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On certain ray class invariants of real quadratic fields

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Introduction.

0-1. In his papers [**10**], [**11**] and [**12**], H. M. Stark introduced certain ray class invariants for totally real fields in terms of the value at $s=0$ of the derivative of some L-series of the fields. Then he presented (with numerical evidences) a striking conjecture on the arithmetic nature of the invariants. In this paper, we show that, for each given real quadratic field, the invariants are described in terms of special values of a certain special function. The function is closely related to the *double gamma function* of E. W. Barnes. Then we prove the conjecture for a very special (but non-trivial) case.

0-2. For a pair $\omega=(\omega_1, \omega_2)$ of *positive* numbers we denote by $\Gamma_2(z, \omega)$ the double gamma function introduced by E. W. Barnes (for the definition and basic properties of the double gamma function, see [**2**] and [**7**]). Set

$$\boldsymbol{F}(z, \omega)=\Gamma_2(z, \omega)/\Gamma_2(\omega_1+\omega_2-z, \omega).$$

Then $\boldsymbol{F}$ is a meromorphic function of z which satisfies the following equalities (0-1) and (0-2).

(0-1)
$$\boldsymbol{F}(z+\omega_1, \omega)=2\sin(\pi z/\omega_2)\boldsymbol{F}(z, \omega),$$
$$\boldsymbol{F}(z+\omega_2, \omega)=2\sin(\pi z/\omega_1)\boldsymbol{F}(z, \omega).$$

(0-2)
$$\boldsymbol{F}((\omega_1+\omega_2)/2, \omega)=1.$$

If ω_2/ω_1 is irrational, properties (0-1) and (0-2) characterize $\boldsymbol{F}$ as a meromorphic function of z. Let F be a real quadratic field embedded in the real number field $\boldsymbol{R}$. For an integral ideal $\mathfrak{f}$ of F, denote by $H_F(\mathfrak{f})$ the group of *narrow* ray classes modulo $\mathfrak{f}$ of F. Assume that $\mathfrak{f}$ satisfies the following condition (0-3):

(0-3) For any totally positive unit u of F, $u+1\notin\mathfrak{f}$.

Take a totally positive integer ν of F with the property $\nu+1\in\mathfrak{f}$. Denote by the same letter ν the narrow ray class modulo $\mathfrak{f}$ represented by the principal ideal (ν). Then ν is an element of order 2 of the group $H_F(\mathfrak{f})$. Choose integral ideals $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{h_0}$ of F so that they form a complete set of *narrow* ideal classes of F. For each $c\in H_F(\mathfrak{f})$, there is a unique index $j(1\leqq j\leqq h_0)$ such that c and $\mathfrak{a}_j\mathfrak{f}$ are in the same narrow ideal class of F. Denote by ε a fundamental totally positive unit of F and set

$$(0\text{-}4)\qquad R(\varepsilon, c)=\left\{\begin{array}{l} z=x+y\varepsilon\in(\mathfrak{a}_j\mathfrak{f})^{-1};\ x, y\in\boldsymbol{Q}, \\ 0<x\leqq 1,\ \ 0\leqq y<1,\ \ (z)\mathfrak{a}_j\mathfrak{f}=c \text{ in } H_F(\mathfrak{f}) \end{array}\right\}.$$

Then $R(\varepsilon, c)$ is a *finite* subset of $(\mathfrak{a}_j\mathfrak{f})^{-1}$. Set

$$(0\text{-}5)\qquad X_{\mathfrak{f}}(c)=\prod_{z\in R(\varepsilon,c)}\{\boldsymbol{F}(z,(1,\varepsilon))\boldsymbol{F}(z',(1,\varepsilon'))\},$$

where z' (resp. ε') is the conjugate of z (resp. ε). The invariant $X_{\mathfrak{f}}(c)$ is *positive* for each $c\in H_F(\mathfrak{f})$. Set $\zeta_F(s, c)=\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$, where the summation with respect to $\mathfrak{a}$ is over all integral ideals of F which are in the same narrow ray class modulo $\mathfrak{f}$ as c. It is known that $\zeta_F(s, c)$ is holomorphic except for a simple pole at $s=1$. The following theorem guarantees that $X_{\mathfrak{f}}(c)$ is independent of the choice of $\mathfrak{a}_1, \cdots, \mathfrak{a}_{h_0}$.

THEOREM 1. *The notation and assumptions being as above.*

$$\zeta_F'(0, c)-\zeta_F'(0, c\nu)=\log X_{\mathfrak{f}}(c).$$

We further assume that $\mathfrak{f}$ satisfies the following condition (0-6):

(0-6) There is no unit of F such that $u>0$, $u'<0$ and $u-1\in\mathfrak{f}$.

Let μ be an integer of F such that $\mu<0$, $\mu'>0$ and $\mu-1\in\mathfrak{f}$. Denote by the same letter μ the narrow ray class modulo $\mathfrak{f}$ represented by the principal ideal (μ). Then μ is an element of order at most two of the group $H_F(\mathfrak{f})$. Let G be a subgroup of $H_F(\mathfrak{f})$. *Assume that μ is in G but ν is not in G.* Set

$$X_{\mathfrak{f}}(c, G)=\prod_{g\in G} X_{\mathfrak{f}}(cg).$$

Then $X_{\mathfrak{f}}(c, G)$ is an invariant for $c\in H_F(\mathfrak{f})/G$. Denote by $K_F(\mathfrak{f})$ the maximal narrow ray class field over F with conductor $\mathfrak{f}$ and denote by σ the Artin canonical isomorphism from $H_F(\mathfrak{f})$ onto the Galois group of $K_F(\mathfrak{f})$ with respect to F. Furthermore, let $K_F(\mathfrak{f}, G)$ be the subfield of $\sigma(G)$-fixed elements of $K_F(\mathfrak{f})$. In view of Theorem 1, Stark's conjecture in [10]-[12] implies the following (cf. [8]):

(0-7) **Conjecture** (modified version of the Stark conjecture). There exists a positive rational integer m such that the following assertions (i) and (ii) hold:

(i) For each $c\in H_F(\mathfrak{f})/G$, $X_{\mathfrak{f}}(c, G)^m$ is a unit of $K_F(\mathfrak{f}, G)$.
Moreover, $\{X_{\mathfrak{f}}(c, G)^m\}^{\sigma(c_0)}=X_{\mathfrak{f}}(cc_0, G)^m \qquad (\forall c_0\in H_F(\mathfrak{f}))$.

(ii) A system of invariants

(0-8)
$$\bigcup_{\mathfrak{f}_0} \{X_{\mathfrak{f}_0}(c, \tilde{G})^m;\ c \in H_F(\mathfrak{f}_0)/\tilde{G}\}$$

generates $K_F(\mathfrak{f}, G)$ over F.
In (0-8), $\mathfrak{f}_0'$s are divisors of $\mathfrak{f}$ with properties (0-3) and (0-6) and $\tilde{G}$ is the image of G under the natural homomorphism from $H_F(\mathfrak{f})$ onto $H_F(\mathfrak{f}_0)$.
Without loss of generality, *we may assume that* $\mathfrak{f}$ *is invariant under the non-trivial automorphism* ι *of* F. In fact, if $\mathfrak{f} \neq \iota(\mathfrak{f})$, we may replace $\mathfrak{f}$ by $\mathfrak{f} \cap \iota(\mathfrak{f})$ and G by its inverse image under the natural homomorphism from $H_F(\mathfrak{f} \cap \iota(\mathfrak{f}))$ onto $H_F(\mathfrak{f})$.

We prove the conjecture under the following assumption (0-9).

(0-9) The field $K_F(\mathfrak{f}, G)$ is a quadratic extension of its maximal absolutely abelian subfield. Moreover exactly one of two infinite primes of F (one which corresponds to the prescribed embedding of F into $\boldsymbol{R}$) splits in $K_F(\mathfrak{f}, G)$.

Denote by K the normal closure of $K_F(\mathfrak{f}, G)$ with respect to $\boldsymbol{Q}$. Then (0-9) implies that K is a quadratic extension of $K_F(\mathfrak{f}, G)$ contained in $K_F(\mathfrak{f})$. Let G_1 be the subgroup of $H_F(\mathfrak{f})$ which corresponds to K. Then G_1 is invariant under ι and is a subgroup of index 2 of G. Furthermore, G is generated by μ and G_1. Set

$$(H_F(\mathfrak{f})/G_1)_0 = \{c \in H_F(\mathfrak{f})/G_1;\ \iota(c) = c\}\ .$$

Then (0-9) implies that $(H_F(\mathfrak{f})/G_1)_0$ is a subgroup of index two of $H_F(\mathfrak{f})/G_1$.

Thus, it is now easy to see that the condition (0-9) is equivalent to the following condition (0-9)′ on G:

(0-9)′ There exists a subgroup G_1 of G with index 2 invariant under ι such that

$$[H_F(\mathfrak{f})/G_1,\ (H_F(\mathfrak{f})/G_1)_0] = 2\,.$$

THEOREM 2. *Under the assumption* (0-9) (*which is equivalent to* (0-9)′), *the conjecture is true.*

0-3. The present paper consists of three sections. In § 1 we summarize some results of Ramachandra [5] for later applications. In § 2, we first recall certain results of [7] and prove Theorem 1. In fact, Theorem 1 is implicit in Corollary 2 to Theorem 1 of [7]. Then we show that, under the assumptions of Theorem 2, $L_F(s, \chi)$, where χ is a character of the group $H_F(\mathfrak{f})/G$ such that $\chi(\nu) = -1$, coincides with an L function of a suitable imaginary quadratic field. Applying results of Ramachandra, we can express $X_{\mathfrak{f}}(c, G)^m$ in terms of singular values of elliptic modular functions and prove Theorem 2. We must emphasize that assumptions imposed on Theorem 2 are quite restrictive. Moreover, the expression (0-5) for $X_{\mathfrak{f}}(c)$ in terms of the function $\boldsymbol{F}$ plays no role in our proof of Theorem 2. However, (0-5) is quite useful for numerical computations of $X_{\mathfrak{f}}(c)$. In § 3, we report on a few numerical experiments based on (0-5) which support the conjecture when Theorem 2 is not applicable.

0-4. When the author was writing down the first version of the present paper, Stark's papers [**10**]-[**12**] were unknown to him. A summary of the first version was announced in [8].

Notation.

As usual, we denote by $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$ and by $\boldsymbol{Z}$ the field of complex numbers, the field of real numbers, the field of rational numbers and the ring of rational integers, respectively. For $x \in \boldsymbol{R}-\{0\}$, $\mathrm{sgn}(x)=x/|x|$. For a complex number z, $\mathrm{Re}(z)$ (resp. $\mathrm{Im}(z)$) denotes the real part (resp. imaginary part) of z. For a finite set S, $|S|$ is the cardinality of S. For a given group G, $\langle g_1, g_2, \cdots, g_m\rangle$ $(g_1, g_2, \cdots, g_m \in G)$ is the subgroup of G generated by $g_1, \cdots, g_m$. For a finite algebraic number field k, $\mathfrak{O}_k$ denotes the ring of integers of k and $\mathfrak{d}_k$ denotes the differente of k. For $t \in k-\{0\}$, (t) is the principal ideal of k generated by t. For a fractional ideal $\mathfrak{a}$ of k, $N(\mathfrak{a})$ is the (absolute) norm of $\mathfrak{a}$. For any integral ideal $\mathfrak{f}$ of k, $H_k(\mathfrak{f})$, the group of narrow ray classes with conductor $\mathfrak{f}$, is the quotient group $I_k(\mathfrak{f})/P_+(\mathfrak{f})$, where $I_k(\mathfrak{f})$ is the group of fractional ideals of k prime to $\mathfrak{f}$ and $P_+(\mathfrak{f})$ is the group of principal ideals of k generated by totally positive numbers t of k such that the numerator of $(t-1)$ is divisible by $\mathfrak{f}$.

If $\mathfrak{f}_0$ is a divisor of $\mathfrak{f}$, the natural injection of $I_k(\mathfrak{f})$ into $I_k(\mathfrak{f}_0)$ induces a surjective homomorphism from $H_k(\mathfrak{f})$ onto $H_k(\mathfrak{f}_0)$. The homomorphism is called the natural homomorphism from $H_k(\mathfrak{f})$ onto $H_k(\mathfrak{f}_0)$. For a character χ of the group $H_k(\mathfrak{f})$,

$$L_k(s, \chi)=\sum \chi(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

where the summation with respect to $\mathfrak{a}$ is over all the integral ideals of k which are prime to $\mathfrak{f}$.

For a normal extension K of k, $\mathrm{Gal}(K/k)$ denotes the Galois group of K with respect to k.

The gamma function is denoted by $\Gamma(s)$ and the m-th Bernoulli Polynomial is denoted by $B_m(x)$.

§ 1.

1. For real numbers u, v and for a complex number z with positive imaginary part, set

$$\Phi_0\left(\binom{v}{u}, z\right)=\exp\{-i\pi u(v-uz)\}\frac{\vartheta_1(v-uz, z)}{\eta(z)},$$

where

$$\vartheta_1(w, z)=-i\sum_{n\in \boldsymbol{Z}}(-1)^n\exp\left\{i\pi z\left(n+\frac{1}{2}\right)^2+2\pi iw\left(n+\frac{1}{2}\right)\right\}$$

and

$$\eta(z)=\exp\left(\frac{\pi iz}{12}\right)\prod_{n=1}^{\infty}(1-e^{2n\pi iz}).$$

It is known that for any integral unimodular matrix $M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z})$, Φ_0 satisfies the following transformation formula:

$$\Phi_0\left(\binom{av+bu}{cv+du}, \frac{az+b}{cz+d}\right)=\varepsilon(M)^2\Phi_0\left(\binom{v}{u}, z\right), \tag{1}$$

where $\varepsilon(M)^2$ is a twelfth root of unity which depends only on M. It is easy to see that

$$\Phi_0\left(\binom{v+1}{u}, z\right)=-e^{-i\pi u}\Phi_0\left(\binom{v}{u}, z\right) \quad \text{and}$$

(2)

$$\Phi_0\left(\binom{v}{u+1}, z\right)=-e^{i\pi v}\Phi_0\left(\binom{v}{u}, z\right).$$

For details, see 3 of [**5**].

