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A Proof of Schauder's Theorem

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1. Introduction. The purpose of this note is to give a simple proof to the following theorem of J. Schauder[1]: A bounded linear operator T defined on a Banach space X is completely continuous if and only if the adjoint operator T^* of T defined on the conjugate space X^* of X is completely continuous. We shall give a formulation of Schauder's theorem (Theorem 2) in which X and X^* (and hence T and T^*) appear as a dual pair. (It is to be observed that X^* has no need to be the conjugate space of X in Theorem 2). Since T and T^* play equivelent roles in our formulation, the "if" part of the theorem is an equivalent proposition to the "only if" part.

Our proof of Schauder's theorem is based on the following well-known theorem of G. Arzelà: A uniformly bounded, equi-continuous family $F=\{f(x)\}$ of real-valued continuous functions $f(x)$ defined on a totally bounded metric space X is totally bounded with respect to the metric

$$d(f_1, f_2)=\sup_{x\in X} |f_1(x)-f_2(x)|. \tag{1}$$

We shall give a formulation of a special case of Arzela's theorem (Theorem 1) in which X and F play equivalent roles so that the total boundedness of X is also necessary for the total boundedness of F. The notion of totally bounded functions introduced in section 2 will be helpful in making arguments simpler.

2. Totally bounded functions. Let $X=\{x\}$, $Y=\{y\}$ be two sets. Let $f(x, y)$ be a bounded real-valued function defined for all $x\in X$ and for all $y\in Y$.

Lemma 1. The following three conditions are mutually equivalent: (i) *for any* $\varepsilon>0$ *there exists a decomposition* $X=\bigcup_{i=1}^{m} A_i$ *of* X *into a finite number of subsets* A_i, $i=1, \cdots, m$, *such that*

$$|f(x_1, y)-f(x_2, y)|<\varepsilon \tag{2}$$

for all $x_1, x_2\in A_i$ (*same* i), $i=1, \cdots, m$, *and for all* $y\in Y$. (ii) *for any* $\varepsilon>0$ *there exists a decomposition* $Y=\bigcup_{j=1}^{n} B_j$ *of* Y *into a finite number of subsets* B_j, $j=1, \cdots, n$, *such that*

$$|f(x, y_1)-f(x, y_2)|<\varepsilon \tag{3}$$

for all $x\in X$ *and for all* $y_1, y_2\in B_j$ (*same* j), $j=1, \cdots, n$. (iii) *for any* $\varepsilon>0$ *there exist decompositions* $X=\bigcup_{i=1}^{m} A^i$, $Y=\bigcup_{j=1}^{n} B_j$ *of* X *and* Y *into a finite* *number of subsets A_i, $i=1, \cdots, m$, and B_j, $j=1, \cdots, n$, such that*

$$|f(x_1, y_1)-f(x_2, y_2)|<\varepsilon \tag{4}$$

for all x_1, $x_2 \in A_i$ (same i), $i=1, \cdots, m$, and for all $y_1, y_2 \in B_j$ (same j), $j=1, \cdots, n$.

The proof of this lemma is easy and so omitted. In case one (and hence all) of the conditions of Lemma 1 is satisfied, we say that the function $f(x, y)$ is *totally bounded* on X and Y.

3. Arzela's theorem. Let $X=\{x\}$, $Y=\{y\}$ and $f(x, y)$ be the same as in section 2. For any x_1, $x_2 \in X$, let us put

$$d^{(1)}(x_1, x_2; f)=\sup_{y\in Y}|f(x_1, y)-f(x_2, y)|. \tag{5}$$

Then $d^{(1)}(x_1, x_2; f)$ is a quasi-metric defined on X (i.e. $d^{(1)}(x_1, x_2; f)$ satisfies all axioms of a metric except possibly the separation axiom: $d^{(1)}(x_1, x_2; f)>0$ if $x_1 \neq x_2$). X is called totally bounded with respect to $d^{(1)}(x_1, x_2; f)$ if for any $\varepsilon>0$ there exists a decomposition $X=U_{i=1}^{m} A_i$ of X into a finite number of subsets A_i, $i=1, \cdots, m$, such that $d^{(1)}(x_1, x_2; f)<\varepsilon$ for all x_1, $x_2 \in A_i$ (same i), $i=1, \cdots, m$. Similarly, if we put for any $y_1, y_2 \in Y$

$$d^{(2)}(y_1, y_2; f)=\sup_{x\in X}|f(x, y_1)-f(x, y_2)|, \tag{6}$$

then $d^{(2)}(y_1, y_2; f)$ is a quasi-metric on Y. The total boundedness of Y with respect to the quasi-metric $d^{(2)}(y_1, y_2; f)$ is defined similarly.

Theorem 1. X is totally bounded with respect to $d^{(1)}(x_1, x_2; f)$ if and only if Y is totally bounded with respect to $d^{(2)}(y_1, y_2 : f)$.

Proof. Theorem 1 follows immediately from Lemma 1 if we observe that the total boundedness of X with respect to $d^{(1)}(x_1, x_2; f)$ is equivalent to the condition (i) of Lemma 1 and that the total boundedness of Y with respect to $d^{(2)}((y_1, y_2; f)$ is equivalent to the condition (ii) of Lemma 1.

