenstein ring if and only if $K_R \# K_S \cong T(d)$ for some integer d.*

COROLLARY (4.3.3). *We assume that R and S are Gorenstein rings, $r \geqq 2$ and $s \geqq 2$. If T is a Macaulay ring and if $a(R)=a(S)$, T is a Gorenstein ring.*

PROOF. If we put $a=a(R)=a(S)$, $K_R \# K_S \cong R(a) \# S(a) \cong T(a)$.

COROLLARY (4.3.4). *If R is a Gorenstein ring, $r \geqq 2$ and if $R \# R$ is a Macaulay ring, $R \# R$ is a Gorenstein ring.*

REMARK (4.3.5). If $r=1$, $s \geqq 2$ and if R is Macaulay ring, there is an exact sequence

$$0 \longrightarrow K_R \# K_S \longrightarrow K_T \longrightarrow R^* \# K_S \longrightarrow 0 .$$

PROOF. Let $0 \to R \to R^0 \to R^1 \to 0$ be the Cousin complex of R in the category of graded R-modules. As R is a Macaulay ring, this complex is exact and $\underline{H}^1_{\mathfrak{m}}(R) \cong R^1$. By (4.1.7), $\underline{H}^s_{\mathfrak{P}}(T) \cong R^0 \# \underline{H}^s_{\mathfrak{n}}(S)$. So there is an exact sequence

$$0 \longrightarrow R \# \underline{H}^s_{\mathfrak{n}}(S) \longrightarrow \underline{H}^s_{\mathfrak{P}}(T) \longrightarrow \underline{H}^1_{\mathfrak{m}}(R) \# \underline{H}^s_{\mathfrak{n}}(S) \longrightarrow 0 .$$

If we take the dual of this exact sequence, we have the desired result by (4.0.3).

4. Segre product of R and S which satisfy the conditions $R=k[R_1]$ and $S=k[S_1]$.

In this section, we assume that $R=k[R_1]$ and $S=k[S_1]$. We put $r=\dim R$, $s=\dim S$ and assume that $r \geqq 1$, $s \geqq 1$.

LEMMA (4.4.1). *$(\underline{H}^r_{\mathfrak{m}}(R))_n \neq 0$ for $n \leqq a(R)$.*

PROOF. If depth $R=0$, we put $\bar{R}=R/\underline{H}^0_{\mathfrak{m}}(R)$. Then depth $\bar{R}>0$ and $\underline{H}^r_{\mathfrak{m}}(R) = H^r_{\mathfrak{m}}(\bar{R})$. So we may assume that depth $R>0$. Also, we may assume that k is an infinite field. If we take an R-regular element $x \in R_1$, the exact sequence

$$0 \longrightarrow R(-1) \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0$$

induces the following exact sequence of local cohomology groups

$$\underline{H}^{r-1}_{\mathfrak{m}}(R/xR) \longrightarrow \underline{H}^r_{\mathfrak{m}}(R)(-1) \xrightarrow{x} \underline{H}^r_{\mathfrak{m}}(R) \longrightarrow 0 .$$

This exact sequence shows that $[(\underline{H}^r_{\mathfrak{m}}(R))_n : k] \geqq [(\underline{H}^r_{\mathfrak{m}}(R))_{n+1} : k]$ for every integer n. As $(\underline{H}^r_{\mathfrak{m}}(R))_{a(R)} \neq 0$ by the definition of $a(R)$, the result follows.

LEMMA (4.4.2). $\dim (R(a) \# S(b))=r+s-1$ *for every integer a and b.*

PROOF. If $r \geqq 2$ and $s \geqq 2$, then by (4.1.5), (4.1.6) and (4.4.1), $\underline{H}^{r+s-1}_{\mathfrak{P}}(R(a) \# S(b)) \cong \underline{H}^r_{\mathfrak{m}}(R)(a) \# \underline{H}^s_{\mathfrak{n}}(S)(b) \neq 0$ and $\underline{H}^q_{\mathfrak{P}}(R(a) \# S(b))=0$ if $q \geqq r+s$. So $\dim R(a) \# S(b) = r+s-1$ by (4.2.1). If $r=1$, we may assume that R is a Macaulay ring by the same argument as in (4.4.1). Then putting $M=R(a)$ in (4.1.7), we have $\dim R(a) \# S(b)=s$.

PROPOSITION (4.4.3). *If M and N are finitely generated and if neither is an Artinian module,* $\dim M \# N = \dim M + \dim N - 1$.

