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# Infinitesimal variations of hypersurfaces of a Kaehlerian manifold

Dedicated to Prof. Shôkichi Iyanaga on his seventieth birthday

By Kentaro YANO

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## § 0. Introduction.

B. Y. Chen [**1, 2, 3**], C. S. Houh [**2**], S. Tachibana [**10**] and the present author [**3, 9, 10**] have recently studied infinitesimal variations of submanifolds of Riemannian and Kaehlerian manifolds. (See also [**7**].)

The main purpose of the present paper is to study infinitesimal variations of hypersurfaces of a Kaehlerian manifold, to obtain variations of structure tensors of the almost contact metric structure induced on the hypersurfaces from the Kaehlerian structure of the ambient manifold and to prove theorems on variations of Sasakian hypersurfaces with $f$-sectional curvature $\alpha^2$. (For the definition of a Sasakian manifold with $f$-sectional curvature $\alpha^2$, see § 1.)

In § 1 we state some preliminaries on almost contact metric structure induced on a hypersurface from the Kaehlerian structure of the ambient manifold.

In § 2 we consider infinitesimal variation of a hypersurface of a Kaehlerian manifold and obtain variations of structure tensors of almost contact metric structure induced on the hypersurface.

§ 3 is devoted to the study of what the author calls parallel variation and § 4 to that of variation of the second fundamental form of the hypersurface.

In the last § 5, we consider variations of a Sasakian hypersurface with $f$-sectional curvature $\alpha^2$ and obtain the condition that an infinitesimal variation carries a Sasakian hypersurface with $f$-sectional curvature $\alpha^2$ into a hypersurface of the same kind.

## § 1. Preliminaries.

Let $M^{2n+2}$ $(n>1)$ be a real $(2n+2)$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and $F_i{}^h$ be its almost complex structure tensor and $g_{ji}$ its almost Hermitian metric tensor, where and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1', 2', \cdots, (2n+2)'\}$. Then we have

$$F_i{}^t F_t{}^h = = -\delta_i^h \tag{1.1}$$

and

$$F_j{}^t F_i{}^s g_{ts} = g_{ji} . \tag{1.2}$$

Let $M^{2n+1}$ be a $(2n+1)$-dimensional orientable Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and $g_{cb}$ be its fundamental metric tensor, where and in the sequel the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$. We assume that $M^{2n+1}$ is isometrically immersed into $M^{2n+2}$ by the immersion $i: M^{2n+1} \to M^{2n+2}$ and identify $i(M^{2n+1})$ with $M^{2n+1}$. We represent the immersion by

$$x^h = x^h(y^a) \tag{1.3}$$

and put

$$B_b{}^h = \partial_b x^h , \qquad (\partial_b = \partial/\partial y^b) . \tag{1.4}$$

Then $B_b{}^h$ are $2n+1$ linearly independent vectors of $M^{2n+2}$ tangent to $M^{2n+1}$. Since the immersion is isometric we have

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i . \tag{1.5}$$

We represent the unit normal to $M^{2n+1}$ by $C^h$. We then have

$$g_{ji} B_b{}^j C^i = 0 \tag{1.6}$$

and

$$g_{ji} C^j C^i = 1 . \tag{1.7}$$

Now the transform $F_i{}^h B_b{}^i$ of $B_b{}^i$ by $F_i{}^h$ can be written as

$$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h + f_b C^h , \tag{1.8}$$

where $f_b{}^a$ is a tensor field of type (1, 1) and $f_b$ a 1-form of $M^{2n+1}$ and the transform $F_i{}^h C^i$ of $C^i$ by $F_i{}^h$, being orthogonal to $C^h$, can be written as

$$F_i{}^h C^i = -f^a B_a{}^h , \tag{1.9}$$

where $f^a = f_b g^{ba}$ is a vector field of $M^{2n+1}$, $g^{ba}$ being contravariant components of $g_{cb}$.

Applying $F$ to the both sides of (1.8) and using (1.1), (1.8) and (1.9), we find

$$f_b{}^e f_e{}^a = -\delta_b^a + f_b f^a , \qquad f_b{}^e f_e = 0 . \tag{1.10}$$

Applying $F$ to the both sides of (1.9) and using (1.1), (1.8) and (1.9), we find

$$f_e{}^a f^e = 0\,, \qquad f_e f^e = 1\,. \tag{1.11}$$

From (1.2), (1.5), (1.6), (1.7) and (1.8), we have

$$f_c{}^e f_b{}^d g_{ed} = g_{cb} - f_c f_b\,. \tag{1.12}$$

Equations (1.10), (1.11) and (1.12) show that the set $(f_b{}^a, g_{cb}, f_b)$ defines the so-called *almost contact metric structure* on $M^{2n+1}$ (Tashiro [**6**]).

We denote by $\Gamma^h_{ji}$ the Christoffel symbols formed with $g_{ji}$ and by $\Gamma^a_{cb}$ those formed with $g_{cb}$. Then it is well known that $\Gamma^h_{ji}$ and $\Gamma^a_{cb}$ are related by

$$\Gamma^a_{cb} = (\partial_c B_b{}^h + \Gamma^h_{ji} B^{ji}_{cb}) B^a{}_h\,, \tag{1.13}$$

where $B^{ji}_{cb} = B_c{}^j B_b{}^i$ and $B^a{}_h = B_b{}^i g^{ba} g_{ih}$. We define the van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ along $M^{2n+1}$ by

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \Gamma^h_{ji} B^{ji}_{cb} - \Gamma^a_{cb} B_a{}^h \tag{1.14}$$

and that of $C^h$ by

$$\nabla_c C^h = \partial_c C^h + \Gamma^h_{ji} B_c{}^j C^i\,. \tag{1.15}$$

Then equations of Gauss can be written as

$$\nabla_c B_b{}^h = h_{cb} C^h\,, \tag{1.16}$$

where $h_{cb}$ is the second fundamental tensor of $M^{2n+1}$ and equations of Weingarten as

$$\nabla_c C^h = -h_c{}^a B_a{}^h\,, \tag{1.17}$$

where $h_c{}^a = h_{cb} g^{ba}$.

