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On projective normality of abelian varieties II

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This is a continuation of my paper [7]. In the previous paper, we proved: the model of an abelian variety X projectively embedded by means of $\Gamma(L^a)$ with an ample invertible sheaf L of separable type and $a \geqq 3$ is projectively normal, if the ground field is not of characteristic $p=2, 3, 5$. In the present paper, we shall generalize the methods in the above paper to every characteristic case, and prove the above statement affirmatively without exceptional characteristic cases.

Section 0 will be devoted to showing that the facts, stated in § 0 of [7], are true even in the inseparable case. In Section 1, we shall discuss the Koizumi's rank theorem without any restriction on the characteristic, whose proof almost follows the previous manner. In the last section, we shall give a proof of our main result in every characteristic case.

The theta structure theorem in a positive characteristic case stated in Section 0 is due to Mumford and appears in [6], but he has never published his proof anywhere. He is kind enough to send me his note including a sketch of the proof, and gave me the permission of reconstructing his proof in the appendix of this paper. The author expresses his hearty thanks to Professor D. Mumford for his generosity.

We follow the previous paper [7] in notation in this paper, for example, X, Y denote abelian varieties of dimension g over an algebraically closed field k of characteristic p, etc.

0. We shall freely use the fundamental facts on theta-groups given by D. Mumford [4]. Throughout the paper, we denote by B any k-algebra.

In general, for every morphism $f: S \rightarrow T$ of ringed spaces and for a sheaf $\mathcal{F}$ of $\mathcal{O}_T$-modules, we can define a natural di-homomorphism

$$f^*: H^p(T, \mathcal{F}) \longrightarrow H^p(S, f^*\mathcal{F});$$

or abbreviating the terminology,

$$f^*: H^p(\mathcal{F}) \longrightarrow H^p(f^*\mathcal{F}),$$

for every p. (Cf. E. G. A. III, Chap. 0, 12.1.3.)

PROPOSITION 0.1. *Let $f: X\to Y$ be an isogeny with $K=\ker f$. Let L and M be non-degenerate invertible sheaves on X and Y such that there exists an isomorphism $\alpha: f^*M \xrightarrow{\sim} L$. Then L and M have the same index, which we say i. Let K^* be the level subgroup corresponding to the isomorphism α; $\mathcal{G}(M)^*$ be the centralizer of K^* in $\mathcal{G}(L)$; and $\bar{f}: \mathcal{G}(M)^*\to\mathcal{G}(M)$ be the canonical map. Then for any B-valued point z of $\mathcal{G}(M)^*$, we have a commutative diagram:*

$$\begin{array}{ccc} H^i(M)\otimes B & \xrightarrow{f^*} & H^i(L)\otimes B \\ {\scriptstyle U_{\bar{f}(z)}}\downarrow & & \downarrow{\scriptstyle U_z} \\ H^i(M)\otimes B & \xrightarrow{f^*} & H^i(L)\otimes B\,. \end{array}$$

In fact, we can prove this in the same way as in the separable case.

Moreover, the next two propositions can also be seen by considering B-valued points.

PROPOSITION 0.2. *Let L and M be non-degenerate invertible sheaves on X and Y. Let i_L and i_M be the canonical inclusions of $\boldsymbol{G}_m$ into $\mathcal{G}(L)$ and $\mathcal{G}(M)$ respectively; ι denote the inverse morphism of $\boldsymbol{G}_m$; and D be the image of the morphism $(i_L, i_M\circ\iota): \boldsymbol{G}_m\to\mathcal{G}(L)\times\mathcal{G}(M)$. Then we have a canonical isomorphism:*

$$\mathcal{G}(p_1^*L\otimes p_2^*M) \xrightarrow{\sim} \mathcal{G}(L)\times\mathcal{G}(M)/D\,.$$

REMARK 0.3. Frequently we identify every subgroup H (resp. H') of $\mathcal{G}(L)$ (resp. $\mathcal{G}(M)$) with the canonical image of $H\times\{1\}$ (resp. $\{1\}\times H'$) in $\mathcal{G}(p_1^*L\otimes p_2^*M)$.

PROPOSITION 0.4. *Let L and M be non-degenerate invertible sheaves on X and Y of indices p and q respectively. Then for any B-valued point $z=(z_1, z_2)$ of $\mathcal{G}(L)\times\mathcal{G}(M)$, if we denote by $\bar{z}$ its canonical image in $\mathcal{G}(p_1^*L\otimes p_2^*M)$, we have a commutative diagram:*

$$\begin{array}{ccc} H^p(L)\otimes H^q(M)\otimes B & \xrightarrow[\text{Künneth decom.}]{\sim} & H^{p+q}(p_1^*L\otimes p_2^*M)\otimes B \\ {\scriptstyle U_{z_1}\otimes U_{z_2}}\downarrow & & \downarrow{\scriptstyle U_{\bar{z}}} \\ H^p(L)\otimes H^q(M)\otimes B & \xrightarrow[\text{Künneth decom.}]{\sim} & H^{p+q}(p_1^*L\otimes p_2^*M)\otimes B\,. \end{array}$$

The next theorem, which is mentioned by D. Mumford in [**6**], gives a foundation of our argument.

THETA-STRUCTURE THEOREM (D. Mumford). *If L is a non-degenerate invertible sheaf on X of index i, then the theta-group scheme $\mathcal{G}(L)$ is a non-degenerate extension and $H^i(L)$ is its unique irreducible representation, with $\boldsymbol{G}_m$ acting naturally.*

PROPOSITION 0.5. *Let L be a non-degenerate invertible sheaf on X of index* *i with $|\chi(L)|=1$. Let m, n be two positive integers such that $(m, n)=1$, and let $j=j(L^{mn})$. Under these situations, if W is a $j^{-1}(X_n)$-stable non-trivial subspace in $H^i(L^{mn})$, then we have*

$$\dim W \geqq n^g .$$

PROOF. Let H be a maximal isotropic subgroup of $K(L^m)=X_m$, and $\pi: X \to X/H=Y$ be the canonical projection. Then there exists an invertible sheaf M on Y such that $\pi^*M \cong L^{mn}$, i.e., the diagram:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\phi_{L^{mn}}"'] & Y \arrow[d, "\phi_M"] \\ \hat{X} & \hat{Y} \arrow[l, "\hat{\pi}"'] \end{tikzcd}$$

commutes. Since $(m, n)=1$, we have that

$$K(M)=Y_n \quad \text{and} \quad \pi^{-1}(Y_n)=X_n \oplus H .$$

Therefore we have

$$\mathcal{G}(M)^*=j^{-1}(\pi^{-1}(Y_n))=j^{-1}(X_n)+j^{-1}(H) \supset H^* ,$$

i.e.,

$$\mathcal{G}(M) \cong \mathcal{G}(M)^*/H^* \cong j^{-1}(X_n) .$$

Hence by the theta-structure theorem, we obtain our assertion. Q.E.D.

1. First of all we notice that for any integers a, b, if we define a homomorphism $\xi: X\times X \to X\times X$ by $(x, y) \mapsto (x-by, x+ay)$, we have an isomorphism

$$\text{(0)} \qquad \xi^*(p_1^*(L^a \otimes P_\alpha) \otimes p_2^*(L^b \otimes P_\beta)) \overset{\phi}{\cong} p_1^*(L^{a+b} \otimes P_{\alpha+\beta}) \otimes p_2^*(L^{ab(a+b)} \otimes P_{a\beta-b\alpha})$$

for any symmetric invertible sheaf L on X and any $\alpha, \beta \in \hat{X}$. (Cf. [7], Prop. 1.2.)

