J. Math. Soc. Japan
Vol. 28, No. 2, 1976

On almost-primes in arithmetic progressions

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(Received May 24, 1975)

§ 1. Introduction.

The problem of the distribution of almost-primes in arithmetic progressions has been investigated by various authors. The hitherto most successful study can be found in Jurkat-Richert [1] and Richert [5]. To state a special case of their beautiful works we first introduce some notations and conventions.

Let $z \geqq 2$ be arbitrary and

$$P_k(z)=\prod_{\substack{p<z\\ p\nmid k}} p, \qquad \Gamma_k(z)=\prod_{\substack{p<z\\ p\nmid k}}\left(1-\frac{1}{p}\right),$$

where p denotes generally a prime number. We put

$$A(x;k,l;z)=\sum_{\substack{n\leqq x\\ n\equiv l(\mathrm{mod}\ k)\\ (n,P_k(z))=1}} 1, \qquad (k,l)=1.$$

Further let the functions $F(u)$ and $f(u)$ be defined by

$$\begin{aligned} &F(u)=2e^{\gamma}/u, \qquad f(u)=0, \qquad 0<u\leqq 2, \\ &(uF(u))'=f(u-1), \qquad (uf(u))'=F(u-1), \qquad 2\leqq u, \end{aligned} \tag{1.1}$$

where γ is the Euler constant. Then according to [1] we have, for any $z<x/k$,

$$\begin{aligned} A(x;k,l;z) &\leqq \frac{x}{k}\Gamma_k(z)\left\{F\left(\frac{\log\frac{x}{k}}{\log z}\right)+O\left(\left(\log\frac{x}{k}\right)^{-\frac{1}{14}}\right)\right\}, \\ A(x;k,l;z) &\geqq \frac{x}{k}\Gamma_k(z)\left\{f\left(\frac{\log\frac{x}{k}}{\log z}\right)-O\left(\left(\log\frac{x}{k}\right)^{-\frac{1}{14}}\right)\right\}, \end{aligned} \tag{1.2}$$

uniformly for all l, $(k,l)=1$. This has been obtained by the completion of the combinatorial strengthening due to Buchstab of the sieve procedure. To get the result on almost-primes, the idea of Kuhn of attaching certain weights to sieve has played vital role in many works. Also this has been considerably sophisticated in [5], from which we excerpt here the following result: Let, denoting by q a prime number, (1.3) $$W_\zeta(x;k,l;z,w)=\sideset{}{'}\sum_{\substack{n\le x\\ n\equiv l(\mathrm{mod}\,k)\\ (n,P_k(z))=1}}\Big\{1-\zeta\sum_{\substack{z\le q<w\\ q\mid n\\ q\nmid k}}\Big(1-\frac{\log q}{\log w}\Big)\Big\},$$

where the prime of the sum indicates that n is restricted by

$$n\not\equiv 0\ (\mathrm{mod}\,q^2)\qquad \text{for}\quad z\le q<w,\ q\nmid k\,.$$

Then for non-negative constant ζ we have

(1.4) $$W_\zeta(x;k,l;z,w)\ge\frac{x}{k}\Gamma_k(z)\Big\{f(v)-\zeta\int_u^v F\Big(v\Big(1-\frac{1}{t}\Big)\Big)\Big(1-\frac{u}{t}\Big)\frac{dt}{t}-O\Big(\Big(\log\frac{x}{k}\Big)^{-\frac{1}{15}}\Big)\Big\},$$

where

$$u=\frac{\log\frac{x}{k}}{\log w},\qquad v=\frac{\log\frac{x}{k}}{\log z},\qquad 1<u\le v\,.$$

From this it follows that, denoting as usual by P_r the integer which has at most r prime factors counting the multiplicity, there is a P_r such that

$$P_r\le k^{\frac{\Lambda_r}{\Lambda_r-1}+\varepsilon},\qquad P_r\equiv l\ (\mathrm{mod}\,k),\qquad r\ge 2,$$

uniformly for all l, $(k,l)=1$, where

$$\Lambda_r=r+1-\frac{\log\{4/(1+3^{-r})\}}{\log 3}\,.$$

Now, the special case (i.e. if $z=(x/k)^{1/2}$) of the first inequality of (1.2) gives the Brun-Titchmarsh theorem

$$\pi(x;k,l)\le(2+\varepsilon)\frac{x}{\varphi(k)\log\frac{x}{k}},$$

where $\pi(x;k,l)$ denotes as usual the number of primes less than x and congruent to $l \bmod k$. But recently in our papers [3], [4] this has been improved to

$$\pi(x;k,l)\le(2+\varepsilon)\frac{x}{\varphi(k)\log\frac{x}{\sqrt{k}}},$$

for all l, $(k,l)=1$ and for almost all l, $(k,l)=1$ accordingly to $k\le x^{2/5}$ and to $k\le x^{1-\varepsilon}$, respectively. Our result has been obtained by the direct application of a rather routine analytical procedure to the Selberg sieve. Thus it seems natural to expect that our method may give some improvements in the results of Jurkat and Richert. This is the main purpose of the present paper. However, the investigation of the problem on the requirement that the result should hold uniformly for all reduced residue classes turns out rather tedious and also is of little use, since, then, we have to impose the condition $k\leqq x^{2/5}$. Hence we will concentrate our concern to the average behaviour of their sieve procedure. As a whole our present paper can be considered to be the asymptotical study of the works [1] and [5] in the case of arithmetic progressions. Our results are as follows:

THEOREM 1. *Let $A>1$ be an arbitrary number, and let*

$$k\leqq x(\log x)^{-A}.$$

Then for any z with $2\leqq z\leqq x/\sqrt{k}$ we have

$$A(x;k,l;z)\geqq\frac{x}{k}\Gamma_k(z)\left\{f\left(\frac{\log\frac{x}{\sqrt{k}}}{\log z}\right)-O((\log x)^{-\frac{1}{14}})\right\},$$

$$A(x;k,l;z)\leqq\frac{x}{k}\Gamma_k(z)\left\{F\left(\frac{\log\frac{x}{\sqrt{k}}}{\log z}\right)+O((\log x)^{-\frac{1}{14}})\right\},$$

save for at most $k(\log x)^{-(A/2)+23}$ reduced residue classes $l \bmod k$.

THEOREM 2. *Let $A>1$ be an arbitrary number, and let*

$$k\leqq x(\log x)^{-A},\qquad 2\leqq z\leqq w\leqq\frac{x}{\sqrt{k}}(\log x)^{-A}.$$

Then, for any non-negative constant ζ, we have, save for at most $k(\log x)^{-(A/2)+33}$ reduced residue classes $l \bmod k$,

$$W_\zeta(x;k,l;z,w)$$
$$\geqq\frac{x}{k}\Gamma_k(z)\left\{f(v)-\zeta\int_u^v F\left(v\left(1-\frac{1}{t}\right)\right)\left(1-\frac{u}{t}\right)\frac{dt}{t}-O((\log x)^{-\frac{1}{15}})\right\},$$

where

$$u=\frac{\log\frac{x}{\sqrt{k}}}{\log w},\qquad v=\frac{\log\frac{x}{\sqrt{k}}}{\log z},\qquad 1<u\leqq v.$$

Compared with (1.2) and (1.4), our results differ only in that the factor x/k is replaced by $x/\sqrt{k}$ in all expressions. But this is a considerable improvement when application is made to the problem of almost primes.

COROLLARY TO THEOREM 2. *There exist numbers such that*

$$P_2\leqq k^{\frac{11}{10}},\qquad P_2\equiv l \pmod{k},$$

$$P_3\leqq k(\log k)^{70},\qquad P_3\equiv l \pmod{k},$$

save for at most $k(\log k)^{-2}$ reduced residue classes $l \bmod k$.