Let k be the imaginary quadratic field with discriminant d_k. Let $\mathfrak{f}$ be an integral ideal of k. For each $c\in H_k(\mathfrak{f})$, the group of ideal classes modulo $\mathfrak{f}$ of k, we are going to associate ray class invariants $\Phi_{\mathfrak{f}}(c)$ and $Z_{\mathfrak{f}}(c)$ following Ramachandra [**5**]. At first, assume $\mathfrak{f}\neq\mathfrak{O}_k$. For an integral ideal $\mathfrak{b}$ in c^{-1}, choose $\mu\in\mathfrak{b}$ with the congruence property $\mu\equiv 1$ (mod. $\mathfrak{f}$). Let $\{\beta_1, \beta_2\}$ be a $\boldsymbol{Z}$-basis of the ideal $(\mathfrak{b}\mathfrak{f}\mathfrak{d}_k)^{-1}$ such that $\mathrm{Im}(\beta_2/\beta_1)>0$. Let f be the smallest positive integer contained in $\mathfrak{f}$. Set

$$\Phi_{\mathfrak{f}}(c)=\Phi_0\left(\binom{\mathrm{tr}(\mu\beta_2)}{\mathrm{tr}(\mu\beta_1)}, \beta_2/\beta_1\right)^{12f} \tag{3}$$

and

$$Z_{\mathfrak{f}}(c)=|\Phi_{\mathfrak{f}}(c)|^{1/6f}, \tag{4}$$

where tr means the trace over the rational number field $\boldsymbol{Q}$. We note that $\mathrm{tr}(\mu\beta_1)$ and $\mathrm{tr}(\mu\beta_2)$ are not simultaneously equal to rational integers if $\mathfrak{f}\neq\mathfrak{O}_k$. Hence $Z_{\mathfrak{f}}(c)\neq 0$. It follows easily from equalities (1) and (2) that $\Phi_{\mathfrak{f}}(c)$ is independent of the choice of $\mathfrak{b}$, μ, β_1 and β_2. In fact, $\Phi_{\mathfrak{f}}(c)$ coincides with the invariant $\Phi_{\mathfrak{f},\mathfrak{g}}(c)$ for $\mathfrak{g}=\mathfrak{O}_k$ which was introduced by Ramachandra [**5**]. Next, assume $\mathfrak{f}=\mathfrak{O}_k$. Then the group $H_k(\mathfrak{f})$ is the group of absolute ideal classes of k. For each $c\in H_k(\mathfrak{f})$, take an integral ideal $\mathfrak{b}$ in c^{-1} and let $\{\beta_1, \beta_2\}$ be a $\boldsymbol{Z}$-basis for $(\mathfrak{b}\mathfrak{d}_k)^{-1}$, where $\mathrm{Im}(\beta_2/\beta_1)>0$. Let h be the class-number of k. Set

$$(\mathfrak{b}\mathfrak{d}_k)^h=(\alpha) \qquad (\alpha\in\mathfrak{O}_k),$$

(5) $$\Phi_{\mathfrak{f}}(c)=(\beta_1^{12h}\alpha^{12})^{-1}\eta(\beta_2/\beta_1)^{24h}$$

and

(6) $$Z_{\mathfrak{f}}(c)=|(\beta_1^2N(\mathfrak{b}\mathfrak{d}_k))^{-1}\eta(\beta_2/\beta_1)^4|=|\Phi_{\mathfrak{f}}(c)|^{1/6h}.$$

The Ramachandra invariant $\Phi_{\mathfrak{f},\mathfrak{c}}(c)$ for $\mathfrak{g}=\mathfrak{O}_k$ coincides with $|\Phi_{\mathfrak{f}}(c)|^{1/h}$.
For each integral ideal $\mathfrak{f}$ of k, we denote by $K_k(\mathfrak{f})$ the ray class field over k with conductor $\mathfrak{f}$. We denote by $\sigma_{k,\mathfrak{f}}$ the Artin canonical isomorphism from $H_k(\mathfrak{f})$ onto $\mathrm{Gal}(K_k(\mathfrak{f})/k)$. If no confusion is likely, we simply write σ_k instead of $\sigma_{k,\mathfrak{f}}$.

Now we quote the following results on the arithmetic nature of the invariant $\Phi_{\mathfrak{f}}(c)$.

LEMMA 1. (i) *If* $\mathfrak{f}\neq\mathfrak{O}_k$, $\Phi_{\mathfrak{f}}(c)\in K_k(\bar{\mathfrak{f}})$ *for any* $c\in H_k(\mathfrak{f})$, *where* $\bar{\mathfrak{f}}$ *is the ideal conjugate to* $\mathfrak{f}$. *Moreover* $\Phi_{\mathfrak{f}}(c_1)/\Phi_{\mathfrak{f}}(c_2)$ *is a unit for any* $c_1, c_2\in H_k(\mathfrak{f})$. *Furthermore,* $\{\Phi_{\mathfrak{f}}(c)\}^{\sigma_{k,\bar{\mathfrak{f}}}(\bar{c}_0)}=\Phi_{\mathfrak{f}}(cc_0)$ *for any* $c, c_0\in H_k(\mathfrak{f})$.

(ii) *If* $\mathfrak{f}=\mathfrak{O}_k$, $\Phi_{\mathfrak{f}}(c_1)/\Phi_{\mathfrak{f}}(c_2)$ *is a unit in* $K_k(\mathfrak{f})$ *for any* $c_1, c_2\in H_k(\mathfrak{f})$. *Moreover,* $\{\Phi_{\mathfrak{f}}(c_1)/\Phi_{\mathfrak{f}}(c_2)\}^{\sigma_{k,\mathfrak{f}}(c_0)}=\Phi_{\mathfrak{f}}(c_1c_0^{-1})/\Phi_{\mathfrak{f}}(c_2c_0^{-1})$ $(c_1, c_2, c_0\in H_k(\mathfrak{f}))$.

The first part of Lemma 1 follows immediately from Theorem 5 and Theorem 7 of [**5**]. For the proof of the second part of Lemma 1, we refer to [**4**] (13 and 20 in particular) and § 2 of Chap. 2 of [**9**].

2. For $c\in H_k(\mathfrak{f})$, put $\zeta_k(s, c)=\sum N(\mathfrak{a})^{-s}$, where the summation with respect to $\mathfrak{a}$ is over all integral ideals of k which are prime to $\mathfrak{f}$ and are in the class c modulo $\mathfrak{f}$. It is well known that the Dirichlet series $\zeta_k(s, c)$ is absolutely convergent for $\mathrm{Re}\, s>1$ and is extended to an analytic function in $\boldsymbol{C}$ which is holomorphic except for a simple pole at $s=1$. Denote by $\omega(\mathfrak{f})$ the cardinality of the group of units of k which are congruent to 1 modulo $\mathfrak{f}$. The following Proposition is a version of the Kronecker limit formula.

PROPOSITION 1. *The notation being as above,*

(i) $$\omega(\mathfrak{f})\zeta_k(0, c)=\begin{cases} 0 & \text{if } \mathfrak{f}\neq\mathfrak{O}_k, \\ -1 & \text{if } \mathfrak{f}=\mathfrak{O}_k. \end{cases}$$

(ii) $$\omega(\mathfrak{f})\zeta_k'(0, c)=\begin{cases} -\log Z_{\mathfrak{f}}(c) & \text{if } \mathfrak{f}\neq\mathfrak{O}_k, \\ -\log Z_{\mathfrak{f}}(c)-\log 4\pi^2 & \text{if } \mathfrak{f}=\mathfrak{O}_k, \end{cases}$$

(*for notation, see* (4) *or* (6)).

PROOF. Take an integral ideal $\mathfrak{b}$ in the ray class c^{-1}. It follows easily from the definition of $\zeta_k(s, c)$ that $\omega(\mathfrak{f})N(\mathfrak{b})^{-s}\zeta_k(s, c)=\sum_x N(x)^{-s}$, where the summation with respect to x is over all non-zero elements of $\mathfrak{b}$ with the congruence property $x\equiv 1 \pmod{\mathfrak{f}}$. Take a $\mu\in\mathfrak{b}$ such that $\mu\equiv 1 \pmod{\mathfrak{f}}$. Since $\mathfrak{b}$ and $\mathfrak{f}$ are mutually prime,

$$\{x\in\mathfrak{b}\,;\ x-1\in\mathfrak{f}\}=\{\mu+y\,;\ y\in\mathfrak{b}\mathfrak{f}\}\,.$$

Thus, we have

$$\omega(\mathfrak{f})N(\mathfrak{b})^{-s}\zeta_k(s,c)=\sum_{y\in\mathfrak{b}\mathfrak{f}}|\mu+y|^{-2s}\,.$$

Applying the Poisson summation formula, we obtain the following functional equation for $\zeta_k(s,c)$. For $\operatorname{Re} s<0$,

$$\begin{aligned}(7)\qquad &\omega(\mathfrak{f})N(\mathfrak{b})^{-s}\pi^{-s}\Gamma(s)\zeta_k(s,c)\\ &=2N(\mathfrak{b}\mathfrak{f})^{-1}\sqrt{|d_k|}^{-1}(4\pi)^{s-1}\Gamma(1-s)\sum_{0\neq y\in(\mathfrak{b}\mathfrak{f}\mathfrak{d}_k)^{-1}}N(y)^{s-1}\exp(-2\pi i\operatorname{tr}(\mu y))\,.\end{aligned}$$

If $\mathfrak{f}\neq\mathfrak{O}_k$, there exists a $y\in(\mathfrak{b}\mathfrak{f}\mathfrak{d}_k)^{-1}$ such that $\operatorname{tr}(\mu y)\notin\boldsymbol{Z}$. Hence, the second Kronecker limit formula (see (6) of [**5**]) implies that the right side of (7) is holomorphic at $s=0$ and is equal to

$$\frac{-2|\beta_1|^{-2}}{N(\mathfrak{b}\mathfrak{f})\sqrt{|d_k|}}(2\operatorname{Im}(\beta_2/\beta_1))^{-1}\log\left|\Phi_0\left(\begin{pmatrix}\operatorname{tr}(\mu\beta_2)\\ \operatorname{tr}(\mu\beta_1)\end{pmatrix},\ \beta_2/\beta_1\right)\right|,$$

where $\{\beta_1,\beta_2\}$ is a $\boldsymbol{Z}$-basis for $(\mathfrak{b}\mathfrak{f}\mathfrak{d}_k)^{-1}$ such that $\operatorname{Im}(\beta_2/\beta_1)>0$.
Since $2\operatorname{Im}(\beta_2/\beta_1)=\sqrt{|d_k|}N(\mathfrak{b}\mathfrak{f}\mathfrak{d}_k)^{-1}|\beta_1|^{-2}$, we have $\zeta_k(0,c)=0$ and

$$\omega(\mathfrak{f})\zeta_k'(0,c)=-2\log\left|\Phi_0\left(\begin{pmatrix}\operatorname{tr}(\mu\beta_2)\\ \operatorname{tr}(\mu\beta_1)\end{pmatrix},\ \beta_2/\beta_1\right)\right|=-\log Z_{\mathfrak{f}}(c)\,.$$

If $\mathfrak{f}=\mathfrak{O}_k$, $\operatorname{tr}(\mu y)\in\boldsymbol{Z}$ for any $y\in(\mathfrak{b}\mathfrak{d}_k\mathfrak{f})^{-1}$. Hence we have

$$\begin{aligned}&\omega(\mathfrak{f})\zeta_k(s,c)\\ &=2(4\pi)^{s-1}\pi^s\sqrt{|d_k|}^{-s}\frac{\Gamma(1-s)}{\Gamma(s)}\sum_{0\neq(m,n)\in\boldsymbol{Z}}\{2\operatorname{Im}(\beta_2/\beta_1)|m+n\beta_2/\beta_1|^{-2}\}^{1-s}\,.\end{aligned}$$

Thus, the first Kronecker limit formula (see (5) of [**5**]) implies that

$$\omega(\mathfrak{f})\zeta_k(0,c)=-1$$

and

$$\begin{aligned}\omega(\mathfrak{f})\zeta_k'(0,c)&=-(2\log\pi+\log 4-\log\sqrt{|d_k|}+2\gamma)\\ &\qquad+2\gamma-4\log\{(2\operatorname{Im}(\beta_2/\beta_1))^{1/4}|\eta(\beta_2/\beta_1)|\}\\ &=-\log 4\pi^2-\log\{|\beta_1|^{-2}N(\mathfrak{b}\mathfrak{d}_k)^{-1}|\eta(\beta_2/\beta_1)|^4\}\\ &=-\log Z_{\mathfrak{f}}(c)-\log 4\pi^2,\end{aligned}$$

where γ is the Euler constant.