From now on let L be a principal symmetric invertible sheaf on X, and a, b are positive integers such that $(a, b)=1$. From the above isomorphism ϕ, we have the injection

$$\xi^*: \Gamma(L^a \otimes P_\alpha) \otimes \Gamma(L^b \otimes P_\beta) \longrightarrow \Gamma(L^{a+b} \otimes P_{\alpha+\beta}) \otimes \Gamma(L^{ab(a+b)} \otimes P_{a\beta-b\alpha}) .$$

Now we fix non-zero sections u and v in $\Gamma(L^a \otimes P_\alpha)$ and $\Gamma(L^b \otimes P_\beta)$ respectively. Let $\{s_1, \cdots, s_l\}$ and $\{t_1, \cdots, t_m\}$ be bases of $\Gamma(L^{a+b} \otimes P_{\alpha+\beta})$ and $\Gamma(L^{ab(a+b)} \otimes P_{a\beta-b\alpha})$, where $l=(a+b)^g$ and $m=\{ab(a+b)\}^g$. Then we obtain an equation

$$(*) \qquad \xi^*(u \otimes v)=\sum_{\substack{1\leqq u\leqq l \\ 1\leqq \nu\leqq m}} c_{\mu\nu} s_\mu \otimes t_\nu \qquad \text{for some} \quad c_{\mu\nu} \in k .$$

The isomorphism ϕ defines a lifting of the group $K=\ker \xi$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_m & \longrightarrow & \mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})) & \xrightarrow{j} & X_{a+b}\times X_{ab(a+b)} & \longrightarrow & 0\,. \\ & & & & \cup & & \cup & & \\ & & & & K^* & \xrightarrow{\quad\sim\quad} & K & & \end{array}$$

In the same way as in the separable case, we obtain

LEMMA 1.1. *We have* $\operatorname{rank}(c_{\mu\nu})=l$, *i. e.*, $=(a+b)^g$ *for* $c_{\mu\nu}$*'s in* $(*)$. *Therefore, after choosing suitable independent sections* $t_1, \cdots, t_l$ *of* $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$, *we can express* $\xi^*(u\otimes v)$ *in the form:*

$$\xi^*(u\otimes v)=\sum_{i=1}^{l} s_i\otimes t_i\,, \tag{1}$$

for any given basis $\{s_1, \cdots, s_l\}$ *of* $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$. (Cf. [7], Prop. 1.3, (0).)

In the case of $p\nmid a+b$, we have more detail expression of $\xi^*(u\otimes v)$. In fact, in the case ξ becomes separable and we have a Göpel decomposition of $K(L^{a+b}\otimes P_{\alpha+\beta})$:

$$K(L^{a+b}\otimes P_{\alpha+\beta})=X_{a+b}=H(a+b)_1\oplus H(a+b)_2\,,$$

i. e., a decomposition of $K(L^{a+b}\otimes P_{\alpha+\beta})$ into maximal isotropic subgroups $H(a+b)_1$ and $H(a+b)_2$. Let

$$\pi:\ \mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})\times\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})\rightarrow\mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))$$

be the canonical map given in Proposition 0.2. If we put

$$H(a+b)_i^\Delta=\{(by, y)\mid y\in H(a+b)_i\}$$

for each $i=1, 2$, it is a subgroup of $K=\{(by, y)\mid y\in X_{a+b}\}$. Therefore it is automatically lifted up to a subgroup $H(a+b)_i^{\Delta *}$ of K^*. If we take a level subgroup $H(a+b)_i^*$ in $\mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})$ of $H(a+b)_i$, there exists a level subgroup $H'(a+b)_i^*$ in $\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ of $H(a+b)_i$ and the subgroup $H(a+b)_i^{\Delta *}$ defines an isomorphism $'$: $H(a+b)_i^*\rightarrow H'(a+b)_i^*$ by the relation $\pi(\lambda, \lambda')\in H(a+b)_i^{\Delta *}$ for every $\lambda\in H(a+b)_i^*$. Let $s\in\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$ be a non-zero $H(a+b)_2^*$-invariant section. Then $\{U_\lambda s\}_{\lambda\in H(a+b)_1^*}$ becomes a basis of $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$. Under these notation we obtain

LEMMA 1.2. *In the case of* $p\nmid a+b$, *there exists an* $H'(a+b)_2^*$*-invariant element* θ *in* $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ *such that*

$$\xi^*(u\otimes v)=\sum_{\lambda\in H(a+b)_1^*}U_\lambda s\otimes U_{\lambda'}\theta\,. \tag{2}$$

PROOF. From Lemma 1.1, we have the following expression of $\xi^*(u\otimes v)$:

$$\xi^*(u\otimes v)=\sum_{\lambda\in H(a+b)_1^*}U_\lambda s\otimes\theta_\lambda\,,$$

where θ_λ's are linearly independent sections of $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$. Since $\xi^*(u\otimes v)$ is K^*-invariant, for any $z\in H(a+b)_i^*$, we obtain an equality

$$\xi^*(u\otimes v)=\sum_{\lambda\in H(a+b)_1^*}U_\lambda s\otimes\theta_\lambda=\sum_{\lambda\in H(a+b)_1^*}U_z U_\lambda s\otimes U_{z'}\theta_\lambda .$$

If we take z from $H(a+b)_1^*$, $U_{z'}\theta_\lambda=\theta_{\lambda+z}$, i.e., $U_{z'}\theta_0=\theta_z$. If we take z from $H(a+b)_2^*$, $U_zU_\lambda s=e^{L^{a+b}}(j'(z), j'(\lambda))U_\lambda U_z s=e^{L^{a+b}}(j'(z), j'(\lambda))U_\lambda s$, where $j'=j(L^{a+b}\otimes P_{\alpha+\beta})$. Therefore from the above equation, we obtain the equalities:

$$\theta_\lambda=e^{L^{a+b}}(j'(z), j'(\lambda))U_{z'}\theta_\lambda \qquad \text{for any} \quad \lambda\in H(a+b)_1^*.$$

In particular, we have

$$\theta_0=U_{z'}\theta_0 \qquad \text{for any} \quad z\in H(a+b)_2^*.$$

Hence we obtain the requiring expression

$$\xi^*(u\otimes v)=\sum_{\lambda\in H(a+b)_1^*}U_\lambda s\otimes U_{\lambda'}\theta_0 . \qquad \text{Q.E.D.}$$

Returning to general case, let $H(a)$ and $H(b)$ be any maximal isotropic subgroup schemes of $K(L^a)$ and $K(L^b)$, and we denote by $H(a)^*$ and $H(b)^*$ the level subgroups in $\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$. In the following arguments, we identify the subgroups $H(a)^*$ and $H(b)^*$ of $\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ with subgroups of

$$\mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))$$

by the way in Remark 0.3. Moreover, we put $H(ab)^*=H(a)^*+H(b)^*$. If we denote by $\mathcal{G}^*$ the centralizer of K^* in

$$\mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})),$$

we have easily

$$\mathcal{G}^*\supset H(ab)^* .$$

Since $(ab, a+b)=1$, we have furthermore

$$H(a)^*\cap K^*=1 \qquad \text{and} \qquad H(b)^*\cap K^*=1$$

scheme-theoretically. Therefore the subgroups $H(a)^*$ and $H(b)^*$ are canonically isomorphic to subgroups of

$$\mathcal{G}^*/K^*\cong\mathcal{G}(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))\cong\mathcal{G}(L^a\otimes P_\alpha)\times\mathcal{G}(L^b\otimes P_\beta)/D ,$$

which we denote by $\bar{H}(a)^*$ and $\bar{H}(b)^*$ respectively. These subgroups can be identified with subgroups of $\mathcal{G}(L^a\otimes P_\alpha)$ and $\mathcal{G}(L^b\otimes P_\beta)$ by the way of Remark 0.3, because $(a, b)=1$. For any element $z\in H(a)^*\cup H(b)^*$, we denote by $\bar{z}$ its canonical image in $\bar{H}(a)^*\cup\bar{H}(b)^*$. Now we put

$$W_{\circ}=\langle t_1, \cdots, t_l\rangle \subset \Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}),$$

where $t_1, \cdots, t_l$ are elements given in the equality (1). Under these notation, we have the key proposition in the same way as in the separable case.

