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On some additive divisor problems

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§ 1. Introduction.

Let $f(n)$ be a multiplicative function, and let $d(n)$ and $r(n)$ be the number of divisors and the number of representations as a sum of two squares of n, respectively. We consider the problem to find the asymptotic behaviour of the sums

$$(1.1)\qquad \sum_{n\leqq N} d(n)f(n+a),$$

$$\text{(as } N\to\infty)$$

$$(1.2)\qquad \sum_{n\leqq N} r(n)f(n+a),$$

where a is an arbitrary non-zero integer. The conjugate sums

$$(1.3)\qquad \sum_{n\leqq N} d(n)f(N-n),$$

$$\text{(as } N\to\infty)$$

$$(1.4)\qquad \sum_{n\leqq N} r(n)f(N-n),$$

where N runs over integers, may also be considered.

These problems have been treated by various authors mainly in the case $f(n)=d_k(n)$ the number of representations of n as a product of k factors. The first general result was obtained by Linnik [**4**], who proved the asymptotic formula for the sum (1.1) in the case of $a=1$ and $f(n)=d_k(n)$ with arbitrary $k\geqq 2$ by appealing to his own powerful 'Dispersion Method'. In his method the work of A. Weil on the Kloosterman sum plays vital part. Later his result was extended by Bredikhin [**2**].

The complexity and difficulty of the dispersion method compelled us to seek a different way, and we have found in [**6**] that the improved large sieve method due to Bombieri [**1**] enables us in some cases to simplify the proof as a whole as well as to dispense with Weil's work. This way of investigation has been further developed and strengthened by Wolke [**10**] [**11**], who has solved the fairly general problem (1.1) in which f is restricted only by the size of $f(n)$ and by the average behaviour of $f(p)$, p a prime. In both Wolke's and our works the main concern is the proof of the analogue for $f(n)$ of the mean prime number theorem of Bombieri [1] (see also Vinogradov [9]). This in turn is reduced to the investigation of the function

$$F(s, \chi)=\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s}} f(n),$$

where χ is a Dirichlet character and $s=\sigma+it$ a complex variable. In our case $F(s, \chi)$ was a power of $L(s, \chi)$ the Dirichlet L-function, and we needed a good estimation of the sum

$$\sum_{q \leqq Q} \sum_{\chi(\bmod q)}^{*}\left|L\left(\frac{1}{2}+it, \chi\right)\right|^{2k} \qquad (k=1, 2, \cdots), \tag{1.5}$$

where $\sum^{*}$ denotes the sum over all primitive characters mod q. But this has been settled only for $k \leqq 4$ (e. g. [8]), and our method failed to give a general result. But Wolke took an ingenious way in which he divided $F(s, \chi)$ and the range of s into several parts and then estimated $F(s, \chi)$ by appealing to the large sieve method.

Now the purpose of the present paper is to show that there is a considerably simpler way of investigation than most of previous works in this field. Our method needs only an easy estimate of

$$\sum_{\chi(\bmod q)}\left|L\left(\frac{1}{2}+it, \chi\right)\right|^{2}$$

instead of (1.5), and also the large sieve method is applied in a very simple manner. To make this clear we shall first give a quick proof of the asymptotic formula of Linnik. And later we shall outline the proof of an iteration formula from which the analogue of Bombieri's theorem for $f(n)$ follows immediately, provided that $f(n)$ satisfies a general (but not so general as Wolke's) condition. As for applications we shall show several asymptotic formulas without proof.

§ 2. Linnik's asymptotic formula.

We shall use following lemmas:

LEMMA 1. *For any complex numbers a_n we have*

$$\sum_{q \leqq Q} \sum_{\chi(\bmod q)}^{*} \left|\sum_{n \leqq N} a_n \chi(n)\right|^{2} \ll (Q^{2}+N) \sum_{n \leqq N} |a_n|^{2}.$$

LEMMA 2.

$$\sum_{\chi(\bmod q)}\left|L\left(\frac{1}{2}+it, \chi\right)\right|^{2} \ll q(|t|+1) \log^{2}(q(|t|+1)).$$

LEMMA 3. *Let*

$$\sqrt{x} \leqq y < x, \qquad q \leqq y^{1-\varepsilon}, \qquad (q, l)=1.$$

Then we have, for any positive integer α,

$$\sum_{\substack{n\equiv l \pmod q\\ x-y<n\leqq x}} d_k^{\alpha}(n) \ll \frac{y}{q}(\log x)^{k^{\alpha}-1},$$

where ε $(<1/2)$ *is an arbitrary positive constant.*

Lemma 1 is the large sieve inequality, and for its simple proof see [3]. Lemma 2 is an easy fact which can be proved only by the Pólya-Vinogradov theorem and the usual character sum method. Lemma 3 is one of the fundamental tools in the theory of additive divisor problems, and this has been essentially proved in [**5**].