This should be compared with the result of Turán [6]: On the extended Riemannian Hypothesis there exists a prime number p such that

$$p \leqq k(\log k)^{2+\varepsilon}, \qquad p \equiv l \pmod{k},$$

for almost all l, $(k, l)=1$.

Notations: x is a sufficiently large positive variable, and it is supposed that $k \leqq x$. $\bar{n}$ means that $n \cdot \bar{n} \equiv 1 \pmod{k}$ for any integer n, $(k, n)=1$. $\varphi(k)$ is the number of reduced residue classes mod k. χ is a Dirichlet character mod k, and χ_0 denotes the principal character. $L(s, \chi)$ is the Dirichlet L-function attached to χ and s a complex variable. The symbol $[d_1, d_2]$ stands for the least common multiple of d_1 and d_2. The constants implied by the symbols "O" and "$\ll$" depend only on A and ε a sufficiently small positive constant.

§ 2. Fundamental inequality.

Our whole calculations depend on a combinatorial identity similar to the formula (2.1) of [1] which maximize the effect of the Selberg sieve. Before stating this identity we introduce here some conventions to simplify the notations in what follows.

We take two numbers Δ and $A>1$ which are to be determined later and should be considered to be fixed and sufficiently large, for a while. Let $\{p_1, \cdots, p_a\}$ be a set of prime numbers with arbitrary $a \geqq 0$. For the positive variable x we write

$$x_0 = \frac{x}{\sqrt{k}}(\log \Delta)^{-A}, \qquad x_a = \frac{x}{\sqrt{k}\, p_1 \cdots p_a}(\log \Delta)^{-A}, \tag{2.1}$$

and we impose following conditions to $\{p_1, \cdots, p_a\}$:

$$\begin{aligned} &C_1(x) = C_1(x; \Delta, A) \equiv \{p_j \nmid k,\ p_j < \sqrt{x_j}\ \ j=1, 2, \cdots, a\}, \\ &C_2(x) = C_2(x; \Delta, A) \equiv \{p_j \nmid k,\ p_j < \sqrt{x_j}\ \ j=1, 2, \cdots, a-1;\ p_a \nmid k,\ \sqrt{x_a} \leqq p_a < x_a\}. \end{aligned} \tag{2.2}$$

Then our basic identity is

LEMMA 2.1. *Let z be restricted by $z \leqq \sqrt{x_0}$. Then for an arbitrary integer $r \geqq 1$ and for any z_1, $2 \leqq z_1 \leqq z$, we have*

$$\begin{aligned} A(x; k, l; z) = &\sum_{0 \leqq i \leqq r-1} (-1)^i \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} A\Big(\frac{x}{p_1 \cdots p_i}; k, l\overline{p_1 \cdots p_i}; z_1\Big) \\ &+ (-1)^r \sum_{\substack{z_1 \leqq p_r < \cdots < p_1 < z \\ C_1(x)}} A\Big(\frac{x}{p_1 \cdots p_r}; k, l\overline{p_1 \cdots p_r}; p_r\Big) \\ &+ \sum_{1 \leqq i \leqq r} (-1)^i \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_2(x)}} A\Big(\frac{x}{p_1 \cdots p_i}; k, l\overline{p_1 \cdots p_i}; p_i\Big). \end{aligned}$$

PROOF. This can be shown by the induction on r as Theorem 1 of [1].

Let introduce one more basic parameter z_0 such that

$$2 \leqq z_0 \leqq z_1,$$

and hereafter let z stand for the triple (z_0, z_1, z) when it appears in formulas. By the simple application of Buchstab's identity (see (2.2) of [1]) we have, for both $\nu=1$ and 2,

$$(2.3)\quad \begin{aligned} &\sum_{0\leqq i\leqq r-1}(-1)^i \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}} A\left(\frac{x}{p_1\cdots p_i}\,;\,k,\,l\overline{p_1\cdots p_i}\,;\,z_1\right)\\ &=(-1)^{\nu}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}} A\left(\frac{x}{p_1\cdots p_i}\,;\,k,\,l\overline{p_1\cdots p_i}\,;\,z_1\right)\\ &\quad+(-1)^{\nu+1}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}} A\left(\frac{x}{p_1\cdots p_i}\,;\,k,\,l\overline{p_1\cdots p_i}\,;\,z_0\right)\\ &\quad+(-1)^{\nu}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\ \sum_{\substack{z_0\leqq p_0<z_1\\ p_0\nmid k}} A\left(\frac{x}{p_0p_1\cdots p_i}\,;\,k,\,l\overline{p_0p_1\cdots p_i}\,;\,p_0\right). \end{aligned}$$

Further we put

$$(2.4)\quad \begin{aligned} &A_0^{(\nu)}(x\,;\,l\,;\,z\,;\,r)\\ &=(-1)^{\nu+1}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}} A\left(\frac{x}{p_1\cdots p_i}\,;\,k,\,l\overline{p_1\cdots p_i}\,;\,z_0\right), \end{aligned}$$

and

$$(2.5)\quad \begin{aligned} &A_1^{(\nu)}(x\,;\,l\,;\,z\,;\,r)\\ &=\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}} A\left(\frac{x}{p_1\cdots p_i}\,;\,k,\,l\overline{p_1\cdots p_i}\,;\,z_1\right)\\ &\quad+\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\ \sum_{\substack{z_0\leqq p_0<z_1\\ p_0\nmid k}} A\left(\frac{x}{p_0p_1\cdots p_i}\,;\,k,\,l\overline{p_0p_1\cdots p_i}\,;\,p_0\right)\\ &\quad+\sum_{\substack{z_1\leqq p_r<\cdots<p_1<z\\ C_1(x)}} A\left(\frac{x}{p_1\cdots p_r}\,;\,k,\,l\overline{p_1\cdots p_r}\,;\,p_r\right)\\ &\quad+\sum_{\substack{1\leqq i\leqq r\\ i\equiv\nu\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_2(x)}} A\left(\frac{x}{p_1\cdots p_i}\,;\,k,\,l\overline{p_1\cdots p_i}\,;\,p_i\right). \end{aligned}$$

Then from Lemma 2.1 and by (2.3) we get easily the following fundamental inequalities:

LEMMA 2.2. *If r is even,*

$$A(x; k, l; z) \leqq A_0^{(2)}(x; l; z; r) + A_1^{(2)}(x; l; z; r).$$

Also if r is odd,

$$A(x; k, l; z) \geqq A_0^{(1)}(x; l; z; r) - A_1^{(1)}(x; l; z; r).$$

To $A_0^{(\nu)}(x; l; z; r)$ and to $A_1^{(\nu)}(x; l; z; r)$ we shall apply the Eratosthenes-Legendre sieve and the Selberg sieve, respectively.

§ 3. Selberg's sieve.

In this paragraph we summarise some results from the theory of the Selberg sieve, and later we modify Lemma 2.2.