PROPOSITION 1.3. *We put* $j''=j(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$.

(0) W_0 *is* $j''^{-1}(X_{a+b})$*-stable subspace of* $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ *of* $\dim l$.

(i) *If* v *is* $\bar{H}(b)^*$*-invariant,* W_0 *is not only* $j''^{-1}(X_{a+b})$*-stable, but* $H(b)^*$*-invariant.*

(ii) *If* $p \nmid a$ *and* $\{U_{\bar{\lambda}}u\}_{\lambda\in H(a)^*}$ *is a basis of* $\Gamma(L^a\otimes P_\alpha)$*; and we put*

$$W=\sum_{\lambda\in H(a)^*} U_\lambda W_0 \qquad in \quad \Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}),$$

then W *is the direct sum of* $U_\lambda W_0$*'s.*

(iii) *If* $p \nmid ab$*, and* $\{U_{\bar{\lambda}}u\}_{\lambda\in H(a)^*}$ *(resp.* $\{U_{\bar{\mu}}v\}_{\mu\in H(b)^*}$*) be a basis of* $\Gamma(L^a\otimes P_\alpha)$ *(resp.* $\Gamma(L^b\otimes P_\beta)$*), then we have*

$$\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})=\bigoplus_{(\lambda,\mu)\in H(a)^*\times H(b)^*} U_{\lambda+\mu}W_0\,.$$

THEOREM 1.4. (The rank theorem; cf. [2], Th. 2.5 and [7], Th. 1.4.) *Let* Y *be any abelian variety of* $\dim g$ *and* M *be any principal invertible sheaf on* Y*. Let* c, d *be positive integers such that* $b=|c-d|>0$, $p\nmid c$ *and* $(c, d)=1$*. Let* $K(M^c)=H(c)_1\oplus H(c)_2$ *be a Göpel decomposition. Then* $H(c)_i$ $(i=1.2)$ *are lifted up to level subgroups:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{G}_m & \longrightarrow & \mathcal{G}(M^{cd}) & \longrightarrow & Y_{cd} & \longrightarrow & 0\,. \\ & & & & \cup & & \cup & & \\ & & & & H(c)_i^{**} & \cong & H(c)_i & & \end{array}$$

Let $\{\theta_1, \cdots, \theta_l\}$ *be a basis of* $\Gamma(M^{cd})^{H(c)_2^{**}}$*, where* $l=d^g$*. Then we have the equality*

$$\operatorname{rank}(U_\lambda\theta_i(y))_{(\lambda,i)\in H(c)_1^*\times\{1,\cdots,l\}}=\operatorname{Min}(c^g, d^g)$$

for any closed point $y\in Y$.

PROOF. Without loss of generality, we may assume $y=0$. For the assumption is not essential in the following proof. In the same way as in the proof of Lemma 1.1 in [7], there exists an abelian variety X, an isogeny $\pi: X\to Y$ and a principal symmetric invertible sheaf L on X such that

$$\pi^*(M^{cd})\cong L^{bcd}\otimes P_\gamma \qquad \text{for some} \quad \gamma\in\hat{X},$$

and $\ker\pi$ is a maximal isotropic subgroup scheme $H(b)$ of $K(L^b)=X_b$. The above isomorphism defines a lifting of the group $H(b)$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_m & \longrightarrow & \mathcal{G}(L^{bcd}\otimes P_\gamma) & \xrightarrow{j''} & X_{bcd} & \longrightarrow & 0\,. \\ & & & & \cup & & \cup & & \\ & & & & H(b)^* & \cong & H(b) & & \end{array}$$

Moreover if we denote by $\mathcal{G}(M^{cd})^*$ the centralizer of $H(b)^*$, we have the canonical isomorphism $\mathcal{G}(M^{cd})\cong\mathcal{G}(M^{cd})^*/H(b)^*$. Since $H(b)^*$ is contained in the center of $\mathcal{G}(M^{cd})^*$ and $(cd, b)=1$, the given level subgroup $H(c)_i^{**}$ in $\mathcal{G}(M^{cd})$ is naturally isomorphic to a subgroup $H(c)_i^*$ of $\mathcal{G}(M^{cd})^*$ for each $i=1, 2$. By the map π^*, $\Gamma(M^{cd})$ is isomorphic to $H(b)^*$-invariant subspace $\Gamma(L^{bcd}\otimes P_\gamma)^{H(b)^*}$, and the isomorphism is compatible with the actions of $\mathcal{G}(M^{cd})$ and $\mathcal{G}(M^{cd})^*$. There fore we have been able to reduce our assertion to the equality

$$\operatorname{rank}\,(U_\lambda\theta_i(0))_{(\lambda,i)\in H(c)_1^*\times\{1,\cdots,l\}}=\operatorname{Min}\,(c^g, d^g)$$

for a basis $\{\theta_1, \cdots, \theta_l\}$ of $\Gamma(L^{bcd}\otimes P_\gamma)^{H(b)^*+H(c)_2^*}$.

Here we put $\operatorname{Min}(c, d)=a$, and $\operatorname{Max}(c, d)=a+b$. Let $\alpha, \beta\in\hat{X}$ be a solution of the equation $a\beta-b\alpha=\gamma$, and $\xi: X\times X\to X\times X$ be the homomorphism defined by $(x, y)\mapsto(x-by, x+ay)$. Under the notation in Proposition 1.3, let v be a non-zero $\bar{H}(b)^*$-invariant section of $\Gamma(L^b\otimes P_\beta)$.

First of all, we consider the case of $c<d$, i.e., $c=a$ and $d=a+b$. Under the notation in the same proposition, we take a non-zero $\bar{H}(a)_2^*$-invariant section u in $\Gamma(L^a\otimes P_\alpha)$. Then we have

$$\xi^*(u\otimes v)\in\Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes W_0\subset\Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_\gamma)\,,$$

where W_0 is $j''^{-1}(X_{a+b})$-stable and invariant under the action of $H(b)^*$ and $H(a)_2^*$. Moreover, since $\dim W_0=l$, we obtain

$$W_0=\Gamma(L^{ab(a+b)}\otimes P_\gamma)^{H(b)^*+H(a)_2^*}\,,$$

i.e., $\{\theta_1, \cdots, \theta_l\}$ is a basis of W_0. By Lemma 1.1, for suitable basis $\{s_1, \cdots, s_l\}$ of $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$, we have

$$\xi^*(u\otimes v)=\sum_{i=1}^{l}s_i\otimes\theta_i\,.$$

On the other hand, we have a commutative diagram:

$$\begin{array}{ccc} \Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta) & \xrightarrow{\xi^*} & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_\gamma) \\ \Big\downarrow U_{\bar{z}} & & \Big\downarrow U_z \\ \Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta) & \xrightarrow{\xi^*} & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_\gamma) \end{array}$$

for each $z\in H(a)_1^*\cup H(b)^*\subset\mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_\gamma))$. Hence, applying this diagram to the above equality, we obtain

$$\xi^*(U_{\bar{\lambda}}u\otimes v)=\sum_{i=1}^{l}s_i\otimes U_\lambda\theta_i$$

for every $\lambda\in H(a)_1^*$. Moreover the commutative diagram:

$$\begin{tikzcd} X\times\{0\} \arrow[d, "\iota"'] \arrow[dr, "\Delta"] & \\ X\times X \arrow[r, "\xi"] & X\times X \end{tikzcd}$$

leads us to the equality

$$(\Delta^*(U_{\bar{\lambda}}u\otimes v))_{\lambda\in H(a)_1^*}=(s_i)_{1\leq i\leq l}(U_\lambda\theta_i(0))_{(i,\lambda)\in\{1,\cdots,l\}\times H(a)_1^*},$$

where $(\ )_{\lambda\in H(a)_1^*}$ and $(\ )_{1\leq i\leq l}$ mean row vectors. Since the components of the left side of this equality are linearly independent, we obtain the requiring equality:

$$\operatorname{rank}(U_\lambda\theta_i(0))_{(i,\lambda)\in\{1,\cdots,l\}\times H(a)_1^*}=a^g.$$