We now start the proof of Linnik's asymptotic formula. Let

$$D_k(x;q)=\sum_{\substack{n\equiv 1 \pmod q\\ n\leqq x}} d_k(n),$$

and let A be a positive constant to be determined later. Then we have

$$\begin{aligned}
&\sum_{p\leqq N} d(n)d_k(n+1)\\
(2.1)\qquad &=2\sum_{q\leqq\sqrt{N}\log^{-A}N} D_k(N;q)+O\Big(\sum_{\sqrt{N}\log^{-A}N\leqq q\leqq\sqrt{N}\log^{A}N} D_k(N;q)\Big)\\
&=2S_k(N)+O(N(\log N)^{k-1}\log\log N), \qquad \text{say},
\end{aligned}$$

where we used Lemma 3. We decompose $D_k(N;q)$ into two parts as follows:

$$\begin{aligned}
D_k(N;q)&=\sum_{\substack{n\equiv 1 \pmod q\\ n\leqq N}}\ \sum_{mu=n} d_{k-1}(m)\\
&=\sum_{\substack{n\equiv 1 \pmod q\\ n\leqq N}}\Big\{\sum_{\substack{mu=n\\ m\leqq N\log^{-2A}N}} d_{k-1}(m)+\sum_{\substack{mu=n\\ N\log^{-2A}N<m\leqq N}} d_{k-1}(m)\Big\}\\
&=D_k^{(1)}(N;q)+D_k^{(2)}(N;q), \qquad \text{say}.
\end{aligned}$$

In the sum $D_k^{(2)}(N;q)$ we have $u\leqq(\log N)^{2A}$, and so

$$\sum_{q\leqq\sqrt{N}\log^{-A}N} D_k^{(2)}(N;q)\leqq \sum_{\substack{(u,q)=1\\ u\leqq\log^{2A}N}}\ \sum_{q\leqq\sqrt{N}}\ \sum_{\substack{m\equiv\bar{u} \pmod q\\ m\leqq N/u}} d_{k-1}(m),$$

where $\bar{u}u\equiv 1 \pmod q$. Thus again by Lemma 3 this is

$$\begin{aligned}
&\ll \sum_{u\leqq\log^{2A}N}\ \sum_{q\leqq\sqrt{N}} \frac{N}{qu}(\log N)^{k-2}\\
&\ll N(\log N)^{k-1}\log\log N.
\end{aligned}$$

This gives

(2.2)
$$S_k(N)=\sum_{q\leqq\sqrt{N}\log^{-A}N} D_k^{(1)}(N\,;\,q)+O(N(\log N)^{k-1}\operatorname{loglog} N)$$
$$=S_k^{(1)}(N)+O(N(\log N)^{k-1}\operatorname{loglog} N)\,,\qquad \text{say}\,.$$

We put

(2.3)
$$T_k^{(0)}(y\,;\,q)=\sum_{\substack{(n,q)=1\\ n\leqq y}}\ \sum_{\substack{mu=n\\ m\leqq N\log^{-2A}N}} d_{k-1}(m)\,,$$
$$T_k(y\,;\,q)=\sum_{\substack{n\equiv 1\ (\mathrm{mod}\ q)\\ n\leqq y}}\ \sum_{\substack{mu=n\\ m\leqq N\log^{-2A}N}} d_{k-1}(m)\,,$$

and

$$\varDelta_k(y\,;\,q)=T_k(y\,;\,q)-\frac{1}{\varphi(q)}T_k^{(0)}(y\,;\,q)\,,$$

where $\varphi(q)$ is the Euler function. Then obviously we have

$$D_k^{(1)}(N\,;\,q)=T_k(N\,;\,q)\,,$$

and so we may write

(2.4)
$$S_k^{(1)}(N)=\sum_{q\leqq\sqrt{N}\log^{-A}N}\frac{1}{\varphi(q)}T_k^{(0)}(N\,;\,q)+\sum_{q\leqq\sqrt{N}\log^{-A}N}\varDelta_k(N\,;\,q)$$
$$=S_k^{(0)}(N)+S_k^{(2)}(N)\,,\qquad \text{say}\,.$$