PROOF. By (4.1.5), (4.1.6), (4.1.7) and (4.2.1), $\dim M \# N \leqq \dim M + \dim N - 1$. On the other hand, there is a cyclic R-submodule M' of M (resp. cyclic S-submodule N' of N) with $\dim M = \dim M'$ (resp. $\dim N' = \dim N$). By (4.4.2), $\dim M' \# N' = \dim M + \dim N - 1$. As $M' \# N'$ is a submodule of $M \# N$, $\dim M \# N = \dim M + \dim N - 1$.

THEOREM (4.4.4). *We assume that R and S are Macaulay rings.*

(i) *If* $r\geqq 2$ *and* $s\geqq 2$, T *is a Macaulay ring if and only if* $a(R)<0$ *and* $a(S)<0$.

(ii) *If* $r=1$ *and* $s\geqq 2$, T *is a Macaulay ring if and only if* R *is a polynomial ring.*

PROOF. (i) By (4.2.3), T is Macaulay if and only if $R\,\#\,\underline{H}_{\mathfrak{n}}^{s}(S)=0$ and $\underline{H}_{\mathfrak{m}}^{r}(R)\,\#\,S=0$. So T is Macaulay if and only if $(\underline{H}_{\mathfrak{m}}^{r}(R))_d=0$ and $(\underline{H}_{\mathfrak{n}}^{s}(S))_d=0$ for every $d\geqq 0$. The latter condition is equivalent to say that $a(R)<0$ and $a(S)<0$ by (4.4.1).

(ii) By (4.2.3), T is Macaulay if and only if $a(R)<0$. This condition is then equivalent to say that R is a polynomial ring by the following lemma.

LEMMA (4.4.5). *If* R *is a Macaulay ring,* $\dim R=r$ *and if* $R=k[R_1]$, *then* $a(R)\geqq -r$ *and* $a(R)=-r$ *if and only if* R *is a polynomial ring over* k.

PROOF. If $r=0$, $a(R)=t(R)\geqq 0$ (cf. (3.1.4)) and $a(R)=0$ if and only if $R=k$. If $r\geqq 1$, we may assume that k is an infinite field and we may take an R-regular element $x\in R_1$. Then $a(R/xR)=a(R)+1$ by (3.1.6). R is a polynomial ring if and only if R/xR is a polynomial ring. Thus we can proceed by induction on r.

LEMMA (4.4.6). *If* $a\geqq b$, $\nu_T(R(a)\,\#\,S(b))=[R_{a-b}:k]$.

PROOF. $R(a)\,\#\,S(b)$ is generated by the elements of $(R(a)\,\#\,S(b))_{-b}=R(a)_{-b}\otimes_k S(b)_{-b}=R_{a-b}$.

THEOREM (4.4.7). *We assume that* R, S *are Gorenstein rings,* $r\geqq 2$, $s\geqq 2$ *and that* T *is a Macaulay ring. We put* $a=a(R)$, $b=a(S)$ *and we assume that* $a\geqq b$. *Then*

$$r(T)=[R_{a-b}:k].$$

In particular, T is a Gorenstein ring if and only if $a=b$.

PROOF. $r(T)=\nu_T(K_T)=\nu_T(K_R\,\#\,K_S)=\nu_T(R(a)\,\#\,S(b))=[R_{a-b}:k]$ by (4.4.6), (4.3.1) and (2.1.8).

EXAMPLE (4.4.8). We put $R=k[X_1,\cdots,X_r]$ and $S=k[Y_1,\cdots,Y_s]$ and assume that $r\geqq s$. Then T is a Macaulay ring with

$$r(T)=\binom{r-1}{s-1}.$$

PROOF. By (2.2.8), $K_R=R(-r)$ and $K_S=S(-s)$. T is a Macaulay ring by (4.4.4) and by (4.4.7),

$$r(T)=[S_{r-s}:k]=\binom{r-1}{s-1}.$$

THEOREM (4.4.9). *We assume that* R, S *are Macaulay rings and that* $r\geqq 2$, $s\geqq 2$. *If* T *is a Gorenstein ring, then* R *and* S *are Gorenstein rings and* $a(R)=a(S)<0$.

To prove this theorem, we need two lemmas.

LEMMA (4.4.10). *We take* $x\in M_n$, $y\in N_n$ *and assume that* $N_m\neq 0$ *for* $m\geqq n$.

If $x \# y$ generates $M \# N$ over T, $R_d x = M_{n+d}$ for every $d \geqq 0$.

PROOF. If $R_d x \neq M_{n+d}$ for some $d \geqq 0$, then $T_d(x \# y) = xR_d \otimes_k S_d y \neq (M \# N)_{n+d}$ since $N_{n+d} \neq 0$.