Secondly, we consider the case of $d<c$, i.e., $a=d$ and $c=a+b$. In this case, we take the given $H(c)_i^*$ as $H'(a+b)_i^*$ in Lemma 1.2. Let $\{u_1,\cdots,u_l\}$ be a basis of $\Gamma(L^a\otimes P_\alpha)$. Then by the same lemma and Proposition 1.3, there exists a $H'(a+b)_2^*+H(b)^*$-invariant section θ_i' in $\Gamma(L^{ab(a+b)}\otimes P_\gamma)$ such that

$$\xi^*(u_i\otimes v)=\sum_{\lambda\in H(a+b)_1^*}U_\lambda s\otimes U_{\lambda'}\theta_i'$$

for each $i=1,\cdots,l$. Therefore by the same argument in the first case, we obtain the equality

$$\operatorname{rank}(U_\lambda\theta_i'(0))_{(\lambda,i)\in H(a+b)_1^*\times\{1,\cdots,l\}}=l.$$

Moreover, since $\theta_1',\cdots,\theta_l'$ are linearly independent, they form a basis of

$$\Gamma(L^{ab(a+b)}\otimes P_\gamma)^{H(b)^*+H(a+b)_2^*}.$$

Hence we have a linear relation:

$$(\theta_1,\cdots,\theta_l)=(\theta_1',\cdots,\theta_l')P,$$

where P is a non-singular matrix of size l, i.e.,

$$(U_\lambda\theta_i)_{(\lambda,i)\in H(a+b)_1^*\times\{1,\cdots,l\}}=(U_\lambda\theta_i')_{(\lambda,i)\in H(a+b)_1^*\times\{1,\cdots,l\}}P.$$

Therefore from the above equality, we obtain our assertion. Q.E.D.

2. In the section, a, b, d denote positive integers such that $(ad, a+b)=1$, $abd>a+b$ and $p \nmid d$. As in Section 1, we define a homomorphism $\xi: X\times X \to X\times X$ by $(x, y)\mapsto(x-by, x+ay)$.

PROPOSITION 2.1. (Cf. [2], Prop. 3.2, and [7], Prop. 2.1.) *Let L be a symmetric principal invertible sheaf on X, and α, β be two closed points on $\hat{X}$. Let $\hat{H}(d)$ be a maximal isotropic direct summand of $K((\phi_L^{-1})^*L^d)$. Then*

$$\sum_{\gamma\in\hat{H}(d)} \Gamma(L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(L^b\otimes P_{\beta+\gamma}) \longrightarrow \Gamma(L^{a+b}\otimes P_{\alpha+\beta})$$

is surjective.

PROOF. Let $\hat{Y}=\hat{X}/\hat{H}(d)$; $\hat{\pi}: \hat{X}\to\hat{Y}$ be the canonical map; and let $\pi: Y\to X$ be the dualized map of $\hat{\pi}$. Then there exists a principal invertible sheaf M on Y such that $\pi^*L\cong M^d$ and ker π is a maximal isotropic direct summand of $K(M^d)$, which we put $H(d)$. Here we have an isomorphic relation:

$$\begin{aligned}\pi_*(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}) &\cong \pi_*(\pi^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))\\ &\cong \sum_{\gamma\in\hat{H}(d)} L^{ab(a+b)}\otimes P_{a\beta-b\alpha+\gamma},\end{aligned}$$

which leads us to a decomposition

$$\sum_{\gamma\in\hat{H}(d)} \Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+\gamma}) \xrightarrow[\sim]{\pi^*} \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}). \tag{1}$$

First of all we shall consider the case of $p\nmid ab$. The proof, except the last part, will follow the one of [7], Prop. 2.1. In this case, there exists a Göpel decomposition of Y_{abd}:

$$K(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})=Y_{abd(a+b)}\supset Y_{abd}=H(abd)_1\oplus H(abd)_2$$

such that $H(abd)_2\supset H(d)$. Here we put $H(a)_1=H(abd)_1\cap Y_a$ and $H(b)_1=H(abd)_1\cap Y_b$. Let $H(abd)_2^{**}\supset H(d)^{**}$ and $H(abd)_1^{**}\supset H(a)_1^{**}$, $H(b)_1^{**}$ be level subgroups of them in $\mathcal{G}(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})$, such that $H(d)^{**}$ corresponds to the isomorphism $M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}\cong\pi^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$. Let $\mathcal{G}^{**}$ be the centralizer of $H(d)^{**}$ in $\mathcal{G}(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})$. Then we have the canonical isomorphism

$$\mathcal{G}^{**}/H(d)^{**}\cong\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}),$$

and $H(abd)_2^{**}$, $H(a)_1^{**}$, $H(b)_1^{**}$ are contained in $\mathcal{G}^{**}$. We denote by $H(ab)_2^*$ the image of $H(abd)_2^{**}$ by this canonical map, and we put

$$H(a)_2^*=H(ab)_2^*\cap X_a,\quad H(b)_2^*=H(ab)_2^*\cap X_b.$$

Moreover, in view of the fact that $(H(a)_1^{**}\cup H(b)_1^{**})\cap H(d)^{**}=\{1\}$ scheme-theoretically, $H(a)_1^{**}$ and $H(b)_1^{**}$ are canonically isomorphic to subgroups of

$\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$, which we denote by $H(a)_1^*$ and $H(b)_1^*$ respectively. By Proposition 0.1, for any $\lambda\in H(a)_1^{**}\cup H(b)_1^{**}\cup H(abd)_2^{**}$, we obtain a commutative diagram:

$$
\begin{tikzcd}
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \arrow[r, "\pi^*"] \arrow[d, "U_{\lambda'}"'] & \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}) \arrow[d, "U_\lambda"] \\
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \arrow[r, "\pi^*"] & \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})
\end{tikzcd}
\tag{2}
$$

where λ' is the canonical image of λ in $H(a)_1^*\cup H(b)_1^*\cup H(ab)_2^*$. Furthermore, for any $\mu=(y, \phi)\in H(abd)_i^{**}$, we obtain a commutative diagram:

$$
\begin{tikzcd}
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \arrow[r, "\pi^*"] \arrow[d, "T^*_{\pi y}"] \arrow[dd, bend right=60, dashed, "U'_\mu"'] & \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}) \arrow[d, "T^*_y"] \arrow[dd, bend left=60, "U_\mu"] \\
\Gamma(T^*_{\pi y}L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \arrow[r, "\pi^*"] \arrow[d, "\wr"'] & \Gamma(T^*_y M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}) \arrow[d, "\wr"', "\phi^{-1}"] \\
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+ab(a+b)\phi_L(\pi y)}) \arrow[r, "\pi^*"] & \Gamma(M^{abd(a+b)}\otimes P_{\alpha\hat{\pi}\beta-a\hat{\pi}\alpha}) .
\end{tikzcd}
\tag{3}
$$