Further we put

$$\tilde{\varDelta}_k(y\,;\,q)=\int_1^y \varDelta_k(w\,;\,q)\frac{dw}{w}\,.$$

Then, by the standard way of smoothening, we get, for any $0<\delta<1$,

$$|\varDelta_k(y\,;\,q)|\ll\delta^{-1}|\tilde{\varDelta}_k(ye^{\delta}\,;\,q)-\tilde{\varDelta}_k(y\,;\,q)|+|T_k(ye^{\delta}\,;\,q)-T_k(y\,;\,q)|$$
$$+\frac{1}{\varphi(q)}|T_k^{(0)}(ye^{\delta}\,;\,q)-T_k^{(0)}(y\,;\,q)|\,.$$

Thus we have, by Lemma 3,

$$|\varDelta_k(N\,;\,q)|\ll\delta^{-1}\max_{N\leqq y\leqq Ne^{\delta}}|\tilde{\varDelta}_k(y\,;\,q)|+\delta\frac{N}{q}(\log N)^k\,,$$

provided

(2.5)
$$(\delta N)^{1-\varepsilon}\geqq\sqrt{N}\,.$$

From this and (2.4), we have

(2.6)
$$S_k^{(2)}(N)\ll\delta^{-1}\sum_{q\leqq\sqrt{N}\log^{-A}N}\ \max_{N\leqq y\leqq Ne^{\delta}}|\tilde{\varDelta}_k(y\,;\,q)|+\delta N(\log N)^{k+1}$$
$$=\delta^{-1}S_k^{(3)}(N)+\delta N(\log N)^{k+1}\,,\qquad \text{say}\,.$$

Now we see easily that

$$\tilde{\Delta}_k(y\,;\,q)=\frac{1}{2\pi i\varphi(q)}\sum_{\chi\neq\chi_0\,(\mathrm{mod}\,q)}\int_{(2)}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\Big(\sum_{\substack{mu=n\\ m\leqq N\log^{-2A}N}}d_{k-1}(m)\Big)\frac{y^s}{s^2}\,ds$$

$$=\frac{1}{2\pi i\varphi(q)}\sum_{\chi\neq\chi_0\,(\mathrm{mod}\,q)}\int_{(2)}L(s,\chi)M_k(s,\chi)\frac{y^s}{s^2}\,ds\,,$$

where χ_0 is the principal character mod q, and

$$M_k(s,\chi)=\sum_{m\leqq N\log^{-2A}N}\frac{\chi(m)}{m^s}d_{k-1}(m)\,.$$

By an obvious reason we can change the line of integration from $\mathrm{Re}(s)=2$ to $\mathrm{Re}(s)=1/2$. So we have

$$\max_{N\leqq y\leqq Ne^{\delta}}|\tilde{\Delta}_k(y\,;\,q)|\ll\sqrt{N}\log N\frac{1}{q}\sum_{\chi\neq\chi_0\,(\mathrm{mod}\,q)}\int_{(\frac{1}{2})}|L(s,\chi)M_k(s,\chi)|\frac{|ds|}{|s|^2}\,.$$

From this and (2.6), we get

$$(2.7)\quad \begin{aligned} S_k^{(3)}(N)&\ll\sqrt{N}\log N\int_{(\frac{1}{2})}\sum_{q\leqq\sqrt{N}\log^{-A}N}\frac{1}{q}\sum_{\chi\neq\chi_0\,(\mathrm{mod}\,q)}|L(s,\chi)M_k(s,\chi)|\frac{|ds|}{|s^2|}\\ &=\sqrt{N}\log N\int_{(\frac{1}{2})}H(s,N)\frac{|ds|}{|s|^2},\qquad \text{say}\,.\end{aligned}$$

Let $\chi(\mathrm{mod}\,q)$ be induced by the primitive character $\chi^*(\mathrm{mod}\,q^*)$. Then we have $\chi=\chi^*\chi_{q_0}$, where χ_{q_0} is the principal character mod q_0, $q_0=q/q^*$. And we have

$$H(s,N)\ll\sum_{q^*q_0\leqq\sqrt{N}\log^{-A}N}\sum_{\chi^*\chi_{q_0}}\frac{1}{q^*q_0}|L(s,\chi^*\chi_{q_0})H(s,\chi^*\chi_{q_0})|.$$