LEMMA (4.4.11). *If R is a Macaulay ring, $r \geqq 2$ and if $n > -a(R)$, $[(K_R)_n : k] \geqq 2$.*

PROOF. By (2.1.8), K_R is a Macaulay R-module of depth r. Extending the base field, if necessary, we may assume that k is an infinite field. Then we can take a parameter system $(x_1, \cdots, x_r)$ of K_R from R_1. If we put $R' = k[x_1, \cdots, x_r]$, R' is a polynomial ring over k and K_R is a free R'-module. If we take a k-basis of $(K_R)_{-a(R)}$, then we may assume these elements form a subset of a free basis of K_R over R'. Thus we have an inequality, $[(K_R)_n : k] \geqq [(R')_{n+a(R)} : k] \geqq 2$ if $n > -a(R)$.

PROOF OF (4.4.9). We assume that T is a Gorenstein ring. Then $K_T = K_R \# K_S$ is a cyclic T-module. If we put $b = -a(T)$,

$$1 = \nu_T(K_R \# K_S) = [(K_R)_b : k][(K_S)_b : k] .$$

So, by (4.4.11), $b = -a(R) = -a(S)$ and by (4.4.10), K_R and K_S are cyclic modules. Thus R and S are Gorenstein rings.

PROPOSITION (4.4.12). *We assume that R and S are Macaulay rings of dimension 1 and that R is not a polynomial ring. If T is a Gorenstein ring, then S is a polynomial ring and R is a Gorenstein ring.*

PROOF. We may assume that k is an infinite field. We take a T-regular element $z \in T_1$. Then T/zT is an Artinian Gorenstein ring. So, if we put $n = t(T/zT)$, $H(n, T/zT) = 1$. But on the other hand, $H(n, T/zT) = H(n, T) - H(n-1, T) = H(n, R) \cdot H(n, S) - H(n-1, R) \cdot H(n-1, S)$. It is easy to show that the equality $1 = H(n, R) \cdot H(n, S) - H(n-1, R) \cdot H(n-1, S)$ is impossible unless $H(n, S) = 1$. Thus we have proved that S is a polynomial ring. Then $R \cong T$ is a Gorenstein ring.

EXAMPLE (4.4.13). If R and S are Macaulay rings, $r \geqq 2$, $s \geqq 2$ and $a(R) \geqq 0$, T is not a Macaulay ring by (4.4.4). But if $a(S) \leqq -a(R) - 2$ and if we choose an integer n such that $-a(S) > n > a(R)$, then $R(n) \# S$ is a Macaulay T-module and depth $(R(n) \# S) = \dim T$.

EXAMPLE (4.4.14). If R or S is not generated by homogeneous elements of degree 1 over k, (4.4.4) is not true. If we put $R = k[X, Y, Z]/(X^2 + Y^3 + Z^n)$ where n is an integer prime to 6 and $n \geqq 7$. R is a graded ring over k if we put $\deg(X) = 3n$, $\deg(Y) = 2n$ and $\deg(Z) = 6$. In this case, $R \# R$ is a Macaulay (and also a Gorenstein) ring but $a(R) = n - 6 > 0$.

Chapter 5. Geometric Backgrounds.

1. Proj(R) of a class of graded rings.

Proj(R) of a graded ring R is discussed in [8]. But, there, most part is written under the assumption that "R_+ is generated by R_1". But this condition is too strong for us. Instead, we assume the following condition for a graded ring $R=\bigoplus_{n\geqq 0} R_n$.

(♯) There exists an integer d_0 such that for every $d\geqq d_0$, $R^{(d)}$ is generated by $R_d=[R^{(d)}]_1$ over R_0.

(5.1.1). We use the following notations in this section.

$R=\bigoplus_{n\geqq 0} R_n$ is a Noetherian graded ring which satisfies the condition (♯).

$X=\mathrm{Proj}(R)=\{\mathfrak{p}\,;\ \mathfrak{p}$ is a graded prime ideal of R, $\mathfrak{p}\not\supset R_+\}$.

$D_+(f)=\{\mathfrak{p}\in X\,;\ f\notin\mathfrak{p}\}\qquad (f\in R_d,\ d>0)$.

$R_{(f)}=\left\{\frac{r}{f^n}\in R_f\,;\ r\in R_{nd}\right\}=(R_f)_0\qquad (f\in R_d,\ d>0)$.

We know that $D_+(f)=\mathrm{Spec}(R_{(f)})$.