Here we denote by U'_μ the composite of the left vertical arrows. On the other hand, for any $x\in X$, the diagram:

$$
\begin{tikzcd}
X\times X \arrow[r, "\xi"] \arrow[d, "T_{(0,x)}"'] & X\times X \arrow[d, "T_{(-bx,ax)}"] \\
X\times X \arrow[r, "\xi"] & X\times X
\end{tikzcd}
$$

commutes. Hence we have an isomorphism

$$\xi^*(T^*_{(-bx,ax)}(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta)))\cong T^*_{(0,x)}\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta)),$$

i. e.,

$$\begin{aligned}
&\xi^*(p_1^*(L^a\otimes P_{\alpha-ab\phi_L(x)})\otimes p_2^*(L^b\otimes P_{\beta+ab\phi_L(x)})) \\
&\qquad\cong p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+ab(a+b)\phi_L(x)}) .
\end{aligned}$$

Therefore we obtain a commutative diagram:

$$
(4)\quad
\begin{tikzcd}
\Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta) \arrow[r, "\xi^*"] \arrow[d, "T^*_{(-b\pi y, a\pi y)}"'] \arrow[dd, bend right=60, dashed, "U''_\mu"'] & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \arrow[d, "1\otimes T^*_{\pi y}"] \arrow[dd, bend left=60, "1\otimes U'_\mu"] \\
\Gamma(T^*_{-b\pi y}L^a\otimes P_\alpha)\otimes\Gamma(T^*_{a\pi y}L^b\otimes P_\beta) \arrow[r, "\xi^*"] \arrow[d, "\wr"'] & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(T^*_{\pi y}L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \arrow[d, "\wr"'] \\
\Gamma(L^a\otimes P_{\alpha-ab\phi_L(\pi y)})\otimes\Gamma(L^b\otimes P_{\beta+ab\phi_L(\pi y)}) \arrow[r, "\xi^*"] & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+ab(a+b)\phi_L(\pi y)})\,.
\end{tikzcd}
$$

Similarly, we denote by U''_μ the composite of the left vertical arrows. The subgroups $H(a)^*_i$ and $H(b)^*_i$ of $\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ are canonically isomorphic to subgroups $\bar{H}(a)^*_i$ and $\bar{H}(b)^*_i$ of $\mathcal{G}(L^a\otimes P_\alpha)$ and $\mathcal{G}(L^b\otimes P_\beta)$ respectively, and we denote by $\bar{z}$ the canonical image in $\bigcup_{i=1}^{2}(\bar{H}(a)^*_i\cup\bar{H}(b)^*_i)$ of an element

$$z\in\bigcup_{i=1}^{2}(H(a)^*_i\cup H(b)^*_i)\,.$$

Now we take a non-zero $\bar{H}(a)^*_2$-invariant section u and $\bar{H}(b)^*_2$-invariant one v from $\Gamma(L^a\otimes P_\alpha)$ and $\Gamma(L^b\otimes P_\beta)$ respectively. Then by Lemma 1.1 and Proposition 1.3, for a basis $\{s_1,\cdots,s_l\}$ of $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$ and linearly independent $H(ab)^*_2$-invariant sections $\theta_1,\cdots,\theta_l$ of $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$, we have

$$\xi^*(u\otimes v)=\sum_{i=1} s_i\otimes\theta_i\,.$$

Applying the diagram (3) and (4) to this equality, we obtain

$$
\begin{aligned}
(5)\qquad & (((1\otimes\pi^*)\xi^*U''_\mu(u\otimes v))(x,y))_{\mu\in H(abd)_1^{**}} \\
&\quad =(s_i(x))_{1\leqq i\leqq l}((U_\mu(\pi^*\theta_i))(y))_{(i,\mu)\in\{1,\cdots,l\}\times H(abd)_1^{**}}.
\end{aligned}
$$

On the other hand, by the commutative diagram (2),

$$\langle\pi^*\theta_1,\cdots,\pi^*\theta_l\rangle=\Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})^{H(abd)_2^{**}}.$$

Therefore by the rank theorem, we obtain the equality

$$\operatorname{rank}(U_\mu(\pi^*\theta_i)(0))_{(i,\mu)\in\{1,\cdots,l\}\times H(abd)_1^{**}}=(a+b)^g\,.$$

Here we obtain our assertion, putting $y=0$ in (5).

Secondly, we shall consider the case of $p\nmid a+b$. Combining the homomorphism ξ^* and the isomorphism (1), we have

$$\sum_{\gamma\in\hat{H}(d)}\Gamma(L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(L^b\otimes P_{\beta+\gamma})\xrightarrow{\xi^*}$$

$$\Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes(\sum_{\gamma\in\hat{H}(d)}\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+(a+b)\gamma}))$$

$$\overset{1\otimes\pi^*}{\cong}\Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}).$$

Let $\{u_1^{(\gamma)},\cdots,u_m^{(\gamma)}\}$ and $\{v_1^{(\gamma)},\cdots,v_n^{(\gamma)}\}$ be bases of $\Gamma(L^a\otimes P_{\alpha-\gamma})$ and $\Gamma(L^b\otimes P_{\beta+\gamma})$ for each $\gamma\in\hat{H}(d)$, where $m=a^g$ and $n=b^g$. From the assumption $p\nmid a+b$, we can decompose $K(L^{a+b}\otimes P_{\alpha+\beta})$ into maximal isotropic subgroups:

$$K(L^{a+b}\otimes P_{\alpha+\beta})=H(a+b)_1\oplus H(a+b)_2.$$

We denote by $H(a+b)_i^*$ the level subgroups of them. By Lemma 1.2, for each $\gamma\in\hat{H}(d)$, there exist level subgroups

$$H_\gamma(a+b)_1^*,\qquad H_\gamma(a+b)_2^*\subset\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+(a+b)\gamma})$$

and an isomorphism $'$: $H(a+b)_i^*\to H_\gamma(a+b)_i^*$. Moreover there exists $H_\gamma(a+b)_2^*$-invariant element $\theta_{i,j}^{(\gamma)}$ in $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+(a+b)\gamma})$ for each pair (i,j), such that

$$\xi^*(u_i^{(\gamma)}\otimes v_j^{(\gamma)})=\sum_{\lambda\in H(a+b)_1^*}U_\lambda s\otimes U_{\lambda'}\theta_{i,j}^{(\gamma)}.$$

Now, for any row vectors $a_1,\cdots,a_r$ and matrices $M_1,\cdots,M_r$ of same size, we put

$$(a_i)_{1\leq i\leq r}=(a_1,\cdots,a_r),\qquad {}^t(a_i)_{1\leq i\leq r}=\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix}$$

and

$$(M_i)_{1\leq i\leq r}=(M_1,\cdots,M_r):\text{ the new matrix}.$$

Then from the above equality, we have

$$\text{(6)}\qquad \Big(\big((((1\otimes\pi^*)\xi^*(u_i^{(\gamma)}\otimes v_j^{(\gamma)}))(x,0))_{1\leq i\leq m}\big)_{1\leq j\leq n}\Big)_{\gamma\in\hat{H}(d)}$$

$$=(U_\lambda s(x))_{\lambda\in H(a+b)_1^*}\,{}^t\Big(\big(((U_{\lambda'}\pi^*\theta_{i,j}^{(\gamma)}(0))_{1\leq i\leq m})_{1\leq j\leq n}\big)_{\gamma\in\hat{H}(d)}\Big)_{\lambda'\in H(a+b)_1^*}$$

Here since $(d,a+b)=1$, $H_\gamma(a+b)_2^*$'s are naturally lifted up to only one subgroup, which we also denote by $H(a+b)_2^*$, in $\mathcal{G}(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})$. Therefore noting the $H_\gamma(a+b)_2^*$-invariantness of $\theta_{i,j}^{(\gamma)}$, we have

$$\langle\{\pi^*\theta_{i,j}^{(\gamma)}\}_{\substack{i=1,\cdots,m,\gamma\in\hat{H}(d)\\ j=1,\cdots,n}}\rangle=\Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})^{H(a+b)_2^*}.$$

Hence by the rank theorem, the rank of the matrix in the right side of the equality (6) is $(a+b)^g$, which implies our assertion in the second case.