Let η_1, η_2 be arbitrary positive numbers, and let

$$\lambda_d(\eta_1, \eta_2) = \begin{cases} \mu(d)\dfrac{d}{\varphi(d)} Y_k^{-1}(\eta_1, \eta_2) \sum\limits_{\substack{(m,d)=1 \\ m \leqq \eta_2/d \\ m | P_k(\eta_1)}} \dfrac{1}{\varphi(m)}, & \text{for } (k, d)=1,\ d \leqq \eta_2, \\ 0 & , \text{otherwise}. \end{cases}$$

Here

$$Y_k(\eta_1, \eta_2) = \sum_{\substack{m \leqq \eta_2 \\ m | P_k(\eta_1)}} \frac{1}{\varphi(m)}.$$

Then as is well-known we have

$$|\lambda_d(\eta_1, \eta_2)| \leqq 1, \tag{3.1}$$

and

$$Y_k^{-1}(\eta_1, \eta_2) = \sum_{\substack{d_\nu | P_k(\eta_1) \\ \nu=1,2}} \frac{\lambda_{d_1}(\eta_1, \eta_2)\lambda_{d_2}(\eta_1, \eta_2)}{[d_1, d_2]}. \tag{3.2}$$

Further, if $\eta_1 > \eta_0$, where η_0 is a sufficiently large constant, we have

$$Y_k^{-1}(\eta_1, \eta_2) \leqq \Gamma_k(\eta_1)(1 + O(e^{-2\frac{\log \eta_2}{\log \eta_1}})) \tag{3.3}$$

(see p. 224 of [1]), and also we have

$$Y_k^{-1}(\eta_1, \eta_2) \leqq \frac{k}{\varphi(k)}(\log \eta_1)^{-1}. \tag{3.4}$$

Now, we put, for any set of prime numbers $\{p_1, \cdots, p_a\}$, $p_j \nmid k$ $(j=1, \cdots, a)$,

$$\psi_{p_1 \cdots p_a}(u; l; \eta_1, \eta_2) = \sum_{\substack{n \equiv \overline{l p_1 \cdots p_a} \pmod k \\ n \leqq u/p_1 \cdots p_a}} \Big(\sum_{\substack{d | n \\ d | P_k(\eta_1)}} \lambda_d(\eta_1, \eta_2) \Big)^2. \tag{3.5}$$

Then we have by the standard application of the Selberg sieve

$$A\Big(\frac{u}{p_1\cdots p_a}\,;\,k,\,l\overline{p_1\cdots p_a}\,;\,\eta_1\Big)\leqq\phi_{p_1\cdots p_a}(u\,;\,l\,;\,\eta_1,\,\eta_2)\,. \tag{3.6}$$

Next, we put

$$\begin{aligned}&\Psi_1^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r)\\&=\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\phi_{p_1\cdots p_i}(u\,;\,l\,;\,z_1,\,\sqrt{x_i})\\&+\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1(\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\ \sum_{\substack{z_0\leqq p_0<z_1\\ p_0\nmid k}}\phi_{p_0p_1\cdots p_i}\Big(u\,;\,l\,;\,p_0,\,\sqrt{\frac{x_i}{p_0}}\Big)\\&+\sum_{\substack{z_1\leqq p_r<\cdots<p_1<z\\ C_1(x)}}\phi_{p_1\cdots p_r}(u\,;\,l\,;\,p_r,\,\sqrt{x_r})\\&+\sum_{\substack{1\leqq i\leqq r\\ i\equiv\nu\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_2(x)}}\phi_{p_1\cdots p_i}(u\,;\,l\,;\,\sqrt{x_i},\,\sqrt{x_i})\,.\end{aligned} \tag{3.7}$$

From (2.5) and (3.6) we see easily

$$A_1^{(\nu)}(x\,;\,l\,;\,z\,;\,r)\leqq\Psi_1^{(\nu)}(x\,;\,l\,;\,x,\,z\,;\,r)\,. \tag{3.8}$$

Also, if we put

$$\begin{aligned}&\Psi_0^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r)\\&=(-1)^{\nu+1}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}A\Big(\frac{u}{p_1\cdots p_i}\,;\,k,\,l\overline{p_1\cdots p_i}\,;\,z_0\Big),\end{aligned} \tag{3.9}$$

obviously from (2.4) we have, for both $\nu=1$ and 2,

$$A_0^{(\nu)}(x\,;\,l\,;\,z\,;\,r)=\Psi_0^{(\nu)}(x\,;\,l\,;\,x,\,z\,;\,r)\,. \tag{3.10}$$

Therefore, if we put, for $\nu=1, 2$,

$$B_\nu(u\,;\,l\,;\,x,\,z\,;\,r)=\Psi_0^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r)+(-1)^\nu\Psi_1^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r)\,, \tag{3.11}$$

we get from Lemma 2.2 and from (3.8), (3.10),

LEMMA 3.1. *If r is even,*

$$A(x\,;\,k,\,l\,;\,z)\leqq B_2(x\,;\,l\,;\,x,\,z\,;\,r)\,.$$

Also if r is odd,

$$A(x\,;\,k,\,l\,;\,z)\geqq B_1(x\,;\,l\,;\,x,\,z\,;\,r)\,.$$

§ 4. Application of analytic method.

Our next problem is the asymptotical estimation of $B_\nu(u\,;\,l\,;\,x,\,z\,;\,r)$. For this sake we introduce

$$(4.1)\qquad \tilde{B}_\nu(u\,;\,l\,;\,x,\,z\,;\,r)=\int_1^u B_\nu(w\,;\,l\,;\,x,\,z\,;\,r)\frac{dw}{w},$$

which, we shall see later, has an analytic representation. From (3.11) we have

$$(4.2)\qquad \tilde{B}_\nu(u\,;\,l\,;\,x,\,z\,;\,r)=\tilde{\Psi}_0^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r)+(-1)^\nu\tilde{\Psi}_1^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r),$$

where, as is easily seen,

$$(4.3)\qquad \begin{aligned}&\tilde{\Psi}_0^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r)\\ &=(-1)^{\nu+1}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\ \sum_{\substack{n\equiv l\overline{p_1\cdots p_i}\ (\mathrm{mod}\ k)\\ (n,P_k(z_0))=1\\ n\leqq u/p_1\cdots p_i}}\log\left(\frac{u}{np_1\cdots p_i}\right),\end{aligned}$$

and $\tilde{\Psi}_1^{(\nu)}(u\,;\,l\,;\,x,\,z\,;\,r)$ is defined to be the sum which is obtained by replacing $\phi_{p_1\cdots p_i}(u\,;\,l\,;\,\eta_1,\,\eta_2)$'s in the expression (3.7) by

$$(4.4)\qquad \begin{aligned}&\tilde{\phi}_{p_1\cdots p_i}(u\,;\,l\,;\,\eta_1,\,\eta_2)\\ &=\sum_{\substack{n\equiv l\overline{p_1\cdots p_i}\ (\mathrm{mod}\ k)\\ n\leqq u/p_1\cdots p_i}}\Big(\sum_{\substack{d\mid n\\ d\mid P_k(\eta_1)}}\lambda_d(\eta_1,\,\eta_2)\Big)^2\log\left(\frac{u}{np_1\cdots p_i}\right).\end{aligned}$$

Now, let $\alpha>1$, then we have

$$(4.5)\qquad \begin{aligned}&\sum_{\substack{n\equiv l\overline{p_1\cdots p_i}\ (\mathrm{mod}\ k)\\ (n,P_k(z_0))=1\\ n\leqq u/p_1\cdots p_i}}\log\left(\frac{u}{np_1\cdots p_i}\right)\\ &=\frac{1}{\varphi(k)}\sum_{\chi\ (\mathrm{mod}\ k)}\bar{\chi}(l\overline{p_1\cdots p_i})\frac{1}{2\pi i}\int_{(\alpha)}L(s,\,\chi)\prod_{p\leqq z_0}\left(1-\frac{\chi(p)}{p^s}\right)\left(\frac{u}{p_1\cdots p_i}\right)^s\frac{ds}{s^2},\end{aligned}$$

which is nothing other than the application of the Eratosthenes-Legendre sieve. Also we have

$$(4.6)\qquad \begin{aligned}&\tilde{\phi}_{p_1\cdots p_i}(u\,;\,l\,;\,\eta_1,\,\eta_2)\\ &=\frac{1}{\varphi(k)}\sum_{\chi\ (\mathrm{mod}\ k)}\bar{\chi}(l\overline{p_1\cdots p_i})\frac{1}{2\pi i}\int_{(\alpha)}L(s,\,\chi)H(s,\,\chi\,;\,\eta_1,\,\eta_2)\left(\frac{u}{p_1\cdots p_i}\right)^s\frac{ds}{s^2},\end{aligned}$$

where

$$(4.7)\qquad H(s,\,\chi\,;\,\eta_1,\,\eta_2)=\sum_{\substack{d_\nu\mid P_k(\eta_1)\\ \nu=1,2}}\chi([d_1,\,d_2])\frac{\lambda_{d_1}(\eta_1,\,\eta_2)\lambda_{d_2}(\eta_1,\,\eta_2)}{[d_1,\,d_2]^s}.$$