COROLLARY TO PROPOSITION 1. *For any non-principal character ξ of the group $H_k(\mathfrak{f})$,*

$$\left(\frac{d}{ds}L_k(s,\xi)\right)_{s=0}=-\frac{1}{\omega(\mathfrak{f})}\sum_{c\in H_k(\mathfrak{f})}\xi(c)\log\{Z_{\mathfrak{f}}(c)\},$$

and

$$L_k(0,\xi)=0.$$

3. We are going to introduce another invariant $W_{\mathfrak{f}}(c)$ ($c\in H_k(\mathfrak{f})$) which is closely related to the invariant $Z_{\mathfrak{f}}(c)$. Denote by $\mathfrak{P}(\mathfrak{f})$ the set of prime divisors of $\mathfrak{f}$. For each subset S of $\mathfrak{P}(\mathfrak{f})$, denote by $\mathfrak{f}(S)$ the intersection of all the divisors of $\mathfrak{f}$ which are prime to any $\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S$. In other words, if $\mathfrak{f}=\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})}\mathfrak{p}^{\nu(\mathfrak{p})}$ ($\nu(\mathfrak{p})$ is a positive integer), $\mathfrak{f}(S)$ is given by $\mathfrak{f}(S)=\prod_{\mathfrak{p}\in S}\mathfrak{p}^{\nu(\mathfrak{p})}$. Further, put

$$n(S)=\omega(\mathfrak{f}(S))\,|H_k(\mathfrak{f})|/|H_k(\mathfrak{f}(S))| \tag{8}$$

and

$$W_{\mathfrak{f}}(c)=\prod_S Z_{\mathfrak{f}(S)}\Big(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}(\tilde{\mathfrak{p}})^{-1}\Big)^{1/n(S)}, \tag{9}$$

where the product is over all subsets S of $\mathfrak{P}(\mathfrak{f})$. In (9), for each S, $\tilde{c}$ (resp. $\tilde{\mathfrak{p}}$) means the ray class modulo $\mathfrak{f}(S)$ represented by c (resp. $\mathfrak{p}$). For each character ξ of the group $H_k(\mathfrak{f})$, we denote by $\mathfrak{f}_\xi$ the conductor of ξ and by $\tilde{\xi}$ the primitive character of the group $H_k(\mathfrak{f}_\xi)$ which corresponds to ξ in a natural manner.

PROPOSITION 2. *The notation being as above, for each non-principal character ξ of the group $H_k(\mathfrak{f})$,*

$$L_k'(0,\tilde{\xi})=-\sum_{c\in H_k(\mathfrak{f})}\xi(c)\log W_{\mathfrak{f}}(c). \tag{10}$$

PROOF. It follows from (9) that the right side of (10) is equal to

$$-\sum_S\frac{1}{n(S)}A(S,\xi), \tag{11}$$

where we put

$$A(S,\xi)=\sum_{c\in H_k(\mathfrak{f})}\xi(c)\log\Big\{Z_{\mathfrak{f}(S)}\Big(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}\tilde{\mathfrak{p}}^{-1}\Big)\Big\}.$$

In (11), the summation with respect to S is over all subsets of $\mathfrak{P}(\mathfrak{f})$. Denote by $\mathfrak{P}(\xi)$ the set of prime divisors of $\mathfrak{f}_\xi$. Assume $\mathfrak{P}(\xi)$ is not a subset of S. Then the restriction of the character ξ to the kernel of the natural homomorphism from $H_k(\mathfrak{f})$ onto $H_k(\mathfrak{f}(S))$ is non trivial. Thus $A(S,\xi)=0$.

Now assume $S\supset\mathfrak{P}(\xi)$ and denote by ξ_S the character of $H_k(\mathfrak{f}(S))$ which corresponds to ξ in a natural manner. In view of (8),

$$\frac{A(S,\xi)}{n(S)}=\frac{1}{\omega(\mathfrak{f}(S))}\xi_S\Big(\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}\tilde{\mathfrak{p}}\Big)\sum_{c\in H_k(\mathfrak{f}(S))}\xi_S(c)\log\{Z_{\mathfrak{f}(S)}(c)\}.$$

On the other hand, it is easy to see that

$$L_k(s, \tilde{\xi})=L_k(s, \xi)\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-\mathfrak{P}(\xi)}\Big(1-\frac{\tilde{\xi}(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-1}.$$

Furthermore, for each subset S of $\mathfrak{P}(\mathfrak{f})$ which contains $\mathfrak{P}(\xi)$,

$$L_k(s, \xi_S)=L_k(s, \xi)\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}\Big(1-\frac{\xi_S(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-1}.$$

Recall the following identity:

$$1+\sum_{i=1}^{n}\frac{x_i}{1-x_i}+\sum_{1\leq i<j\leq n}\frac{x_ix_j}{(1-x_i)(1-x_j)}+\cdots$$

$$+\frac{x_1\cdots x_n}{(1-x_1)\cdots(1-x_n)}=\frac{1}{(1-x_1)\cdots(1-x_n)}.$$

In view of the identity, it is now easy to see that

$$L_k(s, \tilde{\xi})=\sum_S\{\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}\xi_S(\mathfrak{p})N(\mathfrak{p})^{-s}\}\,L_k(\xi_S, k),$$

where the summation with respect to S is over all subsets of $\mathfrak{P}(\mathfrak{f})$ which contains $\mathfrak{P}(\xi)$. Since ξ is non-principal, it follows from Corollary to Proposition 1 that

$$L'_k(0, \tilde{\xi})=\sum_S\frac{-1}{\omega(\mathfrak{f}(s))}\{\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}\xi_S(\mathfrak{p})\}\times\sum_{c\in H_k(\mathfrak{f}(S))}\xi_S(c)\log\{Z_{\mathfrak{f}(S)}(c)\},$$

where the summation with respect to S is over all subsets of $\mathfrak{P}(\mathfrak{f})$ which contain $\mathfrak{P}(\xi)$. Thus, the Proposition follows.

§ 2.

1. For a pair $\omega=(\omega_1, \omega_2)$ of positive numbers and a positive number z, let $\zeta_2(s, \omega, z)$ be the *double zeta function* given by

$$\zeta_2(s, \omega, z)=\sum_{n,m=0}^{\infty}(z+m\omega_1+n\omega_2)^{-s}.$$

It is known that $\zeta_2(s, \omega, z)$ is absolutely convergent for $\mathrm{Re}\, s>2$ and is extended to a meromorphic function of s in $\boldsymbol{C}$ which is holomorphic except for simple poles at $s=2$ and $s=1$. Furthermore, there uniquely exists a meromorphic function $\Gamma_2(z, \omega)$ of z, positive on the positive real axis which satisfies the following equalities:

$$\Big\{\frac{d}{ds}\zeta_2(s, \omega, z)\Big\}_{s=0}=\log\Big\{\frac{\Gamma_2(z, \omega)}{\rho_2(\omega)}\Big\},$$

where $\rho_2(\omega)$ is a positive constant independent of z,

$$(z\Gamma_2(z, \omega))_{z=0}=1 .$$

It satisfies the following difference equations.

$$\Gamma_2(z+\omega_1, \omega)=\sqrt{2\pi}\,\Gamma\Big(\frac{z}{\omega_2}\Big)^{-1}\Gamma_2(z, \omega)\exp\Big\{\Big(\frac{1}{2}-\frac{z}{\omega_2}\Big)\log\omega_2\Big\}$$

(12)

$$\Gamma_2(z+\omega_2, \omega)=\sqrt{2\pi}\,\Gamma\Big(\frac{z}{\omega_1}\Big)^{-1}\Gamma_2(z, \omega)\exp\Big\{\Big(\frac{1}{2}-\frac{z}{\omega_1}\Big)\log\omega_1\Big\} .$$

Set $\boldsymbol{F}(z, \omega)=\Gamma_2(z, \omega)/\Gamma_2(\omega_1+\omega_2-z)$ (cf. Proposition 5 of [7]). It follows easily from (12) that $\boldsymbol{F}$ satisfies difference equations (0-1). If ω_2/ω_1 is *irrational*, zeros (resp. poles) of $\boldsymbol{F}(z, \omega)$ are all simple and are situated at $z=m\omega_1+n\omega_2$ ($m, n=1, 2, \cdots$) (resp. $z=-(m\omega_1+n\omega_2)$, $m, n=0, 1, 2, \cdots$).

2. Let F be a real quadratic field embedded in the real field $\boldsymbol{R}$. For each $x\in F$, x' is the conjugate of x. Let $\mathfrak{f}$ be an integral ideal of F. *We always assume that* $\mathfrak{f}$ *satisfies the condition* (0-3).

Let χ be a character of the group $H_F(\mathfrak{f})$. Then, for an integral principal ideal (μ) of F, $\chi((\mu))$ is given by one of the following four formulas:

$$\chi((\mu))=\chi_0(\mu) ,$$

(13) $$\chi((\mu))=\chi_0(\mu)\operatorname{sgn}(\mu) ,$$

(14) $$\chi((\mu))=\chi_0(\mu)\operatorname{sgn}(\mu') ,$$

$$\chi((\mu))=\chi_0(\mu)\operatorname{sgn}(\mu\mu') ,$$

where χ_0 is a character of the group of invertible residue classes modulo $\mathfrak{f}$. The condition (0-3) for the ideal $\mathfrak{f}$ is equivalent to the following:

(15) The group $H_F(\mathfrak{f})$ has a character of type (13) or (14).

Take a totally positive integer ν of F such that $\nu\equiv-1 \bmod \mathfrak{f}$. Denote by $\nu(\mathfrak{f})$ the ray class modulo $\mathfrak{f}$ represented by the integral principal ideal (ν). The condition (0-3) implies that $\nu(\mathfrak{f})$ is an element of order 2 of the group $H_F(\mathfrak{f})$. If there is no fear of confusion we write simply ν instead of $\nu(\mathfrak{f})$. Let $\varepsilon>1$ be the fundamental totally positive unit of F. Choose *integral* ideals $\mathfrak{a}_1, \cdots, \mathfrak{a}_{h_0}$ of F *so that they form a complete set of representatives for narrow ideal classes of* F. For each $c\in H_F(\mathfrak{f})$, we define $X_{\mathfrak{f}}(c)$ by (0-5). We are now ready to prove Theorem 1.

Proof of Theorem 1. Take the index j such that $\mathfrak{a}_j\mathfrak{f}$ is in the same narrow ideal class as c. Let $E_+(F)$ be the group of totally positive units of F

and let $\boldsymbol{Z}_+$ be the set of non-negative rational integers. Then it is easy to see that the mapping:

$$(z, m, n, u) \longmapsto u(z+m+n\varepsilon)$$

establishes a bijection from the set $R(\varepsilon, c)\times \boldsymbol{Z}_+\times \boldsymbol{Z}_+\times E_+(F)$ (for the definition of $R(\varepsilon, c)$, see (0-4)) onto the set

$$\{x\in(\mathfrak{a}_j\mathfrak{f})^{-1};\ x, x'>0,\ (x)\mathfrak{a}_j\mathfrak{f}=c\}\ .$$

Thus, we have

$$\zeta_F(s, c)=N(\mathfrak{a}_j\mathfrak{f})^{-s}\sum_{z\in R(\varepsilon,c)}\sum_{m,n=0}^{\infty}N(z+m+n\varepsilon)^{-s}\ .$$

It follows from Corollary to Proposition 1 of [6] that

$$\zeta_F(0, c)=\sum_z\left\{\frac{1}{4}(\varepsilon+\varepsilon')B_2(x)+B_1(x)B_1(y)+\frac{1}{4}(\varepsilon+\varepsilon')B_2(y)\right\} \tag{16}$$

where the summation is over all $z=x+y\varepsilon$ $(x, y\in\boldsymbol{Q})$ of $R(\varepsilon, c)$. Furthermore, Proposition 3 of [7] implies that

$$\begin{aligned}\left\{\frac{d}{ds}\zeta_F(s, c)\right\}_{s=0} &= \sum_{z=x+y\varepsilon\in R(\varepsilon,c)}\left[\log\left\{\frac{\Gamma_2(z, \boldsymbol{\varepsilon})\Gamma_2(z', \boldsymbol{\varepsilon}')}{\rho_2(\boldsymbol{\varepsilon})\rho_2(\boldsymbol{\varepsilon}')}\right\}+\frac{\varepsilon-\varepsilon'}{4}\log\left(\frac{\varepsilon'}{\varepsilon}\right)B_2(x)\right] \\ &\qquad -\zeta_F(0, c)\log\{N(\mathfrak{a}_j\mathfrak{f})\}\ ,\end{aligned} \tag{17}$$

where we put $\boldsymbol{\varepsilon}=(1, \varepsilon)$ and $\boldsymbol{\varepsilon}'=(1, \varepsilon')$.