Q. E. D.

COROLLARY 2.2. *Let L be a principal invertible sheaf on X, and $(R, \boldsymbol{M})$ be any local ring over k with the residue field k. Let α, β be two R-valued points of $\hat{X}$ for which there exist R-valued points u, v, w of X such that $\phi_{L^a}(u)=\alpha$, $\phi_{L^b}(v)=\beta$ and $\phi_{L^{a+b}}(w)=\alpha+\beta$. Moreover, let $\hat{H}(d)$ be a maximal isotropic direct summand of $K((\phi_L^{-1})^*L^d)$. Then*

$$\sum_{\gamma\in\hat{H}(d)} \Gamma(p_1^*L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(p_1^*L^b\otimes P_{\beta+\gamma}) \longrightarrow \Gamma(p_1^*L^{a+b}\otimes P_{\alpha+\beta})$$

is surjective, where $p_1: X\times\mathrm{Spec}(R)\to X$ is the projection to the first factor.

PROOF. Since every invertible sheaf is algebraically equivalent to a symmetric invertible sheaf, by slight modification, if necessary, of α and β, we may assume that our L is symmetric. Let ι: $\mathrm{Spec}(R/\boldsymbol{M})\to\mathrm{Spec}(R)$ be the canonical inclusion, and we denote by $\bar{x}$ the $(R/\boldsymbol{M})$-valued point $x\circ\iota$, for any R-valued point x. Then, from the commutative diagram:

$$\begin{array}{ccc}
\Gamma(p_1^*L^a\otimes P_\alpha) & \xrightarrow{(1_X\times\iota)^*} & \Gamma(L^a\otimes P_{\bar{\alpha}}) \\
T_u^*\uparrow & & \\
\Gamma(p_1^*L^a) & & \uparrow T_{\bar{u}}^* \\
\wr\| & & \\
\Gamma(L^a)\otimes R & \xrightarrow{(1_X\times\iota)^*} & \Gamma(L^a)\otimes_k(R/\boldsymbol{M})\cong\Gamma(L^a)\,,
\end{array}$$

and the surjectivity of the bottom arrow in this diagram,

$$(1_X\times\iota)^*:\ \Gamma(p_1^*L^a\otimes P_\alpha)\longrightarrow\Gamma(L^a\otimes P_{\bar{\alpha}})$$

is surjective and $\Gamma(p_1^*L^a\otimes P_\alpha)$ a free R-module of rank $\dim\Gamma(L^a)$. Similarly, these assertions are true for $\Gamma(p_1^*L^b\otimes P_\beta)$ and $\Gamma(p_1^*L^{a+b}\otimes P_{\alpha+\beta})$. Moreover, we have a commutative diagram:

$$\begin{array}{ccc}
\sum_{\gamma\in\hat{H}(d)} \Gamma(p_1^*L^a\otimes P_{\alpha-\gamma})\otimes_k\Gamma(p_1^*L^b\otimes P_{\beta+\gamma}) & \xrightarrow{\tau} & \Gamma(p_1^*L^{a+b}\otimes P_{\alpha+\beta}) \\
(1_X\times\iota)^*\otimes(1_X\times\iota)^*\downarrow & & \downarrow(1_X\times\iota)^* \\
\sum_{\gamma\in\hat{H}(d)} \Gamma(L^a\otimes P_{\bar{\alpha}-\gamma})\otimes_k\Gamma(L^b\otimes P_{\bar{\beta}+\gamma}) & \longrightarrow & \Gamma(L^{a+b}\otimes P_{\bar{\alpha}+\bar{\beta}})\,.
\end{array}$$

The bottom arrow is surjective by Proposition 2.1, and $(1_X\times\iota)^*$'s are surjective. Therefore there exist l elements $f_1,\cdots,f_l$ of

$$\sum_{\gamma\in\hat{H}(d)} \Gamma(p_1^*L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(p_1^*L^b\otimes P_{\beta+\gamma})$$

whose images in $\Gamma(L^{a+b}\otimes P_{\bar{\alpha}+\bar{\beta}})$ make a basis of it, where $l=(a+b)^g$. Let $e_1,\cdots,e_l$ be a basis of the free R-module $\Gamma(p_1^*L^{a+b}\otimes P_{\alpha+\beta})$. Then $\tau(f_i)$'s can be written in the form:

$$\tau(f_i)=\sum_{j=1}^{l} a_{ij}e_j \qquad \text{with} \quad a_{ij}\in R\,,$$

for $i=1, \cdots, l$. By the way of choice of f_i's, $\det(\bar{a}_{ij})\neq 0$ in $R/\boldsymbol{M}$, i.e., $\det(a_{ij})$ is a unit in R. If we consider

$$\sum_{\gamma\in \hat{H}(d)} \Gamma(p_1^*L^a\otimes P_{\alpha-\gamma})\otimes_k \Gamma(p_1^*L^b\otimes P_{\beta+\gamma})$$

as an R-module by the multiplication of elements of R to the first factor, τ becomes an R-module homomorphism. Hence we obtain the surjectivity of τ.

Q.E.D.

THEOREM 2.3. *Let L be any ample invertible sheaf on X, and α, β be two closed points on $\hat{X}$. Let H be a maximal isotropic subgroup of $K(L)$, and we put $\pi: X\to Y=X/H$: the canonical map. Let $H(d)$ be a maximal isotropic subgroup of $K(L^d)$ such that $\pi(H(d))$ is a maximal isotropic direct summand of $K(M^d)$, where M is a principal invertible sheaf on Y such that $\pi^*M\cong L$. Moreover we put $\hat{H}(d)=\phi_L(H(L^d))$. Then*

$$\sum_{\gamma\in\hat{H}(d)} \Gamma(L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(L^b\otimes P_{\beta+\gamma}) \xrightarrow{\tau} \Gamma(L^{a+b}\otimes P_{\alpha+\beta})$$

is surjective.

PROOF. In the same way as in the separable case, we can reduce our assertion to the case where $\chi(L)$ is a power of p. So we suppose that $\chi(L)$ is a power of p. We put $W=\operatorname{Im}\tau$, and let R be any local ring over k with the residue field k. Here we take two points α', β' from $\hat{\pi}^{-1}(\alpha)$ and $\hat{\pi}^{-1}(\beta)$ respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\displaystyle\sum_{\gamma\in\hat{H}(d)} \Gamma(L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(L^b\otimes P_{\beta+\gamma}) & \longrightarrow & \Gamma(L^{a+b}\otimes P_{\alpha+\beta}) \\
\pi^*\otimes\pi^* \big\uparrow & & \big\uparrow \pi^* \\
\displaystyle\sum_{\gamma'\in\phi_M\pi(H(L^d))} \Gamma(M^a\otimes P_{\alpha'-\gamma'})\otimes\Gamma(M^b\otimes P_{\beta'+\gamma'}) & \longrightarrow & \Gamma(M^{a+b}\otimes P_{\alpha'+\beta'})\,.
\end{array}$$

Therefore, if we show the inclusive relation

$$U_u(\pi^*(\Gamma(M^{a+b}\otimes P_{\alpha'+\beta'}))\otimes R)\subset W\otimes R$$

for any R-valued point u of $K(L^{a+b}\otimes P_{\alpha+\beta})$, by the theta-structure theorem, we have the equality $W=\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$. Here we consider $K(L^{a+b}\otimes P_{\alpha+\beta})$ as a subscheme of $\mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})$, embedding it by a cross-section of the canonical map $j: \mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})\to K(L^{a+b}\otimes P_{\alpha+\beta})$.