Remark. Theorem 1 is symmetric in X and Y. Hence the "if" part and the "only if" part of Theorem 1 are essentially the same propositions. Further, it is easy to see that this proposition is a consequence of Arzelà's theorem. In fact, if we put $f_y(x)=f(x, y)$, then $F=\{f_y(x) \mid y\in Y\}$ is a uniformly bounded, equi-continuous family of continuous functions $f_y(x)$ defined on a set X with a quasi-metric $d^{(1)}(x_1, x_2; f)$; and we have

$$\begin{aligned} d(f_{y_1}, f_{y_2}) &=\sup_{x\in X}|f_{y_1}(x)-f_{y_2}(x)| \\ &=\sup_{x\in X}|f(x, y_1)-f(x, y_2)| \\ &=d^{(2)}(y_1, y_2; f). \end{aligned} \tag{7}$$

4. Normed pair. Let $X=\{x\}$, $Y=\{y\}$ be two normed linear spaces with $\|x\|_1$, $\|y\|_2$ as norms. Assume that there is a real-valued bilinear functional (x, y) defined for all $x \in X$ and for all $y \in Y$ such that

(8) $$\|x\|_1 = \sup_{\|y\|_2 \leq 1} |(x, y)|$$

(9) $$\|y\|_2 = \sup_{\|x\|_1 \leq 1} |(x, y)|$$

for all $x \in X$ and for all $y \in Y$. X and Y are called a *normed pair* with respect to the *inner product* (x, y).

Let X, Y be a normed pair with respect to the inner product (x, y). Let T, T^* be two bounded linear operators defined on X, Y respectively. T and T^* are called an *adjoint pair* if

(10) $$(Tx, y) = (x, T^*y)$$

for all $x \in X$ and for all $y \in Y$. It is easy to see that

(11) $$\begin{aligned} \|T\|_1 &= \sup_{\|x\|_1 \leq 1} \|Tx\|_1 \\ &= \sup_{\|x\|_1 \leq 1} \sup_{\|y\|_2 \leq 1} |(Tx, y)| \\ &= \sup_{\|y\|_2 \leq 1} \sup_{\|x\|_1 \leq 1} |(x, T^*y)| \\ &= \sup_{\|y\|_2 \leq 1} \|T^*y\|_2 = \|T^*\|_2. \end{aligned}$$

An example of a normed pair is given by a Banach space X and its conjugate space X^* if we define the inner product (x, x^*) as the value of a bounded linear functional x^* at a point x. Similarly, the conjugate space X^* of X and the second conjugate space X^{**} of X (i.e. the conjugate space of X^*) form a normed pair. (It is also easy to see that, if we consider X as a linear subspace of X^{**} and if Y is a linear subspace of X^{**} which contains X, then X^* and Y form a normed pair). Further, a bounded linear operator T (defined on X) and the adjoint operator T^* (defined on X^*) form an adjoint pair. Similarly, the adjoint operator T^* of T and the second adjoint operator T^{**} of T (defined on X^{**}) form an adjoint pair.

From Theorem 1 follows immediately:

Lemma 2. Let X, Y be a normed pair with respect to the inner product (x, y), and let A, B be two bounded subsets of X, Y respectively. Then the following three conditions are mutually equivalent:

(i) A *is totally bounded with respect to the quasi-metric*

(12) $$d^{(1)}(x_1, x_2; B) = \sup_{y \in B} |(x_1, y) - (x_2, y)|$$

(ii) B *is totally bounded with respect to the quasi-metric*

(13) $$d^{(2)}(y_1, y_2; A) = \sup_{x \in A} |(x, y_1) - (x, y_2)|$$

(iii) *the inner product (x, y) is totally bounded on A and B,* If we consider the special case when B is the unit sphere S_Y of Y, then we obtain:

Lemma 3. Let X, Y be a normed pair with respect to the inner product (x, y), and let A be a bounded subset of X. Then the following three conditions are mutually equivalent: (i) *A is totally bounded with respect to the metric $d^{(1)}(x_1, x_2)=\|x_1-x_2\|_1$.* (ii) *the unit sphere S_Y of Y is totally bounded with respect to the quasi-metric $d^{(2)}(y_1, y_2; A)$ (or equivalently, from any sequence $\{y_n \mid n=1, 2, \cdots\}$ of elements of S_Y it is possible to find a subsequence $\{y_{n_k} \mid k=1, 2, \cdots\}$ for which the inner product (x, y_{n_k}) converges uniformly on A).* (iii) *the inner product (x, y) is totally bounded on A and S_Y.*

5. Schauder's theorem. A bounded linear operator T defined on a normed linear space X is called *completely continuous* on X if the image $T(S_X)$ of the unit sphere S_X of X by T is totally bounded with respect to the metric $d^{(1)}(x_1, x_2)=\|x_1-x_2\|_1$.

Theorem 2. Let X, Y be a normed pair, and let T, T^* be an adjoint pair of bounded linear operators defined on X, Y, respectively. Then T is completely continuous on X if and only if T^* is completely continuous on Y.

Proof. Let S_X, S_Y be the unit spheres of X, Y respectively; and consider the function

$$f(x, y)=(Tx, y)=(x, T^*y) \tag{14}$$

defined for all $x \in S_X$ and for all $y \in S_Y$. Theorem 2 follows immediately from Lemma 3 if we observe that the following five conditions are mutually equivalent: (i) $T(S_X)$ is totally bounded with respect to the metric $d^{(1)}(x_1, x_2)=\|x_1-x_2\|_1$; (ii) the inner product (x, y) is totally bounded on $T(S_X)$ and S_Y; (iii) the function $f(x, y)$ is totally bounded on S_X and S_Y; (iv) the inner product (x, y) is totally bounded on S_X and $T^*(S_Y)$; (v) $T^*(S_Y)$ is totally bounded with respect to the metric $d^{(2)}(y_1, y_2)=\|y_1-y_2\|_2$.

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Bibliography

1) J. Schauder, Über lineare vollstetige Operatoren, Studia Math., **2** (1930), 183–196.