For $z=x+y\varepsilon\in R(\varepsilon, c)$, set

$$\overline{-z}=\begin{cases} 1-x+(1-y)\varepsilon & \text{if } 0<x,\ y<1\,, \\ 1-x & \text{if } y=0,\ 0<x<1\,, \\ 1+(1-y)\varepsilon & \text{if } x=1,\ 0<y<1\,. \end{cases}$$

It is easy to see that the mapping $z\mapsto\overline{-z}$ establishes a bijection from $R(\varepsilon, c)$ onto $R(\varepsilon, c\nu)$. Furthermore the mapping $x\mapsto 1+x\varepsilon$ establishes a bijection from the set $\{x\in R(\varepsilon, c)\,;\, x\in\boldsymbol{Q},\ 0<x<1\}$ onto the set $\{(1+y\varepsilon)\in R(\varepsilon, c)\,;\, y\in\boldsymbol{Q},\ 0<y<1\}$. If follows now easily from (16) and (17) that $\zeta_F(0, c)=\zeta_F(0, c\nu)$ and that

$$\left\{\frac{d}{ds}\zeta_F(s, c)-\frac{d}{ds}\zeta_F(s, c\nu)\right\}_{s=0} = \sum_{z\in R(\varepsilon,c)}\log\left\{\frac{\Gamma_2(z, \boldsymbol{\varepsilon})\Gamma_2(z', \boldsymbol{\varepsilon}')}{\Gamma_2(\overline{-z}, \boldsymbol{\varepsilon})\Gamma_2((\overline{-z})', \boldsymbol{\varepsilon}')}\right\}.$$

If $z=x+\varepsilon y\in R(\varepsilon, c)$ and $0<x,\ y<1$ $(x, y\in \boldsymbol{Q})$,

$$\frac{\Gamma_2(z, \varepsilon)\Gamma_2(z', \varepsilon')}{\Gamma_2(\overline{-z}, \varepsilon)\Gamma_2((\overline{-z})', \varepsilon')}=\boldsymbol{F}(z, \varepsilon)\boldsymbol{F}(z', \varepsilon').$$

If $z=x\in R(\varepsilon, c)$ and $0<x<1$ $(x\in \boldsymbol{Q})$, the difference equations (12) for Γ_2 imply that

$$\frac{\Gamma_2(x, \varepsilon)\Gamma_2(x, \varepsilon')\Gamma_2(1+\varepsilon x, \varepsilon)\Gamma_2(1+\varepsilon' x, \varepsilon')}{\Gamma_2(1-x, \varepsilon)\Gamma_2(1-x, \varepsilon')\Gamma_2(1+\varepsilon(1-x), \varepsilon)\Gamma_2(1+\varepsilon'(1-x), \varepsilon')}$$

$$=\boldsymbol{F}(x, \varepsilon)\boldsymbol{F}(x, \varepsilon')\boldsymbol{F}(1+\varepsilon x, \varepsilon)\boldsymbol{F}(1+\varepsilon' x, \varepsilon')$$

(cf. the proof of Corollary 2 to Theorem 1 of [7]). The proof of Theorem 1 is now complete.

COROLLARIES TO THEOREM 1.

(i) $X_{\mathfrak{f}}(c\nu)=X_{\mathfrak{f}}(c)^{-1}$.

(ii) *If $\mathfrak{f}'$ is the conjugate of $\mathfrak{f}$ and c' is the conjugate of c,*

$$X_{\mathfrak{f}}(c)=X_{\mathfrak{f}'}(c').$$

3. We are going to introduce another invariant $Y_{\mathfrak{f}}(c)$ for $c\in H_F(\mathfrak{f})$. As in 2, we assume that the integral ideal $\mathfrak{f}$ of F satisfies the condition (0-3). Denote by $\mathfrak{P}(\mathfrak{f})$ the set of prime divisors of $\mathfrak{f}$. For each subset S of $\mathfrak{P}(\mathfrak{f})$, denote by $\mathfrak{f}(S)$ the intersection of all the divisors of $\mathfrak{f}$ which are prime to $\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}\mathfrak{p}$. Further put $n(S)=|H_F(\mathfrak{f})|/|H_F(\mathfrak{f}(S))|$. For each $c\in H_F(\mathfrak{f})$, set

$$Y_{\mathfrak{f}}(c)=\prod_S X_{\mathfrak{f}(S)}(\tilde{c}\prod_{\mathfrak{p}\in\mathfrak{P}(\mathfrak{f})-S}(\tilde{\mathfrak{p}})^{-1})^{1/n(S)}, \tag{18}$$

where the product is over all subsets of $\mathfrak{P}(\mathfrak{f})$ *such that* $\mathfrak{f}(S)$ *satisfies* the condition (0-3). In (18), for each S, $\tilde{c}$ (resp. $\tilde{\mathfrak{p}}$) is the ray class modulo $\mathfrak{f}(S)$ represented by c (resp. $\mathfrak{p}$). For each character χ of the group $H_F(\mathfrak{f})$, we denote by $\mathfrak{f}_\chi$ the conductor of χ and by $\tilde{\chi}$ the primitive character of the group $H_F(\mathfrak{f}_\chi)$ which corresponds to χ in a natural manner. The first half of the next proposition is an immediate consequence of Theorem 1. The proof of the second half is quite similar to that of Proposition 2.

PROPOSITION 3. *The notation being as above, let χ be a character of the group $H_F(\mathfrak{f})$ such that $\chi(\nu)=-1$.*(*)

(i) $$\left\{\frac{d}{ds}L_F(s, \chi)\right\}_{s=0}=\sum_{c\in H_F(\mathfrak{f})/<\nu>}\chi(c)\log X_{\mathfrak{f}}(c),$$

(ii) $$\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0}=\sum_{c\in H_F(\mathfrak{f})/<\nu>}\chi(c)\log Y_{\mathfrak{f}}(c),$$

(*) In other words, χ is of type (13) or (14).

where $\langle\nu\rangle$ *is the subgroup of* $H_F(\mathfrak{f})$ *generated by* ν.

COROLLARY TO PROPOSITION 3. *Let* $\mathfrak{f}_0$ *be a divisor of* $\mathfrak{f}$ *which satisfies the condition* (0-3). *Then for each* $c\in H_F(\mathfrak{f}_0)$, $Y_{\mathfrak{f}_0}(c)=\prod_x Y_{\mathfrak{f}}(x)$, *where the product is over all* $x\in H_F(\mathfrak{f})$ *whose image under the natural homomorphism from* $H_F(\mathfrak{f})$ *onto* $H_F(\mathfrak{f}_0)$ *coincides with* c.

4. In this paragraph, we assume that the group $H_F(\mathfrak{f})$ ($\mathfrak{f}$ is an integral ideal of F) has a character of type (14).

It is easy to see that $H_F(\mathfrak{f})$ has a character of type (14) if and only if it satisfies the condition (0-3) and the condition (0-6).

Take an integer μ of F such that $\mu<0$, $\mu'>0$ and $\mu\equiv 1 \bmod \mathfrak{f}$. We denote by $\mu(\mathfrak{f})$ the element of $H_F(\mathfrak{f})$ represented by the principal ideal (μ). Then $\mu(\mathfrak{f})$ is an element of order at most 2 of the group $H_F(\mathfrak{f})$. When there is no fear of confusion, we write simply μ instead of $\mu(\mathfrak{f})$. Let G be a subgroup of $H_F(\mathfrak{f})$ which *contains* μ *but does not contain* ν. For each $c\in H_F(\mathfrak{f})$, set

$$X_{\mathfrak{f}}(c, G)=\prod_{g\in G} X_{\mathfrak{f}}(cg)\,. \qquad \text{(cf. (0-5))} \tag{19}$$

Then $X_{\mathfrak{f}}(c, G)$ is an invariant for $c\in H_F(\mathfrak{f})/G$. We also set

$$Y_{\mathfrak{f}}(c, G)=\prod_{g\in G} Y_{\mathfrak{f}}(cg)\,. \qquad \text{(cf. (18))} \tag{20}$$

Let $\mathfrak{f}_0$ be a divisor of $\mathfrak{f}$ which satisfies conditions (0-3) and (0-6). Let $\tilde{G}$ be the image of G under the natural homomorphism from $H_F(\mathfrak{f})$ onto $H_F(\mathfrak{f}_0)$. Corollary to Proposition 3 implies the following:

LEMMA 2. *The notation being as above, for any* $c_0\in H_F(\mathfrak{f}_0)/\tilde{G}$,

$$Y_{\mathfrak{f}_0}(c_0, \tilde{G})=\prod_c Y_{\mathfrak{f}}(c, G)\,,$$

where the product with respect to c is over all $c\in H_F(\mathfrak{f})/G$, *whose image under the natural homomorphism from* $H_F(\mathfrak{f})/G$ *onto* $H_F(\mathfrak{f}_0)/\tilde{G}$ *coincides with* c_0.

We note that the Stark invariant $\varepsilon_m(c)$ introduced in [**10**] is given by the following formula:

$$\varepsilon_m(c)=\begin{cases} X_{\mathfrak{f}}(c)^m X_{\mathfrak{f}}(c\mu)^m & \text{if } \mu\neq 1\,, \\ X_{\mathfrak{f}}(c)^m & \text{if } \mu=1\,. \end{cases} \tag{21}$$

5. The remaining part of the present paper is devoted to the proof of Theorem 2 stated in the introduction. We use the notation given there without further comment. We assume that $\mathfrak{f}$ is a *self conjugate* integral ideal of F which satisfies the condition (0-3). Then $\mathfrak{f}$ satisfies also the condition (0-6). Moreover $\mu=\mu(\mathfrak{f})$ is an element of order 2 of the group $H_F(\mathfrak{f})$. We denote by ι the non-trivial automorphism of the real quadratic field F given by $\iota(x)=x'$.

Then ι operates naturally on the group $H_F(\mathfrak{f})$ as an automorphism of order 2. We put $\iota(c)=c'$ for any $c\in H_F(\mathfrak{f})$.

LEMMA 3. *If* $\mathfrak{f}'=\iota(\mathfrak{f})=\mathfrak{f}$, $\mu\mu'=\nu$.

PROOF. The ray class μ is represented by an integral principal ideal (μ_0) generated by an integer μ_0 such that $\mu_0<0$, $\mu_0'>0$ and $\mu_0-1\in\mathfrak{f}$. Then $-\mu_0\mu_0'$ is a totally positive integer and $-\mu_0\mu_0'\equiv-1 \bmod \mathfrak{f}$. Thus, the ray class represented by the principal ideal generated by $-\mu_0\mu_0'$ is ν. Thus $\mu\mu'=\nu$.

Assumptions on Thheorem 2 implies the existence of a subgroup G_1 of G which is invariant under ι and satisfies the following conditions (i) and (ii) (cf. (0-9')).

(i) The group G is generated by μ and G_1.

(ii) $[H_F(\mathfrak{f})/G_1\,;\,(H_F(\mathfrak{f})/G_1)_0]=2$.

To simplify the notation, we put $K=K_F(\mathfrak{f}, G_1)$, where $K_F(\mathfrak{f}, G_1)$ is the subfield of $\sigma(G_1)$-fixed elements of $K_F(\mathfrak{f})$. Then the Artin map σ establishes an isomorphism from $H_F(\mathfrak{f})/G_1$ onto $\mathrm{Gal}(K/F)$. Let L be the subfield of $\sigma((H_F(\mathfrak{f})/G_1)_0)$-fixed elements of K. It follows from the assumption (ii) of Theorem 2 that L is a quadratic extension of F.

LEMMA 4. *The notation and assumptions being as above, L is a composition of F with a suitable imaginary quadratic field k. Moreover, K is an abelian extension of k.*

PROOF. Since $\mathfrak{f}$ and G_1 are invariant under the non-trivial automorphism ι of F, K is normal over the rational number field $\boldsymbol{Q}$. Furthermore, since $((H_F(\mathfrak{f})/G_1)_0$ is an ι-invariant subgroup of $H_F(\mathfrak{f})/G_1$, L is also normal over $\boldsymbol{Q}$. Thus, the group $\mathrm{Gal}(L/\boldsymbol{Q})$ is either isomorphic to a cyclic group of order 4 or to a direct product of cyclic groups of order 2. If $\mathrm{Gal}(L/\boldsymbol{Q})$ were cyclic, there would exist a rational prime p which remains to be a prime ideal in L. Then (p) is a prime ideal of F which is invariant under ι. Thus $(p)\in(H_F(\mathfrak{f})/G_1)_0$. Hence (p) splits in L. Contradiction! Thus, L is a composition of F with a suitable quadratic field k. Since $\mu\notin(H_F(\mathfrak{f})/G_1)_0$, L is not a totally real quadratic extension of F. Hence k is an imaginary quadratic field. The field K, being normal over $\boldsymbol{Q}$, is also normal over k. The group $\mathrm{Gal}(K/L)$ is an abelian normal subgroup of index 2 of the group $\mathrm{Gal}(K/k)$. Take an element λ of $\mathrm{Gal}(K/k)$ which is not in $\mathrm{Gal}(K/L)$. Then $\mathrm{Gal}(K/k)$ is generated by λ and $\mathrm{Gal}(K/L)$. To prove that $\mathrm{Gal}(K/k)$ is abelian it is sufficient to prove that λ commutes with each element of $\mathrm{Gal}(K/L)$. Since λ induces a non-trivial automorphism on L which is generated by k and F over $\boldsymbol{Q}$, λ induces the non-trivial automorphism ι on F. Take $\gamma\in\mathrm{Gal}(K/L)$. Then there exists a $c\in H_F(\mathfrak{f})$ such that $\gamma=\sigma(c)$. Then $\lambda\gamma\lambda^{-1}=\sigma(c')$. Since $c\in(H_F(\mathfrak{f})/G_1)_0$, $c'c^{-1}\in G_1$. Thus $\sigma(c')=\sigma(c)$ in $\mathrm{Gal}(K/k)$. Hence $\lambda\gamma\lambda^{-1}=\gamma$ and λ commutes with γ.