Here it should be remarked that we can write

$$(4.8)\qquad H(s,\,\chi\,;\,\eta_1,\,\eta_2)=\sum_{d\leqq\eta_2^2}\frac{\chi(d)}{d^s}\rho_d(\eta_1,\,\eta_2),$$

where

$$\rho_d(\eta_1, \eta_2) = \sum_{\substack{d=[d_1,d_2] \\ d_\nu \mid P_k(\eta_1) \\ \nu=1,2}} \lambda_{d_1}(\eta_1, \eta_2)\lambda_{d_2}(\eta_1, \eta_2),$$

and that by (3.1)

$$|\rho_d(\eta_1, \eta_2)| \leqq \sum_{d=[d_1,d_2]} 1 \leqq \tau_3(d). \tag{4.9}$$

Hence, collecting (3.7), (4.2), (4.5) and (4.6), we find

$$\begin{aligned} &\tilde{B}_\nu(u\,;l\,;x,z\,;r) \\ &= \frac{1}{2\pi i\varphi(k)} \sum_{\chi \pmod k} \bar{\chi}(l) \int_{(\alpha)} L(s,\chi) M_\nu(s,\chi\,;x,z\,;r) \frac{u^s}{s^2} ds, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} &M_\nu(s,\chi\,;x,z\,;r) \\ &= (-1)^{\nu+1} \sum_{\substack{0\leqq i\leqq r-1 \\ i\equiv \nu+1 \pmod 2}} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \frac{\chi(p_1\cdots p_i)}{(p_1\cdots p_i)^s} \prod_{p\leqq z_0}\left(1-\frac{\chi(p)}{p^s}\right) \\ &+ (-1)^\nu \sum_{\substack{0\leqq i\leqq r-1 \\ i\equiv \nu+1 \pmod 2}} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \sum_{\substack{z_0 \leqq p_0 < z_1 \\ p_0 \nmid k}} \frac{\chi(p_0 p_1\cdots p_i)}{(p_0 p_1\cdots p_i)^s} H\left(s,\chi\,;p_0, \sqrt{\frac{x_i}{p_0}}\right) \\ &+ (-1)^\nu \sum_{\substack{0\leqq i\leqq r-1 \\ i\equiv \nu \pmod 2}} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \frac{\chi(p_1\cdots p_i)}{(p_1\cdots p_i)^s} H(s,\chi\,;z_1,\sqrt{x_i}) \\ &+ (-1)^\nu \sum_{\substack{z_1 \leqq p_r < \cdots < p_1 < z \\ C_1(x)}} \frac{\chi(p_1\cdots p_r)}{(p_1\cdots p_r)^s} H(s,\chi\,;p_r,\sqrt{x_r}) \\ &+ (-1)^\nu \sum_{\substack{1\leqq i\leqq r \\ i\equiv \nu \pmod 2}} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_2(x)}} \frac{\chi(p_1\cdots p_i)}{(p_1\cdots p_i)^s} H(s,\chi\,;\sqrt{x_i},\sqrt{x_i}). \end{aligned} \tag{4.11}$$

This complicated expression can be simplified as follows:

LEMMA 4.1. *We have*

$$M_\nu(s,\chi\,;x,z\,;r) = \sum_{m\leqq M} \frac{\chi(m)}{m^s} \Phi_r^{(\nu)}(m\,;x,z),$$

where

$$|\Phi_r^{(\nu)}(m\,;x,z)| \leqq 4\tau_4(m),$$

and under the condition

$$z_0^{z_0} \leqq z_1^2 \tag{4.12}$$

we have

$$M \leqq \frac{x}{\sqrt{k}} (\log \Delta)^{-A}.$$

PROOF. From (4.8) and (4.11) we see

$$\Phi_r^{(\nu)}(m\,;x,z)=(-1)^{\nu+1}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{m=vp_1\cdots p_i\\ z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)\\ v\mid P_k(z_0)}}\mu(v)$$

$$+(-1)^{\nu}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu+1\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{m=vp_0p_1\cdots p_i\\ z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)\\ z_0\leqq p_0<z,\, p_0\nmid k\\ v\leqq x_i/p_0}}\rho_v\Big(p_0,\sqrt{\frac{x_i}{p_0}}\Big)$$

$$+(-1)^{\nu}\sum_{\substack{0\leqq i\leqq r-1\\ i\equiv\nu\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{m=vp_1\cdots p_i\\ z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)\\ v\leqq x_i}}\rho_v(z_1,\sqrt{x_i}\,)$$

$$+(-1)^{\nu}\sum_{\substack{m=vp_1\cdots p_r\\ z_1\leqq p_r<\cdots<p_1<z\\ C_1(x)\\ v\leqq x_r}}\rho_v(p_r,\sqrt{x_r})$$

$$+(-1)^{\nu}\sum_{\substack{1\leqq i\leqq r\\ i\equiv\nu\ (\mathrm{mod}\ 2)}}\ \sum_{\substack{m=vp_1\cdots p_i\\ z_1\leqq p_i<\cdots<p_1<z\\ C_2(x)\\ v\leqq x_i}}\rho_v(\sqrt{x_i}\,,\sqrt{x_i}\,)\cdot$$

In the first double sum m is restricted by

$$m\leqq z_0^{z_0}p_1\cdots p_i\,,$$

since obviously $v\leqq z_0^{z_0}$, and on the other hand we have by the condition $C_1(x)$

$$\frac{x}{\sqrt{k}\,p_1\cdots p_i}(\log\varDelta)^{-A}\geqq p_i^2\geqq z_1^2\,.$$

So we have

$$m\leqq\frac{z_0^{z_0}}{z_1^2}\cdot\frac{x}{\sqrt{k}}(\log\varDelta)^{-A}\,.$$

In other sums it is easy to see that by the conditions $C_1(x)$ or $C_2(x)$ readily we have

$$m\leqq\frac{x}{\sqrt{k}}(\log\varDelta)^{-A}\,.$$

This proves the assertion of the lemma on the size of M. Now, by (4.9), we get

$$|\Phi_r^{(\nu)}(m\,;x,z)|\leqq\sum_{0\leqq i\leqq r-1}\ \sum_{\substack{m=vp_1\cdots p_i\\ p_i<\cdots<p_1}}1$$

$$+\sum_{0\leqq i\leqq r-1}\ \sum_{\substack{m=vp_0p_1\cdots p_i\\ p_0<p_i<\cdots<p_1}}\tau_3(v)+\sum_{0\leqq i\leqq r-1}\ \sum_{\substack{m=vp_1\cdots p_i\\ p_i<\cdots<p_1}}\tau_3(v)$$

$$+\sum_{\substack{m=vp_1\cdots p_r\\ p_r<\cdots<p_1}}\tau_3(v)+\sum_{1\leqq i\leqq r}\ \sum_{\substack{m=vp_1\cdots p_i\\ p_i<\cdots<p_1}}\tau_3(v)$$

$$\leqq 4 \sum_{1\leqq i\leqq r} \sum_{\substack{v\mid m \\ \Omega\left(\frac{m}{v}\right)=i}} \tau_3(v),$$

which proves the lemma.