So let u be any R-valued point of $K(L^{a+b}\otimes P_{\alpha+\beta})$. Then

$$T_{\pi u}^*(p_1^*M^{a+b}\otimes P_{\alpha'+\beta'})\cong p_1^*M^{a+b}\otimes P_{\phi_M(\pi u')+\alpha'+\beta'}\,,$$

where $u'=(a+b)u\in K(L)$. Since $(a, b)=1$, either a or b, say a, is prime to $\chi(L)$. Therefore there exists an R-valued point u'' of $K(L)$ such that $au''=u'$, and we have

$$T^*_{\pi u''}p_1^*(M^a\otimes P_{\alpha'-\gamma'})\cong p_1^*M^a\otimes P_{a\phi_M(\pi u'')+\alpha'-\gamma'}$$
$$\cong p_1^*M^a\otimes P_{\phi_M(\pi u')+\alpha'-\gamma'}\,.$$

Since $u''\in K(L)\subset K(L^a)$,

$$\Gamma(p_1^*M^a\otimes P_{\phi_M(\pi u')+\alpha'-\gamma'})\xrightarrow{\pi^*}\Gamma(L^a\otimes P_{\alpha-\gamma})\otimes R\,.$$

Moreover we have a commutative diagram:

$$\begin{array}{ccc}
\sum\limits_{\gamma\in\hat{H}(d)}(\Gamma(L^a\otimes P_{\alpha-\gamma})\otimes R)\otimes_k(\Gamma(L^b\otimes P_{\beta+\gamma})\otimes R) & \longrightarrow & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes R \\
\uparrow{\scriptstyle \pi^*\otimes\pi^*} & & \uparrow{\scriptstyle \pi^*} \\
\sum\limits_{\gamma'\in\phi_M\pi(H(L^d))}\Gamma(p_1^*M^a\otimes P_{\phi_M(\pi u')+\alpha'-\gamma'})\otimes\Gamma(p_1^*M^b\otimes P_{\beta'+\gamma'}) & \longrightarrow & \Gamma(p_1^*M^{a+b}\otimes P_{\phi_M(\pi u')+\alpha'+\beta'}).
\end{array}$$

By Corollary 2.2, the bottom arrow is surjective. Therefore we have the required relation

$$U_u(\pi^*(\Gamma(M^{a+b}\otimes P_{\alpha'+\beta'}))\otimes R)=\pi^*\Gamma(p_1^*M^{a+b}\otimes P_{\phi_M(\pi u')+\alpha'+\beta'})\subset W\otimes R\,.$$

Q. E. D.

Inductively applying Theorem 2.3, we have

THEOREM 2.4. *Let L be any ample invertible sheaf on X; and α, β be two closed points on $\hat{X}$. Then*

$$\Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta)\longrightarrow\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$$

is surjective for all integers a, b such that $a\geqq 2$, $b\geqq 3$.

APPENDIX (this is reconstructed from Mumford's note). Here we shall consider only group schemes over k, and we denote by B any k-algebra. We assume the fundamental results on theta groups given in [5], § 23.

DEFINITION A.1. A theta group $1\to\boldsymbol{G}_m\to\mathcal{G}\to K\to 0$ is a *finite Heisenberg group* if it is non-degenerate and K is a finite group scheme.

Hereafter we fix a finite Heisenberg group $1\to\boldsymbol{G}_m\xrightarrow{i}\mathcal{G}\xrightarrow{j}K\to 0$. We note that the order of K becomes d^2 for some integer d.

DEFINITION A.2. Let V be a finite dimensional k-vector space, and l be an integer. A representation of $\mathcal{G}$ on V of *weight* l is a homomorphism σ: $\mathcal{G}\to GL(V)$ such that $\sigma(\lambda)=\lambda^l\cdot 1_V$ for $\lambda\in\boldsymbol{G}_m$.

Let $\tau: K\to\mathcal{G}$ be a cross-section of j, and we put $K=\operatorname{Spec}R$ and $\mathcal{G}=\operatorname{Spec}A$. Then A is a k-algebra and R is a finite dimensional k-algebra. Let

$\mu: \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ (resp. $m: K\times K\to K$) be the multiplication morphism of $\mathcal{G}$ (resp. of K). Here we notice that A and $\mathrm{Hom}_k(\mathcal{G}, \boldsymbol{A}^1)$ are canonically identified. So, frequently we consider an element of A a function defined over $\mathcal{G}$. The restricted multiplication morphism $\mu|_{\boldsymbol{G}_m\times\mathcal{G}}: \boldsymbol{G}_m\times\mathcal{G}\to\mathcal{G}$ defines a homomorphism

$$s: A \longrightarrow k[t, t^{-1}]\otimes_k A.$$

For any element $a\in A$, let $s(a)=\sum_{i=-\infty}^{\infty} t^i\otimes\pi_i(a)$. Then $\pi_i: A\to A$ is a k-linear map for each i. Let $\mathrm{Im}(\pi_i)=A_i$. Then we have

LEMMA A.3.

(i) $A=\bigoplus_{i=-\infty}^{\infty} A_i$.

(ii) *For $f\in A$, $f\in A_i \Leftrightarrow f(\lambda x)=\lambda^i f(x)$ for any B-valued points $\lambda\in\boldsymbol{G}_m(B)$ and $x\in G(B)$.*

(iii) *According to the co-multiplication $\mu^*: A\to A\otimes A$, $\mu^*(A_i)\subset A_i\otimes A_i$ for each i.*

(iv) *$A_0\cong R$ and there exists an element $t\in A$ such that $A_i=A_0t^i$ for each i.*

PROOF. Let e: $\mathrm{Spec}\, k\to\boldsymbol{G}_m$ be the identity morphism. Then we have commutative diagrams:

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[dr, "1_A"'] & k[t, t^{-1}]\otimes A \arrow[d, "e^*\otimes 1_A"] \\ & A \end{tikzcd}$$

and

$$\begin{tikzcd} & k[t, t^{-1}]\otimes A \arrow[dr, "(\mu|_{\boldsymbol{G}_m\times\boldsymbol{G}_m})^*\otimes 1_A"] & \\ A \arrow[ur, "s"] \arrow[dr, "s"'] & & k[t, t^{-1}]\otimes k[t, t^{-1}]\otimes A. \\ & k[t, t^{-1}]\otimes A \arrow[ur, "1\otimes s"'] & \end{tikzcd}$$

Therefore we have equalities

$$a=\sum_{i=-\infty}^{\infty}\pi_i(a)$$

and

$$\sum_{i,j} t^i\otimes t^j\otimes\pi_j\pi_i(a)=\sum_i t^i\otimes t^i\otimes\pi_i(a) \qquad \text{for all} \quad a\in A,$$

i.e.,

$$\pi_j\pi_i(a)=\begin{cases}\pi_i(a) & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$$

These imply the assertion (i).

For (ii), let $f\in A_i$ and λ be any B-valued point of $\boldsymbol{G}_m$. Then the right translation R_λ by λ is given by $R_\lambda=(p_1, \mu\circ(\lambda\times 1_{\mathcal{G}}))$: $\operatorname{Spec} B\times\mathcal{G}\rightarrow\operatorname{Spec} B\times\mathcal{G}$, where p_1: $\operatorname{Spec} B\times\mathcal{G}\rightarrow\operatorname{Spec} B$ is the projection to the first factor. Therefore we have a commutative diagram:

$$\begin{array}{ccc} B\otimes A & \xleftarrow{1_B\otimes f^*} & B[t] \\ {\scriptstyle (R_\lambda)^*}\uparrow & & \uparrow{\scriptstyle \text{mult. by } \lambda^*(t)^i,} \\ B\otimes A & \xleftarrow[1_B\otimes f^*]{} & B[t] \end{array}$$

which implies the required equality $f(\lambda x)=\lambda^i f(x)$ for all B-valued points $x\in\mathcal{G}(B)$ and $\lambda\in\boldsymbol{G}_m(B)$. The converse is a direct consequence of (i) and of this result.