$\mathcal{O}_X$ is the structure sheaf of X. $\tilde{M}$ is the $\mathcal{O}_X$-module associated to M if M is a graded R-module. We know that $\tilde{M}|_{D_+(f)}=(\tilde{M}_{(f)})$ on $D_+(f)=\mathrm{Spec}(R_{(f)})$.

$\mathcal{O}_X(n)=\widetilde{R(n)}$ for a integer n.

LEMMA (5.1.2). *If M and N are finitely generated graded R-module, then the homomorphisms*

$$\lambda:\ \tilde{M}\otimes_{\mathcal{O}_X}\tilde{N}\longrightarrow (M\otimes_R N)^{\sim}$$

$$\mu:\ (\underline{\mathrm{Hom}}_R(M,N))^{\sim}\longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\tilde{M},\tilde{N})$$

defined respectively in (2.5.11) *and* (2.5.12) *of* [8] *are isomorphisms. In particular,*

$$\mathcal{O}_X(n)\otimes_{\mathcal{O}_X}\mathcal{O}_X(m)\cong\mathcal{O}_X(n+m)$$

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(n),\mathcal{O}_X(m))\cong\mathcal{O}_X(m-n)$$

and $\mathcal{O}_X(n)$ is an invertible $\mathcal{O}_X$-Module for every integer n.

PROOF. First, we prove a sublemma.

SUBLEMMA (5.1.3). *Assume that R satisfies the condition* (♯). *If we take $f\in R_d$ $(d>0)$ such that $R_f\neq 0$, then every finitely generated graded R_f-module M is generated over M_0 as an R_f-module.*

PROOF. Take a graded prime $\mathfrak{p}$ of R such that $\mathfrak{p}\cdot R_f$ is an H-maximal ideal of R_f. Then, by (1.1.1), $R_f/\mathfrak{p}\cdot R_f\cong K[T,T^{-1}]$ where $K=(R_f/\mathfrak{p}R_f)_0$ is a field and T is an indeterminate over K. By (♯), there exists an element $g\in R_e$, $(d,e)=1$, $g\notin\mathfrak{p}$. So, $\deg(T)=1$. Now, let M' be the submodule of M generated by

M_0 over R_f. Then, by (1.1.1), $M/\mathfrak{p}M$ is a free $R_f/\mathfrak{p}R_f$-module and so is generated by M_0. So $M/(M'+\mathfrak{p}M)=0$ and thus $(M/M')_{\mathfrak{p}}=0$ by Nakayama's lemma. As $\mathfrak{p}R_f$ is arbitrary H-maximal ideal of R_f, we have $M=M'$.

PROOF OF (5.1.2). We take $f\in R_d$ $(d>0)$ such that $R_f\neq 0$. We have to show that

$$M_{(f)}\otimes_{R_{(f)}}N_{(f)}\cong(M\underline{\otimes}_R N)_{(f)}\cong(M_f\underline{\otimes}_{R_f}N_f)_0\,.$$

As M_f and N_f satisfy the condition of (5.1.3), we have $(M_f)_n=(R_f)_n\cdot(M_f)_0$ and $(N_f)_{-n}=(R_f)_{-n}\cdot(N_f)_0$ for every integer n. Thus $(M_f)_n\otimes_{R_{(f)}}(N_f)_{-n}=M_{(f)}\otimes_{R_{(f)}}N_{(f)}$ and we have proved that λ is an isomorphism. The same argument shows that μ is an isomorphism.

NOTATION (5.1.4). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-Module. We put $\mathcal{F}(n)=\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_X(n)$ and $\Gamma_*(\mathcal{F})=\bigoplus_{n\in Z}\Gamma(X, \mathcal{F}(n))$. $\Gamma_*(\mathcal{F})$ is a finitely generated graded R-module if $\mathrm{Ass}(\mathcal{F})$ has no component of dimension 0.

LEMMA (5.1.5). *The homomorphism*

$$\beta:\widetilde{\Gamma_*(\mathcal{F})}\longrightarrow\mathcal{F}$$

defined in (2.7.5) *of* [8] *is an isomorphism.*

PROOF. The proof of [8], (2.7.5) works in this case, too.

REMARK. If R does not satisfy the condition (♯), then (5.1.2) and (5.1.5) are not true in general. For example, if R is generated by R_d $(d\geqq 2)$ over R_0, then $\mathcal{O}_X(n)=0$ if n is not a multiple of d.

