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A note on Ξ-product

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§ 1. Introduction.

Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces. We denote the box product space by $B_\alpha X_\alpha$. For $p\in B_\alpha X_\alpha$ let Ξ_p be the subspace $\{x\in B_\alpha X_\alpha : x_\alpha\neq p_\alpha$ for at most finitely many $\alpha\}$ of $B_\alpha X_\alpha$.

Recently E. K. van Douwen [4] showed Ξ_p is stratifiable if each X_α is a metrizable space and p is any point of $B_\alpha X_\alpha$.

In this paper we shall show the followings.

(A) *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of metrizable spaces and p be any point of $B_\alpha X_\alpha$. Then Ξ_p is an M_1-space* (Corollary 3.3).

(B) *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of M_2-spaces and p be any point of $B_\alpha X_\alpha$. Then Ξ_p is an M_2-space* (Corollary 3.4).

(C) *There exists an M_1-space X and a closed subset A of X such that (X, A) is not semi-canonical* (Example 4.1).

Both (A) and (B) strengthen the theorem of E. K. van Douwen [4]. (C) shows that the Lemma of M. R. Cauty [3] is false.

Throughout of this paper all topological space are T_1 and N denote the natural numbers.

The auther thanks the referee for pointing out that (B) follows from our original version.

§ 2. Preliminaries.

The symbol $\Pi_\alpha X_\alpha$ denotes the set product of the family of the sets $\{X_\alpha\}_{\alpha\in A}$.

LEMMA 2.1. *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces and let $\mathfrak{P}_\alpha$, $\alpha\in A$, be a cushioned collection* (see [3]) *consisting of ordered pairs of subsets $P_\alpha=(P_\alpha^1, P_\alpha^2)$ of X_α. Then for any subspace Y of $B_\alpha X_\alpha$ $\{(\Pi_\alpha P_\alpha^1\cap Y, \Pi_\alpha P_\alpha^2\cap Y) : P=(P_\alpha^1, P_\alpha^2)\in\Pi_\alpha\mathfrak{P}_\alpha\}$ is a cushioned collection of Y.*

PROOF. For a given arbitrary subfamily $\mathfrak{P}$ of $\Pi_\alpha\mathfrak{P}_\alpha$, we have to show that

$$cl_Y(\cup\{\Pi_\alpha P_\alpha^1\cap Y : P\in\mathfrak{P}\})\subset\cup\{\Pi_\alpha P_\alpha^2\cap Y : P\in\mathfrak{P}\}\,.$$

Let p be any point of $Y-\cup\{\Pi_\alpha P_\alpha^2\cap Y : P\in\mathfrak{P}\}$. For any $\alpha\in A$, let

$$\mathfrak{Q}_\alpha=\{P\in\mathfrak{P}: p_\alpha\notin P_\alpha^2\}\ .$$

Then clearly $\mathfrak{P}=\cup_\alpha\mathfrak{Q}_\alpha$. For every $\alpha\in A$, let $U_\alpha=X_\alpha-cl_{X_\alpha}(\cup\{P_\alpha^1: P\in\mathfrak{Q}_\alpha\})$. Then U_α is a neighborhood, because $p_\alpha\in X_\alpha-\cup\{P_\alpha^2: P\in\mathfrak{Q}_\alpha\}$ and $cl_{X\alpha}(\cup\{P_\alpha^1: P\in\mathfrak{Q}_\alpha\})\subset\cup\{P_\alpha^2: P\in\mathfrak{Q}_\alpha\}$.

So $U=\Pi_\alpha U_\alpha$ is a neighborhood of p. Since $\mathfrak{P}=\cup_\alpha\mathfrak{Q}_\alpha$, $U\cap\cup\{\Pi_\alpha P_\alpha^1\cap Y: P\in\mathfrak{P}\}=\emptyset$. This completes the proof.

For a *closure preserving* collection (see [**3**]), the lemma does not generally holded (see Example 2.3). We have to restrict to a subspace Y of the following special type.

Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces, $p\in B_\alpha X_\alpha$ and λ any infinite cardinal number. We denote by $\Xi_p(\lambda)$ the subspace $\{x\in B_\alpha X_\alpha$: cardinality of $\{\alpha\in A: x_\alpha\neq p_\alpha\}$ is less than $\lambda\}$. Clearly $\Xi_p=\Xi_p(\aleph_0)$.