Equations of Gauss and Codazzi are respectively written as

$$K_{dcb}{}^a = K_{kji}{}^h B^{kjia}_{dcbh} + h_d{}^a h_{cb} - h_c{}^a h_{db} \tag{1.18}$$

and

$$K_{kji}{}^h B^{kji}_{dcb} C_h = \nabla_d h_{cb} - \nabla_c h_{db}\,, \tag{1.19}$$

$K_{dcb}{}^a$ and $K_{kji}{}^h$ being curvature tensors of $M^{2n+1}$ and $M^{2n+2}$ respectively, where $B^{kjia}_{dcbh} = B_d{}^k B_c{}^j B_b{}^i B^a{}_h$, $B^{kji}_{dcb} = B_d{}^k B_c{}^j B_b{}^i$ and $C_h = C^i g_{ih}$.

We now assume that the ambient manifold $M^{2n+2}$ is Kaehlerian, that is,

$$\nabla_j F_i{}^h = 0\,, \tag{1.20}$$

where $\nabla_j$ denotes the operator of covariant differentiation with respect to $\Gamma^h_{ji}$.

Differentiating (1.8) covariantly along $M^{2n+1}$ and using (1.16), (1.17) and

(1.20), we find

(1.21) $$\nabla_c f_b{}^a = -h_{cb} f^a + h_c{}^a f_b$$

and

(1.22) $$\nabla_c f_b = -h_{ce} f_b{}^e .$$

Differentiating (1.9) covariantly along $M^{2n+1}$ and using (1.16), (1.17) and (1.20), we find

(1.23) $$\nabla_c f^a = h_c{}^e f_e{}^a .$$

Equations (1.22) and (1.23) are equivalent because of the relation $f_{ba} = f_b{}^e g_{ea} = -f_{ab}$.

We now consider the tensor field $S_{cb}{}^a$ defined by

(1.24) $$S_{cb}{}^a = N_{cb}{}^a + (\nabla_c f_b - \nabla_b f_c) f^a ,$$

where $N_{cb}{}^a$ is the Nijenhuis tensor formed with $f_b{}^a$:

(1.25) $$N_{cb}{}^a = f_c{}^e \nabla_e f_b{}^a - f_b{}^e \nabla_e f_c{}^a - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f_e{}^a .$$

When the tensor field $S_{cb}{}^a$ vanishes identically, the almost contact metric structure $(f_b{}^a, g_{cb}, f_b)$ is said to be *normal.*

Substituting (1.21) and (1.22) into (1.24), we find

(1.26) $$S_{cb}{}^a = (f_c{}^e h_e{}^a - h_c{}^e f_e{}^a) f_b - (f_b{}^e h_e{}^a - h_b{}^e f_e{}^a) f_c .$$

Thus if the almost contact metric structure is normal, we have

$$(f_c{}^e h_e{}^a - h_c{}^e f_e{}^a) f_b - (f_b{}^e h_e{}^a - h_b{}^e f_e{}^a) f_c = 0 ,$$

from which, using (1.10) and (1.11),

(1.27) $$f_c{}^e h_e{}^a - h_c{}^e f_e{}^a = 0$$

and

(1.28) $$h_e{}^a f^e = h f^a ,$$

where $h = h_{cb} f^c f^b$.

When the almost contact metric structure $(f_b{}^a, g_{cb}, f_b)$ satisfies

(1.29) $$\nabla_c f_b - \nabla_b f_c = 2 f_{cb} ,$$

the structure is said to be *contact.* Substituting (1.22) into (1.29), we obtain

(1.30) $$f_c{}^e h_e{}^a + h_c{}^e f_e{}^a = 2 f_c{}^a .$$

When the almost contact metric structure is normal and contact, we call the structure a *Sasakian structure* [**5**]. For a Sasakian structure, we have