LEMMA 5. *Let τ_0 be an embedding of K into $\boldsymbol{C}$ which extends the prescribed embedding of F into $\boldsymbol{R}$. Then for any $x\in K$, $\overline{\tau_0(x)}=\tau_0(x^{\sigma(\mu)})$, where $\overline{}$ denotes* *the complex conjugation.*

PROOF. The field K is a quadratic extension of the field $K_F(\mathfrak{f}, G)$. The field $K_F(\mathfrak{f}, G)$ is the subfield of $\sigma(\mu)$-fixed elements of K. Set $\lambda(x)=\tau_0^{-1}(\overline{\tau_0(x)})$. Since K is totally imaginary, λ is an element of order 2 of $\mathrm{Gal}(K/F)$. On the other hand, as an abelian extension of F, $K_F(\mathfrak{f}, G)$ is unramified at the Archimedean prime which the prescribed embedding of F into $\boldsymbol{R}$ determines. Thus, λ induces the trivial automorphism on $K_F(\mathfrak{f}, G)$. Hence $\sigma(\mu)=\lambda$.

Since K is abelian over k, there exists an integral ideal $\mathfrak{c}$ of k such that K is a class field over k with conductor $\mathfrak{c}$. Let H_1 be the subgroup of $H_k(\mathfrak{c})$ to which K corresponds. Since K is normal over $\boldsymbol{Q}$, both $\mathfrak{c}$ and H_1 are invariant under the non-trivial automorphism κ of k. Set

$$(H_k(\mathfrak{c})/H_1)_0=\{c\in H_k(\mathfrak{c})/H_1,\ \kappa(c)=c\}\ .$$

Denote by σ_k the Artin canonical isomorphism from $H_k(\mathfrak{c})/H_1$ onto $\mathrm{Gal}(K/k)$.

LEMMA 6. *The notation being as above, the subfield of* $\sigma_k(H_k(\mathfrak{c})/H_1)_0$*-fixed elements of* K *coincides with* L.

PROOF. Denote by H the subgroup of $H_k(\mathfrak{c})$ which corresponds to L. Then $H\supset H_1$ and H/H_1 is a subgroup of index 2 of the group $H_k(\mathfrak{c})/H_1$. For each $c\in H$, $\sigma_k(c)$, which is in $\mathrm{Gal}(K/L)\subset\mathrm{Gal}(K/F)$, commutes with $\sigma(\mu)$. Since $\sigma(\mu)$ induces the non-trivial automorphism on k, $\sigma_k(\kappa c)=\sigma(\mu)\sigma_k(c)\sigma(\mu)^{-1}=\sigma_k(c)$. Thus $(\kappa c)c^{-1}\in H_1$ and $c\in(H_k(\mathfrak{c})/H_1)_0$. Hence $(H_k(\mathfrak{c})/H_1)_0\supset H/H_1$. Since K is not abelian over $\boldsymbol{Q}$, $(H_k(\mathfrak{c})/H_1)_0\neq H_k(\mathfrak{c})/H_1$. Hence $(H_k(\mathfrak{c})/H_1)_0=H/H_1$ and the Lemma follows.

Lemma 6 implies that $\sigma_k^{-1}\sigma$ induces an isomorphic mapping from the group $(H_F(\mathfrak{f})/G_1)_0$ onto the group $(H_k(\mathfrak{c})/H_1)_0$. For each $c\in(H_F(\mathfrak{f})/G_1)_0$, we put

$$\dot{c}=\sigma_k^{-1}\sigma(c)\ . \tag{22}$$

LEMMA 7. *For* $c\in H_k(\mathfrak{c})/H_1$, $c^{-1}\kappa(c)=1$ *or* $\dot{\nu}$ *according as* $c\in(H_k(\mathfrak{c})/H_1)_0$ *or not.*

PROOF. In view of Lemma 3 and the assumption (ii) of Theorem 2, a system of generators for the commutator subgroup of $\mathrm{Gal}(K/\boldsymbol{Q})$ is given by

$$\{\sigma(c\iota(c)^{-1})\ ;\ c\in H_F(\mathfrak{f})/G_1\}=\{1,\sigma(\nu)\}\ .$$

It is also given by $\{\sigma_k(c\kappa(c)^{-1}): c\in H_k(\mathfrak{c})/H_1\}$. Thus $\sigma_k(c\kappa(c)^{-1})=1$ or $\sigma(\nu)$ according as $c\in(H_k(\mathfrak{c})/H_1)_0$ or not.

In the remaining part of the proof of Theorem 2, the following situation (23) should be always kept in mind.

(23) The field K is the class field over F with conductor $\mathfrak{f}$ which corresponds to the subgroup G_1 of $H_F(\mathfrak{f})$. At the same time, K is the class field over k with conductor $\mathfrak{c}$ which corresponds to the subgroup H_1 of $H_k(\mathfrak{c})$.

The following proposition plays a key role in the proof of Theorem 2.

PROPOSITION 4. *For each* $c\in(H_F(\mathfrak{f})/G_1)_0$,

$$Y_{\mathfrak{f}}(c, G)=\prod_{h\in H_1}(W_c(\dot{c}\dot{\nu}h)/W_c(\dot{c}h))$$

(*for notation see* (9), (18) *and* (20)).

PROOF. Recall that the integral ideal $\mathfrak{f}$ of F satisfies the conditions (0-3) and (0-6), and that the subgroup G of $H_F(\mathfrak{f})$ contains μ but does not contain ν. Thus, the group $H_F(\mathfrak{f})/G$ has a character χ of type (14). Then $\chi(\nu)=-1$. Denote by $\mathfrak{f}_\chi$ the conductor of χ and by $\tilde{\chi}$ the primitive character of the group $H_k(\mathfrak{f}_\chi)$ which corresponds to χ in a natural manner. Then Proposition 3 implies that

$$\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0}=\sum_{c\in H_F(\mathfrak{f})/\langle\nu\rangle}\chi(c)\log Y_{\mathfrak{f}}(c)$$

$$=\sum_{c\in H_F(\mathfrak{f})/\langle G,\nu\rangle}\chi(c)\log\{\prod_{g\in G}Y_{\mathfrak{f}}(cg)\}$$

$$=\sum_{c\in H_F(\mathfrak{f})/\langle G,\nu\rangle}\chi(c)\log\{Y_{\mathfrak{f}}(c, G)\}\ ,$$

where $\langle G, \nu\rangle$ is the subgroup of $H_F(\mathfrak{f})$ generated by G and ν. Recall that G is generated by G_1 and μ and that $(H_F(\mathfrak{f})/G_1)_0$ is a subgroup of index 2 of $H_F(\mathfrak{f})/G_1$ such that $\nu\in(H(\mathfrak{f})/G_1)_0$ and $\mu\notin(H(\mathfrak{f})/G_1)_0$. Thus, $H_F(\mathfrak{f})/\langle G, \nu\rangle$ is naturally identified with $(H_F(\mathfrak{f})/G_1)_0/\langle\nu\rangle$ and

$$\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0}=\sum_{c\in(H_F(\mathfrak{f})/G_1)_0/\langle\nu\rangle}\chi(c)\log\{Y_{\mathfrak{f}}(c, G)\}\ . \tag{24}$$

Denote by χ' the character of $H_F(\mathfrak{f})$ given by $\chi'(c)=\chi(c')=\chi(\iota(c))$. Since $\chi(\nu)=\chi(\mu(\mu')^{-1})=-1$,

$$\chi\neq\chi'\ . \tag{25}$$

Via the Artin canonical isomorphism σ, identify χ with a character of Gal(K/F) and denote by ψ_χ the character of Gal$(K/\boldsymbol{Q})$ induced from χ. Then (25) implies that ψ_χ is irreducible and of degree 2. Denote by $L(s, \psi_\chi, K/\boldsymbol{Q})$ the Artin L-function of K associated with character ψ_χ. Then a well-known result in the theory of Artin L-function implies

$$L(s, \psi_\chi, K/\boldsymbol{Q})=L_F(s, \tilde{\chi})\ . \tag{26}$$

The group Gal(K/k) is an abelian subgroup of index 2 of Gal$(K/\boldsymbol{Q})$. Since ψ_χ is irreducible and of degree 2, the restriction of ψ_χ to Gal(K/k) is a direct sum of two distinct non-trivial one dimensional characters ξ_χ and ξ'_χ of Gal(K/k). Furthermore the character of Gal$(K/\boldsymbol{Q})$ induced from ξ_χ coincides with ψ_χ. We note that

(27) $$\chi=\chi'=\xi_\chi=\xi'_\chi \quad \text{on} \quad \mathrm{Gal}(K/L).$$

Identify ξ_χ and ξ'_χ with characters of the group $H_k(\mathfrak{c})$ via the Artin canonical isomorphism σ_k. Then

(28) $$\xi'_\chi(\mathfrak{c})=\xi_\chi(\kappa(c)) \qquad (c\in H_k(\mathfrak{c})),$$

where κ is the non-trivial automorphism of k. Lemma 6, Lemma 7 and equalities (27) and (28) imply the following:

(29) $$\xi_\chi(c)=\xi'_\chi(c) \qquad (\forall c\in(H_k(\mathfrak{c})/H_1)_0)$$

$$\xi_\chi(c)=-\xi'_\chi(c) \qquad (\forall c\in H_k(\mathfrak{c})/H_1-(H_k(\mathfrak{c})/H_1)_0).$$

Denote by $\mathfrak{c}_\chi$ (resp. $\mathfrak{c}_{\chi'}$) the conductor of ξ_χ (resp. ξ'_χ) and let $\tilde{\xi}_\chi$ (resp. $\tilde{\xi}'_\chi$) be the primitive character of the group $H_k(\mathfrak{c}_\chi)$ (resp. $H_k(\mathfrak{c}_{\chi'})$) which corresponds to ξ_χ (resp. ξ'_χ) in a nutural manner. Then

(30) $$L(s, \psi_\chi, K/\boldsymbol{Q})=L_k(s, \tilde{\xi}_\chi)=L_k(s, \tilde{\xi}'_\chi).$$

Proposition 2 implies that

(31) $$\left\{\frac{d}{ds}L_k(s, \tilde{\xi}_\chi)\right\}_{s=0}=-\sum_{c\in H_k(\mathfrak{c})}\xi_\chi(c)\log W_\mathfrak{c}(c)$$

$$=-\sum_{c\in H_k(\mathfrak{c})/H_1}\xi_\chi(c)\log\{\prod_{h\in H_1}W_\mathfrak{c}(ch)\}$$

$$=\left\{\frac{d}{ds}L_k(s, \tilde{\xi}'_\chi)\right\}_{s=0}$$

$$=-\sum_{c\in H_k(\mathfrak{c})/H_1}\xi'_\chi(c)\log\{\prod_{h\in H_1}W_\mathfrak{c}(ch)\}.$$

The equalities (29) and (31) now imply that

$$\left\{\frac{d}{ds}L_k(s, \tilde{\xi}_\chi)\right\}_{s=0}=-\sum_{c\in(H_k(\mathfrak{c})/H_1)_0}\xi_\chi(c)\log\{\prod_{h\in H_1}W_\mathfrak{c}(ch)\}.$$

The mapping $\sigma_k^{-1}\sigma$ induces an isomorphism: $c\mapsto\dot{c}$ from $(H_F(\mathfrak{f})/G_1)_0$ onto $(H_k(\mathfrak{c})/H_1)_0$ such that $\xi_\chi(\dot{c})=\chi(c)$. It follows from the above equality and equalities (24), (26) and (30) that

(32) $$\sum_{c\in(H_F(\mathfrak{f})/G_1)_0/\langle\nu\rangle}\chi(c)\log Y_\mathfrak{f}(c, G)$$

$$=-\sum_{c\in(H_F(\mathfrak{f})/G_1)_0}\chi(c)\log\{\prod_{h\in H_1}W_\mathfrak{c}(\dot{c}h)\}$$

$$=\sum_{c\in(H_F(\mathfrak{f})/G_1)_0/\langle\nu\rangle}\chi(c)\log\left\{\prod_{h\in H_1}\frac{W_\mathfrak{c}(\dot{c}\dot{\nu}h)}{W_\mathfrak{c}(\dot{c}h)}\right\}.$$

The equality (32) holds for *any character* χ *of* $H_F(\mathfrak{f})/G$ of type (14). Now let χ_1 be a character of $H_F(\mathfrak{f})/G$ of type (14). Then the mapping: $\eta\mapsto\chi_1\eta$ establishes a bijection from the set of characters of $H_F(\mathfrak{f})/\langle G, \nu\rangle$ onto the set of characters of $H_F(\mathfrak{f})/G$ of type (14). It is easy to see that the group $H_F(\mathfrak{f})/\langle G, \nu\rangle$ is isomorphic to the group $(H_F(\mathfrak{f})/G_1)_0/\langle\nu\rangle$. Thus the equality (32) now implies

$$Y_{\mathfrak{f}}(c, G)=\prod_{h\in H_1}\frac{W_{\mathfrak{c}}(\dot{c}\dot{\nu}h)}{W_{\mathfrak{c}}(\dot{c}h)} \qquad \text{for any} \quad c\in(H_F(\mathfrak{f})/G_1)_0\,.$$

The proof of Proposition 4 is now complete.

PROPOSITION 5. *For a suitable positive rational integer m, the following assertions* (i), (ii) *and* (iii) *hold.*

(i) *$Y_{\mathfrak{f}}(c, G)^m$ $(c\in H_F(\mathfrak{f})/G)$ is a unit in $K_F(\mathfrak{f}, G)$ and generates $K_F(\mathfrak{f}, G)$ over F.*

(ii) $\{Y_{\mathfrak{f}}(c, G)^m\}^{\sigma_F(c_0)}=Y_{\mathfrak{f}}(cc_0, G)^m$ $(\forall c_0\in H_F(\mathfrak{f}))$.

(iii) *Let τ be an embedding of $K_F(\mathfrak{f}, G)$ into $\boldsymbol{C}$ inducing the non-trivial automorphism on F, then $\tau(Y_{\mathfrak{f}}(c, G)^m)$ is a complex number of modulus* 1.