LEMMA 2.2. *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces and for all $\alpha\in A$ let $\mathfrak{V}_\alpha$ be a closure preserving collection of X_α. Then for any $p\in B_\alpha X_\alpha$ and any infinite cardinal number λ, $\{\Pi_\alpha V_\alpha\cap\Xi_p(\lambda):(V_\alpha)_{\alpha\in A}\in\Pi_\alpha\mathfrak{V}_\alpha\}$ is a closure preserving collection of $\Xi_p(\lambda)$.*

PROOF. For all $V=(V_\alpha)_{\alpha\in A}\in\Pi_\alpha\mathfrak{V}_\alpha$, let $A(V)=\{\alpha\in A: p_\alpha\notin V_\alpha\}$.

Then it is easy to show the followings.

(1) If $\Pi_\alpha V_\alpha\cap\Xi_p(\lambda)\neq\emptyset$, then cardinality of $A((V_\alpha)_{\alpha\in A})$ is less than λ.

(2) $\Xi_p(\lambda)$ is a closed subspace of $B_\alpha X_\alpha$.

Let $\mathfrak{V}_0=\{V=(V_\alpha)_{\alpha\in A}\in\Pi_\alpha V_\alpha:\Pi_\alpha V_\alpha\cap\Xi_p(\lambda)\neq 0\}$. We have to show that

$$\{\Pi_\alpha V_\alpha\cap\Xi_p(\lambda): V=(V_\alpha)_{\alpha\in A}\in\mathfrak{V}_0\}$$

is a closure preserving collection of $\Xi_p(\lambda)$. For this end, we have to show

$$cl(\cup\{\Pi_\alpha V_\alpha\cap\Xi_p(\lambda): V\in\mathfrak{V}\})\subset\cup\{cl(\Pi_\alpha V_\alpha\cap\Xi_p(\lambda)): V\in\mathfrak{V}\}$$

for any subcollection $\mathfrak{V}$ of $\mathfrak{V}_0$ (by (2), we may consider that the closure is taken in $B_\alpha X_\alpha$).

Let q be any point of $\Xi_p(\lambda)-\cup\{cl(\Pi_\alpha V_\alpha\cap\Xi_p(\lambda)): V\in\mathfrak{V}\}$. For $\alpha\in A$, let

$$\mathfrak{W}_\alpha=\{V\in\mathfrak{V}: q_\alpha\notin cl V_\alpha\}\ .$$

Then we have $\mathfrak{V}=\cup_\alpha\mathfrak{W}_\alpha$. Because, assume $V\in\mathfrak{V}-\cup_\alpha\mathfrak{W}_\alpha$ and put

$$B=\{\alpha\in A: q_\alpha\neq p_\alpha\}\cup A(V)\ .$$

By (1), cardinality of B is less than λ and hence

$$q\in\Pi_{\alpha\in B}cl V_\alpha\times\Pi_{\alpha\in A-B}\{p_\alpha\}=cl(\Pi_{\alpha\in B}V_\alpha\times\Pi_{\alpha\in A-B}\{p_\alpha\})\subset cl(\Pi_\alpha V_\alpha\cap\Xi_p(\lambda))\ .$$

But this is absurd. Now, for each $\alpha\in A$, let $U_\alpha=X_\alpha-cl(\cup\{V_\alpha: V\in\mathfrak{W}_\alpha\})$. Then $U=\Pi_\alpha U_\alpha$ is a neighborhood of q and $U\cap\cup\{\Pi_\alpha V_\alpha\cap\Xi_p(\lambda): V\in\mathfrak{V}\}=\emptyset$. This completes the proof.

EXAMPLE 2.3. *There is a space X and a closed subspace A of X and there* *exists a closure preserving open collection* $\mathfrak{V}$ *of* X *such that* $\{V\cap A : V\in\mathfrak{V}\}$ *is not a closure preserving collection of* A.

Let I be the unit interval. We denote by X the quotient space obtained from the product $I\times N$ by identifying $\{0\}\times N$ to a point (denoted by *). Give X a natural metric topology. Finally let $A=\{*\}\cup\{1/n, n) : n\in N\}$ and

$$\mathfrak{V}=\{\{x : 0<x\leqq 1\}\times\{n\} : n\in N\} .$$

Then A and $\mathfrak{V}$ have desired properties.