(1.27) and (1.30) and consequently

(1.31) $$f_c{}^e h_e{}^a = f_c{}^a ,$$

from which, using (1.10), (1.11) and (1.28),

(1.32) $$h_{cb} = g_{cb} + (h-1) f_c f_b .$$

For a Sasakian structure, (1.21), (1.22) and (1.23) reduce respectively to

(1.33) $$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c^a f_b ,$$

(1.34) $$\nabla_c f_b = f_{cb}$$

and

(1.35) $$\nabla_b f^a = f_b{}^a .$$

But if, instead of (1.29), we assume

(1.36) $$\nabla_c f_b - \nabla_b f_c = 2\alpha f_{cb} ,$$

$\alpha$ being a function, we obtain, substituting (1.22) into (1.36),

(1.37) $$f_c{}^e h_e{}^a + h_c{}^e f_e{}^a = 2\alpha f_c{}^a .$$

Thus, from (1.27) and (1.37), we find

(1.38) $$f_c{}^e h_e{}^a = \alpha f_c{}^a ,$$

from which, using (1.10), (1.11) and (1.28),

(1.39) $$h_{cb} = \alpha g_{cb} + (h-\alpha) f_c f_b .$$

Thus (1.21), (1.22) and (1.23) reduce respectively to

(1.40) $$\nabla_c f_b{}^a = \alpha(-g_{cb} f^a + \delta_c^a f_b) ,$$

(1.41) $$\nabla_c f_b = \alpha f_{cb}$$

and

(1.42) $$\nabla_b f^a = \alpha f_b{}^a .$$

From (1.36) and (1.40), we have

$$\nabla_d(\nabla_c f_b - \nabla_b f_c) = 2(\nabla_d \alpha) f_{cb} + 2\alpha^2(-g_{dc} f_b + g_{db} f_c) ,$$

from which, using the Ricci formula and the Bianchi identity, we obtain

$$(\nabla_d \alpha) f_{cb} + (\nabla_b \alpha) f_{dc} + (\nabla_c \alpha) f_{bd} = 0 .$$

Thus if $n>1$, we have

(1.43) $$\alpha = \text{constant} .$$

For a Sasakian structure, we have, from (1.33) and (1.35),

$$K_{cbe}{}^{a}f^{e}=\delta_{c}^{a}f_{b}-\delta_{b}^{a}f_{c}\,, \tag{1.44}$$

which shows that the sectional curvature with respect to a plane section containing $f^a$ is always 1.

But for the structure under consideration, we have, from (1.40) and (1.42),

$$K_{cbe}{}^{a}f^{e}=\alpha^{2}(\delta_{c}^{a}f_{b}-\delta_{b}^{a}f_{c})\,, \tag{1.45}$$

which shows that the sectional curvature with respect to a plane section containing $f^a$ is always $\alpha^2$.

So we call an almost metric structure satisfying (1.40), (1.41) and (1.42) a Sasakian structure with $f$-sectional curvature $\alpha^2$ (Okumura [**4**]).

## § 2. Infinitesimal variation of a hypersurface of a Kaehlerian manifold.

We now consider an infinitesimal variation of the hypersurface $M^{2n+1}$ in $M^{2n+2}$ given by

$$\bar{x}^{h}=x^{h}+\xi^{h}(y)\varepsilon\,, \tag{2.1}$$

$\xi^h$ being a vector field of $M^{2n+2}$ defined along $M^{2n+1}$, where $\varepsilon$ is an infinitesimal. We then have

$$\bar{B}_{b}{}^{h}=B_{b}{}^{h}+(\partial_{b}\xi^{h})\varepsilon\,, \tag{2.2}$$

where $\bar{B}_b{}^h=\partial_b\bar{x}^h$ are $2n+1$ linearly independent vectors tangent to the varied hypersurface at the varied point $(\bar{x}^h)$.

We displace vectors $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point $(x^h)$ and obtain

$$\tilde{B}_{b}{}^{h}=\bar{B}_{b}{}^{h}+\Gamma_{ji}^{h}(x+\xi\varepsilon)\xi^{j}\bar{B}_{b}{}^{i}\varepsilon\,,$$

or

$$\tilde{B}_{b}{}^{h}=B_{b}{}^{h}+(\nabla_{b}\xi^{h})\varepsilon\,, \tag{2.3}$$

neglecting the terms of order higher than one with respect to $\varepsilon$, where

$$\nabla_{b}\xi^{h}=\partial_{b}\xi^{h}+\Gamma_{ji}^{h}B_{b}{}^{j}\xi^{i}\,. \tag{2.4}$$

In the sequel we always neglect terms of order higher than one with respect to $\varepsilon$.

Putting

$$\delta B_{b}{}^{h}=\tilde{B}_{b}{}^{h}-B_{b}{}^{h}\,, \tag{2.5}$$

we have

(2.6) $$\delta B_b{}^h=(\nabla_b\xi^h)\varepsilon .$$

If we put

(2.7) $$\xi^h=\xi^a B_a{}^h+\lambda C^h ,$$

$\xi^a$ and $\lambda$ being respectively a vector field and a scalar function on $M^{2n+1}$, we have

(2.8) $$\nabla_b\xi^h=(\nabla_b\xi^a-\lambda h_b{}^a)B_a{}^h+(\nabla_b\lambda+h_{be}\xi^e)C^h .$$

We denote by $\bar{C}^h$ the unit normal to the varied hypersurface. We displace $\bar{C}^h$ parallelly from the point $(\bar{x}^h)$ to $(x^h)$ and obtain

(2.9) $$\tilde{C}^h=\bar{C}^h+\Gamma^h_{ji}(x+\xi\varepsilon)\xi^j\bar{C}^i\varepsilon .$$

We put

(2.10) $$\delta C^h=\hat{C}^h-C^h .$$

Then $\delta C^h$, being orthogonal to $C^h$, is of the form

(2.11) $$\delta C^h=\eta^a B_a{}^h\varepsilon ,$$

$\eta^a$ being a vector field on $M^{2n+1}$. Thus from (2.9), (2.10) and (2.11), we have

(2.12) $$\bar{C}^h=C^h-\Gamma^h_{ji}\xi^j C^i\varepsilon+\eta^a B_a{}^h\varepsilon .$$

