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On the Measure-Preserving Flow on the Torus[1]

Toshiya SAITO

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1. Let us consider the one-parameter stationary flow S_t on the euclidean plane defined by the following system of differential equations

$$(1)\qquad \left\{\begin{aligned} \frac{dx}{dt}&=X(x,y),\\ \frac{dy}{dt}&=Y(x,y), \end{aligned}\right.$$

where X and Y are assumed to be real-valued functions having continuous first derivatives. If we moreover assume X and Y to be periodic functions of period 1 with respect to their arguments, they can be expanded into uniformly convergent Fourier series in the following way.

$$(2)\qquad \left\{\begin{aligned} X&=\sum a_{mn}e^{2\pi i(mx+ny)}\\ Y&=\sum b_{mn}e^{2\pi i(mx+ny)}. \end{aligned}\right.$$

Let us then suppose that our flow is measure-preserving, or, in other words, differential equations (1) admit an integral invariant

$$\iint dx\,dy.$$

In this case, we have

$$(3)\qquad \frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=0.$$

Then, by termwise differentiation, this relation can be written in the form

$$ma_{mn}+nb_{mn}=o,\quad m,\ n=o,\ \pm1,\ \pm2,\cdots.$$

Hence we can find a sequence $\{c_{mn}\}$ such that

$$a_{mn}=nc_{mn},\quad b_{mn}=-mc_{mn},\quad (m,\ n)\neq(o,\ o).$$

Consequently we can write

$$(2')\qquad \left\{\begin{aligned} X&=a_{00}+\sum nc_{mn}e^{2\pi i(mx+ny)},\\ Y&=b_{00}-\sum mc_{mn}e^{2\pi i(mx+ny)}. \end{aligned}\right.$$

If we identify all the points P_{mn}: $(x+m, y+n)$, $m, n=o, \pm 1, \pm 2, \cdots$ on the plane, differential equations (1) can be regarded as defining a measure-preserving flow on a torus Ω. The object of this paper is to establish a criterion for the ergodicity of this flow.

2. Let $P=(x_0, y_0)$ be a singular point of our flow (i.e. a point where $X=Y=o$). According to Poincaré, singular points of 2-dimensional flow are classified into four categories which are respectively called "*noeud*", "*foyer*", "*centre*", and "*col*".[2] He has also proved that if S_t is a stationary flow on a compact 2-dimensional manifold and N_1, N_2, N_3, N_4, are respectively the numbers of *noeuds*, *foyers*, *centres*, and *cols* of this flow, we have

$$N_1+N_2+N_3-N_4=2-2p$$

where p is the genus of the manifold.[3]

Now let us consider a small circle

$$C=\{Q=(x, y)\ ;\ \text{dist.}(Q, P)=\sqrt{(x-x_0)^2+(y-y_0)^2}=\delta>o\}$$

around P. If P is a *noeud* or a *foyer*, dist.$(S_tQ,P)\to o$ as $t\to+\infty$ or $t\to-\infty$. For example let us suppose that dist.$(S_tQ,P)\to o$ as $t\to+\infty$. Then there exists a finite positive number $T(Q)$ such that

$$\text{dist.}(S_tQ,P)<\delta, \text{ for } t>T(Q).$$

Since C is compact and $T(Q)$ is finite for every Q on C, there exists a finite positive number T such that $T>T(Q)$ for every Q on C. Then for $t>T$, dist.$(S_tQ,P)<\delta$ for every Q on C. Thus, if we denote by V the domain bounded by the circle C, $S_t V$ is entirely contained in V for $t>T$. But this contradicts with the assumption that our flow is measure-preserving. Hence P cannot be a *noeud* or a *foyer*. Evidently we obtain the same result when dist.$(S_tQ,P)\to o$ as $t\to-\infty$. So, in our case, N_1 and N_2 must be zero and

$$N_3-N_4=2-2p.$$

Moreover, Ω being a torus, $p=1$ and

$$N_3=N_4.$$

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1) The content of this work is roughly stated in Sûgaku, Vol. 1, No. 4, 1949 (in Japanese)

2) Poincaré, Sur les courbes définies par les équations différentielles, Chap. II and XI, Oeuvre t.I.

3) Poincaré, loc. cit. Chap. XIII.

Hence if our flow has singular points, there exists at least one *centre*-type singular point. As the neighborhood of a *centre* is filled out with periodic trajectories, such a flow is obviously non-ergodic. So we hereafter assume that X and Y have no common zeros.

We consider the real-valued function

$$H=a_{00}y-b_{00}x+\frac{1}{2\pi i}\sum c_{mn}e^{2\pi i\,(mx+ny)}\ .$$

Since Fourier series in this expression converges uniformly, we have, by termwise differentiation,

$$\frac{\partial H}{\partial x}=-Y,\qquad \frac{\partial H}{\partial y}=X.$$

Consequently

$$\frac{\partial H}{\partial x}X+\frac{\partial H}{\partial y}Y=o,$$

which shows that H is an integral of the differential equations (1). In general, H is not one-valued on Ω because

$$H(P_{mn})=H(P_{00})+a_{00}\,n-b_{00}m.$$

If $a_{00}=b_{00}=o$, however, H is a one-valued continuous integral of (1) on Ω. Therefore the flow is non-ergodic.

If $a_{00}\neq o$ and b_{00}/a_{00} is a rational number, we can write b_{00}/a_{00} in the form q/p where p and q are both integers. In this case, the function

$$e^{2\pi i\frac{p}{a_{00}}H}$$

is a one-valued continuous integral of our flow. Hence the flow is also non-ergodic.

If $a_{00}=o$, $b_{00}\neq o$, we can show by a similar discussion that the flow is also non-ergodic. Hence we have only to examine the case $a_{00}\neq o$, $b_{00}\neq o$, and a_{00}/b_{00} is an irrational number. For that purpose, we first prove the following theorem.