Here we note

$$\begin{aligned}|L(s,\chi^*\chi_{q_0})|&=\Big|\prod_{p|q_0}\Big(1-\frac{\chi^*(p)}{p^s}\Big)L(s,\chi^*)\Big|\\ &\leqq d(q_0)|L(s,\chi^*)|\end{aligned}$$

for $\mathrm{Re}(s)>0$. Thus we get

$$\begin{aligned}H(s,N)&\ll\sum_{q_0\leqq\sqrt{N}\log^{-A}N}\frac{d(q_0)}{q_0}\sum_{q\leqq\frac{\sqrt{N}}{q_0}\log^{-A}N}\frac{1}{q}\sum_{\chi\,(\mathrm{mod}\,q)}^{*}|L(s,\chi)H(s,\chi\chi_{q_0})|\\ &\ll\log N\sum_{q_0\leqq\sqrt{N}\log^{-A}N}\frac{d(q_0)}{q_0}\max_{Q\leqq\frac{\sqrt{N}}{q_0}\log^{-A}N}Q^{-1}\sum_{q\leqq Q}\sum_{\chi(\mathrm{mod}\,q)}^{*}|L(s,\chi)H(s,\chi\chi_{q_0})|.\end{aligned}$$

On the other hand we have, for $s=\frac{1}{2}+it$,

$$\begin{aligned}&\sum_{q\leqq Q}\sum_{\chi(\mathrm{mod}\,q)}^{*}|L(s,\chi)H(s,\chi\chi_{q_0})|\\ &\leqq\Big\{\sum_{q\leqq Q}\sum_{\chi(\mathrm{mod}\,q)}|L(s,\chi)|^2\Big\}^{\frac{1}{2}}\Big\{\sum_{q\leqq Q}\sum_{\chi(\mathrm{mod}\,q)}^{*}|H(s,\chi\chi_{q_0})|^2\Big\}^{\frac{1}{2}}\end{aligned}$$

$$\ll Q(|t|+1)^{\frac{1}{2}} \log(Q(|t|+1))\Big\{(Q^2+N\log^{-2A}N)\sum_{m\leqq N}\frac{d_{k+1}^2(m)}{m}\Big\}^{\frac{1}{2}},$$

where we have used Lemma 1 and Lemma 2. Thus we get

$$H\Big(\frac{1}{2}+it,\, N\Big)\ll (N(|t|+1))^{\frac{1}{2}}\log(N(|t|+1))(\log N)^{-A+3+\frac{1}{2}(k-1)^2}$$

which, being inserted into (2.7), gives

$$S_k^{(3)}(N)\ll N(\log N)^{-A+5+\frac{1}{2}(k-1)^2}.$$

So, setting

$$\delta^2=(\log N)^{-A+4+\frac{1}{2}(k-1)^2-k},$$

$$A\geqq\frac{1}{2}(k-1)^2-k+5 \qquad (>0),$$

in (2.6), we see that the requirement (2.5) is satisfied by sufficiently large N, and we have

$$S_k^{(2)}(N)\ll N(\log N)^{-\frac{1}{4}(2A-k^2-13)}. \tag{2.8}$$

Next we have to estimate $S_k^{(0)}(N)$. For this sake we note that

$$\sum_{\substack{u\leqq x\\(r,u)=1}}1=\frac{\varphi(r)}{r}x+O(d(r)),$$

$$\sum_{\substack{u\leqq x\\(r,u)=1}}\frac{1}{u}=\frac{\varphi(r)}{r}\Big\{\log x+\gamma-\sum_{p|r}\frac{\log p}{p-1}\Big\}+O\Big(\frac{d(r)}{x}\log x\Big)$$

$$=\frac{\varphi(r)}{r}\{\log x+O(\log\log 3r)\}+O\Big(\frac{d(r)}{x}\log x\Big),$$

where γ is the Euler constant. These are proved by an easy application of Eratosthenes' sieve. Thus we have from (2.3) and (2.4)

$$S_k^{(0)}(N)=\sum_{q\leqq\sqrt{N}\log^{-A}N}\frac{1}{\varphi(q)}\sum_{\substack{(m,q)=1\\m\leqq N\log^{-2A}N}}d_{k-1}(m)\sum_{\substack{u\leqq N/m\\(u,q)=1}}1$$

$$=N\sum_{m\leqq N\log^{-2A}N}\frac{d_{k-1}(m)}{m}\sum_{\substack{q\leqq\sqrt{N}\log^{-A}N\\(m,q)=1}}\frac{1}{q}+O(N(\log N)^{k-2A})$$