Now applying the operator $\delta$ to $B_b{}^j C^i g_{ji}=0$ and using (2.6) and $\delta g_{ji}=0$, we find

$$(\nabla_b\xi^j)C^i g_{ji}+B_b{}^j\eta^a B_a{}^i g_{ji}=0 ,$$

from which, using (2.8),

(2.13) $$\eta_b=-(\nabla_b\lambda+h_{ba}\xi^a) ,$$

where $\eta_b=\eta^c g_{cb}$. Thus (2.11) can be written as

(2.14) $$\delta C^h=-(\nabla^a\lambda+h_b{}^a\xi^b)B_a{}^h\varepsilon ,$$

where $\nabla^a=g^{ab}\nabla_b$ and (2.12) as

(2.15) $$\bar{C}^h=C^h-\Gamma^h_{ji}\xi^j C^i\varepsilon-(\nabla^a\lambda+h_b{}^a\xi^b)B_a{}^h\varepsilon .$$

Now applying the operator $\delta$ to (1.8) and using $\delta F_i{}^h=0$, (2.6) and (2.14), we find

$$F_i{}^h(\nabla_b\xi^i)\varepsilon=(\delta f_b{}^a)B_a{}^h+f_b{}^a(\nabla_a\xi^h)\varepsilon+(\delta f_b)C^h-f_b(\nabla^a\lambda+h_e{}^a\xi^e)B_a{}^h\varepsilon ,$$

from which, using (1.8), (1.9) and (2.8),

$$\begin{aligned}&[(\nabla_b\xi^e-\lambda h_b{}^e)(f_e{}^a B_a{}^h+f_e C^h)-(\nabla_b\lambda+h_{be}\xi^e)f^a B_a{}^h]\varepsilon\\ &\quad=(\delta f_b{}^a)B_a{}^h+[f_b{}^e(\nabla_e\xi^a-\lambda h_e{}^a)B_a{}^h+f_b{}^e(\nabla_e\lambda+h_{ea}\xi^a)C^h]\varepsilon\\ &\qquad+(\delta f_b)C^h-f_b(\nabla^a\lambda+h_e{}^a\xi^e)B_a{}^h\varepsilon\end{aligned}$$

and consequently comparing tangential and normal parts, we have

$$\delta f_b{}^a = [(\nabla_b \xi^e - \lambda h_b{}^e) f_e{}^a - f_b{}^e (\nabla_e \xi^a - \lambda h_e{}^a) - (\nabla_b \lambda + h_{be} \xi^e) f^a + f_b (\nabla^a \lambda + h_e{}^a \xi^e)]\varepsilon \tag{2.16}$$

and

$$\delta f_b = [(\nabla_b \xi^e - \lambda h_b{}^e) f_e - f_b{}^e (\nabla_e \lambda + h_{ea} \xi^a)]\varepsilon . \tag{2.17}$$

Using (1.21), we can write (2.16) in the form

$$\delta f_b{}^a = [\mathcal{L} f_b{}^a + \lambda (f_b{}^e h_e{}^a - h_b{}^e f_e{}^a) + f_b (\nabla^a \lambda) - (\nabla_b \lambda) f^a]\varepsilon , \tag{2.18}$$

where $\mathcal{L} f_b{}^a$ denotes the Lie derivative of $f_b{}^a$ with respect to the vector field $\xi^a$ in $M^{2n+1}$ [8], that is,

$$\mathcal{L} f_b{}^a = \xi^e \nabla_e f_b{}^a + (\nabla_b \xi^e) f_e{}^a - (\nabla_e \xi^a) f_b{}^e . \tag{2.19}$$

Using (1.22) we can write (2.17) in the form

$$\delta f_b = [\mathcal{L} f_b - \lambda h_b{}^e f_e - f_b{}^e \nabla_e \lambda]\varepsilon , \tag{2.20}$$

where $\mathcal{L} f_b$ denotes the Lie derivative of $f_b$ with respect to $\xi^a$, that is,

$$\mathcal{L} f_b = \xi^e \nabla_e f_b + (\nabla_b \xi^e) f_e . \tag{2.21}$$

Next applying the operator $\delta$ to (1.9) and using $\delta F_i{}^h = 0$, (2.6) and (2.14), we find

$$-F_i{}^h (\nabla^a \lambda + h_b{}^a \xi^b) B_a{}^i \varepsilon = -(\delta f^a) B_a{}^h - f^a \nabla_a \xi^h \varepsilon ,$$

or using (1.8) and (2.8),

$$-(\nabla^e \lambda + h_b{}^e \xi^b)(f_e{}^a B_a{}^h + f_e C^h)\varepsilon$$
$$= -(\delta f^a) B_a{}^h - f^e [(\nabla_e \xi^a - \lambda h_e{}^a) B_a{}^h + (\nabla_e \lambda + h_{ec} \xi^c) C^h]\varepsilon ,$$

from which, comparing the tangential parts,

$$\delta f^a = [(\nabla^e \lambda + h_b{}^e \xi^b) f_e{}^a - f^e (\nabla_e \xi^a - \lambda h_e{}^a)]\varepsilon . \tag{2.22}$$