Now, returning to the formula (4.10) we change the line of integration from $\mathrm{Re}(s)=\alpha$ to $\mathrm{Re}(s)=1/2$. This is permissible by the well-known estimate of $L\left(\frac{1}{2}+it, \chi\right)$ and by the boundedness of $M_\nu(s, \chi; x, z; r)$ with respect to s which follows from Lemma 4.1. And we get

$$\tilde{B}_\nu(u; l; x, z; r)=\frac{u}{k} M_\nu(1, \chi_0; x, z; r)$$

$$+\frac{1}{2\pi i\varphi(k)} \sum_{\chi \,(\mathrm{mod}\, k)} \bar{\chi}(l)\int_{\left(\frac{1}{2}\right)} L(s, \chi)M_\nu(s, \chi; x, z; r)\frac{u^s}{s^2} ds.$$

So it follows

$$\sum_{\substack{l=1 \\ (k,l)=1}}^{k} |\tilde{B}_\nu(u; l; x, z; r)-\frac{u}{k}M_\nu(1, \chi_0; x, z; r)|^2$$

$$=\frac{1}{4\pi^2\varphi(k)} \sum_{\chi \,(\mathrm{mod}\, k)}\left|\int_{\left(\frac{1}{2}\right)} L(s, \chi)M_\nu(s, \chi; x, z; r)\frac{u^s}{s^2}ds\right|^2$$

$$\ll\frac{u}{\varphi(k)}\int_{-\infty}^{+\infty}\left\{\sum_{\chi \,(\mathrm{mod}\, k)}\left|L\left(\frac{1}{2}+it, \chi\right)\right|^4\right\}^{\frac{1}{2}}\left\{\sum_{\chi \,(\mathrm{mod}\, k)}\left|M_\nu\left(\frac{1}{2}+it, \chi; x, z; r\right)\right|^4\right\}^{\frac{1}{2}}\frac{dt}{(|t|+1)^2}.$$

Here we have

$$\sum_{\chi \,(\mathrm{mod}\, k)}\left|L\left(\frac{1}{2}+it, \chi\right)\right|^4 \ll k(|t|+1)\log^9(k(|t|+1)), \tag{4.13}$$

(see [2]). Also we have from Lemma 4.1

$$\sum_{\chi \,(\mathrm{mod}\, k)}\left|M_\nu\left(\frac{1}{2}+it, \chi; x, z; r\right)\right|^4$$

$$=\sum_{\chi \,(\mathrm{mod}\, k)}\left|\sum_{m\leqq M^2}\frac{\chi(m)}{m^{\frac{1}{2}+it}}\left(\sum_{m=m_1m_2}\Phi_r^{(\nu)}(m_1; x, z)\Phi_r^{(\nu)}(m_2; x, z)\right)\right|^2$$

$$\ll(k+M^2)\sum_{m\leqq M^2}\frac{1}{m}\left|\sum_{m=m_1m_2}\Phi_r^{(\nu)}(m_1; x, z)\Phi_r^{(\nu)}(m_2; x, z)\right|^2$$

$$\ll(k+M^2)\sum_{m\leqq M^2}\frac{1}{m}(\tau_8(m))^2,$$

which is on the condition (4.12)

$$\ll k\left(1+\left(\frac{x}{k}\right)^2(\log \Delta)^{-2A}\right)(\log x)^{64}.$$

Hence we get, on (4.12),

$$\sum_{\substack{l=1\\(k,l)=1}}^{k}\left|\tilde{B}_\nu(u\,;l\,;x,z\,;r)-\frac{u}{k}M_\nu(1,\chi_0\,;x,z\,;r)\right|^2 \tag{4.14}$$
$$\ll u\left(1+\frac{x}{k}(\log\Delta)^{-A}\right)(\log x)^{40}.$$

Now, since $B_\nu(u\,;l\,;x,z\,;r)$ is a non-decreasing function of u, we have for any δ, $0<\delta<1$,

$$\frac{1}{\delta}\int_{xe^{-\delta}}^{x}B_\nu(u\,;l\,;x,z\,;r)\frac{du}{u}\leqq B_\nu(x\,;l\,;x,z\,;r)$$
$$\leqq\frac{1}{\delta}\int_{x}^{xe^{\delta}}B_\nu(u\,;l\,;x,z\,;r)\frac{du}{u}.$$

Thus we have from (4.14)

LEMMA 4.2. *Uniformly for any integer $r\geqq1$, for each $\nu=1,2$ and for any δ, $0<\delta<1$, we have on the condition* (4.12)

$$\sum_{\substack{l=1\\(k,l)=1}}^{k}\left|B_\nu(u\,;l\,;x,z\,;r)-\frac{u}{k}M_\nu(1,\chi_0\,;x,z\,;r)\right|^2$$
$$\ll\delta\frac{u^2}{k}M_\nu^2(1,\chi_0\,;x,z\,;r)+\delta^{-1}u\left(1+\frac{x}{k}(\log\Delta)^{-A}\right)(\log x)^{40}.$$

§ 5. Main term.

Lemma 4.2 suggests that the main term of $B_\nu(u\,;l\,;x,z\,;r)$ is $\frac{u}{k}M_\nu(1,\chi_0\,;x,z\,;r)$ and that this quantity bounds $A(x\,;k,l\,;z)$ from below and above according to $r\equiv\nu\pmod 2$ $\nu=1$ and 2, respectively. So it is required to calculate $M_\nu(1,\chi_0\,;x,z\,;r)$, and here the functions F and f, which are introduced in the introduction, play the important role.

We put as in [1]

$$g_\nu(u)=\begin{cases}F(u) & \text{if } \nu\equiv0 \pmod 2,\\ f(u) & \text{if } \nu\equiv1 \pmod 2.\end{cases}$$

Also we remark the well-known fact that

$$F(u)=1+O(e^{-u}),\qquad f(u)=1+O(e^{-u}),\qquad u\geqq1. \tag{5.1}$$

We have

LEMMA 5.1. *Let*

$$2\leqq z_1\leqq z\leqq x^{\frac{1}{2}}.$$

Then for any integer $r \geqq 1$, *and for both* $\nu=1$ *and* 2, *we have*

$$\Gamma_k(z) g_\nu\left(\frac{\log x_0}{\log z}\right) = \sum_{0 \leqq i \leqq r-1} (-1)^i \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \frac{\Gamma_k(z_1)}{p_1 \cdots p_i} g_{\nu+i}\left(\frac{\log x_i}{\log z_1}\right)$$

$$+(-1)^r \sum_{\substack{z_1 \leqq p_r < \cdots < p_1 < z \\ C_1(x)}} \frac{\Gamma_k(p_r)}{p_1 \cdots p_r} g_{\nu+r}\left(\frac{\log x_r}{\log p_r}\right)$$

$$+ \sum_{1 \leqq i \leqq r} (-1)^i \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_2(x)}} \frac{\Gamma_k(p_i)}{p_1 \cdots p_i} g_{\nu+i}\left(\frac{\log x_i}{\log p_i}\right)$$

$$+O\left(\Gamma_k(z) \frac{\log^2 z \log\log 3k}{\log^3 z_1}\right).$$

PROOF. This can be shown as Theorem 4 of [**1**].