(5.1.6) (E. G. A. III (2.1.5)). (i) If R is a graded ring defined over a field k, satisfies the condition (♯) and if M is a finitely generated graded R-module, there is an exact sequence of graded R-modules

$$0\longrightarrow\underline{H}^0_{\mathfrak{m}}(M)\longrightarrow M\longrightarrow\Gamma_*(\tilde{M})\longrightarrow\underline{H}^1_{\mathfrak{m}}(M)\longrightarrow 0\quad(\mathfrak{m}=R_+)$$

and isomorphisms of graded R-modules

$$\bigoplus_{n\in Z}H^p(X, \tilde{M}(n))\cong\underline{H}^{p+1}_{\mathfrak{m}}(M)\qquad(p\geqq 1)\,.$$

(ii) If $\dim M\geqq 2$, M is a Macaulay R-module if and only if the following conditions are satisfied.

(a) $M\rightarrow\Gamma_*(M)$ is an isomorphism.

(b) $H^p(X, \tilde{M}(n))=0$ for $0<p<\dim(\mathrm{Supp}(\tilde{M}))$ and for every integer n.

NOTATION (5.1.7). Until the end of this chapter, we use the following notations.

X is a projective variety defined over a field k with $H^0(X, \mathcal{O}_X)=k$.

$\mathcal{L}$ is an ample invertible sheaf on X. We write $\mathcal{L}=\mathcal{O}_X(1)$ and $\mathcal{L}^{\otimes n}=\mathcal{O}_X(n)$.

$R=R_{X,\mathcal{L}}=\bigoplus_{n\geqq 0}H^0(X, \mathcal{O}_X(n))$. Note that R is a graded ring defined over k, satisfies the condition (♯) and that $\mathrm{Proj}(R)=X$. We have $\mathcal{L}=\widetilde{R(1)}$ by (5.1.5).

$\mathcal{F}(n)=\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}(n)$ for an $\mathcal{O}_X$-Module $\mathcal{F}$.

ω_X is the dualizing Module of X. (Cf. Altman-Kleiman [**1**].)

(5.1.8) If K_R is the canonical module of $R=R_{X,\mathcal{L}}$, then $K_R\cong\Gamma_*(\omega_X)$ and $\omega_X\cong\tilde{K}_R$.

Proof. If $\dim X=d$, $\dim R=d+1$ and by the definition, $(K_R)_n=((\underline{H}^{d+1}(R))_{-n})^*\cong(H^d(X, \mathcal{O}_X(-n)))^*\cong H^0(X, \omega_X(n))$. Thus we have the first statement and the second follows from (5.1.5).

(5.1.9) If $R=R_{X,\mathcal{L}}$ is a Macaulay ring, R is a Gorenstein ring if and only if $\omega_X\cong\mathcal{O}_X(n)$ for some integer n.

Proof. This follows from (2.1.3) and (5.1.8).

Lemma (5.1.10). *If R satisfies the condition ($\sharp$) and if R is a Macaulay (resp. Gorenstein) ring, then $X=\mathrm{Proj}(R)$ is a Macaulay (resp. Gorenstein) scheme.*

Proof. Let $f\in R_d$ $(d>0)$ be such that $R_f\neq 0$ and let $\mathfrak{p}$ be a graded prime ideal of R such that $\mathfrak{p}\cdot R_f$ is an H-maximal ideal of R_f. By the condition ($\sharp$), there exists an element $g\in R_e$, $(d, e)=1$, $g\notin\mathfrak{p}$. So the homogeneous localization $R_{(\mathfrak{p})}$ has an invertible element of degree 1 and $R_{(\mathfrak{p})}\cong(R_{(\mathfrak{p})})_0[T, T^{-1}]$. So if R is a Macaulay (resp. Gorenstein) ring, $(R_{(\mathfrak{p})})_0$ is a Macaulay (resp. Gorenstein) ring. As f and $\mathfrak{p}$ are arbitrary, $\mathrm{Proj}(R)$ is a Macaulay (resp. Gorenstein) scheme.

(5.1.11) If $R=R_{X,\mathcal{L}}$ is a Macaulay ring, then $H^p(X, \mathcal{O}_X)=0$ for $0<p<\dim X$. Conversely, if X is a connected Macaulay scheme and if $H^p(X, \mathcal{O}_X)=0$ for $0<p<\dim X$, then $R^{(n)}$ is a Macaulay ring for every sufficiently large n.

Proof. The first statement follows from (5.1.6). As for the second statement, as $\mathcal{L}$ is ample, $H^p(X, \mathcal{O}_X(n))=0$ for $p>0$ and for every sufficiently large n. On the other hand, if n is sufficiently small,

$$H^p(X, \mathcal{O}_X(n))\cong H^{\dim X-p}(X, \omega_X(-n))=0$$

for every $p<\dim X$. (As X is a Macaulay scheme, we can use the Grothendieck duality theorem.)