Using (1.23), we can write (2.22) in the form

$$\delta f^a = [\mathcal{L} f^a + (\nabla^e \lambda) f_e{}^a + \lambda f^e h_e{}^a]\varepsilon , \tag{2.23}$$

where $\mathcal{L} f^a$ denotes the Lie derivative of $f^a$ with respect to $\xi^a$, that is,

$$\mathcal{L} f^a = \xi^c \nabla_c f^a - (\nabla_e \xi^a) f^e . \tag{2.24}$$

Applying the operator $\delta$ to (1.5) and using $\delta g_{ji} = 0$, (2.6) and (2.8), we find

$$\delta g_{cb} = [\mathcal{L} g_{cb} - 2\lambda h_{cb}]\varepsilon , \tag{2.25}$$

where $\mathcal{L} g_{cb}$ denotes the Lie derivative of $g_{cb}$ with respect to $\xi^a$, that is, (2.26) $$\mathcal{L}g_{cb}=\nabla_c\xi_b+\nabla_b\xi_c\,, \qquad (\xi_b=\xi^c g_{cb})\,.$$

Thus summing up we have

THEOREM 2.1. *Under an infinitesimal variation* (2.1) *of a hypersurface of a Kaehlerian manifold, the variations of structure tensors of the almost contact metric structure induced on the hypersurface are given by*

(2.27) $$\begin{cases} \delta f_b{}^a=[\mathcal{L}f_b{}^a+\lambda(f_b{}^e h_e{}^a-h_b{}^e f_e{}^a)+f_b(\nabla^a\lambda)-(\nabla_b\lambda)f^a]\varepsilon\,, \\ \delta f_b\ =[\mathcal{L}f_b-\lambda h_b{}^e f_e-f_b{}^e\nabla_e\lambda]\varepsilon\,, \\ \delta f^a\ =[\mathcal{L}f^a+(\nabla^e\lambda)f_e{}^a+\lambda f^e h_e{}^a]\varepsilon\,, \\ \delta g_{cb}=[\mathcal{L}g_{cb}-2\lambda h_{cb}]\varepsilon\,. \end{cases}$$

COROLLARY 1. *Under an infinitesimal tangential variation* $\bar{x}^h=x^h+\xi^a B_a{}^h\varepsilon$ *of a hypersurface of a Kaehlerian manifold, the variations of the structure tensors of the almost contact metric structure induced on the hypersurface are given by their Lie derivatives with respect to* $\xi^a$.

COROLLARY 2. *Under an infinitesimal normal variation* $\bar{x}^h=x^h+\lambda C^h\varepsilon$ *of a hypersurface of a Kaehlerian manifold, the variations of the structure tensors of the almost contact metric structure induced on the hypersurface are given by*

(2.28) $$\begin{cases} \delta f_b{}^a=[\lambda(f_b{}^e h_e{}^a-h_b{}^e f_e{}^a)+f_b(\nabla^a\lambda)-(\nabla_b\lambda)f^a]\varepsilon\,, \\ \delta f_b\ =-[\lambda h_b{}^e f_e+f_b{}^e\nabla_e\lambda]\varepsilon\,, \\ \delta f^a\ =[(\nabla^e\lambda)f_e{}^a+\lambda f^e h_e{}^a]\,\varepsilon\,, \\ \delta g_{cb}=-2\lambda h_{cb}\varepsilon\,. \end{cases}$$

From (1.27), (1.28), (1.39) and Theorem 2.1, we have

COROLLARY 3. *Suppose that the almost contact metric structure induced on a hypersurface of a Kaehlerian manifold is a Sasakian structure with f-sectional curvature* $\alpha^2$. *Then under an infinitesimal variation* (2.1) *of the hypersurface the variations of the structure tensors are given by*

(2.29) $$\begin{cases} \delta f_b{}^a=[\mathcal{L}f_b{}^a+f_b(\nabla^a\lambda)-(\nabla_b\lambda)f^a]\varepsilon\,, \\ \delta f_b\ =[\mathcal{L}f_b-\lambda h f_b-f_b{}^e\nabla_e\lambda]\varepsilon\,, \\ \delta f^a\ =[\mathcal{L}f^a+(\nabla^e\lambda)f_e{}^a+\lambda h f^a]\varepsilon\,, \\ \delta g_{cb}=[\mathcal{L}g_{cb}-2\lambda\{\alpha g_{cb}+(h-\alpha)f_c f_b\}]\varepsilon\,. \end{cases}$$

## § 3. Parallel variations.

We consider an infinitesimal variation (2.1) of a hypersurface of a Kaehlerian manifold. When the tangent space at a point $(x^h)$ of the original hypersurface and that at the corresponding point $(\bar{x}^h)$ of the varied hypersurface are parallel, we say that the variation is *parallel*.

From (2.5), (2.6) and (2.8), we have

$$\tilde{B}_b{}^h=[\delta_b^a+(\nabla_b\xi^a-\lambda h_b{}^a)\varepsilon]B_a{}^h+(\nabla_b\lambda+h_{be}\xi^e)C^h\varepsilon, \tag{3.1}$$

and cosequently we have

THEOREM 3.1. *In order for an infinitesimal variation* (2.1) *of a hypersurface to be parallel it is necessary and sufficient that*

$$\nabla_b\lambda+h_{be}\xi^e=0. \tag{3.2}$$

COROLLARY 1. *In order for an infinitesimal normal variation* $\bar{x}^h=x^h+\lambda C^h\varepsilon$ *of a hypersurface to be parallel, it is necessary and sufficient that* $\lambda=$*constant.*

From Theorem 2.1 and this corollary, we have

COROLLARY 2. *Under an infinitesimal parallel normal variation* $\bar{x}^h=x^h+\lambda C^h\varepsilon$ ($\lambda=$*constant*), *the variations of the structure tensors are given by*

$$\begin{cases}\delta f_b{}^a=\lambda(f_b{}^e h_e{}^a-h_b{}^e f_e{}^a)\varepsilon,\\ \delta f_b=-\lambda h_b{}^e f_e\varepsilon,\\ \delta f^a=\lambda h_e{}^a f^e\varepsilon,\\ \delta g_{cb}=-2\lambda h_{cb}\varepsilon,\end{cases} \tag{3.3}$$