PROOF. Recall that $(H_F(\mathfrak{f})/G_1)_0$ is a complete set of representatives for $H_F(\mathfrak{f})/G$. Hence it is sufficient to prove Proposition assuming $c, c_0\in(H_F(\mathfrak{f})/G_1)_0$.

For $t\in H_k(\mathfrak{c})$ and for a divisor $\mathfrak{c}_0$ of $\mathfrak{c}$, set

$$\phi(t, H_1, \mathfrak{c}_0)=\prod_{h\in H_1}\frac{\Phi_{\mathfrak{c}_0}(\widetilde{th\,\dot{\nu}})}{\Phi_{\mathfrak{c}_0}(\widetilde{th})} \tag{33}$$

(for notation, see (3) and (5)),

where $\tilde{t}$ is the image of t under the natural homomorphism from $H_k(\mathfrak{c})$ onto $H_k(\mathfrak{c}_0)$. Since both $\mathfrak{c}$ and H_1 are invariant under the non-trivial automorphism κ of k, Lemma 1 implies that

$$\phi(t, H_1, \mathfrak{c}_0)\in K_k(\mathfrak{c}, H_1) \qquad \text{and that} \tag{34}$$

$$\{\phi(t, H_1, \mathfrak{c}_0)\}^{\sigma_k(t')}=\phi(t\kappa(t'), H_1, \mathfrak{c}_0) \qquad (\forall t'\in H_k(\mathfrak{c}))\,. \tag{35}$$

In particular if $t'\in(H_k(\mathfrak{c})/H_1)_0$,

$$\phi(t, H_1, \mathfrak{c}_0)^{\sigma_k(t')}=\phi(tt', H_1, \mathfrak{c}_0)\,. \tag{36}$$

For an element $\alpha=\sum m_i t_i$ $(m_i\in\boldsymbol{Z}, t_i\in H_k(\mathfrak{c}))$ of the group ring $\boldsymbol{Z}[H_k(\mathfrak{c})]$ of $H_k(\mathfrak{c})$ with rational integral coefficients, we put

$$(\alpha\phi_{\mathfrak{c}_0})(t)=\prod_i\phi(tt_i, H_1, \mathfrak{c}_0)^{m_i}\,. \tag{37}$$

It follows from Proposition 4 and equalities (9), (6) and (4) that for a suitable choice of a positive integer m and suitable choices of $\alpha(\mathfrak{c}_0)\in\boldsymbol{Z}[H_k(\mathfrak{c})]$ for each divisor $\mathfrak{c}_0$ of $\mathfrak{c}$, the following equality holds for any $c\in(H_F(\mathfrak{f})/G_1)_0$:

$$Y_{\mathfrak{f}}(c, G)^m=\prod_{\mathfrak{c}_0}(\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0})(\dot{c})\overline{(\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0})(\dot{c})}\,,$$

where $\overline{}$ denotes the complex conjugation and the product with respect to

$\mathfrak{c}_0$ is over all the divisors of $\mathfrak{c}$. It follows now immediately from Lemma 1 that $Y_{\mathfrak{f}}(\mathfrak{c}, G)^m$ is a unit in $K=K_k(\mathfrak{c}, H_1)=K_F(\mathfrak{f}, G_1)$. Lemma 5 together with equality (34) shows that $\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c})$ is in K and that $\overline{(\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0})(\dot{c})}=(\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0})(\dot{c}))^{\sigma(\mu)}$. Thus, $Y_{\mathfrak{f}}(c, G)^m$ is $\sigma(\mu)$-invariant. Since G is generated by μ and G_1, $Y_{\mathfrak{f}}(c, G)^m\in K_F(\mathfrak{f}, G)$. For $c'\in(H_F(\mathfrak{f})/G_1)_0$, it follows from (22), and (36) that

$$\{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c})\overline{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c})}\}^{\sigma(c')}=\{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c})\overline{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c})}\}^{\sigma_k(\dot{c}')}$$

$$=\{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c}\dot{c}')\overline{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c}\dot{c}')}\}\ .$$

Thus, $\{Y_{\mathfrak{f}}(c, G)^m\}^{\sigma(c')}=Y_{\mathfrak{f}}(cc', G)^m$. Hence,

$$\{Y_{\mathfrak{f}}(c, G)^m\ ;\ c\in H_F(\mathfrak{f})/G\}$$

is a system of units of $K_F(\mathfrak{f}, G)$ which are mutually conjugate over F. Set

$$\Gamma_c=\{c_0\in H_F(\mathfrak{f})/G\ ;\ Y_{\mathfrak{f}}(cc_0, G)^m=Y_{\mathfrak{f}}(c, G)^m\}\ .$$

Then Γ_c is a subgroup of $H_F(\mathfrak{f})/G$ which is independent of c. Assume $\nu\in\Gamma_c$, then Corollary to Theorem 1 implies that $Y_{\mathfrak{f}}(c, G)=1$ for any $c\in H_F(\mathfrak{f})/G$. Hence, it follows from Proposition 3 that $\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0}=0$ for any character χ of $H_F(\mathfrak{f})/G$ of type (14).

If $\Gamma_c\neq\{1\}$ and if $\nu\notin\Gamma_c$, there would exist a character χ of the group $H_F(\mathfrak{f})/G$ of type (14) which is non-trivial on Γ_c. Then it follows from Proposition 3 that

$$m\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0}=\sum_{c\in H_F(\mathfrak{f})/\langle G,\nu\rangle}\chi(c)\log Y_{\mathfrak{f}}(c, G)^m=0\,.$$

However it follows immediately from the functional equation for $L_F(s, \tilde{\chi})$ and the inequality $L_F(1, \tilde{\chi})\neq 0$ that $\left\{\frac{d}{ds}L_F(s, \tilde{\chi})\right\}_{s=0}\neq 0$ for any primitive character $\tilde{\chi}$ of type (14). Hence $\Gamma_c=\{1\}$. Thus $Y_{\mathfrak{f}}(c, G)^m$ generates $K_F(\mathfrak{f}, G)$ over F.

To prove (iii), we may put

$$\tau=\sigma_k(c')\qquad (c'\in H_k(\mathfrak{c})/H_1-(H_k(\mathfrak{c})/H_1)_0)\,.$$

Then it follows from Lemma 7 and (35) that

$$\tau(\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c})\overline{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c})})=\overline{\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c}\dot{c}')}\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(\dot{c}\dot{c}'\dot{\nu})\,.$$

In view of (33) it is easy to see that $|\overline{(\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(c)}\alpha(\mathfrak{c}_0)\phi_{\mathfrak{c}_0}(c\dot{\nu})|=1$. Thus,

$$|\tau(Y_{\mathfrak{f}}(c, G)^m)|=1\,.$$

Proof of Theorem 2. Let $\mathfrak{f}_0$ be a divisor of $\mathfrak{f}$ which satisfies the conditions (0-3) and (0-6). Let $G_{\mathfrak{f}_0}$ be the image of G under the natural homomorphism from the group $H_F(\mathfrak{f})$ onto $H_F(\mathfrak{f}_0)$. For each $\alpha=\sum m_i c_i$ ($m_i\in\boldsymbol{Z}$, $c_i\in H_F(\mathfrak{f}_0)$), we put

$$(\alpha Y_{\mathfrak{f}_0})(c, G_{\mathfrak{f}_0})=\prod_i Y_{\mathfrak{f}_0}(cc_i, G_{\mathfrak{f}_0})^{m_i}.$$

In view of Proposition 5 and Lemma 2, to prove Theorem 2, it is sufficient to prove the following Lemma 8:

LEMMA 8. *Take a suitable positive integer* m'. *Furthermore, for each divisor* $\mathfrak{f}_1$ *of* $\mathfrak{f}_0$ *with conditions* (0-3) *and* (0-6), *choose suitable* $\alpha(\mathfrak{f}_1)\in\boldsymbol{Z}[H_F(\mathfrak{f}_1)]$. *Then*

$$X_{\mathfrak{f}_0}(c, G_{\mathfrak{f}_0})^{m'}=\prod_{\mathfrak{f}_1}(\alpha(\mathfrak{f}_1)Y_{\mathfrak{f}_1})(c, G_{\mathfrak{f}_1})^{m}, \qquad (\forall c\in H_F(\mathfrak{f}_0)),$$

where the product is over all divisors $\mathfrak{f}_1$ *of* $\mathfrak{f}_0$ *with conditions* (0-3) *and* (0-6) (*for notation see* (20), (19) *and* (18)).

PROOF. Apply the induction with respect to the number of divisors of $\mathfrak{f}_0$ with properties (0-3) and (0-6).

REMARK. 1. For the following pairs of F and $\mathfrak{f}$, assumptions of Theorem 2 are all satisfied if one puts $G=\langle\mu\rangle$, $G_1=\{1\}$:

$$F=\boldsymbol{Q}(\sqrt{5}), \quad \mathfrak{f}=(11); \quad F=\boldsymbol{Q}(\sqrt{5}), \quad \mathfrak{f}=(3\sqrt{5});$$
$$F=\boldsymbol{Q}(\sqrt{17}), \quad \mathfrak{f}=(4\sqrt{17}); \quad F=\boldsymbol{Q}(\sqrt{21}), \quad \mathfrak{f}=(\sqrt{21});$$
$$F=\boldsymbol{Q}(\sqrt{10}), \quad \mathfrak{f}=(3).$$

2. A coincidence of an L-series of a real quadratic field with an L-series of an imaginary quadratic field was first observed by Hecke in [**14**].

§ 3.

In this section we discuss a few numerical examples. We use previously introduced notation without further comment.

1. Set $F=\boldsymbol{Q}(\sqrt{5})$, $\mathfrak{f}=(4)$. The class number (in a narrow sense) of F is 1. We may put $\nu=(3)$ and $\mu=(3-2\sqrt{5})$. Set $\varepsilon_0=(1+\sqrt{5})/2$ and $\varepsilon=(3+\sqrt{5})/2$. Then ε_0 (resp. ε) is a fundamental (resp. fundamental totally positive) unit of F. It is easy to see that the group $H_F(\mathfrak{f})$ is an abelian group of type $(2, 2)$ generated by μ and ν. Furthermore,

$$H_F(\mathfrak{f})_0=\{c\in H_F(\mathfrak{f})\,;\, c'=c\}=\{1, \nu\}.$$

Thus, $[H_F(\mathfrak{f}), H_F(\mathfrak{f})_0]=2$. We may put $\mathfrak{a}_1=\mathfrak{O}_F$ as a representative for the narrow ideal class of F. By a simple computation, we have

$$X_{\mathfrak{f}}(1)=\boldsymbol{F}(1/4,(1,\varepsilon))\boldsymbol{F}(1+\varepsilon/4,(1,\varepsilon))\boldsymbol{F}((3+3\varepsilon)/4,(1,\varepsilon))\times$$
$$\times\boldsymbol{F}(1/4,(1,\varepsilon'))\boldsymbol{F}(1+\varepsilon'/4,(1,\varepsilon'))\boldsymbol{F}((3+3\varepsilon')/4,(1,\varepsilon'))$$
$$=\boldsymbol{F}(1/4,(1,\varepsilon))^2\boldsymbol{F}(1+\varepsilon/4,(1,\varepsilon))^2\boldsymbol{F}((3+3\varepsilon)/4,(1,\varepsilon))^2$$
$$=\boldsymbol{F}(1/4,(1,\varepsilon'))^2\boldsymbol{F}(1+\varepsilon'/4,(1,\varepsilon'))^2\boldsymbol{F}((3+3\varepsilon')/4,(1,\varepsilon'))^2$$

(see § 3.1 of [**7**]).