Now, from (4.11) we see that, apart from the first double sum, $M_\nu(1, \chi_0; x, z; r)$ is the sum of $H(1, \chi_0; \eta_1, \eta_2)$'s. By (3.2) we have

$$H(1, \chi_0; \eta_1, \eta_2) = Y_k^{-1}(\eta_1, \eta_2).$$

Thus from (3.3), (3.4), (4.11) and (5.1) we get, after some elementary considerations,

$$(-1)^\nu M_\nu(1, \chi_0; x, z; r)$$

$$\leqq - \sum_{\substack{0 \leqq i \leqq r-1 \\ i \equiv \nu+1 \ (\mathrm{mod}\ 2)}} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \frac{\Gamma_k(z_0)}{p_1 \cdots p_i} g_{\nu+i}\left(\frac{\log x_i}{\log z_1}\right)\left\{1+O(e^{-\frac{\log x_i}{\log z_1}})\right\}$$

$$+ \sum_{\substack{0 \leqq i \leqq r-1 \\ i \equiv \nu+1 \ (\mathrm{mod}\ 2)}} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \sum_{\substack{z_0 \leqq p_0 < z_1 \\ p_0 \nmid k}} \frac{\Gamma_k(p_0)}{p_0 p_1 \cdots p_k} g_{\nu+i}\left(\frac{\log x_i}{\log z_1}\right)\left\{1+O(e^{-\frac{\log \frac{x_i}{p_0}}{\log p_0}})\right\}$$

(5.2)

$$+ \sum_{\substack{0 \leqq i \leqq r-1 \\ i \equiv \nu+1 \ (\mathrm{mod}\ 2)}} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \frac{\Gamma_k(z_1)}{p_1 \cdots p_i} g_{\nu+i}\left(\frac{\log x_i}{\log z_1}\right)\left\{1+O(e^{-\frac{\log x_i}{\log z_1}})\right\}$$

$$+ \sum_{\substack{z_1 \leqq p_r < \cdots < p_1 < z \\ C_1(x)}} \frac{\Gamma_k(p_r)}{p_1 \cdots p_r} g_{\nu+r}\left(\frac{\log x_r}{\log p_r}\right)\left\{1+O(e^{-\frac{\log x_r}{\log p_r}})\right\}$$

$$+ \sum_{1 \leqq i \leqq r} (-1)^{\nu+i} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_2(x)}} \frac{\Gamma_k(p_i)}{p_1 \cdots p_i} g_{\nu+i}\left(\frac{\log x_i}{\log p_i}\right)\left\{1+O\left(\frac{\log\log 3k}{\log p_i}\right)\right\},$$

where in the last sum we have used the fact that on $C_2(x)$

$$f\left(\frac{\log x_i}{\log p_i}\right)=0,$$

and also that by Mertens' theorem

$$\frac{k}{\varphi(k)}(\log\sqrt{x_i})^{-1}=\Gamma_k(p_i)F\Big(\frac{\log x_i}{\log p_i}\Big)\Big(1+O\Big(\frac{\log\log 3k}{\log p_i}\Big)\Big).$$

From this we deduce

LEMMA 5.2. *If* $z\leqq x_0$ *and* $r\equiv\nu \pmod 2$, *then we have*

$$(-1)^\nu M_\nu(1, \chi_0; x, z; r)$$

$$\leqq(-1)^\nu\Gamma_k(z)g_\nu\Big(\frac{\log x_0}{\log z}\Big)+O\Big(\Gamma_k(z)\frac{(\log x_0\log\log 3k)^2}{\log^3 z_1}\Big)+E_r(x, z),$$

where

$$E_r(x, z)\ll\sum_{0\leqq i\leqq r-1}\sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\frac{\Gamma_k(z_1)}{p_1\cdots p_i}e^{-\frac{\log x_i}{\log z_1}}$$

$$+\sum_{\substack{z_1\leqq p_r<\cdots<p_1<z\\ C_1(x)}}\frac{\Gamma_k(p_r)}{p_1\cdots p_r}e^{-\frac{\log x_r}{\log p_r}}.$$

PROOF. Noticing that

$$\Gamma_k(z_0)-\sum_{\substack{z_0\leqq p_0<z_1\\ p_0\nmid k}}\frac{1}{p_0}\Gamma_k(p_0)=\Gamma_k(z_1),$$

we see that the first three sums of the right side of (5.2) can be combined into

$$(-1)^\nu\sum_{0\leqq i\leqq r-1}(-1)^i\sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\frac{\Gamma_k(z_1)}{p_1\cdots p_i}g_{\nu+i}\Big(\frac{\log x_i}{\log z_1}\Big)$$

$$+O\Big(\sum_{0\leqq i\leqq r-1}\sum_{\substack{z_1\leqq p_i<\cdots<p_1<z\\ C_1(x)}}\frac{\Gamma_k(z_1)}{p_1\cdots p_i}e^{-\frac{\log x_i}{\log z_1}}\Big).$$

If $r\equiv\nu \pmod 2$, the fourth sum of (5.2) can be written as

$$(-1)^{r+\nu}\sum_{\substack{z_1\leqq p_r<\cdots<p_1<z\\ C_1(x)}}\frac{\Gamma_k(p_r)}{p_1\cdots p_r}g_{\nu+r}\Big(\frac{\log x_r}{\log p_r}\Big)$$

$$+O\Big(\sum_{\substack{z_1\leqq p_r<\cdots<p_1<z\\ C_1(x)}}\frac{\Gamma_k(p_r)}{p_1\cdots p_r}e^{-\frac{\log x_r}{\log p_r}}\Big).$$

So by Lemma 5.1 we get, on the condition of the present lemma,

$$(-1)^\nu M_\nu(1, \chi_0; x, z; r)$$

$$\leqq(-1)^\nu\Gamma_k(z)g_\nu\Big(\frac{\log x_0}{\log z}\Big)+E_r(x, z)$$

$$+O\Big(\sum_{1\leqq i\leqq r}\sum_{z_1\leqq p_i<\cdots<p_1<z}\frac{1}{p_1\cdots p_i}\Big(\frac{\log\log 3k}{\log z_1}\Big)^2\Big).$$

And this O-term is

$$\ll \left(\frac{\log\log 3k}{\log z_1}\right)^2 \sum_{i=0}^{\infty} \frac{1}{i!}\left(\sum_{z_1 \leqq p < z} \frac{1}{p}\right)^i$$

$$\ll \frac{(\log\log 3k)^2}{(\log z_1)^3} \log z\,,$$

which proves the lemma.

§6. Proof of Theorem 1.

Now, in the above considerations we set

$$k \leqq x(\log x)^{-A}\,, \qquad \Delta = x\,,$$

and also we write

(6.1) $$X = \frac{x}{\sqrt{k}}(\log x)^{-A} \qquad (= x_0,\ \text{also})\,.$$

If $\sqrt{X} \leqq z \leqq X$, then, since we have (1.1), the result of the theorem has already been essentially proved in our previous paper [3]. Thus in what follows we may assume that

$$z < \sqrt{X}\ .$$

And as in [1] we set

$$z_0 = O(1)\,,$$

$$z_1 = \exp\left((\log X)^{\frac{7}{10}}\right),$$

(6.2) $$\frac{(\log X)^{\frac{3}{10}}}{3\log\log 3X} \leqq \left(\frac{3}{2}\right)^r \leqq \frac{3(\log X)^{\frac{3}{10}}}{4\log\log 3X}\,.$$

(In particular the condition (4.12) is satisfied). Then we have

$$\log z_1 \leqq \frac{\log x_i}{2\log\log 3x_i}\,, \qquad i = 0, 1, 2, \cdots, r-1$$