$\lambda$ *being a constant.*

COROLLARY 3. *Suppose that the almost contact metric structure induced on a hypersurface of a Kaehlerian manifold is a Sasakian structure with f-sectional curvature* $\alpha^2$. *Then under an infinitesimal parallel normal variation of the hypersurface, the variations of the structure tensors are given by*

$$\begin{cases}\delta f_b{}^a=0,\\ \delta f_b=-\lambda h f_b\varepsilon,\\ \delta f^a=\lambda h f^a\varepsilon,\\ \delta g_{cb}=-2\lambda\{\alpha g_{cb}+(h-\alpha)f_c f_b\}\varepsilon,\end{cases} \tag{3.4}$$

$\lambda$ *being a constant.*

## § 4. Variation of the second fundamental form.

To find the variation of the second fundamental form we have used in [**9**] the formula

$$\delta\nabla_c B_b{}^h-\nabla_c\delta B_b{}^h=K_{kji}{}^h\xi^k B_{cb}^{ji}\varepsilon-(\delta\Gamma_{cb}^a)B_a{}^h,$$

where $\delta\Gamma_{cb}^a$ is the variation of the Christoffel symbols $\Gamma_{cb}^a$ of the hypersurface. But here we shall use another method to find the variation of the second fundamental form.

For the varied hypersurface, we have

(4.1) $$\bar{\nabla}_c \bar{B}_b{}^h = \bar{h}_{cb}\bar{C}^h .$$

Thus denoting by $\Gamma^a_{cb}+\delta\Gamma^a_{cb}$ the Christoffel symbols and by $h_{cb}+\delta h_{cb}$ the second fundamental form of the varied hypersurface, we have

(4.2) $$\partial_c \bar{B}_b{}^h + \Gamma^h_{ji}(x+\xi\varepsilon)\bar{B}_c{}^j \bar{B}_b{}^i - (\Gamma^a_{cb}+\delta\Gamma^a_{cb})\bar{B}_a{}^h = (h_{cb}+\delta h_{cb})\bar{C}^h .$$

Substituting (2.2) and (2.12) into (4.2), we find by a straightforward computation

(4.3) $$(\nabla_c\nabla_b\xi^h + K_{kji}{}^h\xi^k B^{ji}_{cb})\varepsilon = [\delta\Gamma^a_{cb} - h_{cb}(\nabla^a\lambda + h_e{}^a\xi^e)\varepsilon]B_a{}^h + (\delta h_{cb})C^h .$$

On the other hand, using (2.7) and (2.8), we have

(4.4) $$\begin{aligned}\nabla_c\nabla_b\xi^h + K_{kji}{}^h\xi^k B^{ji}_{cb} = {} & [\nabla_c\nabla_b\xi^a - \nabla_c(\lambda h_b{}^a) - h_c{}^a(\nabla_b\lambda + h_{be}\xi^e)]B_a{}^h \\ & + [\nabla_c\nabla_b\lambda + \nabla_c(h_{be}\xi^e) + h_{ce}(\nabla_b\xi^e - \lambda h_b{}^e)]C^h \\ & + K_{kji}{}^h B^{kji}_{dcb}\xi^d + \lambda K_{kji}{}^h C^k B^{ji}_{cb} .\end{aligned}$$

Thus from (4.3) and (4.4) we obtain

(4.5) $$\begin{aligned}&\delta\Gamma^a_{cb} - h_{cb}(\nabla^a\lambda + h_e{}^a\xi^e)\varepsilon \\ &= [\nabla_c\nabla_b\xi^a - \nabla_c(\lambda h_b{}^a) - h_c{}^a(\nabla_b\lambda + h_{be}\xi^e) + K_{kji}{}^h B^{kjia}_{dcbh}\xi^d \\ &\qquad + \lambda K_{kji}{}^h C^k B^{jia}_{cbh}]\varepsilon ,\end{aligned}$$

where $B^{jia}_{cbh} = B_c{}^j B_b{}^i B^a{}_h$ and

(4.6) $$\begin{aligned}\delta h_{cb} = {} & [\nabla_c\nabla_b\lambda + \nabla_c(h_{be}\xi^e) + h_{ce}(\nabla_b\xi^e - \lambda h_b{}^e) \\ & + K_{kji}{}^h B^{kji}_{dcb} C_h \xi^d + \lambda K_{kji}{}^h C^k B^{ji}_{cb} C_h]\varepsilon .\end{aligned}$$

Thus using equations of Gauss (1.18) and those of Codazzi (1.19), we have from (4.5)

(4.7) $$\delta\Gamma^a_{cb} = [\mathcal{L}\Gamma^a_{cb} - \nabla_c(\lambda h_b{}^a) - \nabla_b(\lambda h_c{}^a) + \nabla^a(\lambda h_{cb})]\varepsilon ,$$

where $\mathcal{L}\Gamma^a_{cb}$ denotes the Lie derivative of $\Gamma^a_{cb}$, that is,

(4.8) $$\mathcal{L}\Gamma^a_{cb} = \nabla_c\nabla_b\xi^a + K_{dcb}{}^a\xi^d ,$$

and from (4.6)

(4.9) $$\delta h_{cb} = [\mathcal{L}h_{cb} + \nabla_c\nabla_b\lambda + \lambda(K_{kji}{}^h C^k B^{ji}_{cb} C_h - h_{ce}h_b{}^e)]\varepsilon ,$$

where $\mathcal{L}h_{cb}$ denotes the Lie derivative of $h_{cb}$, that is,

(4.10) $$\mathcal{L}h_{cb} = \xi^d\nabla_d h_{cb} + h_{ce}\nabla_b\xi^e + h_{eb}\nabla_c\xi^e .$$

Thus we have

THEOREM 4.1. *Under an infinitesimal variation* (2.1) *of a hypersurface, the variations of the Christoffel symbols and the second fundamental form are respective given by* (4.7) *and* (4.9).