§ 3. M_i-spaces ($i=1, 2, 3$).

We investigate the space Ξ_p in case each X_α is an M_i-space ($i=1, 2, 3$). See [1] and [2] for M_i-spaces, $i=1, 2, 3$, or stratifiable spaces.

THEOREM 3.1. *Let* $\{X_\alpha\}_{\alpha\in A}$ *be a family of* M_1 *(resp.* M_2*)-spaces and* p *be a point of* $B_\alpha X_\alpha$. *If for each* $\alpha\in A$ p_α *has a closure preserving local open neighborhood base (resp. local neighborhood base), then* Ξ_p *is an* M_1 *(resp.* M_2*)-space.*

PROOF. Ξ_p is clearly a regular space. We have to show that Ξ_p has a σ-closure preserving base (resp. quasi-base).

From the hypothesis of the theorem, X_α has a σ-closure preserving base (resp. quasi-base) $\bigcup_{n\in N}\mathfrak{V}_n^\alpha$ such that

(1) $\mathfrak{V}_n^\alpha$ is closure preserving for each $n\in N$.

(2) $\mathfrak{V}_n^\alpha$ contains a local open neighborhood base (resp. local neighborhood base) of p_α.

Furthermore we can assume

(3) $\mathfrak{V}_n^\alpha\subset\mathfrak{V}_{n+1}^\alpha$ for each $n\in N$.

Now we let

$$\mathfrak{V}_n=\{\Pi_\alpha V_\alpha\cap\Xi_p : (V_\alpha)_{\alpha\in A}\in\Pi_\alpha\mathfrak{V}_n^\alpha\} .$$

Then by (1) and the lemma 2.2, $\mathfrak{V}_n$ is a closure preserving collection of Ξ_p. By (2) and (3), it is easy to show that $\bigcup_{n\in N}\mathfrak{V}_n$ is a base (resp. quasi-base) of Ξ_p. This completes the proof.

COROLLARY 3.2. *Let* $\{X_\alpha\}_{\alpha\in A}$ *be a family of first countable* M_1*-spaces. Then* Ξ_p *is an* M_1*-space for any* $p\in B_\alpha X_\alpha$.

PROOF. Each point p_α has a decreasing countable open neighborhood base. This collection is clearly closure preserving. Hence the corollary follows from Theorem 3.1.

COROLLARY 3.3. *Let* $\{X_\alpha\}_{\alpha\in A}$ *be a family of metrizable spaces and* p *be any point of* $B_\alpha X_\alpha$. *Then* Ξ_p *is an* M_1*-space.*

COROLLARY 3.4. *Let* $\{X_\alpha\}_{\alpha\in A}$ *be a family of* M_2*-spaces and* p *be any point of* $B_\alpha X_\alpha$. *Then* Ξ_p *is an* M_2*-space.*

PROOF. This follows from Theorem 3.1 and Lemma 7.3 of [3].

It is open whether Ξ_p is an M_1-space if each X_α is M_1, more precisely, whether each point of an M_1-space has a closure preserving local open neighborhood base or not.

In the paper of [4], E. K. van Douwen questioned whether Ξ_p is stratifiable if X_α's are stratifiable. Quite recently, C. J. R. Borges has announced that this result is true in Notice Amer. Math. Soc. **22** (1975). We can give this proof by a similar way to the proof of (A) and (B) as follows.

LEMMA 3.5. *Let X be a stratifiable space and A be a closed subset of X. Then there exists a stratification* ([**1**]) *H of X such that*

(1) *$H(O, n)\subset H(O, n+1)$ for each open subset O and each $n\in N$.*

(2) *$A\subset H(O, 1)$ if $A\subset O$.*

PROOF. Let K be a stratification of X and by the Theorem 2.5 of [**5**], G be a monotone normality operater for X.

If $A\not\subset O$, let $H(O, n)=\bigcup_{i=1}^{n} K(O, i)$ for each $n\in N$.

If $A\subset O$, we define inductively as follows.

$$H(O, 1)=G(A\cup cl(K(O, 1), X-O)$$

and

$$H(O, n+1)=G(cl(H(O, n)\cup\bigcup_{i=1}^{n+1} K(O, i)), X-O)$$

for each $n\in N$.