(5.1.12) If $R_{X,\mathcal{L}}$ is a Gorenstein ring, one of the following cases occurs.

(a) $\omega_X\cong\mathcal{O}_X$, (b) ω_X is ample, (c) ω_X^{-1} is ample.

Proof. As $\mathcal{L}$ is ample, this is obvious by (5.1.9).

Example (5.1.13). Let X be a non-singular projective surface defined over an algebraically closed field k. Assume that $R_{X,\mathcal{L}}$ is a Gorenstein ring. Then by (5.1.12) and the classification theory of surfaces, there are following three cases.

(1) When ω_X is ample, X is a surface of general type without exceptional curves.

(2) When $\omega_X\cong\mathcal{O}_X$, X is a "$K3$-surface" because $H^1(X, \mathcal{O}_X)=0$ by (5.1.11).

(3) When ω_X^{-1} is ample, X is a rational surface. As structure of rational surfaces are well-known (cf. [**22**]), we can determine all cases where $R_{X,\mathcal{L}}$ is a

Gorenstein ring. Let us write the canonical divisor of X by K. $\omega_X=\mathcal{O}_X(K)$. As $-K$ is ample, $KC<0$ for every curve C on X. So, if C is a non-singular rational curve on X, $C^2\geqq -1$. Thus we can see that either X is obtained from $\boldsymbol{P}^2$ by successive blow-ups or $X\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$. If X has an exceptional curve C of the first kind, then $KC=-1$. So, if D is an ample divisor on X with $nD=K$, then $n=-1$. Conversely, if X is a rational surface and $\mathcal{L}$ is an ample invertible sheaf on X with $\mathcal{L}^{\otimes n}=\omega_X^{-1}$ for some positive integer n, then it is not difficult to see $H^1(X, \mathcal{L}^{\otimes m})=0$ for every integer m (if ch$(k)=0$, Kodaira vanishing theorem and Serre duality are sufficient to prove this statement). So, by (5.1.6) and (5.1.9), $R_{X,\mathcal{L}}$ is a Gorenstein ring.

Now, let us make the list of $(X, \mathcal{L})$ and $R_{X,\mathcal{L}}$ where X is a rational surface and $R_{X,\mathcal{L}}$ is a Gorenstein ring.

(a) $X=\boldsymbol{P}^2$, $\mathcal{L}=\mathcal{O}(H)$ (H is a hyperplane of X), $R_{X,\mathcal{L}}=k[T_0, T_1, T_2]$.

(b) $X=\boldsymbol{P}^2$, $\mathcal{L}=\mathcal{O}(3H)=\omega_X^{-1}$, $R_{X,\mathcal{L}}=(k[T_0, T_1, T_2])^{(3)}$.

(c) $X=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $\mathcal{L}=\mathcal{O}(H_1)\otimes_k\mathcal{O}(H_2)$ (H_1 and H_2 are hyperplanes of the first and the second factor, respectively), $R_{X,\mathcal{L}}=k[S_0, S_1]\,\#\,k[T_0, T_1]$.

(d) $X=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $\mathcal{L}=\mathcal{O}(2H_1)\otimes_k\mathcal{O}(2H_2)=\omega_X^{-1}$,
$R_{X,\mathcal{L}}=k[S_0^2, S_0S_1, S_1^2]\,\#\,k[T_0^2, T_0T_1, T_1^2]$.

(e)$_n$ ($n=1, 2, \cdots, 8$) We select n points $P_1, \cdots, P_n$ on $\boldsymbol{P}^2$ satisfying the conditions

(i) no three points lie on a line of $\boldsymbol{P}^2$

(ii) no six points lie on a conic of $\boldsymbol{P}^2$.

Then, we define X to be a surface obtained from $\boldsymbol{P}^2$ by blowing up these n points and we put $\mathcal{L}=\omega_X^{-1}$. In these cases, the ring $R_{X,\mathcal{L}}$ is the subring of $k[T_0, T_1, T_2]$ generated by all homogeneous polynomials of degree $3m$ ($m=1, 2, \cdots$) which vanish m-times at $P_1, \cdots, P_n$.

2. Point divisors on smooth curves.

In this section, let k be an algebraically closed field and X be a complete smooth curve defined over k. We treat point divisors on X and we will find a relationship of $R_{X,\mathcal{L}}$ and a semigroup ring.

NOTATION (5.2.1). X is a complete smooth curve of genus g defined over k. We assume $g\geqq 1$.

P is a closed point on X. We consider P as a divisor on X.