$$=N\Big\{\frac{1}{2}\log N+O(\log\log N)\Big\}\sum_{m\leqq N\log^{-2A}N}\frac{\varphi(m)}{m^2}d_{k-1}(m)$$

$$+O(N(\log N)^{k-2A}).$$

On the other hand we have, for $\mathrm{Re}(s)>1$,

$$\sum_{m=1}^{\infty}\frac{\varphi(m)}{m^{2+s}}d_{k-1}(m)=\zeta(s+1)^{k-1}\prod_p\Big(1-\frac{1}{p}+\frac{1}{p}\Big(1-\frac{1}{p^{s+1}}\Big)^{k-1}\Big),$$

where $\zeta(s)$ is the Riemann zeta-function. So, by the routine complex integration method we get

$$\sum_{m\leq x}\frac{\varphi(m)}{m^2}d_{k-1}(m)=\frac{U_k}{(k-1)!}\log^{k-1}x+O(\log^{k-2}x),$$

where

$$U_k=\prod_p\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^{k-1}\right).$$

Hence we have

$$S_k^{(0)}(N)=\frac{1}{2}N(\log N)^k\frac{U_k}{(k-1)!}+O(N(\log N)^{k-1}\operatorname{loglog}N)+O(N(\log N)^{k-2A}). \tag{2.9}$$

Therefore, collecting (2.1), (2.2), (2.4), (2.8), (2.9), and setting

$$A=k^2/2+7,$$

we obtain the asymptotic formula of Linnik:

THEOREM 1.

$$\sum_{n\leq N}d(n)d_k(n+1)$$
$$=\frac{1}{(k-1)!}N\log^k N\prod_p\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^{k-1}\right)+O(N(\log N)^{k-1}\operatorname{loglog}N).$$

§ 3. Iteration formula.

Now we apply the argument of the preceding paragraph to a general multiplicative function $f(n)$, which is restricted only by the condition

$$f(n)=O(d^C(n)), \tag{3.1}$$

where C is a fixed positive integer. We put

$$g(n)=\sum_{d\mid n}\mu(d)f\left(\frac{n}{d}\right),$$

where $\mu(d)$ is the Möbius function. So we have

$$\sum_{d\mid n}g(d)=f(n), \qquad g(n)=O(d^{C+1}(n)). \tag{3.2}$$

Let

$$\Delta(x;q,l;f)=\sum_{\substack{n\equiv l\ (\mathrm{mod}\ q)\\ n\leq x}}f(n)-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leq x}}f(n).$$

Further let M be a positive number to be determined later, and we put

(3.3)
$$\Delta^{(1)}(x;q,l;f)=\sum_{\substack{n\equiv l\ (\mathrm{mod}\ q)\\ n\leq x}}\Big(\sum_{\substack{mh=n\\ m\leq M}}g(m)\Big)-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leq x}}\Big(\sum_{\substack{mh=n\\ m\leq M}}g(m)\Big)$$
$$\Delta^{(2)}(x;q,l;f)=\sum_{\substack{n\equiv l\ (\mathrm{mod}\ q)\\ n\leq x}}\Big(\sum_{\substack{mh=n\\ m>M}}g(m)\Big)-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leq x}}\Big(\sum_{\substack{mh=n\\ m>M}}g(m)\Big).$$

Then obviously we have, by (3.2),

$$\Delta(x;q,l;f)=\Delta^{(1)}(x;q,l;f)+\Delta^{(2)}(x;q,l;f).$$

Next we put

$$E(x;q;f)=\max_{y\leq x}\max_{\substack{l\\(q,l)=1}}|\Delta(y;q,l;f)|,$$
$$E^{(\nu)}(x;q;f)=\max_{y\leq x}\max_{\substack{l\\(q,l)=1}}|\Delta^{(\nu)}(y;q,l;f)|\qquad(\nu=1,2),$$

and also

(3.4)
$$I(x;Q;f)=\sum_{q\leq Q}E(x;q;f),$$
$$I^{(\nu)}(x;Q;f)=\sum_{q\leq Q}E^{(\nu)}(x;q;f)\qquad(\nu=1,2).$$

The estimation of $I(x;Q;f)$ leads to the analogue of Bombieri's theorem. We note that