COROLLARY 1. *Under an infinitesimal tangential variation* $\bar{x}^h = x^h + \xi^a B_a{}^h \varepsilon$ *of a hypersurface, the variations of the Christoffel symbols and the second-fundamental tensor are given by their Lie derivatives with respect to* $\xi^a$.

COROLLARY 2. *Under an infinitesimal normal variation* $\bar{x}^h = x^h + \lambda C^h \varepsilon$ *of a hypersuaface, the variations of the Christoffel symbols and the second fundamental tensor are respectively given by*

$$\delta \Gamma^a_{cb} = -[\nabla_c(\lambda h_b{}^a) + \nabla_b(\lambda h_c{}^a) - \nabla^a(\lambda h_{cb})]\varepsilon \tag{4.11}$$

*and*

$$\delta h_{cb} = [\nabla_c \nabla_b \lambda + \lambda(K_{kji}{}^h C^k B^{ji}_{cb} C_h - h_{ce} h_b{}^e)]\varepsilon. \tag{4.12}$$

## § 5. Variation of Sasakian hypersurface with $f$-sectional curvature $\alpha^2$.

We now assume that the almost contact metric structure induced on the hypersurface is a Sasakian structure with $f$-sectional curvature $\alpha^2$. Then we have (1.39) and consequently

$$h_{ce} h_b{}^e = \alpha^2 g_{cb} + (h^2 - \alpha^2) f_c f_b. \tag{5.1}$$

Thus (4.9) reduces to

$$\begin{aligned} \delta h_{cb} = [&\alpha(\mathcal{L} g_{cb}) + (\mathcal{L} h) f_c f_b + (h - \alpha)\{(\mathcal{L} f_c) f_b + f_c(\mathcal{L} f_b)\} \\ &+ \nabla_c \nabla_b \lambda + \lambda\{K_{kji}{}^h C^k B^{ji}_{cb} C_h - \alpha^2 g_{cb} - (h^2 - \alpha^2) f_c f_b\}]\varepsilon. \end{aligned} \tag{5.2}$$

On the other hand, we have

$$\delta h = \delta(h_{cb} f^c f^b) = (\delta h_{cb}) f^c f^b + 2 h_{cb} (\delta f^c) f^b.$$

Thus substituting the third equation of (2.29) and (5.2) into the equation above, we find

$$\delta h = [\mathcal{L} h + (\nabla_e \nabla_d \lambda + \lambda K_{kji}{}^h C^k B^{ji}_{ed} C_h) f^e f^d + \lambda h^2]\varepsilon. \tag{5.3}$$

Now in order for the variation to carry the Sasakian hypersurface with $f$-sectional curvature $\alpha^2$ into a Sasakian hypersurface with $f$-sectional curvature $\alpha^2 + \delta\alpha^2$, it is necessary and sufficient that $\delta h_{cb}$ given by (5.2) is equal to

$$\delta h_{cb} = \delta[\alpha g_{cb} + (h - \alpha) f_c f_b], \tag{5.4}$$

that is, to

(5.5) $$\delta h_{cb}=(\delta\alpha)g_{cb}+\alpha(\delta g_{cb})+(\delta h-\delta\alpha)f_c f_b+(h-\alpha)\{(\delta f_c)f_b+f_c(\delta f_b)\}\,.$$

Substituting the second and the fourth equations of (2.29) into (5.5), we see that (5.4) is equivalent to

(5.6) $$\begin{aligned}\delta h_{cb}=[&\alpha(\mathcal{L}g_{cb}-2\lambda\alpha g_{cb})-2\lambda(h^2-\alpha^2)f_c f_b\\ &+(h-\alpha)\{(\mathcal{L}f_c)f_b+f_c(\mathcal{L}f_b)-(f_c{}^e f_b+f_b{}^e f_c)(\nabla_e\lambda)\}]\varepsilon\\ &+(\delta\alpha)(g_{cb}-f_c f_b)+(\delta h)f_c f_b\,.\end{aligned}$$

Comparing (5.2) with (5.6), we find

$$\begin{aligned}&[\nabla_c\nabla_b\lambda+\lambda\{K_{kji}{}^h G^k B^{ji}_{cb}C_h+\alpha^2 g_{cb}+(h^2-\alpha^2)f_c f_b\}\\ &\quad+(h-\alpha)\{f_c{}^e f_b+f_b{}^e f_c\}(\nabla_e\lambda)\\ &\quad-(\delta\alpha)g_{cb}-(\delta h-\mathcal{L}h\varepsilon-\delta\alpha)f_c f_b=0\,,\end{aligned}$$

from which, using (5.3)