$\mathcal{L}=\mathcal{O}_X(P)$ is the invertible sheaf associated to the divisor P. As deg $\mathcal{L}=1$, $\mathcal{L}$ is ample.

$R=R_{X,\mathcal{L}}$. R is a two-dimensional normal domain.

$h(n)=\dim_k H^0(X, \mathcal{O}_X(nP))=\dim_k (R_{X,\mathcal{L}})_n$.

By Riemann-Roch formula, we have the following properties for $h(n)$.

(i) $h(n)=0$ for $n<0$ and $h(n)=n-g+1$ for $n\geqq 2g-1$.

(ii) $h(0)=h(1)=1$. (We have assumed $g\geqq 1$.)

(iii) For every integer n, $h(n)-h(n-1)$ is either 0 or 1.

DEFINITION (5.2.2). We put

$$H=H_{X,P}=\{n\in \boldsymbol{Z}\,;\ h(n)-h(n-1)=1\}.$$

Then H is an additive subsemigroup of $\boldsymbol{N}$. If $n\geqq 2g$, $n\in H$. (Cf. Gunning [30] Section 4.)

NOTATION (5.2.3). The multiplicity $m(H)$ and embedding dimension $\mathrm{emb}(H)$ of a numerical semigroup H was defined by

$m(H)=\min\{n>0\,;\ n\in H\}$

$\mathrm{emb}(H)$ is the number of minimal generators of H.

For a graded ring R defined over k, we write

$m(R)=$the multiplicity of the local ring $R_{\mathfrak{m}}$

$\mathrm{emb}(R)=\nu(R_+)=$embedding dimension of $R_{\mathfrak{m}}$ (where $\mathfrak{m}=R_+$ as usual).

PROPOSITION (5.2.4). *Let t be a non-zero element of R_1. Then t is a prime element of R and $R/tR\cong k[H]$ (the semigroup ring of H).*

PROOF. Let $k(X)$ be the rational function field of X. Then R can be seen as a graded subring of $k(X)[T]$ in the following way.

$$R_n=\{fT^n\in k(X)T^n\ \ ;\ \ v_P(f)\geqq -n \text{ and } v_Q(f)\geqq 0 \text{ for } Q\neq P\}$$

where $v_Q(f)$ is the order of zero (or pole, if $v_Q(f)$ is negative) of f at $Q\in X$. As $R_1=k\cdot T$ by this identification, we may assume that $t=T$. If $fT^n\in R_n$, $fT^n\notin tR$ if and only if $v_P(f)=-n$. Let $fT^n\in R_n$, $gT^m\in R_m$ and assume that $fT^n\notin tR$ and $gT^m\notin tR$. Then $v_P(fg)=-n-m$ and $fgT^{n+m}\notin tR$. Thus tR is a prime ideal. As k is algebraically closed and $\dim R/tR=1$, R/tR is a semigroup ring by (2.2.11). Then it will be clear that the corresponding semigroup is $H=H_{X,P}$.

PROPOSITION (5.2.5). *$m(R_{X,P})=m(H_{X,P})$ and $\mathrm{emb}(R_{X,P})=\mathrm{emb}(H_{X,P})+1$.*

PROOF. We know that $m(k[H])=m(H)$ and that $\mathrm{emb}(k[H])=\mathrm{emb}(H)$ (cf. [14]). As $R/tR=k[H]$, the equality $\mathrm{emb}(R)=\mathrm{emb}(H)+1$ is clear. We will compute $m(R)$. Let $(t, f_1, \cdots, f_e)$ be a minimal generator system of R_+ by homogeneous elements. We assume that $\deg(t)=1$. We put $\mathfrak{q}=(f_1, \cdots, f_e)$ and $\bar{\mathfrak{q}}=(R/tR)_+=(\bar{f}_1, \cdots, \bar{f}_e)$. We will evaluate length $(R/\mathfrak{m}^n)$ for large n. We can write $\mathfrak{m}^n=(t^n, t^{n-1}\mathfrak{q}, \cdots, \mathfrak{q}^n)$. We consider the filtration

$$\mathfrak{m}^n\subset(t^{n-1}, \mathfrak{m}^n)\subset(t^{n-2}, \mathfrak{m}^n)\subset\cdots\subset(t, \mathfrak{m}^n)\subset R\,.$$

As t is a non-zero divisor of R, $t^iR/t^{i+1}R\cong R/tR$ (forgetting the grade) and it is easy to see that $(t^{n-i}, \mathfrak{m}^n)/(t^{n-i+1}, \mathfrak{m}^n)\cong R/(t, \mathfrak{q}^i)\cong k[H]/(\bar{\mathfrak{q}}^i)$. We know that $\mathrm{length}(k[H]/\bar{\mathfrak{q}}^n)=n\cdot m(H)+\text{const.}$ for n sufficiently large. So we can conclude that $m(R)=m(H)$.