(3.5)
$$I(x;Q;f)\leqq I^{(1)}(x;Q;f)+I^{(2)}(x;Q;f).$$

Now from (3.3) we have

$$\Delta^{(2)}(x;q,l;f)=\sum_{\substack{h<x/M\\(h,q)=1}}\Big\{\sum_{\substack{M<m\leq x/h\\ m\equiv l\bar{h}(\mathrm{mod}\ q)}}g(m)-\frac{1}{\varphi(q)}\sum_{\substack{M<m\leq x/h\\(m,q)=1}}g(m)\Big\}$$
$$=\sum_{\substack{h<x/M\\(h,q)=1}}\Big\{\Delta\Big(\frac{x}{h};q,l\bar{h};g\Big)-\Delta(M;q,l\bar{h};g)\Big\}.$$

Thus we get

$$E^{(2)}(x;q;f)\leqq\sum_{h<x/M}\Big\{E\Big(\frac{x}{h};q;g\Big)+E(M;q;g)\Big\},$$

which gives

(3.6)
$$I^{(2)}(x;Q;f)\leqq\sum_{h<x/M}\Big\{I\Big(\frac{x}{h};Q;g\Big)+I(M;Q;g)\Big\}$$
$$\leqq 2\frac{x}{M}I(x;Q;g),$$

provided

$$M\leqq x.$$

As for $\Delta^{(1)}(x;q,l;f)$ we introduce

$$\tilde{\varDelta}^{(1)}(y;q,l;f)=\int_1^y \varDelta^{(1)}(w;q,l;f)\frac{dw}{w},$$

and we put

$$\tilde{E}^{(1)}(x;q;f)=\max_{y\leq x}\max_{\substack{l\\(q,l)=1}}|\tilde{\varDelta}^{(1)}(y;q,l;f)|,$$

$$\tilde{I}^{(1)}(x;Q;f)=\sum_{q\leq Q}\tilde{E}^{(1)}(x;q;f).$$

Then we have, by Lemma 3,

$$E^{(1)}(x;q;f)\leqq 2\delta^{-1}\tilde{E}^{(1)}(xe;q;f)+O\Big(\delta\frac{x}{\varphi(q)}(\log x)^{2^{C+2}}\Big),$$

provided

$$0<\delta<1, \qquad (\delta x)^{1-\varepsilon}\geqq q.$$

From this and (3.4), we get

$$I^{(1)}(x;Q;f)\ll\delta^{-1}I^{(1)}(xe;Q;f)+\delta x(\log x)^{2^{C+2}+1}. \tag{3.7}$$

On the other hand we see easily that

$$\tilde{\varDelta}^{(1)}(y;q,l;f)=\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0 \,(\mathrm{mod}\, q)}\frac{\bar{\chi}(l)}{2\pi i}\int_{(\frac{1}{2})}L(s,\chi)G(s,\chi)\frac{y^s}{s^2}ds,$$

where

$$G(s,\chi)=\sum_{m\leqq M}\frac{\chi(m)}{m^s}g(m).$$

Then, dividing χ into primitive and principal characters as in the preceding paragraph, we get

$$\begin{aligned}&\tilde{I}^{(1)}(xe;Q;f)\\ &\ll x^{\frac{1}{2}}\log Q\sum_{q_0\leqq Q}\frac{1}{q_0}\sum_{q\leqq Q/q_0}\frac{1}{q}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}\int_{(\frac{1}{2})}|L(s,\chi\chi_{q_0})G(s,\chi\chi_{q_0})|\frac{|ds|}{|s|^2}.\end{aligned} \tag{3.8}$$

And here we have, by Lemma 1, Lemma 2 and (3.2),

$$\begin{aligned}&\sum_{q\leqq Q/q_0}\frac{1}{q}\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,q)}|L(s,\chi\chi_{q_0})G(s,\chi\chi_{q_0})|\\ &\qquad\ll d(q_0)\Big(\frac{Q}{q_0}+\sqrt{M}\Big)(|t|+1)^{\frac{1}{2}}(\log M)^{2^{2C+1}}\log^2(Q(|t|+1)),\end{aligned}$$

for $s=1/2+it$. This gives

$$\tilde{I}^{(1)}(xe;Q;f)\ll x^{\frac{1}{2}}(Q+\sqrt{M})(\log x)^{2^{2C+1}+5},$$

provided

$$M,\ Q\ll x.$$

Therefore, setting

$$Q \leqq \sqrt{x} \log^{-2B} x, \qquad M = x \log^{-4B} x,$$

$$\delta = (\log x)^{-B+2+2^{2C}-2^{C+1}}, \qquad B \geqq 2^{2C},$$

we get from (3.7) and (3.8)

$$I^{(1)}(x; Q; f) \ll x(\log x)^{-B+2^{2C+2}}. \tag{3.9}$$

And hence, from (3.5), (3.6) and (3.9), we obtain the iteration formula:

THEOREM 2. *Let $f(n)$ be a multiplicative function such that*

$$f(n) = O(d(n)^C), \qquad C: \text{ a positive integer.}$$

Let $g(n)$ be defined by

$$g(n) = \sum_{d \mid n} \mu(d) f\left(\frac{n}{d}\right).$$

Further let

$$I(x; Q; f) = \sum_{q \leqq Q} \max_{y \leqq Q} \max_{\substack{l \\ (q,l)=1}} \left| \sum_{\substack{n \equiv l \pmod q \\ n \leqq y}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{(n,q)=1 \\ n \leqq y}} f(n) \right|.$$

Then we have

$$I(x; Q; f) \leqq 2(\log x)^{4B} I(x; Q; g) + O(x(\log x)^{-B+2^{2C+2}}).$$

if

$$Q \leqq \sqrt{x} (\log x)^{-2B}, \qquad B \geqq 2^{2^C}.$$

§ 4. Applications.

We now specialize the function f by

$$f(p) = \tau: \text{ a fixed positive integer,}$$

for all primes p. And we write

$$f^{(j)}(n) = \sum_{d_1 d_2 \cdots d_j m = n} \mu(d_1)\mu(d_2) \cdots \mu(d_j) f(m),$$

$$f^{(0)}(n) = f(n).$$

Then we have, from (3.1),

$$f^{(j)}(n) = O(d(n)^{C+j}).$$

Thus, from Theorem 2, we get

$$I(x; Q; f^{(j)}) \leqq 2(\log x)^{4Bj} I(x; Q; f^{(j+1)}) + O(x(\log x)^{-B_j+2^{2(C+j)+2}}), \tag{4.1}$$

provided

$$Q \leqq \sqrt{x}(\log x)^{-2B_j}, \qquad B_j \geqq 2^{2(C+j)}.$$

On the other hand

$$f^{(\tau)}(p)=0,$$

and so the function

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} f^{(\tau)}(n)$$

is regular for $\mathrm{Re}(s)>1/2$. Also it is easy to see that this is

$$\ll (2\sigma-1)^{-E}, \qquad \sigma>\frac{1}{2},$$

where E is a certain constant which depends on τ, C and the constant involved in the 'O' of (3.1) at most. Thus we have uniformly for any q

$$\sum_{\substack{n\equiv l \ (\mathrm{mod}\ q)\\ n\leqq x}} f^{(\tau)}(n) \ll \sqrt{x}(\log x)^{E+1}.$$

From this we have at once

$$I(x; Q; f^{(\tau)}) \ll x(\log x)^{-A}$$

if

$$Q \leqq \sqrt{x}(\log x)^{-A-E-1}.$$

And hence we obtain, from this and (4.1),

THEOREM 3. *Let f be a multiplicative function such that*

$$f(n) \leqq \kappa d(n)^C, \qquad C: \text{ a positive integer,}$$

$$f(p)=\tau: \text{ a fixed positive integer for all primes } p.$$

Then we have

$$\sum_{q\leqq\sqrt{x}\log^{-B}x} \max_{y\leqq x} \max_{\substack{l\\(q,l)=1}} \left| \sum_{\substack{n\equiv l \ (\mathrm{mod}\ q)\\ n\leqq y}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{(n,q)=1\\ n\leqq y}} f(n) \right| \ll x(\log x)^{-A}.$$

Here B is a certain function of A, κ, C, τ, at most.

The above result is certainly weaker than Wolke's, but for most applications this seems sufficient. We also can prove by a slight modification of the present method

THEOREM 4. *We have, uniformly for any positive integers h and α,*

$$\sum_{q\leqq\sqrt{x}\log^{-B}x} \max_{y\leqq x} \max_{\substack{l\\(q,l)=1}} \left| \sum_{\substack{n\equiv l \ (\mathrm{mod}\ q)\\ n\leqq y}} d_k^{\alpha}(hn) - \frac{1}{\varphi(q)} \sum_{\substack{(n,q)=1\\ n\leqq y}} d_k^{\alpha}(hn) \right| \ll d(h)^{3\cdot k^{\alpha}} x(\log x)^{-A}.$$

Here B is a function of A, k, α.