(see pp. 231-232 of [1]), from which it follows that, since $x_i > z_1$,

$$\sum_{0 \leqq i \leqq r-1} \sum_{\substack{z_1 \leqq p_i < \cdots < p_1 < z \\ C_1(x)}} \frac{\Gamma_k(z_1)}{p_1 \cdots p_i} e^{-\frac{\log x_i}{\log z_1}}$$

$$\ll \frac{\Gamma_k(z_1)}{(\log z_1)^2} \sum_{i=0}^{\infty} \frac{1}{i!}\left(\sum_{z_1 \leqq p < z} \frac{1}{p}\right)^i$$

$$\ll \frac{\log z}{(\log z_1)^4} \log\log 3k\,.$$

Also we have as in p. 233 of [**1**]

$$\sum_{\substack{z_1\leqq p_r<\cdots<p_1<z\\ C_1(x)}} \frac{\Gamma_k(p_r)}{p_1\cdots p_r} e^{-\frac{\log x_r}{\log p_r}} \leqq \Gamma_k(z)e^{-\frac{\log X}{\log z}}\theta^r,$$

where

$$\theta=\frac{e}{3}(1+O((\log\log 3k)/(\log X)^{\frac{2}{5}})),$$

and so by (6.2)

$$\theta^r=O((\log X)^{-\frac{1}{14}}).$$

Thus, noticing that by (1.1)

$$g_\nu\Big(\frac{\log X}{\log z}\Big)=g_\nu\Bigg(\frac{\log\frac{x}{\sqrt{k}}}{\log z}\Bigg)+O\Big(\frac{\log\log x}{\log z}\Big),$$

we have from Lemma 5.2

LEMMA 6.1. *On the assumption* (6.2)

$$(-1)^\nu M_\nu(1,\chi_0;x,z;r)\leqq(-1)^\nu\Gamma_k(z)\Bigg\{g_\nu\Bigg(\frac{\log\frac{x}{\sqrt{k}}}{\log z}\Bigg)+O((\log x)^{-\frac{1}{14}})\Bigg\},$$

$$\text{if}\quad r\equiv\nu \pmod 2.$$

On the other hand, putting $u=x$, $\delta=(\log x)^{-\frac{A}{2}+20}$ in Lemma 4.2, we get

$$\sum_{\substack{l=1\\(k,l)=1}}^{k}\Big|B_\nu(x;l;x,z;r)-\frac{x}{k}M_\nu(1,\chi_0;x,z;r)\Big|^2\ll\frac{x^2}{k}(\log x)^{-\frac{A}{2}+20}.$$

Hence, applying the familiar assertion of Čebyšev's inequality to the above result, we see that Theorem 1 follows immediately from Lemma 3.1 and Lemma 6.1.

§ 7. Proof of Theorem 2.

Now for $W_\zeta(x;k,l;z,w)$ which has been defined in the introduction we have

$$W_\zeta(x;k,l;z,w)\geqq A(x;k,l;z)-\sum_{\substack{z\leqq q<w\\ q\nmid k}}\sum_{\substack{n\equiv l \pmod k\\ n\equiv 0 \pmod{q^2}\\ n\leqq x}}1$$

(7.1)

$$-\zeta\sum_{\substack{z\leqq q<w\\ q\nmid k}}\Big(1-\frac{\log q}{\log w}\Big)A\Big(\frac{x}{q};k,l\bar{q};z\Big)$$

$$=A(x;k,l;z)-S(x;l;z,w)-\zeta Q(x;l;z,w),\quad\text{say.}$$

First we treat $S(x;l;z,w)$, but we shall be brief. We have

$$S(x;l;z,w)\leqq\int_1^{xe}S(y;l;z,w)\frac{dy}{y}$$

$$=\frac{1}{\varphi(k)}\sum_{\chi\ (\mathrm{mod}\ k)}\bar{\chi}(l)\int_{(\alpha)}L(s,\chi)\Big\{\sum_{z\leqq q<w}\frac{\chi(q^2)}{q^{2s}}\Big\}\frac{(xe)^s}{s^2}ds.$$

From this we get as in the proof of Lemma 4.2

$$\sum_{\substack{l=1\\(k,l)=1}}^{k}\{S(x;l;z,w)\}^2\ll\frac{1}{k}\Big(\frac{x}{z}\Big)^2$$

$$+x\frac{(\log x)^5}{\sqrt{k}}\int_{(\frac{1}{2})}\Big\{\sum_{\chi\ (\mathrm{mod}\ k)}\Big|\sum_{z\leqq q<w}\frac{\chi(q^2)}{q^{2s}}\Big|^4\Big\}^{\frac{1}{2}}\frac{|ds|}{|s|^{3/2}},$$

where we have used (4.13). Let $a(n)=1$ if n is the square of a prime q, $z\leqq q<w$, and $a(n)=0$ otherwise. And let $b(n)=\sum_{n=n_1n_2}a(n_1)a(n_2)$. Then we have

$$\sum_{\chi\ (\mathrm{mod}\ k)}\Big|\sum_{z\leqq q<w}\frac{\chi(q^2)}{q^{2s}}\Big|^4=\sum_{\chi\ (\mathrm{mod}\ k)}\Big|\sum_{z^4\leqq n<w^4}\frac{\chi(n)}{n^s}b(n)\Big|^2,$$

which, by the familiar way of dividing the range of n, is

$$\ll(\log w)^2\max_{z^4\leqq U\leqq w^4}(k+U)\sum_{U\leqq n<2U}\frac{b(n)^2}{n},\qquad \Big(\mathrm{Re}(s)=\frac{1}{2}\Big),$$

$$\ll(\log w)^2\Big\{\frac{k}{z^2}+w^2\Big\}.$$

So we get

$$\sum_{\substack{l=1\\(k,l)=1}}^{k}\{S(x;l;z,w)\}^2\ll x(\log x)^6\Big\{\frac{1}{z}+\frac{w}{\sqrt{k}}\Big\},$$

from which it follows immediately

LEMMA 7.1. *If*

$$k\leqq x(\log x)^{-A},\qquad w\leqq\frac{x}{\sqrt{k}}(\log x)^{-A},$$

then we have

$$S(x;l;z,w)\ll\frac{x}{k}(\log x)^{-2}$$

save for at most $k(\log x)^{10-A}$ *reduced residue classes* $l\bmod k$.

Next, we have to study the average behaviour of $Q(x;l;z,w)$. We first note that

$$Q(x;l;z,w)\leqq\sum_{\substack{z\leqq q<w\\ z<(\frac{x}{q})^{\frac{1}{2}}\\ q\nmid k}}\Big(1-\frac{\log q}{\log w}\Big)A\Big(\frac{x}{q};k,l\bar{q};z\Big)\tag{7.2}$$

$$+\sum_{\substack{z\leqq q<w\\ z\geqq (\frac{X}{q})^{\frac{1}{2}}\\ q\nmid k}}\left(1-\frac{\log q}{\log w}\right)A\left(\frac{x}{q}\,;\,k,\,l\bar{q}\,;\left(\frac{X}{q}\right)^{\frac{1}{2}}\right),$$

where X is defined by (6.1). We set $\Delta=x$ and let $z(q)$ stand for the triple $(z_0(q), z_1(q), z(q))$, where

$$z_0(q):\ \text{a sufficiently large constant},$$

$$(7.3)\qquad z_1(q)=\exp\left((\log X/q)^{\frac{1}{10}}\right),$$

$$z(q)=z\,.$$

Further the even integer $r(q)$ is to satisfy

$$(7.4)\qquad \frac{\left(\log\frac{X}{q}\right)^{\frac{3}{10}}}{3\log\log 3\frac{X}{q}}\leqq\left(\frac{3}{2}\right)^{r(q)}\leqq\frac{3\left(\log\frac{X}{q}\right)^{\frac{3}{10}}}{4\log\log 3\frac{X}{q}}\,.$$