PROPOSITION (5.2.6). *$R_{X,P}$ is a Gorenstein ring if and only if $H_{X,P}$ is a symmetric semigroup.*

PROOF. As $R/tR=k[H]$, R is a Gorenstein ring if and only if $k[H]$ is a Gorenstein ring. $k[H]$ is a Gorenstein ring if and only if H is a symmetric semigroup by [**14**] (cf. (2.1.9)).

REMARK. We can put another proof of this fact. By (5.1.9), $R_{X,P}$ is a Gorenstein ring if and only if $\omega_X \cong \mathcal{O}_X(nP)$ for some integer. But as $\deg(\omega_X) = 2g-2$, $n=2g-2$. $\omega_X \cong \mathcal{O}_X((2g-2)P)$ if and only if $H^1(X, \mathcal{O}_X((2g-2)P)) \cong H^0(X, \omega_X((2-2g)P)) \neq 0$. By Riemann-Roch theorem, this is equivalent to $h(2g-2)=g$ and by (5.2.1), this is equivalent to $2g-1 \notin H_{X,P}$. As $\sharp\{n \in \boldsymbol{N} \mid n \notin H\}=g$, it is easy to show that $2g-1 \notin H$ if and only if H is symmetric.

EXAMPLE (5.2.7). (i) If $g=1$, $H_{X,P}=\{0, 2, 3, 4, \cdots\}$ for every point P of X. This $H_{X,P}$ is of course symmetric.

(ii) If $g=2$, there are two possibilities for $H_{X,P}$. If $H_{X,P}=\{0, 2, 4, 5, 6, \cdots\}$ this point P is a hyperelliptic point. There are 6 hyperelliptic points on X and for other point P, $H_{X,P}=\{0, 3, 4, 5, \cdots\}$. If P is a hyperelliptic point, $R_{X,P}$ is a Gorenstein ring and if P is not hyperelliptic, $r(R_{X,P})=2=g$. If P is a hyperelliptic point, $R_{X,P}=k[T, U, V]/(F)$ where F is the form

$$F(T, U, V)=V^2-(U-a_1T^2)(U-a_2T^2)\cdots(U-a_5T^2),$$

$$a_i \in k,\ a_i \neq a_j \text{ for } i \neq j,\ \deg(T)=1,\ \deg(U)=2 \text{ and } \deg(V)=5.$$

On the other hand, let X be the smooth curve defined by the equation $x^5=y^2+y$ and $P=(0, 0) \in X$. Then P is not a hyperelliptic point.

$$\begin{aligned} R_{X,P}&=k[T, fT^3, gT^4, hT^5] \\ &\cong k[T, U, V, W]/(V^2-UW, U^3-VW-VT^5, U^2V-WT^5-W^2) \end{aligned}$$

where $f=x^{-3}(y+1)$, $g=x^{-4}(y+1)$ and $h=x^{-5}(y+1)$.

(iii) In general, if P is not a Weierstrass point of X (the set of Weierstrass points on X is a finite set), $H_{X,P}=\{0, g+1, g+2, \cdots\}$ and we have $m(R_{X,P})=g+1$, $\mathrm{emb}(R_{X,P})=g+2$ and $r(R_{X,P})=g$ (we have assumed that $g \geqq 2$).

REMARK (5.2.8). In [**27**], the following theorem was proved.

Theorem. For given integers m and n such that $m-1 \geqq n \geqq 4$, there exists a numerical semigroup H which is symmetric with $\mathrm{emb}(H)=n$, $m(H)=m$ and which is not a complete intersection.

If for every numerical semigroup H, there exist a smooth curve X and a point P on X such that $H_{X,P}=H$, the following is true.

CONJECTURE. For given integers m and n such that $m \geqq n \geqq 5$, there exists a two-dimensional normal Gorenstein local domain R with $m(R)=m$, $\mathrm{emb}(R)=n$

and which is not a complete intersection.

QUESTION. For given positive integers m, n and d such that $m+d-1\geqq n\geqq d+4$, do there exist a smooth d-dimensional projective variety X and ample invertible sheaf $\mathcal{L}$ on X such that $R_{X,\mathcal{L}}$ is a Gorenstein ring with $m(R_{X,\mathcal{L}})=m$, $\mathrm{emb}(R_{X,\mathcal{L}})=n$ and which is not a complete intersection?

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