Then H is a stratification satisfying the required condition. This completes the proof.

THEOREM 3.6 (C. J. R. Borges). *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of stratifiable spaces and p be any point of $B_\alpha X_\alpha$. Then Ξ_p is a stratifiable space.*

PROOF. We have to show that Ξ_p has a σ-cushioned pair-base. For each $\alpha\in A$, by Lemma 3.5, choose a stratification H_α of X_α such that

(1) $H_\alpha(O, n)\subset H_\alpha(O, n+1)$ for each open set O in X_α and $n\in N$.

(2) If $p_\alpha\in O$ then $p_\alpha\in H_\alpha(O, 1)$.

Let $\mathfrak{T}_\alpha$ be the set of all open subsets of X_α. For each $n\in N$ and each $O=(O_\alpha)_{\alpha\in A}\in\Pi_\alpha\mathfrak{T}_\alpha$, let $P_n(O)=\Pi_\alpha H_\alpha(O_\alpha, n)\cap\Xi_p$, $P(O)=\Pi_\alpha O_\alpha\cap\Xi_p$ and

$$\mathfrak{P}_n(O)=(P_n(O), P(O)).$$

Then, by Lemma 3.5. $\mathfrak{P}_n=\{\mathfrak{P}_n(O): O\in\Pi_\alpha\mathfrak{T}_\alpha\}$ forms a cushioned collection of Ξ_p. It is easy to show that $\bigcup_{n\in N}\mathfrak{P}_n$ is a pair-base of Ξ_p. This completes the proof.

§ 4. Semi-canonical pair (X, A).

A pair (X, A) is said to be *semi-canonical* (see [3]) if A is a closed subset of X and there exists an open cover $\mathfrak{W}$ of $X-A$ satisfying the following condition, for any point a of A and any neighborhood V of a in X, there is a neighborhood U in X such that if $U\cap W\neq\emptyset$ for $W\in\mathfrak{M}$, $W\subset V$.

EXAMPLE 4.1. *There exists an M_1-space X and a closed subset A of X such that (X, A) is not semi-canonical.*

Let $Y=B_{n\in N}Q_n$ where $Q_n=\{\text{rationals}\}$ with usual topology.

Let $p\in Y$ be the point whose coordinates are 0 and let $X=\Xi_p$ and $A=\{p\}$. We shall show that for any open cover $\mathfrak{W}$ of $X-A$ there is a neighborhood V of p such that for any neighborhood U of p, there exists $W\in\mathfrak{W}$ satisfying $U\cap W\neq\emptyset$ and $W\not\subset V$.

Since $X-A$ is a countable set, we let

$$X-A=\{x(n)\}_{n\in N} \quad\text{and}\quad x(n)=(x_k(n))_{k\in N}\in B_{n\in N}Q_n\,.$$

Let $\mathfrak{W}$ be any open cover of $X-A$. Then we can choose a sequence $\{e_k(n)\}_{k\in N}$ of positive numbers and an element $W_n\in\mathfrak{W}$ such that

$$x(n)\in\{x\in X: |x_k-x_k(n)|<e_k(n) \text{ for each } k\in N\}\subset W_n\,.$$

Without loss of generality we can assume

(1) $e_n(n)<|x_n(n)|$ if $x_n(n)\neq 0$.

Let $V=\{x\in X: |x_k|<(1/2)e_k(k)$ for each $k\in N\}$, then V is a neighborhood of p, then there is an $n\in N$ such that

(2) $x(n)\in V\cap U$, since p is not an isolated point of X.

Let us show that $x_n(n)=0$. Because, assume $x_n(n)\neq 0$, then (2) implies $|x_n(n)|<(1/2)e_n(n)$, and (1) implies $|x_n(n)|>e_n(n)$. This is absurd. Now, we choose a rational number r such that $(1/2)e_n(n)<r<e_n(n)$ and let $z\in X$ be a point defined as follows

$$z_k=\begin{cases} x_k(n) & \text{if } k\neq n \\ r & \text{if } k=n\,. \end{cases}$$

Then, clearly $x_n(n)\in U\cap W_n$ and $z\in W_n-V$. This completes the proof.

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