With these conventions we put

$$(7.5)\qquad \begin{aligned} R(u\,;\,l\,;\,x,\,z,\,w)=&\sum_{\substack{z\leqq q<w\\ z<(\frac{X}{q})^{\frac{1}{2}}\\ q\nmid k}}\left(1-\frac{\log q}{\log w}\right)B_2\left(\frac{u}{q}\,;\,l\bar{q}\,;\,\frac{x}{q},\,z(q)\,;\,r(q)\right)\\ &+\sum_{\substack{z\leqq q<w\\ z\geqq(\frac{X}{q})^{\frac{1}{2}}\\ q\nmid k}}\left(1-\frac{\log q}{\log w}\right)\phi_q\left(u\,;\,l\,;\left(\frac{X}{q}\right)^{\frac{1}{2}},\left(\frac{X}{q}\right)^{\frac{1}{2}}\right).\end{aligned}$$

Then we see, from Lemma 3.1 and from (3.6), (7.2)

$$(7.6)\qquad Q(x\,;\,l\,;\,z,\,w)\leqq R(x\,;\,l\,;\,x,\,z,\,w)\,.$$

Now, let

$$\tilde{R}(u\,;\,l\,;\,x,\,z,\,w)=\int_1^u R(y\,;\,l\,;\,x,\,z,\,w)\frac{dy}{y}\,,$$

then we have from (4.6), (4.10) and (7.5)

$$\tilde{R}(u\,;\,l\,;\,x,\,z,\,w)$$

$$=\frac{1}{2\pi i\varphi(k)}\sum_{\chi\,(\mathrm{mod}\,k)}\bar{\chi}(l)\int_{(\alpha)}L(s,\,\chi)N(s,\,\chi\,;\,x,\,z,\,w)\frac{u^s}{s^2}ds\,,$$

where

$$N(s, \chi; x, z, w)$$

$$= \sum_{\substack{z \leqq q < w \\ z < (\frac{X}{q})^{\frac{1}{2}}}} \Big(1 - \frac{\log q}{\log w}\Big) M_2\Big(s, \chi; \frac{x}{q}, z(q); r(q)\Big) \frac{\chi(q)}{q^s}$$

$$+ \sum_{\substack{z \leqq q < w \\ z \geqq (\frac{X}{q})^{\frac{1}{2}}}} \Big(1 - \frac{\log q}{\log w}\Big) H\Big(s, \chi; \Big(\frac{X}{q}\Big)^{\frac{1}{2}}, \Big(\frac{X}{q}\Big)^{\frac{1}{2}}\Big) \frac{\chi(q)}{q^s}.$$

From Lemma 4.1 and from (4.8) we have

$$N(s, \chi; x, z, w) = \sum_{m \leqq X} \frac{\chi(m)}{m^s} \Lambda(m; x, z, w),$$

where

$$\Lambda(m; x, z, w) = \sum_{\substack{m = qv \\ z \leqq q < w \\ z < (\frac{X}{q})^{\frac{1}{2}} \\ q \nmid k}} \Big(1 - \frac{\log q}{\log w}\Big) \Phi^{(2)}_{r(q)}\Big(v; \frac{x}{q}, z(q)\Big)$$

$$+ \sum_{\substack{m = qv \\ z \leqq q < w \\ z \geqq (\frac{X}{q})^{\frac{1}{2}} \\ q \nmid k}} \Big(1 - \frac{\log q}{\log w}\Big) \rho_v\Big(\Big(\frac{X}{q}\Big)^{\frac{1}{2}}, \Big(\frac{X}{q}\Big)^{\frac{1}{2}}\Big),$$

and so

$$\Lambda(m; x, z, w) \ll \tau_5(m),$$

since the condition (4.12) is satisfied because of (7.3). Hence as in the proof of Lemma 4.2 we obtain, for any δ, $0 < \delta < 1$,

$$\sum_{\substack{l=1 \\ (k,l)=1}}^{k} \Big| R(u; l; x, z, w) - \frac{u}{k} N(1, \chi_0; x, z, w) \Big|^2$$

(7.7)

$$\ll \delta \frac{u^2}{k} N^2(1, \chi_0; x, z, w) + \delta^{-1} u \Big(1 + \frac{x}{k} (\log x)^{-A}\Big) (\log x)^{60}.$$

Next we have to calculate $N(1, \chi_0; x, z, w)$. By our present assumptions (7.3), (7.4) and from Lemma 6.1, we get

$$N(1, \chi_0; x, z, w)$$

$$\leqq \sum_{\substack{z \leqq q < w \\ z < (\frac{X}{q})^{\frac{1}{2}} \\ q \nmid k}} \Big(1 - \frac{\log q}{\log w}\Big) \frac{1}{q} \Gamma_k(z) \Big\{ F\Big(\frac{\log \frac{X}{q}}{\log z}\Big) + O\Big(\Big(\log \frac{X}{q}\Big)^{-\frac{1}{14}}\Big)\Big\}$$

$$+ \sum_{\substack{z \leqq q < w \\ z \geqq (\frac{X}{q})^{\frac{1}{2}} \\ q \nmid k}} \Big(1 - \frac{\log q}{\log w}\Big) \frac{1}{q} \Gamma_k(z) \Big\{ 2e^{\gamma} \frac{\log z}{\log \frac{X}{q}} + O\Big(\frac{\log \log 3k}{\log \frac{X}{q}}\Big)\Big\}$$

$$=\Gamma_k(z)\int_u^v F\left(v\left(1-\frac{1}{t}\right)\right)\left(1-\frac{u}{t}\right)\frac{dt}{t}+O(\Gamma_k(z)(\log x)^{-\frac{1}{15}}),$$

where

$$u=\frac{\log\dfrac{x}{\sqrt{k}}}{\log w},\qquad v=\frac{\log\dfrac{x}{\sqrt{k}}}{\log z},\qquad 1<u\leqq v.$$

Thus in particular we have from (7.7)

$$\sum_{\substack{l=1\\(k,l)=1}}^{k}\left|R(x;l;x,z,w)-\frac{x}{k}N(1,\chi_0;x,z,w)\right|^2$$

$$\ll\frac{x^2}{k}(\log x)^{-\frac{A}{2}+30},$$

which means that, since (7.6),

$$Q(x;l;z,w)$$

$$\leqq\frac{x}{k}\Gamma_k(z)\left\{\int_u^v F\left(v\left(1-\frac{1}{t}\right)\right)\left(1-\frac{u}{t}\right)\frac{dt}{t}+O((\log x)^{-\frac{1}{15}})\right\},$$

save for at most $k(\log x)^{-\frac{A}{2}+33}$ reduced residue classes $l \bmod k$. Hence from Theorem 1, Lemma 7.1 and the above result, the assertion of Theorem 2 follows immediately.

CONCLUDING REMARK: If we assume the so-called χ-analogue of the Lindelöf hypothesis

$$(*)\qquad L\left(\frac{1}{2}+it,\chi\right)\ll k^{\varepsilon}(|t|+1)^{D},$$

where D is a positive constant, then the above procedure combined with higher Riesz mean would give results similar to Theorem 1 and Theorem 2 but different in that the factor $x/\sqrt{k}$ is replaced by x. So from (*) it follows that there is a P_2 such that

$$P_2\leqq k^{1+\varepsilon'},\qquad P_2\equiv l\pmod{k}$$

for almost all l, $(k,l)=1$, where $\varepsilon'\to 0$ as $\varepsilon\to 0$.

Added in proof: Professors Halberstam and Richert kindly suggested the present author at Oberwolfachmeeting of Analytic Number Theory (Nov. 1975) that the argument of the present paper might give a result concerning the least square-free integer in an arithmetic progression. So we add the following result:

There is a P_5 such that

$$P_5 \equiv l \pmod{k}, \qquad \mu(P_5) \neq 0, \qquad P_5 \leqq k^{1+\varepsilon},$$
$$\textit{for almost all} \quad l, (k, l) = 1.$$

This can be proved along the reasoning of W. Fluch (Monatsch. f. Math., **72** (1968), 427-430).

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