Since $\mu'=\mu\nu$, Corollary to Theorem 1 and the equality $\zeta_F(s,\mu)=\zeta_F(s,\mu')$ imply that $X_{\mathfrak{f}}(\mu)=1$. Set $G=\{1,\mu\}$. Then G is a subgroup of order 2 of $H_F(\mathfrak{f})$. We have

$$X_{\mathfrak{f}}(1,G)=X_{\mathfrak{f}}(1) \quad\text{and}\quad X_{\mathfrak{f}}(\nu,G)=X_{\mathfrak{f}}(1,G)^{-1}.$$

Since there is no proper divisor of $\mathfrak{f}$ with the property (0-3),

$$Y_{\mathfrak{f}}(1,G)=X_{\mathfrak{f}}(1,G)=X_{\mathfrak{f}}(1).$$

It is easy to see that the ray class field $K_F(\mathfrak{f})$ is given by $K=F(\sqrt{\varepsilon_0},\sqrt{\varepsilon_0'})$. The subfield of $\sigma_F(H_F(\mathfrak{f})_0)$-fixed elements of K is given by $L=F(\sqrt{-5})$. Set $k=\boldsymbol{Q}(\sqrt{-5})$. Then K is the ray class field with conductor $\mathfrak{c}=(2)$ over k. The group $H_k(\mathfrak{c})$ is a cyclic group of order 4 generated by $\mathfrak{c}_0=[3,2+\sqrt{-5}]$. Furthermore,

$$H_k(\mathfrak{c})_0=\{c\in H_k(\mathfrak{c})\,;\ \bar{c}=c\}=\{(1),\ (2+\sqrt{-5})\}.$$

By a simple computation, we have

$$Z_{\mathfrak{c}}((1))=\left|\frac{\vartheta_2(\sqrt{-5})}{\eta(\sqrt{-5})}\right|^2 \quad\text{and}\quad Z_{\mathfrak{c}}((2+\sqrt{-5}))=\left|\frac{\vartheta_0(\sqrt{-5})}{\eta(\sqrt{-5})}\right|^2 \quad \text{(cf. (4)), where}$$

$$\vartheta_0(\tau)=\sum_{n\in\boldsymbol{Z}}(-1)^n q^{n^2}, \qquad \vartheta_2(\tau)=\sum_{n\in\boldsymbol{Z}}q^{(n+1/2)^2},$$
$$\vartheta_3(\tau)=\sum_{n\in\boldsymbol{Z}}q^{n^2} \qquad (q=e^{\pi i\tau}).$$

Since $\omega_{\mathfrak{c}}=2$,

$$\frac{W_{\mathfrak{c}}((2+\sqrt{-5}))}{W_{\mathfrak{c}}((1))}=\left\{\frac{Z_{\mathfrak{c}}((2+\sqrt{-5}))}{Z_{\mathfrak{c}}((1))}\right\}^{1/2}=\sqrt{\frac{k'}{k}} \qquad \text{(cf. (9))},$$

where $k=\vartheta_2^2(\sqrt{-5})/\vartheta_3^2(\sqrt{-5})$ and $k'=\vartheta_0^2(\sqrt{-5})/\vartheta_3^2(\sqrt{-5})$.
It is known (see Tabelle 6 of [**13**]) that $4/kk'=(1+\sqrt{5})^3$.
Since $k^2+k'^2=1$ and $k'>k>0$,

$$k'^2=1/2+1/\sqrt{\varepsilon_0^3} \quad\text{and}\quad k^2=1/2-1/\sqrt{\varepsilon_0^3}.$$

Thus, Proposition 4 and the equality $\sqrt{\varepsilon_0^3}=\sqrt{\varepsilon_0}+\sqrt{\varepsilon_0^{-1}}$ imply that

$$Y_{\mathfrak{f}}(1,G)=X_{\mathfrak{f}}(1)=\sqrt{k'/k}=\sqrt{\varepsilon_0}(1+\sqrt{\varepsilon_0}).$$

The equality is consistent with the result of § 3.1 of [**7**].

2. In the remaining part of this section we discuss two numerical examples of conjecture (0-7) for which Theorem 2 is not applicable. We show that numerical computations based on (0-5) provide encouraging evidences which support the conjecture (0-7). We note that, by (0-5), the numerical computations of the invariants $X_{\mathfrak{f}}(c)$ are reduced to those of $\log \Gamma_2(z, (1, \varepsilon'))$ for z sufficiently large and positive. Then, the asymptotic series for $\log \Gamma_2$ given in Proposition 4 of [**7**] is effectively applied. For numerical computations, we made use of HITAC 8700/8800 in the Computer Centre of University of Tokyo. It worked internally with an accuracy of about 33 decimal places. In [**10**]-[**12**], Stark presented several numerical evidences for his conjecture. Our method of computation of the invariant $X_{\mathfrak{f}}(c)$ is different from Stark's. Basic to numerical experiments is the following observation of Stark (see [**10**] and [**11**]):

Conjecture (0-7) implies the following:

If τ is an imbedding of the field $K_F(\mathfrak{f}, G)$ into $\boldsymbol{C}$ inducing the non-trivial isomorphism of F, then $\tau(X_{\mathfrak{f}}(c, G)^m)$ is a complex number of modulus 1.

Set $F=\boldsymbol{Q}(\sqrt{29})$ and $\mathfrak{f}=((3-\sqrt{29})/2)$. Set $\varepsilon_0=(5+\sqrt{29})/2$ and $\varepsilon=(27+5\sqrt{29})/2$. Then ε_0 (resp. ε) is a fundamental (resp. fundamental totally positive) unit of F. We note that $\varepsilon_0'\equiv\varepsilon\equiv 1 \bmod \mathfrak{f}$. The number of (narrow) ideal classes of F is 1. The group $H_F(\mathfrak{f})$ is isomorphic to a cyclic group of order 4 generated by the ray class $c=(2)$. It is easy to see that $\nu=c^2$ and that $\mu=1$. Since no imaginary quadratic field is contained in the normal closure of the field $K_F(\mathfrak{f})$, Theorem 2 is not applicable for this example. We may put $\mathfrak{a}_1=\mathfrak{O}_F$ as a representative for the narrow ideal class of F. By a simple computation, we have

$$R(\varepsilon, 1)=\left\{\frac{6+24\varepsilon}{25}, \frac{1+4\varepsilon}{25}, \frac{21+9\varepsilon}{25}\ \frac{16+14\varepsilon}{25}, \frac{11+19\varepsilon}{25}\right\}, \tag{38}$$

$$R(\varepsilon, c)=\left\{\frac{12+23\varepsilon}{25}, \frac{2+8\varepsilon}{25}, \frac{17+18\varepsilon}{25}, \frac{7+3\varepsilon}{25}, \frac{22+13\varepsilon}{25}\right\}, \tag{39}$$

(for notation, see (0-4)).

To simplify the notation, set

$$\boldsymbol{F}(z)=\boldsymbol{F}(z, (1, \varepsilon')).$$

Then it follows from Corollary to Proposition 2 of [7] that

$$\boldsymbol{F}(z, (1, \varepsilon))=\boldsymbol{F}(\varepsilon' z, (1, \varepsilon'))=\boldsymbol{F}(\varepsilon' z). \tag{40}$$

Set

(41)
$$A_1=\prod_{z\in R(\varepsilon,1)} \boldsymbol{F}(\varepsilon' z), \qquad B_1=\prod_{z\in R(\varepsilon,1)} \boldsymbol{F}(z'),$$
$$A_2=\prod_{z\in R(\varepsilon,c)} \boldsymbol{F}(\varepsilon' z), \qquad B_2=\prod_{z\in R(\varepsilon,c)} \boldsymbol{F}(z').$$

Then it follows from (0-5) and (41) that

$$X_{\mathfrak{f}}(1)=A_1B_1, \qquad X_{\mathfrak{f}}(c^2)=X_{\mathfrak{f}}(1)^{-1}, \qquad X_{\mathfrak{f}}(c)=A_2B_2, \qquad X_{\mathfrak{f}}(c^3)=X_{\mathfrak{f}}(c)^{-1}.$$

Set $Y_m=\sum_{i=0}^{3} X_{\mathfrak{f}}(c^i)^m \qquad (m=1, 2, \cdots)$.

If the conjecture (0-7) is true for this example for $G=\{1\}$, Y_m would be an integer of F whose conjugate is in the interval $(-4, 4)$. Thus, there would exist rational integers α and β such that

$$Y_m=\alpha+\beta\omega \quad \text{and} \quad |\alpha+\beta\omega'|<4,$$

where we put $\omega=(1+\sqrt{29})/2$.

Set $Y'_m=\alpha+\beta\omega'$. Then $\beta=(Y_m-Y'_m)/\sqrt{29}$.

Since $|Y'_m|<4$, $|\beta-[Y_m/\sqrt{29}]|<2$ where $[Y_m/\sqrt{29}]$ denotes the integral part of $Y_m/\sqrt{29}$. Hence, the fractional part of $Y_m-\omega[Y_m/\sqrt{29}]$ must coincide with the fractional part of $-\omega$ or 0 or ω. Now a numerical computation shows that

(42)
$$X_{\mathfrak{f}}(1)=4.6242866\cdots, \qquad X_{\mathfrak{f}}(c)=1.7949175\cdots,$$

and that

$$[Y_1/\sqrt{29}]=1, \quad [Y_2/\sqrt{29}]=4, \quad [Y_3/\sqrt{29}]=19, \quad [Y_4/\sqrt{29}]=86,$$
$$Y_1-\omega=4-(10)^{-26}\times 2.249\cdots,$$
$$Y_2-4\omega=9+\omega-(10)^{-25}\times 1.783\cdots,$$
$$Y_3-19\omega=41+\omega-(10)^{-24}\times 1.088\cdots,$$
$$Y_4-86\omega=190+\omega-(10)^{-24}\times 6.312\cdots.$$

Thus, it is quite probable that the conjecture would be true for $m=1$ and that $X_{\mathfrak{f}}(1)$, $X_{\mathfrak{f}}(c)$, $X_{\mathfrak{f}}(c^2)$, $X_{\mathfrak{f}}(c^3)$ are roots of the following quartic equation:

$$x^4-x^3(9+\sqrt{29})/2+x^2(8+\sqrt{29})-x(9+\sqrt{29})/2+1=0.$$

Set $t=x+x^{-1}$, then t satisfies the following quadratic equation:

$$t^2-t(9+\sqrt{29})/2+6+\sqrt{29}=0.$$

Two roots t_1, t_2 of the equation are given as follows:

$$t_1=\{(9+\sqrt{29})/2+\sqrt{(7+\sqrt{29})/2}\}/2\,,$$

$$t_2=\{(9+\sqrt{29})/2-\sqrt{(7+\sqrt{29})/2}\}/2\,.$$

Taking the equality (42) into account, we infer that the validity of the following equalities is quite probable.

$$(43)\qquad \begin{aligned} &X_{\mathfrak{f}}(1)=(t_1+\sqrt{t_1^2-4})/2\,, && X_{\mathfrak{f}}(c)=(t_2+\sqrt{t_2^2-4})/2\,,\\ &X_{\mathfrak{f}}(c^2)=(t_1-\sqrt{t_1^2-4})/2\,, && X_{\mathfrak{f}}(c^3)=(t_2-\sqrt{t_2^2-4})/2\,. \end{aligned}$$

In the following we assume the validity of (43). Incidentally, numerical computations show that

$$\log B_1=(10)^{-27}\times(-1.829\cdots)\qquad \text{and}$$

$$\log B_2=(10)^{-27}\times(-0.918\cdots)\qquad (\text{cf. } (41))\,.$$

It is quite probable that $B_1=B_2=1$.

Since $(7+\sqrt{29})/2=-\varepsilon_0(3-\sqrt{29})/2$ and $(7+\sqrt{29})/2\equiv\{(9+\sqrt{29})/2\}^2$ mod. 4, t_1 and t_2 are in the subfield of $\sigma(\{1, c^2\})$-fixed elements of $K_F(\mathfrak{f})$.

Note that

$$(44)\qquad (t_1^2-4)(t_2^2-4)=19+2\sqrt{29}=(7+\sqrt{29})(\sqrt{29}-1)^2/8\,.$$

Set $x=X_{\mathfrak{f}}(1)$. Then (43) and (44) show that

$$(45)\qquad \begin{aligned} 2X_{\mathfrak{f}}(c)&=(9+\sqrt{29})/2-(x+x^{-1})\\ &\quad+2^{-1}(\sqrt{29}-1)(x-x^{-1})^{-1}\{2(x+x^{-1})-(9+\sqrt{29})/2\}\,,\\ X_{\mathfrak{f}}(c^2)&=x^{-1}\,,\qquad X_{\mathfrak{f}}(c^3)=X_{\mathfrak{f}}(c)^{-1}\,. \end{aligned}$$

Thus, we see that the field $K=F(X_{\mathfrak{f}}(1), X_{\mathfrak{f}}(c))$ is a quartic normal extension of F. Hence, K is abelian over F. Set $L=F(t_1)$. We have seen that L is the class field over F with conductor $\mathfrak{f}$ which corresponds to the subgroup $\{1, c^2\}$ of $K_F(\mathfrak{f})$. We denote by τ the non-trivial element of Gal(L/K). Since $(\sqrt{29}-1)/2\equiv1$ mod. $\mathfrak{f}$, the prime ideal $((\sqrt{29}-1)/2)$ in F splits in L into a product of two different ideals $\mathfrak{p}$ and $\mathfrak{p}^\tau$. On the other hand, the prime ideal $((7+\sqrt{29})/2)$ in F ramifies to a square of a prime ideal $\mathfrak{q}$ in L ($\mathfrak{q}^\tau=\mathfrak{q}$). We note that $(t_1^2-4)^\tau=(t_2^2-4)$ and that (t_1^2-4) and (t_2^2-4) are different ideals in L (since $(t_1-2)(t_2-2)=1$, it is sufficient to check that $(t_1+2)/(t_2+2)$ is not an algebraic integer). Thus, the equality (44) implies that

$$(t_1^2-4)=\mathfrak{p}^2\mathfrak{q}\,,\qquad (t_2^2-4)=(\mathfrak{p}^\tau)^2\mathfrak{q}\,.$$

Since $t_1^2-4\equiv t_1^2$ mod. 4 in L, the field $K=L(\sqrt{t_1^2-4})$ ramifies only at $\mathfrak{q}$. Hence, the conductor of K with respect to F is a power of $\mathfrak{f}$. Since 4 is prime to $\mathfrak{f}$, the conductor coincides with $\mathfrak{f}$. Thus $K=K_F(\mathfrak{f})$.

The ray class $c=(2)$ of $H_F(\mathfrak{f})$ is represented by a prime ideal

$$\mathfrak{f}'=(\varepsilon_0(3+\sqrt{29})/2).$$

The quotient field $\mathfrak{O}_F/\mathfrak{f}'\cong \boldsymbol{Z}/(5)$ is the finite field with five elements. Set $\sigma=\sigma(c)$. In $\mathfrak{O}_K/\mathfrak{f}'$, σ induces the Frobenius automorphism: $a\longrightarrow a^5$. In $\mathfrak{O}_K/\mathfrak{f}'$, $x=X_{\mathfrak{f}}(1)$ satisfies the equation $x^4-3x^3-3x+1=0$ with coefficients in $\boldsymbol{Z}/(5)$. Thus $\sigma(x)=4x^3+3x^2+3x+2$ in $\mathfrak{O}_K/\mathfrak{f}'$. After some computations, we derive from (45) the equality $X_{\mathfrak{f}}(c)=4x^3+3x^2+3x+2$ in $\mathfrak{O}_K/\mathfrak{f}'$. Hence $X_{\mathfrak{f}}(1)^{\sigma(c)}=X_{\mathfrak{f}}(c)$ in K. For this example, numerical experiment is consistent with the conjecture (0-7).