Theorem 1. *If $a_{00}\neq o$, $b_{00}\neq o$, and a_{00}/b_{00} is an irrational number, no periodic trajectory exists on Ω.*

Proof. If there exists a periodic trajectory C on Ω,

$$\int_C dH=o.$$

since H is an integral of (1). On the other hand, as X and Y are assumed to have no common zeros, no periodic trajectory can be homotopic to zero.[4] Hence no periodic trajectory can be homologous to zero as Ω is a torus. Therefore there must exist a pair of integers $(m, n) \neq (o, o)$ such that

$$ma_{00} + nb_{00} = o.$$

But this is contrary to the assumption of the theorem.

3. To simplify the statement, we here introduce the following definition.

Definition. A simple closed curve on Ω of finite length (in the sense of Lebesgue) is said to be a *circuit without contact* if

(1) for any two different points P, Q on this curve

$$\int_P^Q dH \neq o,$$

(2) for any point P of Ω, the trajectory starting from P at $t=o$ cuts this curve after a finite t-interval.

Lemma *If X and Y have no common zeros on Ω, we can construct a family of circuits without contact* $\{C(\alpha) ; o \leqq \alpha < 1\}$ *such that*

(a) *$C(\alpha)$ and $C(\beta)$ have no points in common if $\alpha \neq \beta$,*

(b) *for any point P on Ω, we can always find a circuit without contact of this family passing through P.*

Proof. If a circuit without contact $C(o)$ has been found, the desired family can easily be constructed. In fact, consider a moving point whose equation of motion is given by (1) where t is regarded as time. Let P_t be the position of such a moving point at t, that starts from P on $C(o)$ at $t=o$. Such a point returns to $C(o)$ after a finite lapse of time. Let $T(P)$ be this time interval. Then the set of points

$$C(\alpha) = \{P_t ; \ t = \alpha T(P), \ \alpha = \text{const.} \ P \in C(o)\}$$

forms a closed curve which is also a circuit without contact. Varying α from o to 1, we obtain a family of closed curves. We can easily show that this family of curves satisfies the properties (a) and (b).

Thus, to complete the proof, we have only to construct $C(o)$. For

4) Bendixon, Acta Math., 24, (1901), pp. 1–88. esp. Théorème III of Chap. I.

that purpose, however, we can adopt the method given by Siegel in his paper on the differential equations on the torus.[5)]

4. We now prove the following theorem which permits us to establish a criterion of ergodicity.

Theorem 2. *If* $a_{00} \neq o$, $b_{00} \neq o$, *and* a_{00}/b_{00} *is an irrational number, our flow is ergodic.*

Proof. Let P_0 be an arbitrary fixed point on $C(\alpha)$, and consider the function

$$\int_{P_0}^{P} dH$$

where $P \in C(\alpha)$ and integration is always made along $C(\alpha)$ and in the increasing sense of the function H. (H is monotone on $C(\alpha)$ because of the property (1) of the circuit without contact.) The above function is not uniquely determined since it admits the period

$$\int_{C(\alpha)} dH.$$

To avoid the ambiguity, we always take its minimum value. If we put

$$\mu(P) = \int_{P_0}^{P} dH \Big/ \int_{C(\alpha)} dH,$$

$\mu(P)$ is a Lebesgue measurable function on $C(\alpha)$. Let us introduce on $C(\alpha)$ a completely additive measure by putting

$$\mu(M) = \int_{M} d\mu(P),$$

for every Lebesgue measurable subset M. Evidently, from the definition of the circuit without contact, every set of positive Lebesgue measure has positive μ-measure.

Let P' be the first intersection point of the trajectory passing through $P \in C(\alpha)$ with $C(\alpha)$. We define an automorphism U of $C(\alpha)$ by putting

$$P' = U(P).$$

It is easy to see that for every interval I on $C(\alpha)$

$$\mu(I) = \mu(U(I))$$

since H is an integral of our flow. According to the complete additivity

5) Siegel, Annals of Math., 46, (1945), pp. 423–428.

of μ-measure, the above formula is also valid for every measurable set I. Hence for any $P \in C(u)$

$$\mu(I)=\gamma, \quad I=[P, U(P)]$$

where γ is a constant independent of P. Therefore, by use of μ-measure, the automorphism U is reduced to the rotation of a circle by the angle $2\pi\gamma$.

By Theorem 1, $U^n(P)$, $n=0, \pm 1, \pm 2, \cdots$ must be all different. Consequently γ must be an irrational number. In such a case, it is well known that the μ-measure of the U-invariant subset must be equal to $\mu(C(u))=1$ as long as it is positive.[6] Therefore the Lebesgue measure of such a set must be equal to the total length of $C(u)$ as long as it is positive.

If our flow leaves invariant a measurable subset A of positive (2-dimensional) Lebesgue measure, $A \cap C(u)$ is a measurable subset of $C(u)$ invariant under U. Hence from the fact stated above, its length must be equal to the total length of $C(u)$ as long as it is positive. So, by the theorem of Fubini, the area of A must be equal to that of Ω. This proves the ergodicity of the flow.

We have thus arrived at a criterion for ergodicity which can be stated as follows.

For the ergodicity of our flow, it is necessary and sufficient that

(1) *X and Y have no common zeros, and*

(2) $a_{00}=\int_0^1\int_0^1 X\,dx\,dy \neq o$, $b_{00}=\int_0^1\int_0^1 Y\,dx\,dy \neq o$, *and*

a_{00}/b_{00} is an irrational number.

Tokyo Institute of Technology

6) For example, see von Neumann, Annals of Math., 33, (1932), pp. 587-642.