(5.7) $$\begin{aligned}&[\nabla_c\nabla_b\lambda+\lambda\{K_{kji}{}^h C^k B^{ji}_{cb}C_h+\alpha^2(g_{cb}-f_c f_b)\}\\ &-(\nabla_e\nabla_d\lambda+\lambda K_{kji}{}^h C^k B^{ji}_{ed}C_h)f^e f^d f_c f_b+(h-\alpha)(f_c{}^e f_b+f_b{}^e f_c)(\nabla_e\lambda)]\varepsilon\\ &-(\delta\alpha)(g_{cb}-f_c f_b)=0\,.\end{aligned}$$

Conversely if $\lambda$ satisfies an equation of the form

(5.8) $$\begin{aligned}&\nabla_c\nabla_b\lambda+\lambda\{K_{kji}{}^h C^k B^{ji}_{cb}C_h+\alpha^2(g_{cb}-f_c f_b)\}\\ &\quad-(\nabla_e\nabla_d\lambda+\lambda K_{kji}{}^h C^k B^{ji}_{ed}C_h)f^e f^d f_c f_b\\ &\quad+(h-\alpha)(f_c{}^e f_b+f_b{}^e f_c)(\nabla_e\lambda)=\alpha'(g_{cb}-f_c f_b)\,,\end{aligned}$$

$\alpha'$ being a function, then (5.2) becomes

$$\begin{aligned}\delta h_{cb}=[&\alpha(\mathcal{L}g_{cb}-2\lambda\alpha g_{cb})-2\lambda(h^2-\alpha^2)f_c f_b\\ &+(h-\alpha)\{(\mathcal{L}f_c)f_b+f_c(\mathcal{L}f_b)-(f_c{}^e f_b+f_b{}^e f_c)(\nabla_e\lambda)\}\\ &+(\nabla_e\nabla_d\lambda+\lambda K_{kji}{}^h C^k B^{ji}_{ed}C_h)f^e f^d+\mathcal{L}h+\lambda h^2\}f_c f_b\\ &+\alpha'(g_{cb}-f_c f_b)]\varepsilon\,,\end{aligned}$$

which shows, together with (5.3), that (5.6) is satisfied, if we put $\delta\alpha=\alpha'\varepsilon$. Thus we have

THEOREM 5.1. *In order for an infinitesimal variation* (2.1) *to carry a Sasakian hypersurface with f-sectional curvature* $\alpha^2$ *into a Sasakian hypersurface with f-sectional curvature* $\alpha^2+\delta\alpha^2$, *it is necessary and sufficient that the function* $\lambda$ *satisfies* (5.7).

COROLLARY 1. *In order for an infinitesimal normal parallel variation* $\bar{x}^h=$

$x^h+\lambda C^h\varepsilon$, $\lambda=$const. $\neq 0$, *to carry a Sasakian hypersurface with f-sectional curvature* $\alpha^2$ *into a Sasakian hypersurface with f-sectional curvature* $\alpha^2+\delta\alpha^2$, *it is necessary and sufficient that the original hypersurface satisfies*

$$\text{(5.9)} \qquad \lambda[\{K_{kji}{}^h C^k B^{ji}_{cb} C_h+\alpha^2(g_{cb}-f_c f_b)\} - K_{kji}{}^h C^k B^{ji}_{ed} C_h f^e f^d f_c f_b]\varepsilon-(\delta\alpha)(g_{cb}-f_c f_b)=0\,.$$

COROLLARY 2. *An infinitesimal normal parallel variation* $\bar{x}^h=x^h+\lambda C^h\varepsilon$ *of a Sasakian hypersurface with f-sectional curvature* $\alpha^2$ *of a flat Kaehlerian manifold carries it into a Sasakian hypersurface with f-sectional curvature* $\alpha^2+\delta\alpha^2$, *if and only if* $\lambda\alpha^2\varepsilon=\delta\alpha$.

COROLLARY 3. *An infinitesimal normal parallel variation never carries a Sasakian hypersurface* (*with f-sectional curvature* 1) *of a flat Kaehlerian manifold into a Sasakian hypersurface.*

Suppose that the ambient Kaehlerian manifold $M^{2n+2}$ is of constant holomorphic sectional curvature $k$, that is,

$$K_{kji}{}^h=\frac{k}{4}[\delta^h_k g_{ji}-\delta^h_j g_{ki}+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_{kj}F_i{}^h]\,,$$

where $F_{ji}=F_j{}^t g_{ti}$. Then (5.9) gives

$$\left\{\lambda\left(\frac{k}{4}+\alpha^2\right)\varepsilon-\delta\alpha\right\}(g_{cb}-f_c f_b)=0\,.$$

Thus we have

COROLLARY 4. *If an infinitesimal normal parallel variation* $\bar{x}^h=x^h+\lambda C^h\varepsilon$ *carries a Sasakian hypersurface with f-sectional curvature* $\alpha^2$ *of a Kaehlerian manifold of constant holomorphic sectional curvature* $k$ *into a Sasakian hypersurface with f-sectional curvature* $\alpha^2+\delta\alpha^2$, *then we have*

$$\delta\alpha=\lambda\left(\frac{k}{4}+\alpha^2\right)\varepsilon\,.$$

COROLLARY 5. *If an infinitesimal normal parallel variation carries a Sasakian hypersurface* (*with f-sectional curvature* 1) *of a Kaehlerian manifold of constant holomorphic sectional curvature* $k$ *into a Sasakian hypersurface, then we have* $k=-4$.

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Kentaro YANO
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo, Japan