3. Set $F=\boldsymbol{Q}(\sqrt{11})$ and $\mathfrak{f}=(3)$. The fundamental unit ε of F is given by $\varepsilon=10+3\sqrt{11}$. The class number of F is 1. Set $c_0=(4+\sqrt{11})$. We may put $\nu=c_0^4$ and $\mu=(1-3\sqrt{11})$. It is easy to see that the group $H_F(\mathfrak{f})$ is isomorphic to a direct product of a cyclic group of order 8 generated by c_0 and a cyclic group of order 2 generated by μ:

$$H_F(\mathfrak{f})\cong\langle c_0\rangle\times\langle\mu\rangle.$$

Since $c_0'=c_0^3$ and $\mu'=\mu c_0^4$ in $H_F(\mathfrak{f})$, we see

$$H_F(\mathfrak{f})_0=\{c\in H_F(\mathfrak{f})\,;\ c'=c\}=\{1,\ \mu c_0^2,\ c_0^4,\ \mu c_0^6\}.$$

Thus, $[H_F(\mathfrak{f}), H_F(\mathfrak{f})_0]=4$. Theorem 2 is not applicable for this example.

Set $\mathfrak{a}_1=\mathfrak{O}_F$, $\mathfrak{a}_2=(3+\sqrt{11})$. Then $\{\mathfrak{a}_1, \mathfrak{a}_2\}$ is a complete set of representatives for the narrow ideal classes of F. After some computations, we see that

$$R(\varepsilon, 1)=\{1+\varepsilon/3,\ (2+2\varepsilon)/3,\ 1/3\},$$

$$R(\varepsilon, c_0)=\{(2+\varepsilon)/9,\ (8+4\varepsilon)/9,\ (5+7\varepsilon)/9\},$$

$$R(\varepsilon, c_0^2)=\{(7+2\varepsilon)/9,\ (4+5\varepsilon)/9,\ (1+8\varepsilon)/9\},$$

$$R(\varepsilon, c_0^3)=\{(1+2\varepsilon)/9,\ (7+5\varepsilon)/9,\ (4+8\varepsilon)/9\},$$

$$R(\varepsilon, \mu)=\{(8+\varepsilon)/9,\ (13+5\varepsilon)/18,\ (5+4\varepsilon)/9,\ (1+17\varepsilon)/18,\ (7+11\varepsilon)/18,\ (2+7\varepsilon)/9\},$$

$$R(\varepsilon, c_0\mu)=\{(5+\varepsilon)/9,\ (8+7\varepsilon)/9,\ (2+4\varepsilon)/9,\ (7+5\varepsilon)/18,\ (13+17\varepsilon)/18,\ (1+11\varepsilon)/18\},$$

$$R(\varepsilon, c_0^2\mu)=\{(1+\varepsilon)/6, (5+3\varepsilon)/6, (3+5\varepsilon)/6,$$
$$1+\varepsilon/3, (2+2\varepsilon)/3, 1/3\},$$
$$R(\varepsilon, c_0^3\mu)=\{(2+\varepsilon)/9, (8+4\varepsilon)/9, (5+7\varepsilon)/9,$$
$$(1+5\varepsilon)/18, (13+11\varepsilon)/18, (7+17\varepsilon)/18\},$$

(cf. (0-4)).

To simplify the notation, set $\boldsymbol{F}(z)=\boldsymbol{F}(z, (1, \varepsilon))$. Further, set

$$S=\boldsymbol{F}(1+\varepsilon/3)^2\boldsymbol{F}((2+2\varepsilon)/3)^2\boldsymbol{F}(1/3)^2,$$
$$T=\boldsymbol{F}((2+\varepsilon)/9)\boldsymbol{F}((8+4\varepsilon)/9)\boldsymbol{F}((5+7\varepsilon)/9)$$
$$\times\boldsymbol{F}((1+2\varepsilon)/9)\boldsymbol{F}((4+8\varepsilon)/9)\boldsymbol{F}((7+5\varepsilon)/9),$$
$$U=\boldsymbol{F}((7+5\varepsilon)/18)\boldsymbol{F}((13+17\varepsilon)/18)\boldsymbol{F}((1+11\varepsilon)/18)$$
$$\times\boldsymbol{F}((5+7\varepsilon)/18)\boldsymbol{F}((17+13\varepsilon)/18)\boldsymbol{F}((11+\varepsilon)/18),$$
$$V=\boldsymbol{F}((1+\varepsilon)/6)^2\boldsymbol{F}((5+3\varepsilon)/6)^2\boldsymbol{F}((3+5\varepsilon)/6)^2.$$

Set $G=\langle\mu\rangle$. Then it follows easily from (0-5), (19) and (40) that

$$X_{\mathfrak{f}}(1, G)=S, \quad X_{\mathfrak{f}}(c_0, G)=U, \quad X_{\mathfrak{f}}(c_0^2, G)=SV, \quad X_{\mathfrak{f}}(c_0^3, G)=T^2U^{-1},$$
$$X_{\mathfrak{f}}(c_0^{i+4}, G)=X_{\mathfrak{f}}(c_0^i, G)^{-1} \qquad (i=0, 1, 2, 3).$$

Set
$$R_i=X_{\mathfrak{f}}(c_0^{i-1}, G)+X_{\mathfrak{f}}(c_0^{i+3}, G) \qquad (i=1, 2, \cdots, 4) \quad \text{and}$$
$$Y_m=\sum_{i=0}^{7} X_{\mathfrak{f}}(c_0^i, G)^m \qquad (m=1, 2, \cdots).$$

If the conjecture (0-7) is true, then Y_m would be an integer of F whose conjugate is in the interval $(-8, 8)$.

Now a numerical computation shows that

(46)
$$X_{\mathfrak{f}}(1, G)=3.564315896\cdots, \qquad X_{\mathfrak{f}}(c_0, G)=0.519601027\cdots,$$
$$X_{\mathfrak{f}}(c_0^2, G)=5.824396333\cdots, \qquad X_{\mathfrak{f}}(c_0^3, G)=5.482353802\cdots,$$
$$R_1=3.844874642\cdots, \qquad R_2=2.444154574\cdots,$$
$$R_3=5.996087946\cdots, \qquad R_4=5.664757207\cdots,$$
$$[Y_1/2\sqrt{11}]=2, \qquad [Y_2/2\sqrt{11}]=12, \qquad [Y_3/2\sqrt{11}]=62,$$
$$[Y_4/2\sqrt{11}]=336,$$
$$Y_1-2\sqrt{11}=8+\sqrt{11}+(10)^{-27}\times 8.281\cdots,$$

$$Y_2-12\sqrt{11}=41+(10)^{-25}\times 1.154\cdots,$$

$$Y_3-62\sqrt{11}=206+\sqrt{11}+(10)^{-24}\times 1.087\cdots,$$

$$Y_4-336\sqrt{11}=1115+(10)^{-24}\times 8.723\cdots.$$

Thus, it is quite probable that the conjecture would be true for $m=1$ and that R_1, R_2, R_3 and R_4 would be the four roots of the following quartic equation:

(47) $$X^4-p_1X^3+p_2X^2-p_3X+p_4=0, \quad \text{where} \quad p_1=8+3\sqrt{11},$$

$$p_2=57+18\sqrt{11}, \quad p_3=164+48\sqrt{11}, \quad p_4=160+48\sqrt{11}.$$

Denote by r_1, r_2, r_3, r_4 roots of the equation (47) and set

$$y_1=r_1r_2+r_3r_4, \quad y_2=r_1r_3+r_2r_4, \quad y_3=r_1r_4+r_2r_3$$

and

$$e_i=(p_2/3-y_i)/4 \quad (i=1, 2, 3).$$

Then e_1, e_2 and e_3 are roots of the following cubic equation:

$$4e^3-15e/4+11/8=4(e-1/2)(e^2+e/2-11/16)=0.$$

Thus, permuting r_1, r_2, r_3, r_4 in a suitable manner if necessary, we have

$$y_2=r_1r_3+r_2r_4=17+6\sqrt{11},$$

$$y_1+y_3=(r_1+r_3)(r_2+r_4)=40+12\sqrt{11}.$$

Since $r_1r_2r_3r_4=160+48\sqrt{11}$ and $r_1+r_2+r_3+r_4=8+3\sqrt{11}$, we have

$$(r_1+r_3-r_2-r_4)^2=3 \quad \text{and} \quad (r_1r_3-r_2r_4)^2=3(2+\sqrt{11})^2.$$

Taking (46) into account, we infer that the following equalities are quite probable:

$$R_1=((8+3\sqrt{11}+\sqrt{3})/2-\sqrt{(15-\sqrt{33})/2})/2,$$

$$R_2=((8+3\sqrt{11}-\sqrt{3})/2-\sqrt{(15+\sqrt{33})/2})/2,$$

$$R_3=((8+3\sqrt{11}+\sqrt{3})/2+\sqrt{(15-\sqrt{33})/2})/2,$$

(48) $$R_4=((8+3\sqrt{11}-\sqrt{3})/2+\sqrt{(15+\sqrt{33})/2})/2,$$

$$X_{\mathfrak{f}}(1, G)=(R_1+\sqrt{R_1^2-4})/2, \quad X_{\mathfrak{f}}(c_0, G)=(R_2-\sqrt{R_2^2-4})/2,$$

$$X_{\mathfrak{f}}(c_0^2, G)=(R_3+\sqrt{R_3^2-4})/2, \quad X_{\mathfrak{f}}(c_0^3, G)=(R_4+\sqrt{R_4^2-4})/2.$$

We assume the validity of the equalities (48).
Set

$$K=F(R_1, R_2, R_2. R_4).$$

It is now easy to see that $K=F(\sqrt{(15+\sqrt{33})/2})$ is a cyclic quartic extension of F. The field K is a quadratic extension of the field $F(\sqrt{3})$. Since $3\equiv(\sqrt{11})^2$ mod. 4, $F(\sqrt{3})$ is the class field over F with conductor $\mathfrak{f}$ which corresponds to the subgroup $\langle\mu, c_0^2\rangle$ of $H_F(\mathfrak{f})$. We note the following identity:

$$(12+3\sqrt{11}+17\sqrt{3}+4\sqrt{33})^2(15+\sqrt{33})/2$$
$$=(8\sqrt{3})^2(81+24\sqrt{11}+40\sqrt{3}+12\sqrt{33}).$$

Since $81+24\sqrt{11}+40\sqrt{3}+12\sqrt{33}\equiv 1$ mod. 4 in $\mathfrak{O}_L$ $(L=F(\sqrt{3}))$, the prime ideal $(3+\sqrt{11})$ of F is unramified in K. Thus, as an abelian extension of F, K ramifies only at (3). Hence K is the class field over F with conductor $\mathfrak{f}$ which corresponds to the subgroup $\{1, \mu, c_0^4, c_0^4\mu\}$.

By direct computations, we see that

$$(R_1^2-4)(R_2^2-4)=(\delta-\sqrt{\alpha+\beta})^2,$$
$$(R_2^2-4)(R_3^2-4)=(-\delta+\sqrt{\alpha-\beta})^2,$$
$$(R_3^2-4)(R_4^2-4)=(\delta+\sqrt{\alpha+\beta})^2,$$
$$(R_4^2-4)(R_1^2-4)=(\delta+\sqrt{\alpha-\beta})^2,$$

where we put

$$\alpha=81+24\sqrt{11}, \qquad \beta=40\sqrt{3}+12\sqrt{33}, \qquad \delta=\sqrt{3}(4+\sqrt{11}).$$

Thus, $K(X_1, X_2, X_3, X_4)=K(X_1)=F(X_1)$ (cf. (48)) where $X_i=X_{\mathfrak{f}}(c^{i-1}, G)$. Furthermore, $F(X_1)$ is an abelian extension of F. We note that the prime ideal $(3+\sqrt{11})$ of F splits in K into a product of two different prime ideals. There exists a prime ideal $\mathfrak{q}$ of K such that $(3+\sqrt{11})=\mathfrak{q}\mathfrak{q}^\tau$, where τ is a generator of Gal(K/F). Note that R_1, R_2, R_3, R_4 are mutually conjugate over F and that $R_1+R_2+R_3+R_4=8+3\sqrt{11}$ is odd. Hence we may assume that R_1 is prime to $\mathfrak{q}$. Then $R_1^2-4\equiv R_1^2$ mod. $\mathfrak{q}^4$ in $\mathfrak{O}_K$. Hence, as a quadratic extension of K, $F(X_1)=K(\sqrt{R_1^2-4})$ is unramified at $\mathfrak{q}$. Since K is abelian over F, $F(X_1)$ is unramified at $(3+\sqrt{11})$. The equality $(R_1^2-4)(R_2^2-4)(R_3^2-4)(R_4^2-4)=48\varepsilon^2$ now implies that $K(X_1)$ (as an abelian extension of F) ramifies only at (3). Since $[F(X_1), F]=8$, $F(X_1)$ is the class field of F with conductor $\mathfrak{f}$ which corresponds to the subgroup $\langle\mu\rangle$ of $H_F(\mathfrak{f})$.

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