From this we can deduce the following asymptotic formulas by an elementary argument:

$$\sum_{n\leqq N} d(n)d_k^\alpha(n+a)=\frac{1}{(k^\alpha-1)!}\ \Psi_k^{(\alpha)}\Phi_k^{(\alpha)}(a)N(\log N)^{k\alpha}+O(N(\log N)^{k\alpha-1}\log\log N),$$

$$\sum_{n\leqq N} d(n)d_k^\alpha(N-n)=\frac{1}{(k^\alpha-1)!}\Psi_k^{(\alpha)}\Phi_k^{(\alpha)}(N)N(\log N)^{k\alpha}$$
$$+O(N(\log N)^{k\alpha-1}(\log\log N)^{2^{4k^\alpha}}),$$

where

$$\Psi_k^{(\alpha)}=\prod_p\Big(1-\frac{1}{p}\Big)^{k^\alpha}\Big(1-\frac{1}{p}+\frac{1}{p}\Big(1-\frac{1}{p}\Big)^{-1}+\Big(1-\frac{1}{p}\Big)\sum_{m=1}^\infty\frac{d_k^\alpha(p^m)}{p^m}\Big),$$

$$\Phi_k^{(\alpha)}(r)=\prod_{p\mid r}\Big(1-\frac{1}{p}+\frac{1}{p}\Big(1-\frac{1}{p}\Big)^{-1}+\Big(1-\frac{1}{p}\Big)\sum_{m=1}^\infty\frac{d_k^\alpha(p^m)}{p^m}\Big)^{-1}$$
$$\times\prod_{\substack{p^\vartheta\parallel r\\(\vartheta\geqq 1)}}\Big(\Big(1-\frac{1}{p}\Big)^{-1}\frac{d_k^\alpha(p^\vartheta)}{p^{\vartheta+1}}+\Big(1-\frac{1}{p}\Big)\sum_{j=0}^{\vartheta}\sum_{m\geqq j}\frac{d_k^\alpha(p^m)}{p^m}\Big).$$

Also we can prove the following asymptotic formulas by following closely the argument of [7]: If a is *odd*,

$$\sum_{n\leqq N} r(n)d_k^\alpha(n-a)=\pi\Lambda_k^{(\alpha)}\Gamma_k^{(\alpha)}(a)N(\log N)^{k\alpha-1}+O(N(\log N)^{k\alpha-1-\theta}).$$

Also if N is *odd*,

$$\sum_{n\leqq N} r(n)d_k^\alpha(N-n)=\pi\Lambda_k^{(\alpha)}\Gamma_k^{(\alpha)}(N)N(\log N)^{k\alpha-1}+O(N(\log N)^{k\alpha-1-\theta}).$$

Here $\theta=0.0179$, and $\Lambda_k^{(\alpha)}$, $\Gamma_k^{(\alpha)}(r)$ are defined by

$$\Lambda_k^{(\alpha)}=\prod_p\Big(1-\frac{1}{p}\Big)^{k^\alpha}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}+\Big(1-\frac{\rho(p)}{p}\Big)\sum_{m=1}^\infty\frac{d_k^\alpha(p^m)}{p^m}\Big),$$

$$\Gamma_k^{(\alpha)}(r)=\Big(1+\rho(r)-\rho(r)(k^\alpha+1)\Big(\sum_{m=1}^\infty\frac{d_k^\alpha(2^m)}{2^m}\Big)^{-1}\Big)$$
$$\times\prod_{p\mid r}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}+\Big(1-\frac{\rho(p)}{p}\Big)\sum_{m=1}^\infty\frac{d_k^\alpha(p^m)}{p^m}\Big)^{-1}$$
$$\times\prod_{\substack{p^\vartheta\parallel r\\(\vartheta\geqq 1)}}\Big(\Big(1-\frac{1}{p}\Big)^{-1}\frac{\rho(p^{\vartheta+1})}{p^{\vartheta+1}}d_k^\alpha(p^\vartheta)+\Big(1-\frac{\rho(p)}{p}\Big)\sum_{j=0}^{\vartheta}\rho(p^j)\sum_{m\geqq j}\frac{d_k^\alpha(p^m)}{p^m}\Big),$$

ρ being the non-principal character mod 4.

Added in proof: Recently the present author has proved an asymptotic series for the sum (2.1) with an error-term $O(N(\log N)^c(\log N)^{-1})$, which solves a problem raised in [2], [4]. A short account of the proof of a little weaker result than this has been published in Proc. Japan Acad., **52** (1976), pp. 279-281. Also he has shown that the argument of § 3 can be modified to prove an induction principle which, among other things, enables one to iterate Bombieri's prime number theorem (for this see Proc. Japan Acad., **52** (1976), pp. 273-275).

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