(iii) is easily induced from (ii).

Moreover, if we put $\phi=\mu\circ(i\times\tau)$: $\boldsymbol{G}_m\times K\rightarrow\mathcal{G}$, then ϕ defines an isomorphism of $\boldsymbol{G}_m$-spaces. Hence, in view of (ii), ϕ^*: $A\rightarrow R\otimes k[t, t^{-1}]$ is an isomorphism of graded rings. So we obtain (iv). Q.E.D.

We define an action of $\mathcal{G}\times\mathcal{G}$ on A_1 by $f(x)\mapsto f(uxv^{-1})$ for any function $f(x)$ in A_1 and any points $u, v\in\mathcal{G}(B)$. Then we have the key theorem

THEOREM A.4. *A_1 is irreducible $\mathcal{G}\times\mathcal{G}$-module.*

For proving this theorem, we need the following lemma.

LEMMA A.5. *Let $\mathcal{G}=\operatorname{Spec} A$ be an affine group scheme over k, with the multiplication morphism μ: $\mathcal{G}\times\mathcal{G}\rightarrow\mathcal{G}$ and the identity morphism e: $\operatorname{Spec} k\rightarrow\mathcal{G}$. Let W be a non-trivial k-subspace in A closed under right translations, i.e., $\mu^*(W)\subset W\otimes A$. Then we have*

(i) *W is closed under any left-invariant k-linear map D: $A\rightarrow A$.*

(ii) *For any k-rational point $x\in\mathcal{G}$, there exists a function $f\in W$ such that $f(x)\neq 0$.*

PROOF. The left-invariantness of D means the commuteness of the diagram:

$$\begin{array}{ccc} A & \xrightarrow{D} & A \\ {\scriptstyle \mu^*}\downarrow & & \downarrow{\scriptstyle \mu^*} \\ A\times A & \xrightarrow[1\otimes D]{} & A\times A. \end{array}$$

Therefore, for any $f\in W$, $\mu^*(D(f))=(1\otimes D)(\mu^*(f))\in W\otimes A$, i. e.,

$$D(f)=(1_A\otimes e^*)\mu^*(D(f))\subset(1_A\otimes e^*)(W\otimes A)=W,$$

which implies (i). Moreover, since W is right-stable and $W\neq 0$, we may assume $x=e$, and there exists a function $f\in W$ such that $f\neq 0$ in $\mathcal{O}_{e,G}$. Therefore there exists a k-linear map $D_0: \mathcal{O}_{e,G}\to k$ such that $D_0(f)\neq 0$. Here we put also D_0: $A\to\mathcal{O}_{e,G}\to k$, and put $D=(1_A\otimes D_0)\circ\mu^*$. Obviously this $D: A\to A$ is a left-invariant k-linear map. Hence, in view of (i), $D(f)$ satisfies our required properties. Q. E. D.

PROOF OF THEOREM A.4. We define an action U of $\mathcal{G}$ on A_1 by $(U_af)(x)=f(axa^{-1})$ for any function $f(x)\in A_1$ and any B-valued point $a\in\mathcal{G}(B)$. Then, from the definition of the skew-symmetric bilinear form $e: K\times K\to \boldsymbol{G}_m$ and Lemma A.3, (ii), $(U_af)(x)=e(j(a), j(x))f(x)$, i. e., $U_a(1\otimes f)=\gamma(j(a))^*(t)(1\otimes f)$ where $\gamma: K\to\hat{K}$ is the isomorphism defined by e. Especially, if we take $\Gamma(\hat{K},\mathcal{O}_{\hat{K}})=A_0^*$ as B and $\tau(\gamma^{-1})$ as a, $\gamma(\gamma^{-1})^*(t)\in A_0^*\otimes A_0$ corresponds to the identity map $1_{A_0}: A_0\to A_0$. Therefore, if we write $\gamma(\gamma^{-1})^*(t)=\sum_i a_i^*\otimes a_i$ with a basis $\{a_i^*\}$ of A_0^*, $\{a_i\}$ is also a basis of A_0.

Now let W be a non-trivial $\mathcal{G}\times\mathcal{G}$-submodule of A_1. Then W is stable under the action U of $\mathcal{G}$. Especially, for any $f\in W$,

$$U_{\tau(\gamma^{-1})}(1\otimes f)=(\sum_i a_i^*\otimes a_i)(1\otimes f)\in A_0^*\otimes W,$$

i. e., we obtain an inclusive relation $A_0W\subset W$.
But, since A_0 and A_1 are isomorphic as A_0-imodule, W is isomorphic to an ideal $\mathcal{J}$ in A_0. Therefore, if $W\subsetneqq A_1$, $\mathcal{J}$ is a proper ideal of A_0. Hence there exists a maximal ideal $\mathcal{M}$ of A_0 containing $\mathcal{J}$. Let x be the point of $K=\operatorname{Spec} A_0$ corresponding to $\mathcal{M}$, and put $y=\tau(x)$. Then, in view of Lemma A.5, (ii), there exists a function $f\in W$ such that $f(y)\neq 0$. This contradicts the fact $\mathcal{J}\subset\mathcal{M}$. Q. E. D.

THEOREM A.6. *Any finite Heisenberg group $\mathcal{G}$ has one and only one irreducible representation of weight* 1, *and all representations of weight* 1 *are completely reducible. Moreover, if* $\operatorname{ord}(K)=d^2$, *then the dimension of the irreducible $\mathcal{G}$-module is d.*

PROOF. Let V_0 be a finite dimensional k-vector space, and $\sigma_0: \mathcal{G}\to GL(V_0)$ be a representation of weight 1. Since $GL(V_0)\subset\operatorname{Hom}(V_0, V_0)=\operatorname{Spec} S(V_0^*\otimes V_0)$ where S denotes the symmetric algebra, the σ_0 defines a $\mathcal{G}\times\mathcal{G}$-module homomorphism $\sigma_0^*: V_0^*\otimes V_0\to A_1$. Therefore, by virtue of Theorem A.4, if V_0 is an irreducible $\mathcal{G}$-module, σ_0^* becomes an isomorphism. Hence, as $\mathcal{G}$-module,

$$A_1=\underbrace{V_0\oplus\cdots\oplus V_0}_{\dim V_0}.$$

We have further $(\dim V_0)^2 = \dim A_1 = \dim A_0 = d^2$, and $\dim V_0 = d$.

Moreover, if V is any $\mathcal{G}$-module of weight 1, and $\sigma: V \to A \otimes V$ is the dual action, we have indeed $\sigma: V \xrightarrow{\sim} \sigma(V) \subset A_1 \otimes V$. If we consider $A_1 \otimes V$ as a $\mathcal{G}$-module by $\mathcal{G}$ acting to the first factor A_1, σ becomes a $\mathcal{G}$-module injection, i.e., $\sigma: V \longrightarrow \underbrace{A_1 \oplus \cdots \oplus A_1}_{\dim V}$ as $\mathcal{G}$-module. Therefore, in view of the above results, we can deduce the complete reducibility of V. Q.E.D.

For a non-degenerate invertible sheaf L of index i on an abelian variety X, $\mathcal{G}(L)$ acts on $H^i(L)$ and the action is of weight 1. Moreover, $\dim H^i(L) = |\chi(L)| = \sqrt{\mathrm{ord}(K(L))}$. Therefore Theorem A.6 is applicable to the case. That is, we have the theta structure theorem stated in Section 0.

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